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The Richberg technique for subsolutions

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Dedicated with great esteem to Karen Uhlenbeck

This note adapts the sophisticated Richberg technique for approximation in pluripotential theory to the F-potential theory associated to a general nonlinear convex subequation $F \subset J^2(X)$ on a manifold X. The main theorem is the following "local to global" result. Suppose u is a continuous strictly F-subharmonic function such that each point $x \in X$ has a fundamental neighborhood system consisting of domains for which a "quasi" form of C^∞ approximation holds. Then for any positive $h \in C(X)$ there exists a strictly F-subharmonic function $w \in C^\infty(X)$ with $u < w < u + h$. Applications include all convex constant coefficient subequations on $\mathbb{R}^n$, various nonlinear subequations on complex and almost complex manifolds, and many more.

1. Introduction

The point of this paper is to extend the classical Richberg technique in pluripotential theory (cf. [19] or [3, Lemma 5.17/Cor. 5.19/Thm. 5.21]) to subsolutions of any convex subequation. Instances of this have appeared in [5], [7], [10], [16], [17], [18], and [20]. Here we present a quite broad result.

The main idea is very general and can be formulated in the context of the potential theory associated with any convex subequation $F \subset J^2(X)$[1] on a manifold X. The space $F(X)$ of **subsolutions** or F-**subharmonic functions** consists of all upper semicontinuous functions $u : X \to \mathbb{R} \cup \{-\infty\}$

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[1] F is a subequation if it satisfies positivity $F + \mathcal{P} \subset F$, negativity $F + \mathcal{N} \subset F$, and a mild topological condition (T) [9, p. 416]. Here $\mathcal{P}_x$ is the set of 2-jets of non-negative functions with critical value xero at x, and $\mathcal{N}_x$ are the jets of non-positive constant functions.

with the property that for any $x \in X$ and any test function[2] φ for u at x, the 2-jet at x of φ lies in F (denoted $J^2_x\varphi \in F$).

A smooth function $v \in F(X)$ is said to be F**-strict** if $J^2_x v \in \mathrm{Int}F$ for all $x \in X$. A general $u \in F(X)$ is said to be F-strict if, given $\varphi \in C^\infty_{\mathrm{cpt}}(X)$, the function $u + \epsilon\varphi$ is F-subharmonic for all $\epsilon > 0$ sufficiently small.

It is useful to further refine this notion as follows. Given a strictly positive $g \in C(X)$, a continuous function $u \in F(X)$ is said to be g**-strict** if u is F^g-subharmonic where F^g is the subequation with fibres

$$F^g_x = \{J \in J^2_x(X) : \mathrm{dist}(J, \sim F_x) \geq g(x)\}.$$

One has that

$$u \text{ is } F\text{-strict} \quad \iff \quad u \text{ is } g\text{-strict for some } g > 0 \text{ on } X. \tag{1.1}$$

See Corollary 7.6 in [9] for the proof of the implication $\Leftarrow$. (We note that the above definition of *F-strict* is implicit in [9].) The proof of the implication $\Rightarrow$ is left to the reader.

The appropriate local approximation hypothesis is the following form of "quasi"-approximation. Let $F^{\mathrm{strict}}_{\mathrm{cont}}(X)$ denote the continuous F-strict functions on X.

Definition 1.1. Suppose $u \in F^{\mathrm{strict}}_{\mathrm{cont}}(X)$.

(a) Given a domain $\Omega \subset\subset X$, we say that **quasi C^∞ approximation holds for u on Ω** if for all compact sets $K \subset \Omega$, there exists $v \in C(\overline{\Omega}) \cap F^{\mathrm{strict}}(\Omega) \cap C^\infty(\Omega)$ such that:

$$\text{(A)}\ \ u < v \text{ on } K \qquad \text{and} \qquad \text{(B)}\ \ u > v \text{ on } \partial\Omega.$$

(b) If for each $h \in C(X)$, with $h > 0$, there exists

$$w \in C^\infty(X) \cap F^{\mathrm{strict}}(X)$$

which satisfies:

$$u < w < u + h$$

on X, then we say that **Richberg approximation holds** for u on X.

Now we can adapt the Richberg technique to subequations to prove the following local-to-global result. (We shall always assume that the subequation F is convex, i.e., the fibre at each point is convex.)

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[2]A C^2-function φ such that $u - \varphi$ has a local max of zero at x

Theorem 1.2. *Suppose $u \in F^{\rm strict}_{\rm cont}(X)$ and that each point $x \in X$ has a fundamental neighborhood system consisting of domains Ω for which quasi C^∞ approximation holds for u. Then Richberg approximation holds for u on X.*

Remark 1.3. If the local C^∞ approximators v_α to u used in the proof can be chosen to be g-strict, for positive $g \in C(X)$, then the function w constructed in the proof can be chosen to be g' strict for any $0 < g' < g$. (For the proof see Lemma 2.2 below.)

We now discuss the applications of Theorem 1.2. The proofs, together with some further results, are given in Section 3.

We begin with a subequation on $\mathbb{R}^n$ which is both constant coefficient and convex.

Theorem 1.4. *Suppose F is a constant coefficient, convex subequation on $\mathbb{R}^n$ and $u \in F^{\rm strict}_{\rm cont}(X)$ for some open subset $X \subset \mathbb{R}^n$. Then Richberg approximation holds for u on X.*

This result extends the pure second-order case provided by Thm. 9.10 in [10].

It applies to all convex subequations

$$F \subset J^2(\mathbb{R}^n) = \mathbb{R}^n \times (\mathbb{R} \oplus \mathbb{R}^n \oplus {\rm Sym}^2(\mathbb{R}^n)).$$

Among these are many highly degenerate subequations which are geometrically interesting – for example, the subequations $F(\mathbb{G})$ defined by a closed subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ of the Grassmannian of p-planes in $\mathbb{R}^n$. These include the subequations coming from calibrations, and those associated to Lagrangian planes (or more generally isotropic planes) in $\mathbb{C}^n$ (see [10], [13] for more details).

For reduced convex subequations F on a manifold X we assume a mild condition on coordinate balls (see Definition 3.12 and Lemma 3.13), and obtain the following two results.

The first is for F a cone, and is based on solving the C^∞ homogeneous Dirichlet problem. In the second result, the cone hypothesis is dropped, but sufficient "monotonicity" is assumed. This second case is again based on solving the C^∞ Dirichlet problem, but this time for the inhomogeneous equation.

Theorem 1.5. (Convex Cone Subequations). *Let F be a reduced*[3] *convex cone subequation on a manifold X. Suppose that F satisfies the coordinate ball condition in Definition 3.12. Assume that on all sufficiently small coordinate balls Ω for a covering family of local coordinates on X one has that the C^∞ homogeneous Dirichlet problem for F is uniquely solvable on Ω (see Def. 3.6). Then for all $u \in F^{\rm strict}_{\rm cont}(X)$ Richberg approximation holds for u on X.*

Theorem 1.6. (Subequations with a Good Monotonicity Cone). *Let F be any reduced subequation with a monotonicity cone M on a manifold X. That is, M is a reduced convex cone subequation such that the fibre-wise sum satisfies*

$$F + M \subset F.$$

Now suppose that M satisfies the coordinate ball condition in Definition 3.12. Assume that on all sufficiently small coordinate balls Ω for a covering family of local coordinates on X one has that the C^∞ inhomogeneous Dirichlet problem for M is uniquely solvable on Ω (see Def. 3.8). Then for all $u \in F^{\rm strict}_{\rm cont}(X)$ Richberg approximation holds for u on X.

A special case of this theorem is when F is a convex cone subequation and one takes $M = F$. See Remark 3.14.

Note that Theorem 1.5 is not a special case of Theorem 1.6 since in 1.6 $\psi \equiv 0$ is excluded.

Example 1.7. (Plurisubharmonics on an Almost Complex Manifold). Let (X, J) be an almost complex n-manifold, and $\mathcal{P}(J)$ the subequation defining the J-plurisubharmonic functions by the condition $J^2_x\varphi \in F_x(J)$ iff $i\partial\overline{\partial}\varphi \geq 0$. Fixing a volume form β on X, there is a natural operator $f(J^2\varphi) = (i\partial\overline{\partial}\varphi)^n/\beta$. It was shown in [15], [2] that the C^∞ inhomogeneous Dirichlet problem is uniquely solvable on small balls in local coordinates. This says that for a smooth function $\psi > 0$, the subequation $\mathbf{F} \equiv \{J : f(J) \geq \psi\}$ satisfies Definition 3.6. Richberg approximation follows from Theorem 1.5. This result was first established in [17].

Example 1.8. (Complex Hessian Equations). Suppose (X, ω) is a Kähler n-manifold, and F is the complex m-Hessian subequation: $J^2_x\varphi \in F_x$ iff $(i\partial\overline{\partial}\varphi)^k \wedge \omega^{n-m} \geq 0$, for all $1 \leq k \leq m$. Fixing a volume form β as above gives an operator $f(J^2\varphi) \equiv (i\partial\overline{\partial}\varphi)^m \wedge \omega^{n-m}/\beta$ for which the C^∞ inhomogeneous Dirichlet Problem is uniquely solvable on small coordinate balls (see

[3]Independent of the value of the function

[18] or [6]). Hence Theorem 1.6 applies to yield Richberg approximation on X.

Example 1.9. (Work of Verbitsky and Greene-Wu). Suppose again that (X, ω) is a Kähler n-manifold, and let F be the subequation: $J^2_x\varphi \in F_x$ iff $i\partial\overline{\partial}\varphi \wedge \omega^k \geq 0$ for fixed k, $0 \leq k \leq n-1$. Equivalently this is the subequation given by demanding that the trace of the complex hessian is nonnegative on every tangent complex $(k+1)$-plane. The Richberg Approximation Theorem was proved in these cases by M. Verbitsky [20]. His argument was based in part on a general local-to-global result due to R. E. Greene and H. Wu [5].

In Theorems 1.5 and 1.6 the subequation is required to be convex (and reduced). An example of a non-convex subequation where nevertheless Richberg approximation holds is discussed in Section 5.

Remark 1.10. (Justifying the Definitions). We point out that in Definitions 3.6. 3.8 the C^∞-regularity for solutions to the Dirichlet problem is only assumed to hold on Ω (given a $C^\infty(\partial\Omega)$ boundary function). Consequently, Theorem 1.5 applies to many more of the examples in the literature than it would if we required C^∞-regularity on $\overline{\Omega}$.

On the other hand, the weakest hypothesis required for adapting the local-to-global technique to subequations (as provided by Theorem 1.2) is the notion of *quasi C^∞ approximation* given in Definition 1.1(a). Theorem 1.5 then follows from Theorem 1.2 by proving that regularity for the Dirichlet problem implies quasi C^∞ approximation.

We introduce another weak notion – that of *approximate C^∞-regularity* (Definition 3.4) – which is implied by the full C^∞-regularity for the homogeneous or inhomogeneous Dirichlet problem (Propositions 3.7, 3.9), but in principle is much easier to establish. This approximate regularity implies our weakest hypothesis of quasi C^∞ approximation (Lemma 3.2).

For the constant coefficient subequations in Theorem 1.4, standard C^∞ approximation holds by the usual convolution techniques (Lemma 3.3). This reduces the proof of Theorem 1.4 to Theorem 1.2 by showing that C^∞ approximation implies quasi C^∞ approximation (Lemma 3.2).

2. The proof of the local to global theorem

The proof of Theorem 1.2 relies on a regularization of the maximum function $M(t) \equiv \max\{t_1, \ldots, t_m\}$ on $\mathbb{R}^m$ by convolution based on an approximate identity which is a product. More specifically, choose $\varphi(s) \in C^\infty_{\rm cpt}(\mathbb{R})$ with $\varphi \geq 0$, supp$\varphi \subset [-1, 1]$, and $\int_{\mathbb{R}} \varphi(s)\, ds = 1$. For each $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$ with

$\epsilon_j > 0$, set $\varphi(y) \equiv \varphi(y_1) \cdots \varphi(y_m)$ and

$$\varphi_\epsilon(t) \equiv \varphi\left(\frac{t_1}{\epsilon_1}\right) \cdots \varphi\left(\frac{t_m}{\epsilon_m}\right) \frac{1}{\epsilon_1 \cdots \epsilon_m}$$

Define

$$M_\epsilon(t) \equiv \int_{\mathbb{R}^m} M(t+y)\varphi_\epsilon(y)\, dy = \int_{\mathbb{R}^m} M(t+\epsilon y)\varphi(y)\, dy, \tag{2.1}$$

using an "abuse of notation" $\epsilon y = (\epsilon_1 y_1, \ldots, \epsilon_m y_m)$. Assume also that φ is an even function, so that

$$\int_{\mathbb{R}^m} y_j\, \varphi_\epsilon(y) \frac{dy_1}{\epsilon_1} \cdots \frac{dy_m}{\epsilon_m} = 0. \tag{2.2}$$

This ensures that the convolution of φ_ϵ with each t_j equals zero.

Properties 2.1.

(1) *$M_\epsilon(t)$ is increasing in all the variables, smooth and convex on $\mathbb{R}^m$, and invariant under permutations of the variables.*

(2) *$M_\epsilon(t + se) = M_\epsilon(t) + s$ where $e = (1, \ldots, 1)$, and hence*

$$\sum_{j=1}^m \frac{\partial M_\epsilon}{\partial t_j} \equiv 1.$$

(3) *$M(t) \le M_\epsilon(t) \le M(t + \epsilon)$.*

(4) *If $t_j + \epsilon_j \le \max_{i \ne j}\{t_i - \epsilon_i\}$, then*

$$M_\epsilon(t) = M_{(\epsilon_1, \ldots, \widehat{\epsilon_j}, \ldots, \epsilon_m)}(t_1, \ldots, \widehat{t_j}, \ldots, t_m).$$

The proof of these properties will be discussed in Section 4, along with a proof of the next result.

Lemma 2.2. *Suppose that F is a convex subequation (not necessarily reduced nor a cone) on a manifold X. Suppose $g \in C(X)$, $g > 0$. If $u_1, \ldots, u_m \in F^g(Y) \cap C^\infty(Y)$ for $Y \subset X$, then we have that*

$$M_\epsilon(u_1, \ldots, u_m) \in F^{g - M(\epsilon)}(Y) \cap C^\infty(Y).$$

Proof of Theorem 1.2. Pick a locally finite open cover $\{\Omega'_\alpha\}$ of X consisting of precompact domains in X with the property that quasi C^∞ approximation holds for u on each Ω'_α. By a standard result in topology we can choose a subordinate open covering $\{V_\alpha\}$ of X with $K_\alpha \equiv \overline{V_\alpha} \subset \Omega'_\alpha$ for all α. By the quasi C^∞-approximation property for u on Ω'_α, there exist functions $v_\alpha \in C(\overline{\Omega}'_\alpha) \cap C^\infty(\Omega'_\alpha) \cap F^{\mathrm{strict}}(\Omega'_\alpha)$ such that

$$(A) \;\; u < v_\alpha \quad \text{on} \;\; K_\alpha \qquad \text{and} \qquad (B) \;\; u > v_\alpha \quad \text{on} \;\; \partial\Omega'_\alpha.$$

Now choose open sets $\Omega_\alpha \supset K_\alpha$ with $\overline{\Omega}_\alpha \subset \Omega'_\alpha$. Then

$$v_\alpha \in C^\infty(\mathrm{nb}\,\overline{\Omega}_\alpha) \cap F^{\mathrm{strict}}(\mathrm{nb}\,\overline{\Omega}_\alpha)$$

(where $\mathrm{nb}\,\overline{\Omega}_\alpha$ denotes a neighborhood of $\overline{\Omega}_\alpha$). Furthermore, by choosing each Ω_α sufficiently large, we can arrange that

$$(A) \;\; u < v_\alpha \quad \text{on} \;\; K_\alpha \qquad \text{and} \qquad (B) \;\; u > v_\alpha \quad \text{on} \;\; \partial\Omega_\alpha.$$

In addition, by choosing the Ω'_α sufficiently small we can assume that

$$\text{(C)} \;\; \sup_{\overline{\Omega}_\alpha} u < \inf_{\overline{\Omega}_\alpha}(u+h).$$

Next we choose ϵ_α so that

$$(\mathrm{A}') \;\; u < v_\alpha - \epsilon_\alpha \text{ on } K_\alpha \qquad \text{and} \qquad (\mathrm{B}') \;\; u > v_\alpha + \epsilon_\alpha \text{ on } \partial\Omega_\alpha.$$

We can define the global function w on X (using the fact that M_ϵ is invariant under permutations) by

$$w(x) \equiv M_\epsilon\{v_\alpha(x) : x \in \Omega_\alpha\} \tag{2.3}$$

It remains to prove that w satisfies all the properties in Theorem 1.2. Note that the smoothness (even the continuity) of w is completely unclear from this definition.

We now fix a smooth function g' with $0 < g' < g$, and we reduce the ϵ_α so that $0 < \epsilon_\alpha \le \sup_{\Omega_\alpha}(g - g')$. We shall first give a proof that:

$$w \in C^\infty(X) \cap F^{g'}(X).$$

Given $x \in X$ there exists a neighborhood of x where

$$M_\epsilon\{v_\alpha(y) : y \in \Omega_\alpha\} = M_\epsilon\{v_\alpha(y) : y \in \Omega_\alpha \text{ and } x \notin \partial\Omega_\alpha\} \tag{2.4}$$

and hence

$$w(y) = M_\epsilon\{v_\alpha(y) : x \in \Omega_\alpha\} \text{ in a neighborhood } Y \text{ of } x. \tag{2.5}$$

To prove (2.4) note that if $x \in \partial\Omega_\alpha$, then since $\{K_\beta\}$ covers X, there exists β with $x \in K_\beta \subset \Omega_\beta$, which by (B) and (A) implies that

$$v_\alpha(x) < u(x) < v_\beta(x). \tag{2.6}$$

Thus $v_\alpha < v_\beta$ is a neighborhood of x. Now by (2.5) we see that w is smooth on the neighborhood Y. Furthermore, Lemma 2.2 applies to prove the g'-strictness of w discussed in Remark 1.3.

Next we prove that $u < w$. Since each point x is contained in $K_\alpha \subset \Omega_\alpha$ for some α, condition (A) implies that $u(x) < v_\alpha(x) \le M\{v_\beta(x) : x \in \Omega_\beta\}$, which by the first part of Property (3) is $\le M_\epsilon\{v_\beta(x) : x \in \Omega_\beta\} \equiv w(x)$.

Finally we prove that $w < u + h$. By (B$'$) and the hypothesis (C) we have

$$\sup_{\partial\Omega_\beta}(v_\beta + \epsilon_\beta) \le \sup_{\partial\Omega_\beta} u \le \inf_{\overline{\Omega}_\beta}(u + h). \tag{2.7}$$

Now

$$w(x) \equiv M_\epsilon\{v_\beta(x) : x \in \Omega_\beta\} \le M\{v_\beta(x) + \epsilon_\beta : x \in \Omega_\beta\}$$

by the second part of Property (3). Since each $v_\beta + \epsilon_\beta$ satisfies the Maximum Principle on Ω_β, we have

$$w(x) \le M\{\sup_{\partial\Omega_\beta}(v_\beta + \epsilon_\beta) : x \in \Omega_\beta\}.$$

Finally, by (2.7) we have $w(x) \le u(x) + h(x)$. □

3. Verifying the local hypothesis

In this section we give the proofs of Theorems 1.4, 1.5 and 1.6 by verifying the local hypothesis of quasi C^∞-approximation so that the local-to-global Theorem 1.2 can be applied. Two distinct methods are used. The first will apply to constant coefficient equations in $\mathbb{R}^n$, yielding Theorem 1.4. The second will apply to a number of important nonlinear equations on manifolds where the Dirichlet problem is sufficiently well understood, and will yield Theorems 1.5 and 1.6.

METHOD 1. Not surprisingly our first method is based on standard convolution. It also employs the following notion.

Definition 3.1 Given $u \in F^{\text{strict}}_{\text{cont}}(X)$ and a domain $\Omega \subset\subset X$, we say that **C^∞ approximation holds for u on Ω** if there exists a constant $c > 0$ and a sequence $\{v_j\}$ of functions v_j which are C^∞ and F^c-strict on a neighborhood of $\overline{\Omega}$ and converge uniformly to u on $\overline{\Omega}$.

Quite naturally, C^∞ approximation implies quasi C^∞ approximation , but there is something to check.

Lemma 3.2 *Suppose C^∞ approximation holds for $u \in F^{\text{strict}}_{\text{cont}}(X)$ on a domain Ω. Then quasi C^∞ approximation holds for u on Ω.*

Proof. Suppose K is a compact subset of Ω. Choose a defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$ (neither the F-subharmonicity of ρ nor the smoothness of $\partial\Omega$ are required in this lemma). Then choose $s > 0$ such that $\rho + s < 0$ on K. Consider the function

$$\overline{v}_j \equiv v_j - \delta(\rho + s) \quad \text{on a neighborhood of } \overline{\Omega}. \tag{3.1}$$

Since each v_j is c-strict, given $0 < c' < c$, there exists $\delta_0 > 0$ such that for $\delta \le \delta_0$

$$\overline{v}_j \ \text{ is } c'\text{-strict on } \overline{\Omega} \text{ for all } j. \tag{3.2}$$

Now $\overline{v}_j - u = -\delta(\rho + s) + v_j - u$. Since $-\delta(\rho + s) > 0$ on K and $-\delta(\rho + s) < 0$ on $\partial\Omega$, by choosing $v_j - u$ small enough on $\overline{\Omega}$ both (A) and (B) hold for $\overline{v}_j$. ☐

This result can be used in the following case.

Convex constant coefficient subequations

Lemma3.3. *Suppose F is a convex constant coefficient subequation on $\mathbb{R}^n$. Given $u \in F^{\text{strict}}_{\text{cont}}(X)$ and a domain $\Omega \subset\subset X$, C^∞ approximation holds for u on Ω.*

Standard convolution can be used, but there are things to prove, and a basic result of [12] is necessary.

Proof. There exists a constant $c > 0$ such that u is c-strict on a neighborhood of $\overline{\Omega}$ (see Section 7 in [9]), i.e., u is F^c-subharmonic on a neighborhood of $\overline{\Omega}$. For any convex set in euclidean space the set of points of distance $\ge c$ to the complement is also convex since minus the log of the distance is convex.

Thus $F^c \subset F \subset \mathbf{J}^2$ is convex. Also note that $F^c = \overline{\mathrm{Int} F^c}$ The condition (P) holds for F^c by Lemma 7.3 in [9]. In summary, F^c is a convex constant coefficient subequation. Consequently, because of the results proved in [12], there is an equivalent approach to subsolutions using distribution theory. Now using the distributional approach, one shows that C^∞ approximation holds on a neighborhood of $\overline{\Omega}$ via standard convolution. □

Proof of Theorem 1.4. Combining Lemmas 3.3 and 3.2 proves that quasi C^∞ approximation holds, so that Theorem 1.2 applies. □

METHOD 2. Our second method is based on assuming certain regularity hypotheses concerning the Dirichlet Problem. Since the discussion is local we can assume that the subequation F is defined on an open set $X \subset \mathbb{R}^n$. We fix a domain $\Omega \subset\subset X$ with smooth boundary $\partial\Omega$ and let $\rho \in C^\infty(\overline{\Omega})$ denote a defining function for $\partial\Omega$. We assume that existence and comparison hold for the Dirichlet Problem (DP) on Ω, and then consider the following form of "regularity".

Definition 3.4. We say that **approximate C^∞-regularity for the (DP) holds on** Ω if, for each boundary function $\varphi \in C^\infty(\partial\Omega)$, the solution H to the (DP) can be uniformly approximated on $\overline{\Omega}$ by a sequence of functions $\{v_k\} \subset C(\overline{\Omega}) \cap F^{\mathrm{strict}}(\Omega) \cap C^\infty(\Omega)$.

Note that v_k is not required to be C^∞ on $\overline{\Omega}$. This kind of regularity is enough for our purposes. Our next result says that the quasi C^∞-approximation hypothesis in Theorem 1.2 can be replaced by this approximate C^∞-regularity hypothesis as long as the subequation is reduced (Thm. 1.2′ below).

Lemma 3.5. *Suppose that F is a reduced subequation on X. If approximate C^∞-regularity for the (DP) holds on Ω, then quasi C^∞-approximation holds for all $u \in F^{\mathrm{strict}}_{\mathrm{cont}}(X)$ on Ω.*

Proof. Suppose a compact subset $K \subset \Omega$ is given. Since u is F-strict on X, for $\epsilon > 0$ sufficiently small

$$u - \epsilon\rho \quad \text{is } F\text{-subharmonic on a neighborhood of } \overline{\Omega}. \tag{3.3}$$

Now choose a constant s with $0 < s < \inf_K(-\rho)$ so that

$$\rho + s < 0 \quad \text{on } K. \tag{3.4}$$

Finally, choose $\varphi \in C^\infty(\partial\Omega)$ with

$$u - \epsilon s < \varphi < u \quad \text{on } \partial\Omega. \tag{3.5}$$

Let H be the solution to the (DP) on Ω with boundary values φ. First we want to prove that

$$\text{(A)} \quad u < H \quad \text{on } K \qquad \text{and} \qquad \text{(B)} \quad H < u \quad \text{on } \partial\Omega$$

Assertion (B) follows since $\varphi < u$ on $\partial\Omega$ by (3.5). Now set $w \equiv u - \epsilon(\rho + s)$ and note that $u < w$ on K by (3.4). To finish the proof of (A) we show that $w \leq H$ on $\overline{\Omega}$ (and therefore on K) by showing that w is in the Perron family $\mathcal{F}(\varphi)$. Note that on $\partial\Omega$ we have $w = u - \epsilon s$, which is $< \varphi$ by (3.5). By (3.3) and the hypothesis that F is reduced, w is F-subharmonic on Ω. Hence $w \in \mathcal{F}(\varphi)$ as claimed.

Now H is neither strict nor C^∞, however by the hypothesis there exists a sequence $\{v_k\} \subset C(\overline{\Omega}) \cap F^{\text{strict}}(\Omega) \cap C^\infty(\Omega)$ converging to H uniformly on $\overline{\Omega}$. Hence (A) and (B) hold with H replaced by v_k for large k, proving that quasi C^∞-approximation holds for u on Ω. $\square$

Theorem 1.2′. *Suppose F is a reduced convex cone subequation on a manifold X. Suppose $u \in F^{\text{strict}}_{\text{cont}}(X)$ and that each point $x \in X$ has a fundamental neighborhood system consisting of domains for which approximate C^∞ regularity holds for the (DP). Then Richberg approximation holds for u on X.*

We now discuss two cases where approximate C^∞-regularity holds so that Theorem 1.2′ can be applied.

C^∞ regularity for the homogeneous F-Dirichlet problem

Definition 3.6. We say that the C^∞ **homogeneous F-Dirichlet problem (DP) is uniquely solvable on** Ω if for each $\varphi \in C^\infty(\partial\Omega)$, there exists a unique $h \in C(\overline{\Omega})$ such that

(a) h is F-harmonic on Ω,

(b) $h\big|_{\partial\Omega} = \varphi$,

(c) $h \in C^\infty(\Omega)$ (but not necessarily in $C^\infty(\overline{\Omega})$).

In the next result we assume that F is a convex cone subequation.

Proposition 3.7. *Assume that there exists a defining function ρ for $\partial\Omega$ which is strictly F-subharmonic on a neighborhood of $\overline{\Omega}$. If the C^∞ homogeneous Dirichlet problem is uniquely solvable on Ω, then approximate C^∞-regularity for the (DP) holds on Ω.*

Proof. Suppose h is the solution of the (DP) on Ω with boundary values $\varphi \in C^\infty(\partial\Omega)$. Then the sequence $v_k \equiv h + \frac{1}{k}\rho$ uniformly approximates h on $\overline{\Omega}$ and $v_k \in C(\overline{\Omega}) \cap F^{\text{strict}}(\Omega) \cap C^\infty(\Omega)$ for each k. □

C^∞ Regularity for the inhomogeneous F-Dirichlet problem given a monotonicity cone

We shall now widen the interesting case above to a broader family of subequations by dropping the assumption that F is a cone. However, we now assume that the subequation F admits a **monotonicity cone** M, that is, M is a convex cone subequation on X for which the fibre-wise sum satisfies

$$F + M \subset F.$$

In addition to the subequations F and M we require weakly elliptic operators [4] $f \in C(F) \cap C^\infty(\mathrm{Int}F)$ and $g \in C^\infty(M)$ which are compatible with the subequations, by which we mean that

$$f \geq 0 \text{ on } F \text{ and } \partial F = \{f = 0\} \quad \text{and} \quad g \geq 0 \text{ on } M \text{ and } \partial M = \{g = 0\}.$$

Finally, we assume that

(1) There is a constant $\delta > 0$ such that if $J \in F$ and $J' \in M$, then

$$f(J + J') \geq \delta g(J').$$

Next we show that interior C^∞ regularity for the inhomogeneous F-Dirichlet Problem implies that approximate C^∞-regularity for the F-(DP) holds on Ω.

Definition 3.8. Assume that there exists a defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$, which is strictly M-subharmonic on Ω. We say that the C^∞ **inhomogeneous Dirichlet Problem (IHDP) is uniquely solvable on** Ω if for all $\varphi \in C^\infty(\partial\Omega)$ and $\psi \in C^\infty(\overline{\Omega}), \psi > 0$, there exists a unique $v \in C(\overline{\Omega})$ satisfying

(a) $f(J^2 v) = \psi \quad$ on Ω,

(b) $v\big|_{\partial\Omega} = \varphi$,

(c) $v \in C^\infty(\Omega)$ (not necessarily in $C^\infty(\overline{\Omega})$), and

[4] On a manifold X weak ellipticity means $f(J + J') \geq f(J)$ for $J \in F$ and $J' \in \mathcal{P}$ (see footnote 1).

(d) v is the Perron function, i.e.,

$$v(x) = \sup\{u(x) : u \in \mathrm{USC}(\overline{\Omega}) \cap F(\Omega),\ u\big|_{\partial\Omega} \le \varphi, \text{ and } f(J^2_{\mathrm{red}} u) \ge \psi \text{ on } \Omega\}$$

Proposition 3.9. *Assume that there exists a defining function ρ for $\partial\Omega$ which is strictly M-subharmonic on a neighborhood of $\overline{\Omega}$. If the C^∞ (IHDP) is uniquely solvable on Ω, then approximate C^∞-regularity for the (DP) holds on Ω.*

Proof. Suppose H is the solution to the homogeneous F-(DP) with boundary values $\varphi \in C^\infty(\partial\Omega)$. Let v_ϵ be the solution to the (IHDP) $f(J^2 v_\epsilon) = g(J^2(\epsilon\rho))$ with boundary values φ. Since $g(J^2(\epsilon\rho)) > 0$ on Ω (by the strict M-subharmonicity of ρ), we see that v_ϵ is F-subharmonic. Since H is the Perron function for the (DP) and $v_\epsilon = \varphi$ on $\partial\Omega$, we have

$$v_\epsilon \le H.$$

On the other hand, since f satisfies (1),

$$f\left(J^2(H + \epsilon\rho)\right) \ge \delta g(J^2(\epsilon\rho)).$$

Therefore $H + \epsilon\rho$ is in the Perron family for the (IHDP) and equals φ on $\partial\Omega$, so that

$$H + \epsilon\rho \le v_\epsilon.$$

Thus, $0 \le H - v_\epsilon \le -\epsilon\rho$ and so $v_\epsilon \to H$ uniformly on $\overline{\Omega}$. By our hypothesis, each v_ϵ is C^∞ on Ω. $\square$

Note 3.10. The solution H to the homogeneous Dirichlet problem in the proof of Proposition 3.9, can simply be a continuous viscosity solution.

Example 3.11. Suppose (X, ω) is a Kähler manifold. Let F be defined by the condition $\{u : i\partial\overline{\partial}u + \omega \ge 0\}$, and define f to be the complex determinant of $i\partial\overline{\partial}u + \omega$. Let M be given by $\{u : i\partial\overline{\partial}u \ge 0\}$, and set $g =$ the complex determinant of $i\partial\overline{\partial}u$.

To complete the proofs of Theorems 1.5 and 1.6 we need a final lemma. The hypothesis of approximate C^∞ regularity in Theorem 1.2$'$ only needs to hold on a family of small domains about each point. For that reason we shall make the following definition below.

We first recall from [9] (or from [11]) that every subequation $F \subset J^2X$ on a manifold X satisfies positivity and negativity conditions. In local coordinates $x = (x_1, \ldots, x_n)$ where we have a canonical trivialization $J_x\mathbb{R}^n =$

$\mathbb{R} \oplus \mathbb{R}^n \oplus \mathrm{Sym}^2(\mathbb{R}^n)$, this means that F and $\mathrm{Int}F$ are preserved under translations by $(r, 0, P)$ with $r \le 0$ and $P \ge 0$.

Definition 3.12. (The Coordinate Ball Condition). We say that a reduced convex cone subequation F satisfies **the coordinate ball condition** if each point $x_0 \in X$ has a local coordinate neighborhood U and $c > 0$ such that the reduced jet

$$(0, cI)_{x_0} \in \mathrm{Int}\left\{F\big|_U\right\}.$$

Note that if this holds for one coordinate neighborhood about a point x_0, then it will hold for all coordinate neighborhood about x_0. To see this, note that after a change of coordinates fixing x_0 the reduced jet $(0, cI)_{x_0}$ becomes $(0, P)$ for $P > 0$. Now there exists a $P_0 \ge 0$ such that $P + P_0 = c'I$ for $c' \ge c$. Since $P \in \mathrm{Int}F$, we have $P + P_0 \in \mathrm{Int}F$.

Lemma 3.13. *Suppose that F is a reduced convex cone subequation on a manifold X. Let U be a local coordinate system about a point $x_0 \in X$. If F satisfies the coordinate ball condition, then there exists a $c_0 > 0$ such that for each $c \ge c_0$, the C^∞ defining function*

$$\rho(x) \equiv \tfrac{c}{2}\left(|x - x_0|^2 - \delta^2\right)$$

for the boundary of the coordinate ball $B_\delta(x_0) \equiv \{x : |x - x_0| < \delta\}$, is strictly F-subharmonic on $B_\delta(x_0)$ for all $\delta > 0$ sufficiently small.

Proof. By the note after Definition 3.12 our condition holds in any given local coordinates. The reduced 2-jet of ρ at x equals $J_x^{2,\mathrm{red}}\rho = c(x - x_0, I)$. At $x = x_0$ this jet belongs to $(\mathrm{Int}F)_{x_0}$ by the hypothesis. Hence, it belongs to $\mathrm{Int}F$ for x near x_0, proving that ρ is F-strict on $B_\delta(x_0)$ for $\delta > 0$ small. $\square$

The upshot of this observation is that in applying any of the above methods for verifying the local hypothesis, we need only consider domains Ω which are very small balls in some local coordinate system.

Proof of Theorem 1.5. In this theorem we are assuming that there is a covering family of local coordinates on X such that for all sufficiently small balls Ω in these coordinates, the C^∞ homogeneous F-Dirichlet problem is uniquely solvable on Ω, that is, Definition 3.6 holds. We are also supposing that F satisfies the coordinate ball condition in Definition 3.12. By the note following Lemma 3.13, we know that Lemma 3.13 applies to all sufficiently small balls in this covering family of coordinates. That is, each such ball has a smooth strictly F-subharmonic defining function ρ.

Now we are able to apply Proposition 3.7 to conclude that approximate C^∞-regularity for the F-(DP) holds on all sufficiently small balls Ω about each point in our family of coordinates. By Theorem 1.2′ (located after proof of Lemma 3.5), we conclude that for all $u \in F^{\text{strict}}_{\text{cont}}(X)$, Richberg approximation holds for u on X. □

Proof of Theorem 1.6. The argument here is parallel to the one above. In this theorem we are assuming that there is a covering family of local coordinates on X such that for all sufficiently small balls Ω in these coordinates, the C^∞ inhomogeneous F-Dirichlet problem (IHDP) is uniquely solvable on Ω, that is, Definition 3.8 holds. We are also supposing that the monotonicity subequation M satisfies the coordinate ball condition in Definition 3.12. By the note following Lemma 3.13, we know that Lemma 3.13 applies to all sufficiently small balls in this covering family of coordinates. That is, each such ball has a smooth strictly M-subharmonic defining function ρ.

Now we are able to apply Proposition 3.9 to conclude that approximate C^∞-regularity for the F-(DP) holds on all sufficiently small balls Ω in our family of coordinates. We then apply Theorem 1.2′ to complete the proof as above. □

Remark 3.14. (A Special Case of Theorem 1.6) Consider the case where F is a reduced convex cone subequation with compatible operator $f \in C(F)$. In this case we can take $M = F$ and the hypothesis (1) before Definition 3.8 and be replaced by the super additive condition:

$$(1') \qquad \text{For all } J, J' \in F, \text{ one has that } f(J+J') \geq f(J) + f(J')$$

because this implies (1) with $\delta = 1$ and $g = f$ (since $f(J) \geq 0\ \forall J \in F$). Note also that (1) $\Rightarrow$ (1)′ with $\delta = 2$ and $g = f$. This is because $f(J) + f(J') \geq 2f(J)$ and $f(J) + f(J') \geq 2f(J')$, when added up, give (1′).

4. Smoothing the maximum function

In this section we discuss the proof of Properties 2.1 and Lemma 2.2. The maximum function $M(t) \equiv \max\{t_1, \ldots, t_m\}$ has the obvious properties that M is convex, invariant under permutations and, with $e = (1, \ldots, 1)$,

$$M(t + se) = M(t) + s. \tag{4.1}$$

These same properties carry over to $M_\epsilon(t)$. This gives Property (1). For Property (2) note that convolution preserves (4.1) since $s * \varphi = s$ by (2.2).

Taking $\frac{d}{ds}\big|_{s=1}$ of both sides of the equation $M_\epsilon(t+se) = M_\epsilon(t)+s$ yields $\sum \frac{\partial M_\epsilon}{\partial t_j} = 1$. Properties (3) and (4) are equally straightforward. This proves Properties 2.1.

Proof of Lemma 2.2. The following is a straightforward calculation.

Lemma 4.1. *With $u_1,\ldots,u_m \in C^\infty(X)$ arbitrary, and $w \equiv M_\epsilon(u_1,\ldots,u_m)$, the 2-jet of w equals*

$$J^2 w = \sum_{j=1}^m \frac{\partial M_\epsilon}{\partial t_j} J^2 u_j + (E_\epsilon, 0, P) \tag{4.2}$$

where

$$E_\epsilon = w - \sum_{j=1}^m \frac{\partial M_\epsilon}{\partial t_j} u_j \qquad \text{and} \qquad P \equiv \sum_{i,j=1}^m \frac{\partial^2 M_\epsilon}{\partial t_i \partial t_j} Du_i \circ Du_j. \tag{4.3}$$

Since M_ϵ is convex, $P \geq 0$. Now the fibres of F and of $F_g^{\rm strict}$ at a point $x \in X$ are convex. Hence, $J_x^2 u_j \in F_x^{g(x)}$ for all $j = 1,\ldots,m$ implies that the convex combination $\sum_j \frac{\partial M_\epsilon}{\partial t_j} J^2 u_j \in F_x^{g(x)}$. By positivity, $\sum_j \frac{\partial M_\epsilon}{\partial t_j} J^2 u_j + (0,0,P) \in F_x^{g(x)}$. The error $E_\epsilon \equiv M_\epsilon(t) - \sum_j t_j \frac{\partial M_\epsilon}{\partial t_j}$ satisfies

$$-M(\epsilon) \leq E_\epsilon(t) \leq M(\epsilon). \tag{4.4}$$

This proves that $J_x^2 w \in F_x^{g(x)-M(\epsilon)}$.

The estimate (4.4) is verified as follows. First compute that

$$\begin{aligned}
E_\epsilon(t) &= \frac{d}{dr}\bigg|_{r=1} \left(rM_\epsilon(t) - M_\epsilon(rt)\right) \\
&= \frac{d}{dr}\bigg|_{r=1} \int_{\mathbb{R}^m} \Big(rM(t+\epsilon y) - M(rt+ey)\Big)\, \varphi(y)\, dy \\
&= \frac{d}{dr}\bigg|_{r=1} \int_{\mathbb{R}^m} \Big(M(rt+r\epsilon y) - M(rt+ey)\Big)\, \varphi(y)\, dy \\
&= \int_{\mathbb{R}^m} \sum_{j=1}^m \epsilon_j y_j \frac{\partial M}{\partial t_j}(t+\epsilon y)\, \varphi(y)\, dy.
\end{aligned}$$

Since $y_j \in \operatorname{supp}(\varphi) \Rightarrow |y_j| \leq 1$, we have $-M(\epsilon) \leq \epsilon_j y_j \leq M(\epsilon)$. Therefore, the convex combination $\sum \epsilon_j y_j \frac{\partial M}{\partial t_j}$ in the integral lies between $-M(\epsilon)$ and $M(\epsilon)$. $\square$

5. An example — non-convex subequations

The problem of smoothing F-subharmonic functions is also interesting for subequations which are not necessarily convex. Diederich and Fornæss [4] show that it is impossible, for general F, to regularize the maximum of a finite number of smooth F-subharmonic functions. On the other hand, they prove that it is possible for the case of smooth n-convex functions, that is, for the subequation $F_n = \widetilde{F}_1 = \widetilde{PSH}$, which is not convex.[5] The following Proposition shows that convexity of a subequation is not a necessary condition for Richberg approximation.

Proposition 5.1. *Let X be a complex manifold. For any continuous strictly n-convex function u on X, and any $h \in C(X)$, $h > 0$, there exists*

$$w \in C^\infty(X) \cap F_n^{\text{strict}}(X)$$

which satisfies:

$$u < w < u + h$$

on X.

Proof. We show first that (locally) approximate C^∞ regularity for the (homogeneous) (DP) holds, that is, any F_n-harmonic function is the limit of a sequence of smooth strictly n-convex functions. Let $\Omega \subset \mathbb{C}^n$ be strictly pseudoconvex and let $H \in C(\Omega)$ be F_n-harmonic. Then $-H$ is a plurisubharmonic function and by [1] there is the sequence $w_k \subset \Omega$ of smooth plurisubharmonic functions which satisfies the following conditions:

(i) w_k converge uniformly to $-H$, and

(ii) $\det\left(\frac{\partial^2 w_k}{\partial z_p \partial \bar{z}_q}\right)_{p,q=1}^n = \left(\frac{1}{2k}\right)^n$.

In particular there is an eigenvalue of $\left(\frac{\partial^2 w_k}{\partial z_p \partial \bar{z}_q}\right)_{p,q=1}^n$ which is smaller than $\frac{1}{k}$ and therefore the function $w_k - \frac{1}{k}|z|^2$ is nowhere plurisubharmonic. Thus the sequence $v_k = \frac{1}{k}|z|^2 - w_k$ is a sequence of smooth strictly n-convex functions which converge uniformly to H.

By Lemma 3.5, approximate C^∞ regularity for the (DP) implies quasi C^∞ approximation. (See [8] for the step in the proof – Lemma 3.1 – where solving the (DP) is required.)

[5] Here n is the complex dimension of the manifold and the subequation F_n is defined by requiring that at least one eigenvalue of the complex hessian is ≥ 0.

Finally, we can choose Ω_α, K_α and v_α as in the proof of Theorem 1.2. The function

$$\tilde{w}(z) = \sup\{v_\alpha(z) : z \in \Omega_\alpha\} > u$$

is, in the terminology of [4], n-convex with corners, and by the Diederich-Fornæss approximation result (Theorem 1 in [4]) it can be approximated by w satisfying the statement in Proposition 5.1. □

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