Communications in
Analysis and Geometry
Volume 28, Number 8, 1773–1786, 2020

On the Hodge conjecture for hypersurfaces in toric varieties

Ugo Bruzzo and Antonella Grassi

To Karen Uhlenbeck, for many more things anyone could possibly say. Her many contributions to mathematics, physics and life would not fit in the margins of a page, would not fit in several volumes.

We show that for very general hypersurfaces in odd-dimensional simplicial projective toric varieties satisfying an effective combinatorial property the Hodge conjecture holds. This gives a connection between the Oda conjecture and Hodge conjecture. We also give an explicit criterion which depends on the degree for very general hypersurfaces for the combinatorial condition to be verified.

1. Introduction

The classical Noether-Lefschetz theorem states that if X is a very general surface in the linear system $|\mathcal{O}_{\mathbb{P}^3}(n)|$, with $n \geq 4$, then the Picard number of X is 1. One of the proofs uses infinitesimal variation of Hodge structure, see for example [6]. A higher dimensional generalization of the infinitesimal variation of Hodge structure argument implies that, under certain hypotheses, a very general, very ample hypersurface X in a smooth projective variety Y of dimension d satisfies the Hodge conjecture if Y does. In particular, the cohomology of X in degree $(p-1, d-p)$, with $1 \leq p \leq d-1$ is the restriction of the cohomology of Y in the same degree (equivalently, the primitive cohomology $PH^{p-1,d-p}(X)$ vanishes) if certain conditions are satisfied. A sufficient condition is that the natural morphism

$$T_X\mathcal{M}_L \otimes PH^{p,d-1-p}(X) \to PH^{p-1,d-p}(X) \tag{1}$$

is surjective, where $\mathcal{M}_L$ is the moduli space of hypersurfaces in Y in a very ample linear system $|L|$, and one also needs some suitable vanishings.

Research partially supported by PRIN "Geometria delle varietà algebriche," GNSAGA-INdAM, and the University of Pennsylvania Department of Mathematics Visitors Fund. U.B. is a member of the VBAC group.

In some cases, as in the classical situation $Y = \mathbb{P}^d$, the map (1) can be expressed as a multiplication morphism between the Jacobian rings of X. We find that the surjectivity of the multiplication between Jacobian rings can in addition give explicit criteria on the class $|L|$ such that a very general hypersurface $X \in |L|$ satisfies the Hodge conjecture.[1]

In this paper we focus on applications to simplicial toric varieties $\mathbb{P}_\Sigma$ and very general hypersurfaces X in a very ample linear system $|L|$. X and $\mathbb{P}_\Sigma$ are rational homology manifolds and we can state the Hodge conjecture as in the smooth case. To our knowledge this is a novelty.

In [4] we showed that the higher dimensional generalization of the "Noether-Lefschetz Theorem" discussed previously holds when $\mathbb{P}_\Sigma$ is a simplicial projective toric variety, even with singularities. We also proved that the map (1) can be expressed as a multiplication morphism between Jacobian rings of $\mathbb{P}_\Sigma$, as in the classical case $Y = \mathbb{P}^d$.

The Hodge conjecture holds for X if the dimension of $\mathbb{P}_\Sigma$ is even (Proposition 3.1). In Theorem 3.2 we give an effective condition on the class $|L|$ to satisfy the surjectivity hypothesis of the morphism (1) if $\dim(\mathbb{P}_\Sigma) = 2d + 1$. Hence the Hodge conjecture holds for a very general hypersurface in the linear system $|L|$, that is any class in $H^{p,p}(X, \mathbb{Q})$ is represented by a linear combination of algebraic cycles.

In [3] we observed that (1) is surjective if the multiplication morphism

$$S_\alpha \otimes S_\beta \to S_{\alpha+\beta}$$

between graded components of the Cox ring S is surjective whenever α and β are an ample and nef class in the class group of Y. We call "Oda varieties" the varieties which satisfy this condition. Oda in fact, in a Oberwolfach meeting in 1988 posed a related question about the Minkowski sums of polytopes [15]. Projective spaces are the easiest examples of Oda's varieties. We give other examples in Section 4 and relate them to the Castenuovo-Mumford regularity.

In Theorem 5.1 we prove an "effective" Hodge result:

Theorem 1.1. *Let $\mathbb{P}_\Sigma$ be an Oda variety of odd dimension $d = 2p + 1$, and let L be a very ample divisor in $\mathbb{P}_\Sigma$ such that $pL + K_{\mathbb{P}_\Sigma}$ is nef. Then the Hodge conjecture with rational coefficients holds for the very general hypersurface in the linear system $|L|$.*

[1] In fact, Huang, Lian Yau and Yu [11] applied our argument of the surjectivity of the Jacobian rings to subsequently prove the Hodge conjecture for very general hypersurfaces in generalized flag varieties G/P.

This gives an explicit criterion, effectively depending on the degree of the hypersurface, for very general hypersurfaces to satisfy the Hodge conjecture. We give examples in Corollary 5.2. We also formulate a partial converse of this results and pose some questions, including generalizations to rational homology manifold.

Acknowledgments. U.B. thanks the Department of Mathematics of the University of Pennsylvania for providing support and hospitality while this work was completed.

2. Hypersurfaces in toric simplicial varieties

We recall some basic facts about hypersurfaces in projective simplicial toric varieties and their cohomology. We mainly follow [2] and [4], also in the notation. All schemes are over the complex numbers.

2.1. Preliminaries and notation

Let N be a free abelian group of rank d. A rational simplicial complete d-dimensional fan Σ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ defines a complete toric variety $\mathbb{P}_\Sigma$ of dimension d with only Abelian quotient singularities. In particular $\mathbb{P}_\Sigma$ is an orbifold; $\mathbb{P}_\Sigma$ is also called *simplicial*. We assume here that $\mathbb{P}_\Sigma$ is also projective.

Let $\{\rho\}$ be the rays of the fan Σ and $\{x_\rho\}$ the homogeneous coordinates associated with them. Let $S = \mathbb{C}[x_\rho]$ be the Cox ring of $\mathbb{P}_\Sigma$, that is the algebra over $\mathbb{C}$ generated by the homogeneous coordinates x_ρ .

Each ray ρ and each coordinate x_ρ determine a torus invariant Weil divisor D_ρ. A monomial $\prod x_\rho^{a_\rho}$ determines a Weil divisor $D = \sum_\rho a_\rho D_\rho$.

S is graded by the class group $Cl(\mathbb{P}_\Sigma)$ of $\mathbb{P}_\Sigma$. Denoting by $\beta = [D]$ the class of D in $Cl(\mathbb{P}_\Sigma)$, one has $S = \oplus_{\beta \in Cl(\mathbb{P}_\Sigma)} S_\beta$.

The Cox ring generalizes the coordinate ring of $\mathbb{P}^d$, in which case S is the polynomial ring in $d+1$ variables over $\mathbb{C}$.

2.2. Hypersurfaces and their cohomology

Let L be a nef divisor. Then $\mathcal{O}_{\mathbb{P}_\Sigma}(L)$ is generated by its global sections [13, Th. 1.6] and a general hypersurface $X \in |L|$ is quasi-smooth, that is, its only singularities are those inherited from $\mathbb{P}_\Sigma$ [14, Lemma 6.6, 6.7]. In particular, if $\mathbb{P}_\Sigma$ is simplicial X is also an orbifold.

We recall some facts from [2] and Section 3 of [4]. For the reader's convenience the Appendix provides the relevant material taken from [4].

Let L be a very ample Cartier divisor and X a general hypersurface $X \in |L|$. The homotopy hyperplane Lefschetz theorem implies that X is also simply connected if $\dim(\mathbb{P}_\Sigma) \geq 3$ [10, Thm. 1.2 Part II]. The hard Lefschetz theorem holds also for projective orbifolds [17, 20]).

The complex cohomology of an orbifold has a pure Hodge structure in each dimension [18, 19]: $H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C})$ and $H^{d-1}(X, \mathbb{C})$ have then pure Hodge structures.

Let $i\colon X \to \mathbb{P}_\Sigma$ be the natural inclusion and $i^*\colon H^\bullet(\mathbb{P}_\Sigma, \mathbb{C}) \to H^\bullet(X, \mathbb{C})$ the associated morphism in cohomology; $i^*\colon H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}) \to H^{d-1}(X, \mathbb{C})$ is injective by Lefschetz's theorem. The morphism i^* is compatible with the Hodge structures.

Definition 2.1. [2, Def. 10.9] The primitive cohomology group $PH^{d-1}(X)$ is the quotient $H^{d-1}(X, \mathbb{C})/\, i^*(H^{d-1}(\mathbb{P}_\Sigma, \mathbb{C}))$.

$PH^{d-1}(X)$ inherits a pure Hodge structure, and one can write

$$PH^{d-1}(X) = \bigoplus_{p=0}^{d-1} PH^{p,d-1-p}(X).$$

Let $\mathscr{X}$ be the open subscheme of $|L|$ parametrizing the quasi-smooth hypersurfaces in $|L|$ and let $\mathcal{M}_\beta$ be the coarse moduli space for the quasi-smooth hypersurfaces in $\mathbb{P}_\Sigma$ with divisor class β.

There is a "Noether-Lefschetz theorem" for odd dimensional simplicial toric varieties as in [6, Theorem 7.5.1]:

Theorem 2.2. *Let $\mathbb{P}_\Sigma$ be a simplicial toric variety of dimension $d = 2p + 1 \geq 3$. If the morphism*

$$\gamma_p\colon T_X\mathcal{M}_\beta \otimes PH^{p+1,d-p-2}(X) \to PH^{p,d-p-1}(X) \tag{2}$$

is surjective, then for z away from a countable union of subschemes of $\mathscr{X}$ of positive codimension one has

$$H^{p,p}(X_z, \mathbb{Q}) = \mathrm{im}[i^*\colon H^{p,p}(\mathbb{P}_\Sigma, \mathbb{Q}) \to H^{2p}(X_z.\mathbb{Q})].$$

Recall that one has $H^{p,p}(X, \mathbb{Q}) = H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Q})$ for any variety X.

2.3. Hypersurfaces in $\mathbb{P}_\Sigma$ and their Jacobian Rings

Let L be an ample, hence very ample, Cartier divisor, of class $[L] = \beta$ on a simplicial toric variety $\mathbb{P}_\Sigma$ of dimension $d = 2p + 1$. Let f be a section of the line bundle $\mathcal{O}_{\mathbb{P}_\Sigma}(L)$ and X be the hypersurface defined by f; it turns out that $f \in S_\beta$.

Definition 2.3. Let $J(f) \subset S$ be the ideal in the Cox ring generated by the derivatives of f. The Jacobian ring $R(f)$ is defined as $R(f) = S/J(f)$; it is naturally graded by the class group $Cl(\mathbb{P}_\Sigma)$.

Theorem 2.4. *[2, Proposition 13.7; Theorem 10.13] Let $\mathbb{P}_\Sigma$ be a simplicial toric variety of dimension $d = 2p + 1$ and L be an ample, Cartier divisor, of class $[L] = \beta$ Let $\beta_0 = -[K_{\mathbb{P}_\Sigma}]$ and $\beta = [L]$. Then:*

(i) $T_X\mathcal{M}_\beta \simeq R(f)_\beta$.

(ii) $PH^{p+1,d-p-2}(X) \simeq R(f)_{p\beta-\beta_0}$.

Corollary 2.5. *[4, Proposition 3.4] The morphism γ_p in equation (2) coincides with the multiplication in the ring $R(f)$:*

$$R(f)_\beta \otimes R(f)_{p\beta-\beta_0} \to R(f)_{(p+1)\beta-\beta_0} \tag{3}$$

3. Hodge Conjecture on Toric varieties

Simplicial toric varieties are orbifolds, and in particular rational homology manifolds; we can state the Hodge conjecture as in the smooth case.

Hodge conjecture for rational homology manifolds. A connected normal complex threefold Y is a rational homology manifold if for every point $p \in Y$, $H_6(Y, Y \setminus p;\ \mathbb{Q}) \simeq \mathbb{Q}$ and $H_i(Y, Y \setminus p;\ \mathbb{Q}) = 0$ for $i \leq 5$. For these manifolds intersection cohomology and the ordinary (simplicial) cohomology coincide [9, 10]; in particular if Y is compact, Poincaré duality and the Hodge decomposition hold.

Proposition 3.1. *Let Y be an even dimensional projective variety of dimension d which is a rational homology manifold and for which the Hodge conjecture holds. Let L be a very ample class line bundle on Y. Then the Hodge conjecture holds for a general variety X which is a rational homology manifold in the linear system $|L|$.*

Proof. The hard Lefschetz theorem holds for intersection cohomology. For rational homology manifolds the Intersection cohomology equals the DeRham cohomology and the decomposition is preserved. Poincaré duality holds for rational homology manifolds. Then by the Lefschetz theorem, the Hodge conjecture holds true for for $p < (d-1)/2$, and by Poincaré duality, also for $p > (d-1)/2$. □

The interesting case is then for odd dimensional varieties $d = 2p + 1$ and for the intermediate cohomology $p = (d-1)/2$.

Theorem 3.2. *Let $\mathbb{P}_\Sigma$ be a simplicial toric variety of dimension $d = 2p + 1 \geq 3$. Assume that morphism*

$$R(f)_\beta \otimes R(f)_{p\beta-\beta_0} \to R(f)_{(p+1)\beta-\beta_0}$$

in equation (3) *is surjective.*

Then the Hodge conjecture holds for a very general hypersurface in the linear system $|L|$, that is any class in $H^{p,p}(X,\mathbb{Q})$ is represented by a linear combination of algebraic cycles.

Proof. We note that the Hodge conjecture holds for $\mathbb{P}_\Sigma$ simplicial, possibly singular, toric varieties, that is, any class in $H^{p,p}(\mathbb{P}_\Sigma,\mathbb{Q})$ is represented by a linear combination of algebraic classes. In fact $\mathbb{P}_\Sigma$ has a cellular decomposition and it satisfies the Hodge conjecture: every cohomology class in $H^{p,p}(\mathbb{P}_\Sigma,\mathbb{Q})$ is a linear combination of algebraic cycles [8, Example 19.1.11]. Since (3) is surjective, (2) is also surjective and Theorem 2.2 implies that any class in $H^{p,p}(X,\mathbb{Q})$ is also represented by a linear combination of algebraic cycles. □

Remark 3.3. Huang, Lian, Yau and Yu [11] applied our argument about the surjectivity of the Jacobian rings to subsequently prove the Hodge conjecture for very general hypersurfaces in flag varieties G/P.

We conclude this section with some questions:

Question 1: Can the analysis of the multiplications maps between the Jacobian rings be applied to other singular varieties, in particular to some odd dimensional rational homology manifolds?

Question 2: How can one determine when the morphism in equation (2) is surjective?

Or equivalently:

Question 3: How can one determine when the morphism in equation (3) is surjective? Note that the morphism of equation (3)

$$R(f)_\beta \otimes R(f)_{p\beta-\beta_0} \to R(f)_{(p+1)\beta-\beta_0}$$

is surjective whenever the morphism

$$S_\beta \otimes S_{p\beta-\beta_0} \to S_{(p+1)\beta-\beta_0}$$

is surjective.

4. Oda varieties

Definition 4.1. A toric variety $\mathbb{P}_\Sigma$ is an Oda variety if the multiplication morphism $S_{\alpha_1} \otimes S_{\alpha_2} \to S_{\alpha_1+\alpha_2}$ is surjective whenever the classes α_1 and α_2 in $\mathrm{Pic}(\mathbb{P}_\Sigma)$ are ample and nef, respectively.

The question of the surjectivity of this map was posed by Oda in [15] under more general conditions. This property can be stated in terms of the Minkowski sum of polytopes, as the integral points of a polytope associated with a line bundle correspond to sections of the line bundle. Definition 4.1 says that the sum $P_{\alpha_1} + P_{\alpha_2}$ of the polytopes associated with the line bundles $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_1)$ and $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_2)$ is equal to their Minkowski sum, that is $P_{\alpha_1+\alpha_2}$, the polytope associated with the line bundle $\mathcal{O}_{\mathbb{P}_\Sigma}(\alpha_1 + \alpha_2)$.

The Oda conjecture is open, even for smooth varieties. Projective spaces are Oda varieties.

Some results by Ikeda can be rephrased as follows.

Theorem 4.2. *[12, Corollary 4.2]*

(i) *A smooth toric variety with Picard number* 2 *is an Oda variety.*

(ii) *The total space of a toric projective bundle over an Oda variety is also an Oda variety.*

In [3] we prove:

Proposition 4.3. *Let $\mathbb{P}_\Sigma$ be a projective toric variety. If $\mathrm{Pic}(\mathbb{P}_\Sigma) = \mathbb{Z}$ and its ample generator η is Castelnuovo-Mumford 0-regular, then $\mathbb{P}_\Sigma$ is an Oda variety.*

5. Oda and Effective Hodge

We then have an effective Hodge result for very general hypersurfaces in toric simplicial Oda varieties:

Theorem 5.1. *Let $\mathbb{P}_\Sigma$ be an Oda variety of dimension $d = 2p + 1 \geq 3$ and $X \in |L|$, a very general element, with L very ample of class $[L] = \beta$, such that $p\beta - \beta_0$ is nef. Then the Hodge conjecture holds for the very general hypersurface in the linear system $|L|$.*

Corollary 5.2. (i) *(Smooth varieties) If $\mathbb{P}_\Sigma$ is smooth Fano or quasi-Fano, with Picard number 2, then the Hodge conjecture holds for X very general in any very ample class β, $\mathbb{P}_\Sigma$, by Ikeda's Theorem 4.2.*

(ii) *(Singular varieties) In particular, if $\mathbb{P}_\Sigma = \mathbb{P}[1, 1, 2, 2, \cdots, 2]$ then the Hodge conjecture holds for X very general in any very ample class $\beta = k\eta$, $k \geq 3$, where η is the very ample generator of the Picard group.*

6. Hodge and Oda

When $p = 1$, and $d = 3$, Theorem 3.2 gives a Noether-Lefschetz theorem on the Neron-Severi group of X.

In the appendix we prove:

Proposition 6.1. *Let $p = 1$, and $d = 3$, and $rk(Cl(X) \otimes \mathbb{Q}) = rk(Cl(\mathbb{P}_\Sigma) \otimes \mathbb{Q})$. Then the morphism* (3)

$$R(f)_\beta \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0}$$

is surjective.

Proof. See Proposition A.4 in the Appendix. □

Proposition 6.1 combined with a result of Ravindra and Srinivas gives:

Corollary 6.2. *Let $p = 1$, $d = 3$. Assume that $K_{\mathbb{P}_\Sigma} + L$ of class $\beta - \beta_0$ is generated by its global section. Then the morphism*

$$R(f)_\beta \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0}$$

is surjective, for a very general $f \in S_\beta$

Proof. [16, Thm. 1] and [5, Prop. 1.3]. □

Remark 6.3. The results above bring to the following:

Question 4: (Noether-Lefschetz and Oda): Let $d=3$. If the Noether-Lefschetz theorem holds, that is, if the morphism

$$R(f)_\beta \otimes R(f)_{\beta-\beta_0} \to R(f)_{2\beta-\beta_0}$$

is surjective, then for a very general $f \in S_\beta$, is the morphism

$$S_\beta \otimes S_{\beta-\beta_0} \to S_{2\beta-\beta_0}$$

surjective?

Question 5: (Hodge and Oda): Let $d=2p+1$. If the Hodge conjecture holds, for very general $X \subset \mathbb{P}_\Sigma$, that is, if the morphism

$$R(f)_\beta \otimes R(f)_{p\beta-\beta_0} \to R(f)_{(p+1)\beta-\beta_0}$$

is surjective, for a very general $f \in S_\beta$, is the morphism

$$S_\beta \otimes S_{p\beta-\beta_0} \to S_{(p+1)\beta-\beta_0}$$

also surjective?

Appendix

Let L be an ample, hence a very ample Cartier divisor, and $X \in |L|$ a quasi-smooth general surface. The definition of the primitive cohomology of $PH^\bullet(X)$ of X was recalled in Section 2.1. Let us also recall that the primitive cohomology $PH^\bullet(X)$ of X has a pure Hodge structure. The primitive cohomology classes $PH^{d-1}(X)$ can be represented by differential forms of top degree on $\mathbb{P}_\Sigma$ with poles along X; for every p with $0 \le p \le d-1$ there is a naturally defined residue map [2]

$$r_p\colon H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)) \to PH^{p,d-p-1}(X)\,. \tag{A.1}$$

Let $\mathscr{Z}$ be the open subscheme of $|L|$ parametrizing the quasi-smooth hypersurfaces in $|L|$, and let $\pi\colon \mathscr{X} \to \mathscr{Z}$ be the tautological family on $\mathscr{Z}$; we denote by X_z the fiber of $\mathscr{X}$ at $z \in \mathscr{Z}$. Let $\mathscr{H}^{d-1}$ be the local system on $\mathscr{Z}$ whose fiber at z is the cohomology $H^{d-1}(X_z, \mathbb{C})$, i.e., $\mathscr{H}^{d-1} = R^{d-1}\pi_*\mathbb{C}$. It defines a flat connection ∇ in the vector bundle $\mathscr{E}^{d-1} = \mathscr{H}^{d-1} \otimes_{\mathbb{C}} \mathcal{O}_{\mathscr{Z}}$, the *Gauss-Manin connection* of $\mathscr{E}^{d-1}$. Since the hypersurfaces X_z are quasi-smooth, the Hodge structure of the fibres $H^{d-1}(X_z, \mathbb{C})$ of $\mathscr{E}^{d-1}$ varies analytically with z [18]. The corresponding filtration defines holomorphic subbundles $F^p\mathscr{E}^{d-1}$, and the graded object of the filtration defines holomophic bundles $Gr^p_F(\mathscr{E}^{d-1})$. The bundles $\mathscr{E}^{p,d-p-1}$ given by the Hodge decomposition are not holomorphic subbundles of $\mathscr{E}^{d-1}$, but are diffeomorphic to $Gr^p_F(\mathscr{E}^{d-1})$, and as such they have a holomorphic structure. The quotient bundles $\mathscr{P}\mathscr{E}^{p,d-p-1}$ of $\mathscr{E}^{p,d-p-1}$ correspond to the primitive cohomologies of the hypersurfaces X_z. Let $\pi_p : \mathscr{E}^{d-1} \to \mathscr{P}\mathscr{E}^{p,d-p+1}$ be the natural projection.

We denote by $\tilde{\gamma}_p$ the cup product

$$\begin{aligned}\tilde{\gamma}_p \colon & H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) \\ & \to H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)).\end{aligned}$$

If z_0 is the point in $\mathscr{Z}$ corresponding to X, the space $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))/\mathbb{C}(f)$, where $\mathbb{C}(f)$ is the 1-dimensional subspace of $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$ generated by f, can be identified with $T_{z_0}\mathscr{Z}$.

The morphism $\tilde{\gamma}_p$ induces in cohomology the Gauss-Manin connection:

Lemma A.1. *Let σ_0 be a primitive class in $PH^{p,d-p-1}(X)$, let $v \in T_{z_0}\mathscr{Z}$, and let σ be a section of $\mathscr{E}^{p,d-p-1}$ along a curve in $\mathscr{Z}$ whose tangent vector at z_0 is v, such that $\sigma(z_0) = \sigma_0$.*

Then

$$\pi_{p-1}(\nabla_v(\sigma)) = r_{p-1}(\tilde{\gamma}_p(\tilde{v} \otimes \tilde{\sigma})) \tag{A.2}$$

where r_p, r_{p-1} are the residue morphisms defined in equation (A.1)*, $\tilde{\sigma}$ is an element in $H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X))$ such that $r_p(\tilde{\sigma}) = \sigma_0$, and $\tilde{v}$ is a pre-image of v in $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$.*

In particular the following diagram commutes:

(A.3)

$$\begin{array}{ccc} H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) & \xrightarrow{\tilde{\gamma}_p} & H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)) \\ \downarrow{\scriptstyle \phi\otimes r_p} & & \downarrow{\scriptstyle r_{p-1}} \\ T_{z_0}\mathscr{Z} \otimes PH^{p,d-1-p}(X) & \xrightarrow{\gamma_p} & PH^{p-1,d-p}(X) \end{array}$$

where γ_p is the morphism that maps $v \otimes \alpha$ to $\nabla_v\alpha$, and ϕ is the projection

$$\phi \colon H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_{z_0}\mathscr{Z}.$$

Lemma A.2. *If α and η are sections of $\mathscr{E}^{p,d-p-1}$ and $\mathscr{E}^{d-p,p-1}$ respectively, then for every tangent vector $v \in T_{z_0}\mathscr{Z}$,*

$$\nabla_v \alpha \cup \eta = -\alpha \cup \nabla_v \eta\,. \tag{A.4}$$

Let $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$ be the subgroup of $\mathrm{Aut}(\mathbb{P}_\Sigma)$ which preserves the grading β. The coarse moduli space $\mathcal{M}_\beta$ for the general quasi-smooth hypersurfaces in $\mathbb{P}_\Sigma$ with divisor class β may be constructed as a quotient

$$U/\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)\,, \tag{A.5}$$

[1, 2], where U is an open subset of $H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X))$, and $\widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma)$ is the unique nontrivial extension

$$1 \to D(\Sigma) \to \widetilde{\mathrm{Aut}}_\beta(\mathbb{P}_\Sigma) \to \mathrm{Aut}_\beta(\mathbb{P}_\Sigma) \to 1\,.$$

By differentiating, we have a surjective map

$$\kappa_\beta \colon H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \to T_X\mathcal{M}_\beta\,,$$

which is an analogue of the Kodaira-Spencer map.

The local system $\mathscr{H}^{d-1}$ and its various sub-systems do not descend to the moduli space $\mathcal{M}_\beta$, because the group $\mathrm{Aut}_\beta(\mathbb{P}_\Sigma)$ is not connected. Nevertheless, this group has a connected subgroup $\mathrm{Aut}^0_\beta(\mathbb{P}_\Sigma)$ of finite order, and, perhaps after suitably shrinking U, the quotient $\mathcal{M}^0_\beta \overset{def}{=} U/\,\mathrm{Aut}^0_\beta(\mathbb{P}_\Sigma)$ is a finite étale covering of $\mathcal{M}_\beta$ [1, 7]. Since we are only interested in the tangent space $T_X\mathcal{M}_\beta$, we can replace $\mathcal{M}_\beta$ with $\mathcal{M}^0_\beta$.

Proposition A.3. *There is a morphism*

$$\gamma_p \colon T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) \to PH^{p-1,d-p}(X) \tag{A.6}$$

such that the diagram

$$\begin{array}{ccc} H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \otimes H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p)X)) & \xrightarrow{\cup} & H^0(\mathbb{P}_\Sigma, \Omega^d_{\mathbb{P}_\Sigma}((d-p+1)X)) \\ \Big\downarrow{\scriptstyle \kappa_\beta\otimes r_p} & & \Big\downarrow{\scriptstyle r_{p-1}} \\ T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) & \xrightarrow{\gamma_p} & PH^{p-1,d-p}(X) \end{array}$$

commutes.

Denote by $H^{d-1}_T(X) \subset H^{d-1}(X)$ the subspace of the cohomology classes that are annihilated by the action of the Gauss-Manin connection. Coefficients may be taken in $\mathbb{C}$ or $\mathbb{Q}$. Note that $H^{d-1}_T(X)$ has a Hodge structure.

The result below is an "infinitesimal Noether-Lefschetz theorem," such as Theorem 7.5.1 in [6].

Proposition A.4. *For a given p with $1 \leq p \leq d-1$, the morphism*

$$\gamma_p \colon T_X\mathcal{M}_\beta \otimes PH^{p,d-1-p}(X) \to PH^{p-1,d-p}(X) \tag{A.7}$$

is surjective if and only if $H^{p,d-1-p}_T(X) = i^*(H^{p,d-1-p}(\mathbb{P}_\Sigma))$.

Proof. The "only if" part was proved in [4]. To prove the "if" part, assume γ_p is not surjective, and decompose $PH^{p-1,d-p}(X)$ as

$$PH^{p-1,d-p}(X) = \operatorname{Im}\gamma_p \oplus (\operatorname{Im}\gamma_p)^\perp.$$

Le $\{\eta_i\}$ be a basis of $PH^{p,d-1-p}(X)$, and $\{t_j\}$ a basis of $T_X\mathcal{M}_\beta$. Fix values for i and j and let $\tau = \gamma_p(t_j \otimes \eta_i)$. If $\alpha \in (\operatorname{Im}\gamma_p)^\perp$, then

$$0 = \langle \alpha, \tau \rangle = \langle \alpha, \gamma_p(t_j \otimes \eta_i) \rangle = \langle \nabla_{t_j}\alpha, \eta_i \rangle \quad \text{for all } i$$

so that $\nabla_{t_j}\alpha = 0$ for all j, i.e., α is a nonzero element in $H^{p-1,d-p}_T(X)$, which implies that $H^{p,d-1-p}_T(X)$ is properly contained in $i^*(H^{p,d-1-p}(\mathbb{P}_\Sigma))$. □

Lemma A.5. *Let $d = 2p+1 \geq 3$, and assume that the hypotheses of the previous Lemma hold for $p = m$. Then for z away from a countable union of subschemes of $\mathscr{Z}$ of positive codimension one has*

$$H^{p,p}(X_z, \mathbb{Q}) = \operatorname{im}[i^* \colon H^{p,p}(\mathbb{P}_\Sigma, \mathbb{Q}) \to H^{2p}(X_z.\mathbb{Q})].$$

Proof. Let $\bar{\mathscr{Z}}$ be the universal cover of $\mathscr{Z}$. On it the (pullback of the) local system $\mathscr{H}^{d-1}$ is trivial. Given a class $\alpha \in H^{p,p}(X)$ we can extend it to a global section of $\mathscr{H}^{d-1}$ by parallel transport using the Gauss-Manin connection. Define the subset $\bar{\mathscr{Z}}_\alpha$ of $\bar{\mathscr{Z}}$ as the common zero locus of the sections $\pi_m(\alpha)$ of $\mathscr{E}^{m,d-1-m}$ for $p \neq m$ (i.e., the locus where α is of type (p,p)).

If $\bar{\mathscr{Z}}_\alpha = \bar{\mathscr{Z}}$ we are done because α is in $H^{d-1}_T(X)$ hence it is in the image of i^* by the previous Lemma. If $\bar{\mathscr{Z}}_\alpha \neq \bar{\mathscr{Z}}$, we note that $\bar{\mathscr{Z}}_\alpha$ is a subscheme of $\bar{\mathscr{Z}}$.

We subtract from $\mathscr{Z}$ the union of the projections of the subschemes $\bar{\mathscr{Z}}_\alpha$ where $\bar{\mathscr{Z}}_\alpha \neq \bar{\mathscr{Z}}$. The set of these varieties is countable because we are considering rational classes. □

References

[1] P. S. Aspinwall, B. R. Greene, and D. R. Morrison, *The monomial-divisor mirror map*, Internat. Math. Res. Notices (1993), 319–337.

[2] V. V. Batyrev and D. A. Cox, *On the Hodge structure of projective hypersurfaces in toric varieties*, Duke Math. J. **75** (1994), 293–338.

[3] U. Bruzzo and A. Grassi, *The Noether-Lefschetz locus of surfaces in toric threefolds*, Commun. Contemp. Math. **20** (2018), no. 5, 1750070, 20 pp.

[4] U. Bruzzo and A. Grassi, *Picard group of hypersurfaces in toric three-folds*, Int. J. Math. **23** (2012), no. 2, 1250028.

[5] U. Bruzzo, A. Grassi, and A. Lopez, *Existence and density of general components of the Noether-Lefschetz locus on normal threefolds*, Int. Math. Res. Notices (2020), rnz358, `DOI:10.1093/imrn/rnz358`.

[6] J. Carlson, S. Müller-Stach, and C. Peters, Period Mappings and Period Domains, Vol. 85 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (2003).

[7] D. A. Cox, R. Donagi, and L. Tu, *Variational Torelli implies generic Torelli*, Invent. Math. **88** (1987), 439–446.

[8] W. Fulton, Intersection Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Vol. 2, Springer-Verlag, Berlin, (1984).

[9] M. Goresky and R. MacPherson, *Intersection homology. II,* Invent. Math. **72** (1983), 77–129.

[10] M. Goresky and R. MacPherson, Stratified Morse Theory, Vol. 14 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, (1988).

[11] A. Huang, B. Lian, S.-T. Yau, and C. Yu, *Jacobian rings for homogenous vector bundles and applications*, `arXiv:1801.08261 [math.AG]`.

[12] A. Ikeda, *Subvarieties of generic hypersurfaces in a nonsingular projective toric variety*, Math. Z. **263** (2009), 923–937.

[13] A. R. Mavlyutov, *Semiample hypersurfaces in toric varieties*, Duke Math. J. **101** (2000), 85–116.

[14] J. P. Mullet, *Toric Calabi-Yau hypersurfaces fibered by weighted $K3$ hypersurfaces*, Comm. Anal. Geom. **17** (2009), 107–138.

[15] T. Oda, *Problems on Minkowski sums of convex lattice polytopes*, arXiv:0812.1418 [math.AG].

[16] G. V. Ravindra and V. Srinivas, *The Noether-Lefschetz theorem for the divisor class group*, J. Algebra **322** (2009), 3373–3391.

[17] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), 221–333.

[18] J. Steenbrink, *Intersection form for quasi-homogeneous singularities*, Compositio Math. **34** (1977), 211–223.

[19] L. Tu, *Macaulay's theorem and local Torelli for weighted hypersurfaces*, Compositio Math. **60** (1986), 33–44.

[20] Z. Z. Wang and D. Zaffran, *A remark on the hard Lefschetz theorem for Kähler orbifolds*, Proc. Amer. Math. Soc. **137** (2009), 2497–2501.

SISSA (Scuola Internazionale Superiore di Studi Avanzati)
Via Bonomea 265, 34136 Trieste, Italia
IGAP (Institute for Geometry and Physics), Trieste, Italia
INFN (Istituto Nazionale di Fisica Nucleare), Sezione di Trieste
Arnold-Regge Center for Algebra, Geometry
and Theoretical Physics, Torino, Italia
E-mail address: bruzzo@sissa.it

Department of Mathematics, University of Pennsylvania
209 S 33rd St., Philadelphia, PA 19104, USA
E-mail address: grassi@sas.upenn.edu

Received May 13, 2018
Accepted February 6, 2019