COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 28, Number 7, 1683–1754, 2020

Extending four dimensional Ricci flows with bounded scalar curvature

MILES SIMON

We consider solutions $(M, g(t)), 0 \le t < T$, to Ricci flow on compact, connected four dimensional manifolds without boundary. We assume that the scalar curvature is bounded uniformly, and that $T < \infty$. In this case, we show that the metric space $(M, d(t))$ associated to $(M, g(t))$ converges uniformly in the C^0 sense to (X, d), as $t \nearrow T$, where (X, d) is a C^0 *Riemannian orbifold* with at most finitely many orbifold points. Estimates on the rate of convergence near and away from the orbifold points are given. We also show that it is possible to continue the flow past (X, d) using the *orbifold* Ricci flow.

1. Introduction

In this paper we consider smooth solutions to Ricci flow, $\frac{\partial}{\partial t}g(t) = -2\mathrm{Rc}(g(t))$ for all $t \in [0,T)$, on closed, connected four manifolds without boundary. We assume that $T < \infty$ and that the scalar curvature satisfies $\sup_{M\times[0,T)} |\mathrm{R}| \leq 1$. In a previous paper, see Theorem 3.6 in [Si1], we showed that this implies

(i) **Integral bounds for the Ricci and Riemannian curvature**

$$\sup_{t\in[0,T)} \int_M |\,\mathrm{Riem}(\cdot,t)|^2 d\mu_{g(t)} \leq c_1 < \infty$$
$$\int_0^T \int_M |\mathrm{Rc}|^4(\cdot,t) d\mu_{g(t)} dt \leq c_2 < \infty$$

for explicit constants $c_1 = c_1(M, g(0), T)$ and $c_2(M, g(0), T)$. An estimate of the first type was independently first proved, using different methods, in the arxiv preprint version of [BZ] (see Theorem 1.8 in [BZ]), which appeared a few months before the arxiv preprint version of [Si1].

In this paper we show the following.

(ii) **Estimates for the singular and regular regions**

A point $p \in M$ is said to be *regular*, if there exists an $r > 0$ such that

$$\int_{{}^tB_r(p)} |\,\mathrm{Riem}\,|^2(\cdot,t) d\mu_{g(t)} \leq \varepsilon_0$$

for all $t \in (0,T)$, for some fixed small ε_0 (not depending on p) which is specified in the proof of Theorem 4.5. In Definition 4.7, an alternative definition of *regular* is given. The *singular* points are those which are not regular. In Theorem 4.5 (and the Corollaries 4.9 and 4.10 thereof) and Theorem 5.1 we obtain estimates for the evolving metric in the singular and regular regions of the manifold.

(iii) **Uniform continuity of the distance function in time.**

Using the estimates mentioned in (ii) we show the following (see Theorem 5.6). For all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|d(x,y,t) - d(x,y,s)| \leq \varepsilon \tag{1.1}$$

for all $x, y \in M$ for all $t, s \in [0,T)$ with $|t - s| \leq \delta$.

(iv) **Convergence of $(M, d(g(t)))$ to a C^0 Riemannian orbifold (X, d) as $t \nearrow T$.**

Using the estimates mentioned in (i),(ii) and (iii), we show that $(M, d(g(t)) \to (X, d_X)$ as $t \nearrow T$ in the Gromov-Hausdorff sense, where (X, d_X) is a C^0-Riemannian orbifold with finitely many orbifold points, and that the Riemannian orbifold metric on X is smooth away from the orbifold points. Also: the convergence is smooth away from the orbifold points (see Lemma 6.2 and Theorems 6.5, 6.6, 8.3).

In [BZ], which appeared a few months before the arxiv preprint version of this paper, the authors also considered Ricci flow of four manifolds with bounded scalar curvature, and they also prove results about the structure of the limiting space one obtains by letting $t \nearrow T$: see Theorem 1.8 and Corollary 1.11 of [BZ] (arxiv version 1).

Note added, March 2018. In the latest version (version 3 on arxiv) of [BZ], which appeared several months after the arxiv preprint version of this paper, the authors have added a proof (see the proof of Corollary 1.11 in version 3 of the arxiv preprint [BZ]) which also shows, using the estimates of their paper and an ε-regularity result of Anderson and a method similar to the one given in the paper [BKN], that the Gromov-Hausdorff limiting space one obtains by letting $t \nearrow T$ is a C^0 Riemannian orbifold.

(v) **The flow may be continued past time T using the orbifold Ricci flow.**

There exists a smooth solution $(N, h(t))_{t \in (0,\hat{T})}$ to the orbifold Ricci flow, such that$(N, d(h(t))) \to (X, d_X)$ in the Gromov-Hausdorff sense as $t \searrow 0$ (see Theorem 9.1).

In order to achieve (v), we find it necessary to explain in depth *how* the convergence in (iv) is occuring, and to give a detailed description of the structure of the metric space near orbifold points. See Theorem 7.4 and Theorem 8.1. In order to flow the limiting orbifold metric, we require not only the description of the orbifold space coming from Theorem 8.1, but also information on how the maps occuring in this description were constructed. This information is contained in Section 8, which in turn uses Theorem 7.4 coming from Section 7.

2. Setup, background, previous results and notation

In this paper we often consider solutions $(M^4, g(t))_{t\in[0,T)}$ which satisfy the following *basic assumptions*.

(a) M^4 is a smooth, compact, connected four dimensional manifold without boundary

(b) $(M^4, g(t))_{t\in[0,T)}$ is a smooth solution to the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ricci}(g(t))$ for all $t \in [0, T)$

(c) $T < \infty$

(d) $\sup_{M^4\times[0,T)} |\mathrm{R}(x,t)| \le 1$

If instead of (d) we only have $\sup_{M\times[0,T)} |\mathrm{R}(x,t)| \le K < \infty$ for some constant $1 < K < \infty$, then we may rescale the solution $\tilde{g}(\cdot,\tilde{t}) := Kg(\cdot, \frac{\tilde{t}}{K})$ to obtain a new solution $(M, \tilde{g}(\tilde{t}))_{t\in[0,\tilde{T})}$, where $\tilde{T} := KT$, which satisfies the basic assumptions. As we mentioned in the introduction, any solution satisfying the basic assumptions also satisfies

$$\sup_{t\in[0,T)} \int_M |\,\mathrm{Riem}(\cdot,t)|^2 d\mu_{g(t)} \le K_0 < \infty \tag{2.1}$$

$$\int_0^T \int_M |\mathrm{Rc}|^4(\cdot,t) d\mu_{g(t)} dt \le c_2 < \infty. \tag{2.2}$$

See Theorem 3.6 in [Si1].

The estimate (2.1) was independently first obtained in the preprint [BZ], which appeared several months before the preprint version of [Si1], (see Theorem 1.8 of that paper), using different methods to those used in [Si1].

There are many papers in which conditions are considered which imply that the solution to Ricci flow defined on $[0,T)$ may be extended. Generally, in the real case, this extension is a smooth extension, and the conditions imply that the solution may be smoothly extended to a time interval $[0, T+\varepsilon)$ for some $\varepsilon > 0$: that is, the solution does not form a singularity as $t \nearrow T$. Here we list some of these conditions. This is by no means an exhaustive list and further references may be found in the papers we have listed here. In the following we assume that $(M^n, g(t))_{t\in[0,T)}$ is a smooth solution to Ricci flow on a compact $n-$dimensional manifold without boundary, and we write the condition which guarantees, that one can extend the solution past time T, followed by an appropriate reference. $\sup_{M^n\times[0,T} |\,\mathrm{Riem}\,| < \infty$ [HaThree]. $\sup_{M^n\times[0,T)} |\mathrm{Ricci}| < \infty$ [Sesum].

$\limsup_{t \nearrow T} |g(t) - h| \le \varepsilon(n)$ for some smooth metric h [SimC0] (see also [KL]). $\sup_{(x,t)\in M^n \times [0,T)} |\operatorname{Riem}(x,t)|(T-t) + |\mathrm{R}(x,t)| < \infty$ [TME] (see also [SesumLe]).

$$\int_0^T \int_{M^n} |\operatorname{Rm}|^{\alpha}(\cdot,t)) d\mu_{g(t)} dt < \infty \text{ for some } \alpha \ge \frac{(n+2)}{2} \text{ [Wang1].}$$
$$\int_0^T \int_{M^n} |\operatorname{Weyl}|^{\alpha}(\cdot,t) + |\mathrm{R}|^{\alpha}(\cdot,t) d\mu_{g(t)} dt < \infty, \text{ where } \alpha \ge \frac{(n+2)}{2} \text{ [Wang1].}$$

See also [Wang1], [Wang2], [ChenWang] for further results on extending Ricci flow.

If one considers solutions to the Kähler Ricci flow, $\frac{\partial}{\partial t} g_{i\bar{j}} = -2 \operatorname{Ric}_{i\bar{j}}$, then the following is known: If $\sup_{M^n \times [0,T)} |\mathrm{R}| < \infty$, then one can extend the flow smoothly past time T [Zhang].

The situation in this paper is somewhat different. We consider solutions with bounded scalar curvature, and we do not rule out the possibility that singularities can form as $t \nearrow T$. However, using our integral curvature estimates (and other estimates) we show that there is a singular limiting space as $t \nearrow T$, and that this singular space is a C^0 Riemannian orbifold which can then be evolved by the orbifold Ricci flow: the limiting space is immediately smoothed out by the orbifold Ricci flow.

The possibility of flowing to a singular time and then continuing with another flow (for example orbifold Ricci flow or a weak Kähler Ricci flow) has been considered in other papers. In the real case, see for example [CTZ].

In the Kähler case see for example Theorem 1.1 in [SongWeinkove2] (see also [SongWeinkove1], [EGZ] and [EGZII] for related papers). Further references can be found in the papers mentioned above.

In [ChenWang], the authors investigate the moduli space of solutions to Ricci flow which have: bounded curvature in the $L^{n/2}$ sense, bounded scalar curvature and are non-collapsed.

There are examples of solutions to Ricci flow which are smooth on $[0,T)$, singular at time T, and then become immediately smooth again after this time: see the neck-pinching examples given in [ACK]. See also [KlLo] and [FIK]. Here, the flow remains the same, but a change in the topology of the manifold occurs at the singular time. This notion of *extending the flow* is once again different to the one we are considering, and different to the notion of smooth extension discussed above

The Orbifold Ricci flow and related flows has been studied in many papers. Here is a (by no means exhaustive) list of some of them: [CTZ], [ChenYWangI], [ChenYWangII], [ChowII], [ChowWu], [HaThreeO], [KLThree], [LiuZhang], [WuLF], [Yin], [YinII].

Notation. We use the Einstein summation convention, and we use the notation of Hamilton [HaThree].

For $i \in \{1, \ldots, n\}$, $\frac{\partial}{\partial x^i}$ denotes a coordinate vector, and dx^i is the corresponding one form.

(M^n, g) is an n-dimensional Riemannian manifold with Riemannian metric g.

$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i})$ is the Riemannian metric g with respect to this coordinate system.

g^{ij} is the inverse of the Riemannian metric ($g^{ij} g_{ik} = \delta_{jk}$).

$d\mu_g$ is the volume form associated to g.

$\mathrm{Rm}(g)_{ijkl} = {}^g\,\mathrm{Riem}_{ijkl} = \mathrm{Riem}(g)_{ijkl} = \mathrm{R}_{ijkl}$ is the full Riemannian curvature Tensor.

$\mathrm{Weyl}(g)_{ijkl}$ is the Weyl Tensor.

${}^g\mathrm{Rc}_{ij} = \mathrm{Ricci}_{ij} = \mathrm{R}_{ij} := g^{kl}\mathrm{R}_{ikjl}$ is the Ricci curvature.

$\mathrm{R} := \mathrm{R}_{ijkl} g^{ik} g^{jl}$ is the scalar curvature.

${}^g\nabla T = \nabla T$ is the covariant derivative of T with respect to g. For example, locally $\nabla_i T^s_{jk} = (\nabla T)(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, dx^s)$ (the first index denotes the direction in which the covariant derivative is taken) if locally $T = T^s_{jk} dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^s}$.

$|T| = {}^g|T|$ is the norm of a tensor with respect to a metric g. For example for $T = T^s_{jk} dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^s}$. $|T|^2 = g^{im} g^{jn} g_{ks} T^s_{ij} T^k_{mn}$.

Sometimes we make it clearer which Riemannian metric we are considering by including the metric in the definition. For example $\mathrm{R}(h)$ refers to the scalar curvature of the Riemannian metric h.

We suppress the g in the notation used for the norm, $|T| = {}^g|T|$, and for other quantities, in the case that is is clear from the context which Riemannian metric we are considering.

A ball of radius $r > 0$ in a metric space (X, d) will be denoted by

$$ {}^dB_r(z) := \{x \in X \mid d(x, z) < r\}. $$

An annulus of inner radius $0 \leq s$ and outer radius $r > s$ on a metric space (X, d) will be denoted by

$$ {}^dB_{r,s}(z) := \{x \in X \mid s < d(x, z) < r\}. $$

Note then that ${}^dB_{0,s}(z) := \{x \in X \mid 0 < d(x, z) < r\} = {}^dB_s(z) \backslash \{z\}$.

The sphere of radius $r > 0$ and centre point p in a metric space (X, d) will be denoted by

$$ {}^{d}S_r(p) := \{x \in X \mid d(x,p) = r\}. $$

$D_{r,R} \subseteq \mathbb{R}^n$ is the standard open annulus of inner radius $r \geq 0$ and outer radius $R \leq \infty$, $(r < R)$ centred at 0:$D_{r,R} = \{x \in \mathbb{R}^n \mid |x| > r, |x| < R\}$.$D_r$ represents the open disc of radius r centred at $0 : D_r := \{x \in \mathbb{R}^n \mid |x| < r\}$. Note $D_{0,R} = \{x \in \mathbb{R}^n \mid |x| > 0, |x| < R\} = D_R\backslash\{0\}$.

$S_r^{n-1}(c) := \{x \in \mathbb{R}^n \mid |x - c| = r\}$ is the $(n-1)$-dimensional sphere of radius $r > 0$ and centre point $c \in \mathbb{R}^n$ in $\mathbb{R}^n$.

ω_n is the volume of a ball of radius one in $\mathbb{R}^n$ with respect to the Lebesgue meaure.

If Γ is a finite subgroup of $O(n)$ acting freely on $\mathbb{R}^n\backslash\{0\}$, then $((\mathbb{R}^n\backslash\{0\})/\Gamma, g)$ is the quotient manifold with the induced (flat) metric coming from $\pi : \mathbb{R}^n\backslash\{0\} \to (\mathbb{R}^n\backslash\{0\})/\Gamma$, $\pi(x) := \{[x] \mid x \in \mathbb{R}^n\backslash\{0\}\}$, where $[x] := \{Gx \mid G \in \Gamma\}$.

$({}^{g}B_{r,s}(0), g) \subseteq ((\mathbb{R}^n\backslash\{0\})/\Gamma, g)$ refers to the set${}^{g}B_{r,s}(0) := \{\pi(x) \mid x \in D_{r,s}\}$ with the Riemannian metric g.

3. Volume control, and the Sobolev inequality

In [Ye] and [Zhang1, Zhang2] the first inequality appearing below was proved, and in [Zhang3] (and in [ChenWang]) the second inequality appearing below was proved.

Theorem 3.1 (R. Ye [Ye], Q. Zhang [Zhang1, Zhang2, Zhang3] (see [ChenWang] also)). *Let $(M^n, g(t))_{t\in[0,T)}$, $T < \infty$, be a smooth solution to Ricci flow on a closed manifold with $\sup_{M\times[0,T)} |\mathrm{R}(x,t)| \leq 1 < \infty$. Then there exist constants $0 < \sigma_0, \sigma_1 < \infty$ depending only on (M, g_0) and T such that*

$$ \sigma_1 \leq \frac{\mathrm{vol}({}^{t}B_r(x))}{r^n} \leq \sigma_2 \text{ for all } x \in M, 0 \leq t < T \text{ and } r \leq 1. \tag{3.1} $$

We use the following notation in this paper which was introduced by Q. Zhang. A solution which satisfies the first inequality is said to be σ_1 *non-collapsed on scales less than* 1. This condition is similar to but stronger than Perelman's *non-collapsing* condition (see [Pe1]), as we make no requirements on the curvature within the balls $B_r(x)$ appearing in (3.1). A solution which satisfies the second inequality is said to be σ_2 *non-inflated on scales less than* 1.

Remark 3.2. Let $(M^n, g(t))_{t\in[0,T)}$ be be a smooth solution to Ricci flow which satisfies the inequalities (3.1), and define $\tilde g(\tilde t) := cg(\cdot, \frac{\tilde t}{c})$ for a constant $c > 0$. Then

$$\sigma_1 \leq \frac{\tilde{\mathrm{vol}}({}^{\tilde t}\tilde B_{\tilde r}(x)}{\tilde r^n}) \leq \sigma_2 \text{ for all } x \in M, 0 \leq \tilde t < \tilde T := cT \text{ and } \tilde r \leq \sqrt{c}, \tag{3.2}$$

that is $(M, \tilde g(\tilde t))_{\tilde t\in[0,\tilde T)}$ is σ_1 *non-collapsed and* σ_2 *non-inflated on scales less than* $\sqrt{c}$. This is because: $\frac{\tilde{\mathrm{vol}}\tilde B_{\tilde r}(x,\tilde t)}{\tilde r^n} = \frac{\mathrm{vol}\, B_r(x,t)}{r^n}$ for $\tilde r := \sqrt{c}r$ and $\tilde t := ct$, and $r = \frac{\tilde r}{\sqrt{c}} \leq 1$ for $\tilde r \leq \sqrt{c}$. Hence, we can't say if the solution $(M, \tilde g(\tilde t))_{\tilde t\in[0,\tilde T)}$ is σ_1 non-collapsed and σ_2 non-inflated on scales less than 1, if we scale by a constant $c < 1$, but the scale improves if we multiply by constants $c > 1$.

In the papers [Ye] and [Zhang1, Zhang2] it is also shown that for any Ricci flow satisfying the basic assumptions a Sobolev inequality holds in which the constants may be chosen to be time independent. Here, we only write down the four dimensional version of their theorem.

Theorem 3.3 (R. Ye [Ye], Q. Zhang [Zhang1, Zhang2]).
Let $(M^4, g(t))_{t\in[0,T)}$, $T < \infty$, *be a smooth solution to Ricci flow satisfying the basic assumptions. Then there exists a constant* $A = A(M, g_0, T) < \infty$ *such that*

$$\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} \leq A\left(\int_M^{g(t)} |\nabla f|^2 d\mu_{g(t)} + \int_M |f|^2 d\mu_{g(t)}\right) \tag{3.3}$$

for all smooth $f : M \to \mathbb{R}$

Note that this Sobolev inequality is not scale invariant, as the last term scales incorrectly. However, we have a scale-invariant version for small balls, as we see in the following:

Corollary 3.4. *Let* $(M^4, g(t))_{t\in[0,T)}$, $T < \infty$ *be a smooth solution to Ricci flow satisfying the basic assumptions. Then there exists a constant* $r^2 = r^2(M, g(0), T) = \frac{1}{2\sqrt{\sigma_2}A} > 0$ *such that*

$$\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} \leq 2A\int_M {}^{g(t)}|\nabla f|^2 d\mu_{g(t)} \tag{3.4}$$

for all smooth $f : M \to \mathbb{R}$ *whose support is contained in a ball* ${}^tB_r(x)$, *for some* $x \in M$, *where* A *is the constant occurring in the Sobolev inequality*

(3.3) *above. If* $\tilde{g}(\cdot,\tilde{t}) := cg(\cdot,\frac{\tilde{t}}{c})$ *is a scaled solution with* $c \geq 1$ *then the estimate*

$$\left(\int_M |f|^4 d\mu_{\tilde{g}(\tilde{t})}\right)^{\frac{1}{2}} \leq 2A \int_M {}^{g(t)}|\nabla f|^2 d\mu_{\tilde{g}(\tilde{t})} \tag{3.5}$$

holds for all $f : M \to \mathbb{R}$ *whose support is contained in a ball* ${}^{\tilde{t}}B_{\tilde{r}}(x)$ *where* $\tilde{r} := r\sqrt{c} \geq r$.

Proof. Let r be chosen so that $r^2\sqrt{\sigma_2} \leq \frac{1}{2A}$, where A is the constant occurring in the Sobolev inequality and σ_2 is the non-inflating constant defined above. Using Hölder's inequality and the above Sobolev inequality we get

$$\begin{aligned}
\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} &\leq A\int_M |\nabla f|^2 d\mu_{g(t)} + A\int_M |f|^2 d\mu_{g(t)}\\
&\leq A\int_M |\nabla f|^2 d\mu_{g(t)} + A\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} (\operatorname{vol} B_r(x,t))^{\frac{1}{2}}\\
&\leq A\int_M |\nabla f|^2 d\mu_{g(t)} + A\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} (\sqrt{\sigma_2}r^2)\\
&\leq A\int_M |\nabla f|^2 d\mu_{g(t)} + \frac{1}{2}\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} \qquad (3.6)
\end{aligned}$$

which implies the result, after subtracting $\frac{1}{2}(\int_M |f|^4 d\mu_{g(t)})^{\frac{1}{2}}$ from both sides of this inequality. The second inequality follows immediately from the fact that

$$\begin{aligned}
&\left(\int_M |f|^4 d\mu_{\tilde{g}(\tilde{t})}\right)^{\frac{1}{2}} - 2A\int_M |\tilde{\nabla} f|^2 d\mu_{\tilde{g}(t)} \qquad (3.7)\\
&= c\left(\left(\int_M |f|^4 d\mu_{g(t)}\right)^{\frac{1}{2}} - 2A\int_M |\nabla f|^2 d\mu_{g(t)}\right)
\end{aligned}$$

if we scale as in the statement of the theorem. □

It is well know that, for a solution satisfying the basic assumptions, the volume of M is changing at a controlled rate:

$$\operatorname{vol}(M,g(t)) \geq -\int_M R d\mu_{g(t)} = \frac{\partial}{\partial t}\operatorname{vol}(M,g(t)) \geq -\operatorname{vol}(M,g(t)) \tag{3.8}$$

($-\int_M R d\mu_{g(t)} = \frac{\partial}{\partial t}\operatorname{vol}(M,g(t))$ was shown in [HaThree]). Integrating in time we see that $e^T \operatorname{vol}(M,g(0)) \geq \operatorname{vol}(M,g(t)) \geq e^{-T}\operatorname{vol}(M,g(0))$.

Notice that the estimates of Peter Topping (see [Topping]) and these volume bounds combined with the non-inflating estimate guarantee that the diameter is bounded from above and below:

Lemma 3.5 (Topping, P. [Topping], Zhang, Q. [Zhang1, Zhang2])). *Let $(M^4, g(t)))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the basic assumptions (in particular $T < \infty$ and $|\mathrm{R}| \leq 1$ at all times and points). Then there exists $d_0 = d_0(M, g_0, T) > 0$ such that*

$$\infty > d_0 \geq \mathrm{diam}(M, g(t)) \geq \frac{1}{d_0} > 0 \tag{3.9}$$

for all $t \in [0, T)$.

Proof. The diameter bound from above follows immediately from Theorem 2.4 (see also Remark 2.5 there) of [Topping] combined with the fact that $\int_M |R|^{\frac{3}{2}} \leq \mathrm{vol}(M, g(0))e^T$ for a solution satisfying the basic assumptions. The diameter bound from below is obtained as follows. Assume that there are times $t_i \in [0, T)$ with $\varepsilon_i := \mathrm{diam}(M, g(t_i)) \to 0$ as $i \to \infty$. Due to smoothness, we must have $t_i \nearrow T$. From the volume estimates above, we must have $\mathrm{vol}(M, g(t)) \geq e^{-T}\,\mathrm{vol}(M, g(0)) =: v_0 > 0$ for all $t \in [0, T)$. Combining this with the non-inflating estimate we get:

$$v_0 \leq \mathrm{vol}(M, g(t_i)) = \mathrm{vol}({}^{t_i}B_{\varepsilon_i}(x_0)) \leq \sigma_2(\varepsilon_i)^4 \to 0$$

as $i \to \infty$, which is a contradiction. □

4. The regular part of the flow

We wish to show that the limit as $t \nearrow T$ (in some to be characterised sense) of $(M, g(t))$ is an C^0 Riemannian orbifold (X, d_X) with at most finitely many orbifold points and that (X, d_X) is smooth away from the orbifold points. In the static case, M. Anderson showed results of this type for sequences of Einstein manifolds whose curvature tensor is bounded in the $L^{n/2}$ sense: see for example Theorem 1.3 in [And1]. Similar results were shown independently by [BKN] (see Theorem 5.5 in [BKN]). See also [Tian]. In the paper [AnCh], the condition that the manifolds have Ricci curvature bounded from above and below or bounded Einstein constant was replaced by the condition that the Ricci curvature is bounded from below. To deal with this situation the authors introduced the $W^{1,p}$ harmonic radius, which we also use here.

To prove the convergence to an orbifold and to obtain information on the orbifold points we require *regularity estimates* for regions where

$\int_{{}^tB_r(x)} |\operatorname{Riem}(g(t))|^2 d\mu_{g(t)}$ is *small*. Regularity estimates in the static case (for example the Einstein case) were shown for example in Lemma 2.1 in [And2]. We show that for certain so called *good times* $t < T$, which are close enough to T, that if $\int_{{}^tB_{r(t)}(x)} |\operatorname{Riem}(g(t))|^2 d\mu_{g(t)} \leq \varepsilon_0$ is *small enough*, where $r(t) = R\sqrt{T-t}$ for some large $R > 0$, then we will have time dependent bounds on the metric on the ball ${}^tB_{r(t)/2}(x)$ for later times s, $t \leq s < T$: see Theorem 4.5 below for the explicit bounds (the constants ε_0, R appearing above, will not depend on x). That is, we have a **fixed set** ${}^tB_{r(t)/2}(x)$ **where we obtain our estimates for later times** $s \in [t, T)$ **(that is, the set** ${}^tB_{r(t)/2}(x)$ **doesn't depend on** s**).** Furthermore, we show that the metric $g(s)$ on the ball ${}^tB_{r(t)/2}(x)$ is C^0 close to the metric $g(l)$ on ${}^tB_{r(t)/2}(x)$ if $s, l \in [t, T)$ and $|s - l|$ is small enough.

In order to obtain our regularity estimates we require a number of ingredients. The estimates from the previous section, a slightly modified version of a result from [And1] and [AnCh] on the $W^{1,p}$ harmonic radius (see also Lemma 4.5 of [Petersen]), a Nash-Moser-de Giorgi argument, and the *Pseudolocality* result of G. Perelman (see Theorem 10.1 of [Pe1]) being the main ones. The Nash-Moser-de Giorgi argument which we use is a modified version of that given in the paper [Li]. The proofs in the paper of [Li] are written for a four dimensional setting, and can be adapted to our setting.

Before stating the theorem we introduce some notation, which we will also use in the subsequent sections of this paper.

Let $(M^4, g(t))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the basic assumptions. In Theorem 3.6 of [Si1], it was shown that

$$\int_S^R \int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} dt \leq K_0 = K_0(M, g_0, T) < \infty \tag{4.1}$$

for $S < R \leq T$. In particular, for any $0 < r < \frac{T}{4}$, and $1 \geq \sigma > 0$, we can find a $t \in [T - (1+\sigma)r, T - r]$ such that

$$\int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} \leq \frac{2K_0}{\sigma r} \tag{4.2}$$

If not, then we can find σ and r such that $\int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} > \frac{2K_0}{\sigma r}$ for all $t \in [T - (1+\sigma)r, T - r]$, and hence

$$\int_{T-(1+\sigma)r}^{T-r} \int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} > \sigma r \frac{2K_0}{\sigma r} = 2K_0$$

which contradicts equation (4.1).

If $t := T - r < T$ is given, where $r < \frac{T}{10}$, then the argument above shows that we can always find a (nearby) $\tilde{t} \in [T - 2r, T - r]$ such that

$$\int_M |\mathrm{Rc}|^4(\cdot, \tilde{t}) d\mu_{g(\tilde{t})} \leq \frac{2K_0}{r} = \frac{2K_0}{T - t} \leq \frac{4K_0}{T - \tilde{t}}. \tag{4.3}$$

A time $\tilde{t}$ which satisfies (4.3) will be known as a $4K_0$ *good time*. More generally, we make the following definition.

Definition 4.1. Let $(M, g(t))_{t \in [0,T)}$ be a smooth solution to Ricci flow. Any $t \in [0, T)$ which satisfies

$$\int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} \leq \frac{C}{T - t} \tag{4.4}$$

$(C > 0)$ shall be called a C-**good time**. If $C = 1$, then we call such a t a **good time**.

By modifying the above argument we see that the following is true.

Lemma 4.2. *Let $(M^4, g(t))_{t \in [0,T)}$ be a solution to Ricci flow satisfying the basic assumptions and let $C > 0$ be given. Then there exists an $\tilde{r} > 0$ such that for all $0 < r < \tilde{r}$ the following holds. For any $\tilde{t} \in [0, T)$ with $r := T - \tilde{t}$ there exists a $t \in [\tilde{t} - r, \tilde{t}] = [T - 2r, T - r]$ which is a C good time.*

Remark 4.3. $\tilde{r}$ will possibly depend on C, $(M, g(0))$ and T as can be seen in the proof below.

Proof. Fix $C > 0$ and assume the conclusion of the theorem doesn't hold. Then we can find a sequence $r_i \to 0$ and $\tilde{t}_i := T - r_i \nearrow T$ such that every $t \in [T - 2r_i, T - r_i]$ is **not** a C good time. That is $\int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} > \frac{C}{T-t}$ for all $t \in [T - 2r_i, T - r_i]$. Integrating in time from $T - 2r_i$ to $T - r_i$ we get

$$\int_{T-2r_i}^{T-r_i} \int_M |\mathrm{Rc}|^4(\cdot, t) d\mu_{g(t)} dt > C \int_{T-2r_i}^{T-r_i} \frac{1}{T - t} dt \geq \frac{C}{2r_i} \int_{T-2r_i}^{T-r_i} dt = \frac{C}{2}.$$

Without loss of generality the intervals $[T-2r_i, T-r_i]_{i\in\mathbb{N}}$ are pairwise disjoint (since $r_i \to 0$). Summing over $i \in \mathbb{N}$ we get

$$\int_0^T \int_M |\mathrm{Rc}|^4(\cdot,t)d\mu_{g(t)}dt \geq \sum_{i=1}^{\infty} \int_{T-2r_i}^{T-r_i} \int_M |\mathrm{Rc}|^4(\cdot,t)d\mu_{g(t)}dt$$
$$\geq \sum_{i=1}^{\infty} \frac{C}{2} = \infty$$

which contradicts the fact that $\int_0^T \int_M |\mathrm{Rc}|^4(\cdot,t)d\mu_{g(t)}dt < \infty$. □

Let $0 < t_i \nearrow T$, $i \in \mathbb{N}$ be a sequence of times approaching T from below. We wish to show that $(M, g(t_i)) \to (X, d)$ as $i \to \infty$ in some to be characterised sense, where (X, d) is a C^0 Riemannian orbifold with only finitely many orbifold points. These orbifold points will be characterised by the fact that they are points where the L^2 integral of curvature concentrates as $t_i \nearrow T$. To explain this more precisely we introduce some notation.

Definition 4.4. Let $(M^4, g(t))_{t\in[0,T)}$ be a solution to Ricci flow with $T < \infty$ satisfying the basic assumptions. A point $p \in M$ is a *regular point in* M (or $p \in M$ is *regular*) if there exists an $r = r(p) > 0$ such that

$$\int_{{}^tB_r(p)} |\,\mathrm{Riem}\,|^2(\cdot,t)d\mu_{g(t)} \leq \varepsilon_0$$

for all times $t \in [0,T)$, where $\varepsilon_0 > 0$ is a small fixed constant depending on $(M^4, g(0))$ and T, which will be specified in the proof of Theorem 4.5 below. A point $p \in M$ is a *singular point in* M (or $p \in M$ is *singular*) if $p \in M$ is not a regular point. In this case, due to smoothness of the flow on $[0,T)$, there must exist a sequence of times $s_i \nearrow T$ and a sequence of numbers $0 < r_i \searrow 0$ as $i \to \infty$ such that $\int_{{}^{s_i}B_{r_i}(p)} |\,\mathrm{Riem}\,|^2 > \varepsilon_0$ for all $i \in \mathbb{N}$. We denote the set of regular points in M by $\mathrm{Reg}(M) := \{p \in M \mid p \text{ is regular }\}$ and the set of singular points in M by $\mathrm{Sing}(M) := \{p \in M \mid p \text{ is singular }\}$.

In this section we obtain information about regular points. In particular we will give another characterisation of the property *regular*. This characterisation is implied by the following theorem (see the Corollary directly after the statement of the Theorem).

Theorem 4.5. *Let $k \in \mathbb{N}$ be fixed, and let $(M, g(t))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the basic assumptions. There exists a (large) constant*

$R > 0$, and (small) constants $v, \varepsilon_0 > 0$, and constants $c_1, \dots, c_k$ such that if

$$\int_{{}^tB_{R\sqrt{T-t}}(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \varepsilon_0 \tag{4.5}$$

for a good time t which satisfies $|T - t| \leq v$, then p is a regular point. We also show that if p, t satisfy these conditions, then

$$\exp\left(-\frac{8|r^{\frac{1}{4}} - s^{\frac{1}{4}}|}{(T-t)^{\frac{1}{4}}}\right) g(r) \leq g(s) \leq \exp\left(\frac{8|r^{\frac{1}{4}} - s^{\frac{1}{4}}|}{(T-t)^{\frac{1}{4}}}\right) g(r), \text{ and} \tag{4.6}$$

$$\frac{1}{2} g(r) \leq g(s) \leq 2g(t) \quad \forall\, t \leq r, s < T, \text{ on } {}^tB_{\frac{R}{2}\sqrt{T-t}}(p) \tag{4.7}$$

$$|\nabla^j \operatorname{Riem}(x, s)|^2_{g(s)} \leq \frac{c_j}{(T-t)^{j+2}} \tag{4.8}$$

$$\forall\, t + \frac{(T-t)}{2} \leq s < T, x \in {}^tB_{\frac{R}{2}\sqrt{T-t}}(p), \tag{4.9}$$

$$\forall\, j \in \{0, \dots, k\}. \tag{4.10}$$

The constants ε_0, R and v depend only on σ_0, σ_1 from (3.1), A from (3.5), and $c(g(0), T)$ from Theorem 4.5, the constants c_j depend only on j, σ_0, σ_1, A and $c(g(0), T)$. That is, all constants depend only on $(M, g(0))$ and T.

For such p and t we therefore have: all $x \in {}^tB_{R\sqrt{T-t}/2}(p)$ are also regular (see the proof for an explanation), and there is a limit in the smooth sense (and hence also in the Cheeger-Gromov sense) of $({}^tB_{\frac{R}{2}\sqrt{T-t}}(p), g(s))$ as $s \nearrow T$.

Remark 4.6. The condition $\int_{{}^tB_{R\sqrt{T-t}}(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \varepsilon_0$ for a good time t which satisfies $|T - t| \leq v$ (v , ε_0 as in the statement of the Theorem above) therefore implies that p *is regular* (see the proof for an explanation). This new condition contains however more information, namely that the estimates appearing in the statement of Theorem 4.5 hold. Furthermore: to show that a point $p \in M$ is regular, we only need to find **one** good time t with $|T - t| < v$ for which $\int_{{}^tB_{R\sqrt{T-t}}(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \varepsilon_0$. We do **not** need to show that $\int_{{}^tB_{r(p)}(p)} |\operatorname{Riem}|^2(\cdot, t) d\mu_{g(t)} \leq \varepsilon_0$ for all $t < T$ for some fixed $r(p) > 0$.

This characterisation is useful when it comes to showing that a limit space (in a sense which will be explained later in this paper) $(X, d_X) := \lim_{t \nearrow T}(M, g(t))$ exists and when it comes to describing its structure.

Definition 4.7. Let $t \in (0,T)$. We say $p \in \mathrm{Reg}_t(M)$ if

$$\int_{{}^tB_{R\sqrt{T-t}}(p)} |\,\mathrm{Riem}\,|^2(\cdot,t)d\mu_{g(t)} \le \varepsilon_0,$$

where ε_0, R are from the above theorem.

Remark 4.8. Notice that this condition is scale invariant: if $(M,\tilde{g}(\tilde{t}))_{\tilde{t}\in[0,\tilde{T})}$ is the solution we get by setting $\tilde{g}(\tilde{t}) := cg(\frac{\tilde{t}}{c})$, $\tilde{T} := cT$, $\tilde{t} = ct$, then

$$\int_{{}^{\tilde{t}}B_{R\sqrt{\tilde{T}-\tilde{t}}}(p)} |\widetilde{\mathrm{Riem}}|^2(\cdot,\tilde{t})d\mu_{\tilde{g}(\tilde{t})} = \int_{{}^tB_{R\sqrt{T-t}}(p)} |\,\mathrm{Riem}\,|^2(\cdot,t)d\mu_{g(t)} \le \varepsilon_0 \tag{4.11}$$

Corollary 4.9. *Theorem 4.5 above shows us that* $\mathrm{Reg}_t(M) \subseteq \mathrm{Reg}(M)$ *for all good times* $t \in (T-v,T)$*. From the definition of* $\mathrm{Reg}(M)$ *we also see: for all* $p \in \mathrm{Reg}(M)$ *there exists a* $T - v < S(p) < T$ *such that* $p \in \mathrm{Reg}_t(M)$ *for all good times* t *with* $t \in (S(p),T)$*. Furthermore, Theorem 4.5 above also tells us, that for every good time* $t \in (T-v,T)$*, and for all* $\varepsilon > 0$*, there exists a* $\delta > 0$ *(depending on* t*), such that*

$$\begin{aligned}&(1-\varepsilon)g(p,s) \le g(p,r) \le (1+\varepsilon)g(p,s)\\ &\quad\forall p \in \mathrm{Reg}_t(M), \quad \text{for all} \quad r,s \in (t,T) \quad \textit{with} \quad |r-s| \le \delta.\end{aligned} \tag{4.12}$$

Corollary 4.10. *For all good times* $t \in (T-v,v)$ *for all* $p \in \mathrm{Reg}_t(M)$*, where* v *and* $\mathrm{Reg}_t(M)$ *are as above, we have*

$$\begin{aligned}&\frac{1}{8}d(x,y,r) \le d(x,y,s) \le 8d(x,y,r)\\ &\qquad \text{for all} \quad r,s \in [t,T), \quad \text{for all} \quad x,y \in {}^tB_{\frac{R}{200}\sqrt{T-t}}(p)\end{aligned} \tag{4.13}$$

Proof of Theorem 4.5. Let $t_i \nearrow T$ be a sequence of good times. We scale (blow up) and shift (in time) the solution g as follows: $g_i(t) := \frac{1}{T-t_i}g(\cdot,T + t(T-t_i))$. Then we have a solution which is defined for $t \in [-A_i := -\frac{T}{T-t_i}, 0)$ and $A_i \to \infty$ as $i \to \infty$. Furthermore, using the fact that the t_i are good times (for the solution before scaling), we see that

$$\begin{aligned}\int_M |\mathrm{Rc}(g_i(-1))|^4 d\mu_{g_i(-1)} &= (T-t_i)^2 \int_M |\mathrm{Rc}(g(t_i)|^4(t)d\mu_{g(t_i)}\\ &\le \delta_i := (T-t_i) \to 0\end{aligned} \tag{4.14}$$

as $i \to \infty$. The scale invariant inequalities (3.1) are also valid for $g_i(-1)$.

Let $B_R(p) = {}^{g_i(-1)}B_R(p) \subseteq M$ be an arbitrary ball with

$$\int_{{}^{g_i(-1)}B_R(p)} |\operatorname{Riem}|^2 d\mu_{g_i(-1)} \leq \varepsilon_0 \quad \text{and} \quad R \geq 4 > 0.$$

Scaling by $\delta = \frac{4}{R^2} < 1$ (*), (that is $\tilde{g}_i(\tilde{t}) := \delta g_i(\frac{\tilde{t}}{\delta})$: we call the solution $\tilde{g}_i(\tilde{t})$ once again $g_i(t)$) we see that

(a) $\int_{{}^{g_i(-\delta)}B_2(p)} |\operatorname{Riem}|^2 d\mu_{g_i(-\delta)} \leq \varepsilon_0$ and $\int_M |\mathrm{Rc}|^4 d\mu_{g_i(-1)} \leq \tilde{\delta}_i$, where $\tilde{\delta}_i := \delta_i/\delta^2 = (T - t_i)/\delta^2 \to 0$ as $i \to \infty$

(b) we have control over non-inflating constants and non-decreasing constants: $\sigma_0 r^4 \leq \operatorname{vol}({}^{g_i(-\delta)}B_r(x)) \leq \sigma_1 r^4$ for all $r \leq r_i \to \infty$ and for all $x \in {}^{g_i(-\delta)}B_2(p)$.

The works of Anderson [And1] and Deane Yang [YangD] imply that $B_1(p)$ is in some $C^{0,\alpha}$ sense close to euclidean space if ε_0 is small enough, and $i \in \mathbb{N}$ is large enough (that is if $\delta_i = (T - t_i)$ is small enough). This is a fact about smooth Riemannian manifolds satisfying (a) and (b), and has nothing to do with the Ricci flow.

We state below a quantitative version of this fact.

Theorem 4.11. *Let (M^4, g) be a smooth connected manifold without boundary (not necessarily complete) and $B_2(p) \subseteq M$ be an arbitrary ball which is compactly contained in M. Assume that*

(a) $\int_{B_2(p)} |\operatorname{Riem}|^2 d\mu_g \leq \varepsilon_0$ *and* $\int_M |\mathrm{Rc}|^4 d\mu_g \leq 1$,

(b) $\sigma_0 r^4 \leq \operatorname{vol}(B_r(x)) \leq \sigma_1 r^4$ *for all* $r \leq 1$, *for all* $x \in B_2(p)$,

where $\varepsilon_0 = \varepsilon_0(\sigma_0, \sigma_1) > 0$ is sufficiently small. Then there exists a constant $V = V(\sigma_0, \sigma_1) > 0$ and an $\alpha = \alpha(\sigma_0, \sigma_1) > 0 \in (0,1)$ such that the following is true. For any $y \in B_{3/2}(p)$, we can find coordinates $\varphi : U \subseteq {}^gB_1(0) \to {}^\delta B_V(0) \subseteq \mathbb{R}^n$, such that $|g_{ij}(x) - \delta_{ij}| \leq |x|^\alpha$ for all $x \in {}^\delta B_V(0)$.

Proof. The claim of the Theorem follows from the fact that harmonic coordinates can be constructed in this setting. See Appendix B of [SiArxiv], for example, for the details. □

Using Perelman's definition of *almost euclidean* (see Theorem 10.1 in [Pe1] for the definition of *almost euclidean*) we see that there is a constant $1 > a = a(V) = a(\sigma_0, \sigma_1) > 0$ such that ${}^{g_i(-\delta)}B_a(y)$ is almost euclidean if $(T - t_i)/\delta^2 \leq 1$. Notice that a doesn't depend on δ and hence, without loss of generality $\delta \ll a$: $\delta = \frac{4}{R^2}$ and $R > 0$ was arbitrary up until this point, so we choose $R^2 \gg \frac{1}{4a}$. Perelman's first Pseudolocality result (Theorem 10.1 in [Pe1]) now tells us that

$$(4.15) \qquad |\operatorname{Riem}(g_i)(x,t)| \leq \frac{1}{\delta + t}, \quad \text{for all} \;\; t \in (-\delta, 0), \;\; x \in {}^{g_i(t)}B_{\tilde{a}}(y)$$

for some constant $\tilde{a} = \tilde{a}(a) > 0$, for all $y \in {}^{-\delta}B_1(p)$. Here we use that $\delta \ll a$ that is $0 < \delta = \delta(V) \ll a(V)$ is chosen small so that the Pseudolocality Theorem applies on the whole time interval $(-\delta, 0)$. Without loss of generality $\delta \ll \tilde{a}$ also. Now $\delta = \delta(V)$ is fixed (and small), that is $R = R(V) = \frac{2}{\sqrt{\delta}} \gg 1$ is fixed (and large). Scaling back to $t = -1$ (that is we set $\tilde{g}_i(\tilde{t}) = \frac{R^2}{4} g_i(\frac{4\tilde{t}}{R^2})$ so that we are dealing with the solution we had before blowing down at the point (*) of the argument above: we call the solution $\tilde{g}_i(\tilde{t})$ once again $g_i(t)$ for ease of reading) we have

$$(4.16) \qquad |\operatorname{Riem}(g_i)(x,t)| \leq \frac{1}{1 + t}, \quad \text{for all} \;\; t \in (-1, 0), \;\; x \in {}^{g_i(t)}B_{\frac{R\tilde{a}}{2}}(y)$$

for all $y \in {}^{g_i(-1)}B_{\frac{R}{2}}(p)$. Using Shi's estimates (see [Shi]), the non-inflating and non-collapsing estimates, the evolution equation $\frac{\partial}{\partial t} g = -2\mathrm{Rc}$, and the injectivity radius estimate of Cheeger-Gromov-Taylor (Theorem 4.3 in [CGT]), we get

$$(4.17) \qquad |\nabla^j \operatorname{Riem}(g_i)(y,t)| \leq A_j, \quad \text{for all} \;\; t \in \left(-\frac{1}{2}, 0\right),$$

for all $0 \leq j \leq K$ where $K \in \mathbb{N}$ is fixed and large and $A_j < \infty$ is a constant, for all $y \in {}^{-1}B_{\frac{R}{2}}(p)$, as long as $R\tilde{a}$ is sufficiently large: as we chose $\delta \ll \tilde{a}$, this is without loss of generality the case. Translating in time and scaling back to the original solution, we obtain the claimed curvature estimates (4.8).

We explain why all $y \in {}^{g_i(-1)}B_{R/2}(p)$, are regular (in particular, p is regular). Choose t close to 0 and $0 < r \leq 1$ small, so that ${}^tB_{10^4 r}(y) \subseteq {}^{g_i(-1)}B_{\frac{R}{2}}(p)$: for every $t < 0$ such an r must exist in view of the fact that the solution is smooth. Then $|\operatorname{Riem}(\cdot, t)| \leq 10$ on ${}^tB_{10^4 r}(y) \subseteq {}^{g_i(-1)}B_{\frac{R}{2}}(p)$ due to (4.16). Then ${}^sB_r(y)$ remains in ${}^tB_{10^4 r}(y) \subseteq {}^{g_i(-1)}B_{\frac{R}{2}}(p)$ for all $s \in [t, 0)$ due to (4.16) and the fact that the metric evolves according to the equation $\frac{\partial}{\partial t} g = -\mathrm{Rc}(g)$, and t is close to 0. Hence $\int_{{}^sB_r(y)} |\operatorname{Riem}(g)|^2(\cdot, s) d\mu_{g(s)} \leq \varepsilon_0$ for all $s \in [t, 0)$, if r is small enough, in view of (4.16) and the non-expanding estimate.

Although these estimates show us that p is a regular point, they do **not** tell us that

$$\int_{{}^tB_{R/2}(p)} |\operatorname{Riem}|^2(\cdot,t)d\mu_{g(t)} \leq \varepsilon_0$$

for all $t \in (-1,0)$: as t gets closer to -1 from above, our estimates on the curvature, (4.16), blow up. However by appropriately modifying the arguments in [Li] we can show that the Riemannian metrics remain close in a C^0 sense to one another on some fixed time independent region within these balls. This fact is useful when it comes to describing (X, d_X) , the limit as $t \nearrow T$ (before scaling) of the solution $(M, g(t))_{t\in[0,T)}$, and how this limit is obtained.

Examining the setup considered in the first part of the paper [Li] of Ye Li, we see that we are *almost* in the same setup: Scale back down to $t = -\delta$ (that is do the step (*) in the argument above again), call the solution g_i again, and consider an arbitrary $y \in {}^{g_i(-\delta)}B_1(p)$ as above.

From the argument above we have

$$|\operatorname{Riem}(g_i)(x,t)| \leq \frac{1}{\delta+t}, \quad \text{for all} \ \ t \in (-\delta, 0) \text{ for all } \ x \in {}^{g_i(t)}B_{\tilde{a}}(y) \tag{4.18}$$

for some constant $\tilde{a} = \tilde{a}(a) > 0$, and $\delta \ll \tilde{a} < a$.

In order to see that we are almost in the same situation as Ye.Li, we shift in time by δ: that is fix i and define $g(t) = g_i(t+\delta)$. This means that the old time 0 (where the flow possibly becomes singular) is now time δ and the old good time $-\delta$ is now the good time 0. Then we have

(i)

$$|\operatorname{Riem}(g)(x,t)| \leq \frac{1}{t}, \quad \text{for all} \ \ t \in (0,\delta) \text{ for all } \ x \in {}^{g(t)}B_{\tilde{a}}(y), \tag{4.19}$$

for all $y \in {}^{g_i(0)}B_1(p)$, where $\tilde{a}$ depends only on a which depends only on σ_0, σ_1, and we have chosen δ so that $\delta \ll \tilde{a} \leq a$. Without loss of generality, we may assume $\tilde{a} = 2$ for this argument. If not, then scale so that it is: we still have $0 < \delta \ll \tilde{a}$ is still as small we we like (but fixed).

This solution also satisfies

(ii) $\int_M |\mathrm{Rc}|^4(\dot{,}0)(g_0) \leq \hat{\delta}_i$, with $\hat{\delta}_i \to 0$ as $i \to \infty$ (by scaling we have changed the constants $\tilde{\delta}_i$ above by a fixed factor: $\hat{\delta}_i = \frac{\tilde{\delta}_i}{(10\tilde{a})^2}$).

(iii) $(1/2)d\mu_{g(r)} \leq d\mu_{g(t)} \leq 2d\mu_{g(s)}$ for all $0 \leq r \leq t \leq s < \delta$ in view of the fact that we are dealing with a solution satisfying the basic assumptions (see the inequalities (3.8)),

(iv) we have a bound on the Sobolev constant $(\int_{B_r(z)} f^4)^{1/2} \leq A \int_{B_r(z)} |\nabla f|^2$ for all ${}^tB_r(z) \subseteq {}^tB_2(y)$ for all $f : B_r(z) \to \mathbb{R}$ which are smooth and have compact support in ${}^tB_r(z)$, for all $0 \leq t < \delta$: see (3.4) and (3.5).

(v)

$$\left(\int_{{}^tB_2(y)} |\operatorname{Riem}|^3\right)^{\frac{1}{3}} \leq \frac{1}{t^{\frac{1}{3}}}\left(\int_M |\operatorname{Riem}|^2\right)^{\frac{1}{3}} \leq \frac{1}{t^{\frac{1}{3}}}(K_0)^{1/3} \tag{4.20}$$

for all $0 \leq t < \delta$ in view of (i) and the bound $\int_M |\operatorname{Riem}|^2 \leq K_0 := c(g(0), M, T)$ from (2.1).

Examining Lemma 1, Lemma 2, Lemma 3 and Theorem 2 of [Li], we see that this is exactly the setup of that paper, call $\mu := (K_0)^{\frac{1}{3}}$, except for the condition $1/2g_0 < g(t) < 2g(s)$ for all $0 < t < s < \delta$, which is also assumed there. We are considering the case that u and f of the paper by Ye Li are $u := |\operatorname{Riem}|$ and $f := |\mathrm{Rc}|$. The argument in the paper of [Li] is a Nash-Moser type iteration argument applied to the parabolic equation satisfied by the function $f = |\mathrm{Rc}|$. The extra assumption $1/2g_0 < g(t) < 2g(s)$ for all $0 < t < s < \delta$ is used in [Li] to construct a time independent cut-off function (in Lemma 3 of [Li], which is also used in Lemma 1 and Lemma 2 of [Li]) for $0 < r' < r$. This cut-off function $\varphi : M \to \mathbb{R}$ is smooth and satisfies $\varphi|_{B_{r'}(y)} = 1$, $\varphi = 0$ on $(B_r(y))^c$, $|\nabla\varphi|_{g_0} \leq \frac{2}{r-r'}$ and $|\nabla\varphi|_{g(t)} \leq 2|\nabla\varphi|_{g_0} \leq \frac{4}{r-r'}$. We will only consider $1 \geq r, r' \geq \frac{1}{4}$. In order to obtain the results of Ye Li, we replace this function by a time dependent cut-off function $\varphi(x,t)$ using the method of Perelman. This new φ satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}\varphi \leq \Delta\varphi + \frac{c}{(r-r')^2} + \frac{c\varphi}{t} \\ &|\nabla\varphi|^2_{g(t)} \leq \frac{c}{(r-r')^2} \\ &\varphi|_{{}^tB_{r'}(y)} = 1, \\ &\varphi|_{({}^tB_r(y))^c} = 0, \end{aligned} \tag{4.21}$$

for all $t \leq S(c_1)$, wherever the function differentiable is, where $S(c_1) > 0$ and $c = c(c_1)$, where c_1 is a constant satisfying $|\operatorname{Riem}| \leq \frac{c_1}{t}$ on ${}^tB_4(y)$: in our case $c_1 = 1$. Using this new φ in the argument given in [Li], we obtain, after making necessary modifications, the following:

$$|\mathrm{Rc}(\cdot,t)| \leq \frac{\delta^4}{t^{3/4}} \text{ on } {}^tB_{3/4}(y), \quad \text{for all } t \in (0,\delta), \tag{4.22}$$

as long as $(T - t_i) \leq \alpha(\sigma_0, \sigma_1, c(g(0), T), A)$ is small enough. See Appendix A in [SiArxiv] for the details. In particular, translating and scaling back to the solution we had before we performed the step (*), we see that $|\mathrm{Rc}(y,t)| \leq \frac{\delta}{(t+1)^{3/4}}$ for all $y \in {}^{g_i(-1)}B_{R/2}(p)$, for all $t \in (-1, 0)$. Hence, integrating the evolution equation $\frac{\partial}{\partial t} g(y,s) = -2\mathrm{Rc}(g)(y,s)$, we get

$$g(y,s)e^{-8\delta|s^{\frac{1}{4}} - r^{\frac{1}{4}}|} \leq g(y,r) \leq g(y,s)e^{8\delta|s^{\frac{1}{4}} - r^{\frac{1}{4}}|} \tag{4.23}$$

for all $r, s \in [-1, 0)$, for all $y \in {}^{g_i(-1)}B_{R/2}(p)$, where $\delta > 0$ is small. Translating in time and scaling back to the original solution, we obtain (4.6). Before scaling back, note that it also implies

$$\frac{1}{2} g(y,s) \leq g(y,r) \leq 2g(y,s) \tag{4.24}$$

for all $r, s \in [-1, 0)$, for all $y \in {}^{g_i(-1)}B_{R/2}(p)$. This condition is scale invariant, so translating and scaling back to the original solution, we obtain (4.7).

For later, notice, that (4.23) implies that: for all $\sigma > 0$, there exists a $\tilde{\delta} > 0$ such that,

$$g(\cdot, s)(1 - \sigma) \leq g(\cdot, r) \leq g(\cdot, s)(1 + \sigma) \tag{4.25}$$

for all $r, s \in (-1, 0]$ with $|r - s| \leq \tilde{\delta}$ on ${}^{-1}B_{\frac{R}{2}}(p)$. Examining the argument above, we see that the results are correct for *any* good time $t_i \in (0, T)$, as long as $(T - t_i) \leq v(\sigma_0, \sigma_1, A, c(g(0), T))$ is small enough. This finishes the proof. □

Proof of the Corollary 4.10. Let $x, y\ t, s$ be as in the statement of the corollary. Scale to the situation as in the proof of Theorem 4.5. Let $\gamma : [0, 1] \to M$ be a length minimising geodesic between x and y with respect to the metric $g(-1)$. The curve doesn't leave ${}^{-1}B_{\frac{R}{10}}(p)$, and hence, using (4.24), $d(x, y, s) \leq L_s(\gamma) \leq 2L_{-1}(\gamma) = 2d(x, y, -1)$. Now let $\sigma : [0, 1] \to M$ be a length minimising geodesic between x and y with respect to $g(s)$. If σ doesn't leave ${}^{-1}B_{\frac{R}{10}}(p)$, then $d(x, y, -1) \leq L_{-1}(\sigma) \leq 2L_s(\sigma) = 2d(x, y, s)$, and hence $d(x, y, s) \geq \frac{1}{2} d(x, y, -1)$ in this case. If σ leaves ${}^{-1}B_{\frac{R}{10}}(p)$, then let m be the first point at which it does so: $\sigma(m) \in \partial({}^{-1}B_{\frac{R}{10}}(p))$, $\sigma(r) \in {}^{-1}B_{\frac{R}{10}}(p)$ for all $r < m$, and consider $\alpha = \sigma|_{[0,m]}$. Then $d(x, y, s) = L_s(\sigma) \geq$

$L_s(\alpha) \geq \frac{1}{2}L_{-1}(\alpha) \geq \frac{1}{100}R = \frac{1}{2}\frac{2R}{100} \geq \frac{1}{2}d(x,y,-1)$. Hence

$$d(x,y,s) \geq \frac{1}{2}d(x,y,-1)$$

in this case as well. □

5. Behaviour of the flow near singular points

In this section we examine the behaviour of the flow near singular points p. We consider a sequence of good times $t_i \nearrow T$. We will show that the singular set $\mathrm{Sing}(M)$ can be covered by L balls $({}^{t_i}B_{R(t_i)}({}^ip_j))_{j=1}^L$ (L being independent of t_i) of radius $R(t_i) = C\sqrt{T-t_i}$ (C a large fixed constant, which is determined in the proof of Theorem 5.1 below) at time t_i, where t_i are good times close enough to T , and that the balls ${}^tB_{R(t_i)}({}^ip_j)$ with $t \in (t_i,T)$ also cover $\mathrm{Sing}(M)$. We say nothing at this stage about the topology of these regions, or how they geometrically look. In the next sections we give more information on how singular regions look like in the limit (as $t \nearrow T$).

The results of this section are used at the end of this section to show that distance is uniformly continuous in the following sense: For all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|d(x,y,t) - d(x,y,s)| \leq \varepsilon$ for all $x,y \in M$ for all $t,s \in [0,T)$ with $|t-s| \leq \delta$. The singular set and the regular set were defined in the previous section: $\mathrm{Reg}(M) := \{p \in M \mid p \text{ is regular}\}$ was defined in Definition 4.4 and $\mathrm{Reg}_t(M)$ was defined in Definition 4.7. $\mathrm{Sing}(M) := \{p \in M \mid p \text{ is } \textbf{not} \text{ regular}\}$. The theorem that we prove in this section is

Theorem 5.1. *Let $(M,g(t))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the basic assumptions. Then there exist (large) constants $0 < J_0, J_1, J_2 < \infty$, a (small) constant $0 < w < \infty$, and a constant $L \in \mathbb{N}$ such that for all good times $s < T$ with $|s-T| \leq w$, there exist $p_1(s),\ldots,p_L(s) \in M$ such that*

$$\begin{aligned}\mathrm{Sing}(M) &= (\mathrm{Reg}(M))^c\\ &\subseteq (\mathrm{Reg}_s(M))^c\\ &\subseteq \cup_{j=1}^L {}^tB_{J_0\sqrt{T-s}}(p_j(s))\\ &\subseteq \cup_{j=1}^L {}^sB_{J_1\sqrt{T-s}}(p_j(s))\\ &\subseteq \cup_{j=1}^L {}^rB_{J_2\sqrt{T-s}}(p_j(s))\end{aligned} \tag{5.1}$$

for all $s \leq t, r < T$.

Remark 5.2. Notice that for fixed s, the sets ${}^{s}B_{J_1\sqrt{T-s}}(p_j(s))$ in the statement of the theorem don't depend on t or r $(s \leq t, r < T)$, but ${}^{t}B_{J_0\sqrt{T-s}}(p_j(s))$ and ${}^{r}B_{J_2\sqrt{T-s}}(p_j(s))$ do.

Remark 5.3. Using the estimates of the previous section and this covering, we will obtain as a corollary, that the distance function is uniformly continuous in time (see Theorem 5.6).

Proof. Let (M,h) be a Riemannian manifold with $\int_M |\operatorname{Riem}(h)|^2 \leq K_0 < \infty$. Let $R > 0$ be given fixed. Assume there is some point p_1 with $\int_{B_R(p_1)} |\operatorname{Riem}|^2 \geq \varepsilon_0$. Then we look for a ball $B_R(p_2)$ which is disjoint from $B_R(p_1)$ and satisfies $\int_{B_R(p_2)} |\operatorname{Riem}|^2 \geq \varepsilon_0$. We continue in this way as long as it is possible to do so. This leads to a family of pairwise disjoint balls $(B_R(p_j))_{j\in\{1,\ldots,L\}}$ such that $\int_{B_R(p_j)} |\operatorname{Riem}|^2 \geq \varepsilon_0$ for all $j \in \{1,\ldots,L\}$. We define

$$\mathcal{B}_R := \mathcal{B}_R(h) := \cup_{j=1}^{L} B_{2R}(p_j)$$

$$\Omega_R := \Omega_R(h) := M \backslash \mathcal{B}_R(h) = M \backslash \cup_{j=1}^{L} B_{2R}(p_j). \tag{5.2}$$

From the definition of Ω_R it follows that $\int_{B_R(x)} |\operatorname{Riem}|^2 \leq \varepsilon_0$ for all $x \in \Omega_R$.

Using $\int_M |\operatorname{Riem}|^2(h)d\mu_h \leq K_0$, we see that we have an upper bound $L \leq \frac{K_0}{\varepsilon_0}$ for the number of balls constructed in this way.

Notice that for fixed R this construction is not unique: by choosing the balls in the construction differently we obtain a different $\mathcal{B}_R$, respectively Ω_R.

If $(M, g(t))_{t\in I}$ is a solution to Ricci flow, I an interval, then ${}^{t}\Omega_R$ will denote $\Omega_R(g(t))$ and ${}^{t}\mathcal{B}_R$ will denote $\mathcal{B}_R(g(t))$ for any $t \in I$. Take a sequence of good times $t_i \nearrow T$, and assume we have scaled as in the proof Theorem 4.5 above, to obtain a solution $(M, g(t))_{t\in(-A_i,0)}$. Using the characterisation of the regular set given in Theorem 4.5, and using the R appearing there, we see that ${}^{-1}\Omega_R \subseteq \operatorname{Reg}(M)$ and hence $\operatorname{Sing}(M) = M\backslash\operatorname{Reg}(M) \subseteq M\backslash{}^{-1}\Omega_R$.

We wish to show that distance is not changing too rapidly near and in $\mathcal{B}_R(g(t=-1))$.

In order to explain this statement more precisely, and to explain the argument which proves the statement, we assume for the moment that there is only one ball ${}^{-1}B_{2R}({}^{i}p_1)$ coming from the above construction of $\mathcal{B}_R(g(-1))$ and we call this ball ${}^{-1}B_{2R}(p)$. Note that for each i, we may obtain a different point ${}^{i}p_1$ depending on i. For the moment we drop the i and the 1 from our notation and simply denote the point ${}^{i}p_1$ by p.

We define $G := {}^{-1}B_{2J}(p)$, for some large $J \gg R$ fixed, and $H := {}^{-1}B_J(p)$. It follows, that $H^c \subseteq ({}^{-1}B_{2R}(p))^c = (M \backslash {}^{-1}B_{2R}(p)) \subseteq \mathrm{Reg}_{-1}(M) \subseteq \mathrm{Reg}(M)$. Hence, using (4.24), we have $\frac{1}{8}g(x,t) \leq g(x,-1) \leq 8g(x,t)$ for all $x \in H^c \cap G$ for all $t \in [-1,0)$.

We may assume that $\mathrm{dist}(g(t=-1))(\partial G, \partial H) = J$ (we are scaling a connected solution to Ricci flow with diameter larger than $\frac{1}{d_0} > 0$ (see (3.9)) by constants c_i which go to infinity, and hence the diameter of the resulting solution is as large as we like at all times).

Note by construction $G \cap H^c \neq \emptyset$ as the diameter of the solutions we are considering is as large as we like, as we just noted. We have $J8 \geq \mathrm{dist}(g(t))(\partial G, \partial H) \geq \frac{J}{8}$ for all $t \in (-1,0)$ as we explain now. Any smooth regular curve $\gamma : [0,1] \to M$ ($\gamma'(s) \neq 0$ for all $s \in [0,1]$) going from ∂H to ∂G which lies completely in $\overline{H^c} \cap \bar{G}$ and satisfies $L_{g(t)}(\gamma) \leq \mathrm{dist}(g(t))(\partial G, \partial H) + \varepsilon$ must satisfy the following:

$$
\begin{aligned}
L_{g(t)}(\gamma) &= \int_0^1 \sqrt{g(\gamma(r),t)(\gamma'(r),\gamma'(r))}dr \\
&\geq \frac{1}{8}\int_0^1 \sqrt{g(\gamma(r),t=-1)(\gamma'(r),\gamma'(r))}dr \\
&\geq \frac{1}{8}\,\mathrm{dist}(g(t=-1))(\partial G, \partial H) = \frac{J}{8} \qquad (5.3)
\end{aligned}
$$

in view of the definition of G and H. Hence

$$
\mathrm{dist}(g(t))(\partial G, \partial H) \geq \frac{J}{8} \qquad (5.4)
$$

for all $t \in [-1,0)$. Notice that this means

$$
{}^{t}B_{J/8}(p) \subseteq G, \qquad (5.5)
$$

since $p \in H$ implies that

$$
\mathrm{dist}(g(t))(p, \partial G) \geq \mathrm{dist}(g(t))(H, \partial G) = \mathrm{dist}(g(t))(\partial H, \partial G)
$$

which is larger than or equal to $J/8$ in view of equation (5.4). Similarly, for $z \in H^c \cap G$, let $\gamma : [0,1] \to M$ be the radial geodesic with respect to the metric at time $t = -1$ coming out of p, starting at z and stopping at ∂G.

We have $\gamma([0,1]) \subseteq H^c \cap \bar{G}$ and hence

$$\begin{aligned} \operatorname{dist}(g(t))(z,\partial G) &\leq L_{g(t)}(\gamma) = \int_0^1 \sqrt{g(\gamma(r),t)(\gamma'(r),\gamma'(r))}dr \\ &\leq 8\int_0^1 \sqrt{g(\gamma(r),t=-1)(\gamma'(r),\gamma'(r))}dr \\ &\leq 8L_{g(-1)}(\gamma) \leq 8J \end{aligned} \tag{5.6}$$

That is,

$$\begin{aligned} &\operatorname{dist}(g(t))(z,\partial G) \leq 8J \text{ for all } z \in H^c \cap G, t \in [-1,0) \quad \text{and} \\ &\operatorname{dist}(g(t))(\partial G,\partial H) \leq 8J \text{ for all } t \in [-1,0) \end{aligned} \tag{5.7}$$

We wish to show that $\operatorname{dist}(p,\partial G,t)$ is bounded by a constant independent of time.

Claim: $\operatorname{dist}(p,\partial G,t) \leq J^5$**.**

Assume that there is some time $t \in (-1,0)$ with $\operatorname{dist}(p,\partial G,t) = N \geq J^5$. Choose $q \in \partial G$ such that $d(p,q,t) = N$. This part of the argument was inspired by the argument given in the proof of Claim 5.1 in the paper [Topping]. Take a distance minimising geodesic $\gamma : [0,N] \to M$ from p to q, at time t, which is parameterised by arclength. Consider points

$$z_0 := \gamma(0), z_1 := \gamma(1), z_2 := \gamma(2), \ldots, z_N := \gamma(N) = q.$$

Without loss of generality $J \in \mathbb{N}$. From the above, we see that the first $N-16J$ points $z_0,\ldots,z_{N-16J}$ must lie in H, as we now explain. If not, then let $z_i = \gamma(i)$ be the first point with $i \leq N-16J$ such that $z_i \notin H$. Then we could join the point $z_{i-1} = \gamma(i-1)$ to ∂G by a geodesic whose length w.r.t to $g(t)$ is less than $8J+2$, in view of (5.7). This would result in a path from p to ∂G at time t whose length is less than N which is a contradiction. Also, using equation (5.4), we see that ${}^tB_1(z_i) \subseteq G$ for all $0 \leq i \leq N-16J-1$ ($z_i \in H$ for such i, so to reach ∂G we must first reach ∂H and then reach ∂G: any such path must have length larger than $J/8 \gg 1$).

For $i \in \{1,\ldots,N-1\}$, the ball ${}^tB_1(z_i)$ is disjoint from all other balls ${}^tB_1(z_j)$, for all $j \in \{0,\ldots,N\}$ except for its two immediate neighbours ${}^tB_1(z_{i-1})$ and ${}^tB_1(z_{i+1})$, since γ is distance minimising implies $\gamma|_I$ is distance minimising for all intervals $I \subseteq [0,N]$. Hence: for $i \neq 0$ we have ${}^tB_1(z_i) \cap {}^tB_1(z_j) = \emptyset$ as long as $j \neq i-1$ and $j \neq i+1$, where $i \in 1,\ldots,N-16J$.

Using the non collapsing estimate we see that

$$\begin{aligned}\mathrm{vol}(G, g(t)) &\geq \mathrm{vol}(\cup_{i=1}^{N-16J-1} ({}^tB_1(z_i))) \\ &\geq \mathrm{vol}(\cup_{i=1}^{(N-16J-1)/2} ({}^tB_1(z_{2i}))) \\ &= \sum_{i=1}^{(N-16J-1)/2} \mathrm{vol}({}^tB_1(z_{2i})) \\ &\geq \sum_{i=1}^{(N-16J-1)/2} \sigma_0 = \sigma_0(N-16J-1)/2 \end{aligned} \tag{5.8}$$

On the other hand, $\mathrm{vol}(G, g(t)) \leq e^2 \mathrm{vol}(G, g(-1)) = e^2 \mathrm{vol}({}^{-1}B_{2J}(p)) \leq e^2\sigma_1 64J^4$ in view of the non-expanding estimate and the fact that G is defined independently of time (here we used the fact that $\frac{\partial}{\partial t} d\mu_{g(t)} \leq d\mu_{g(t)}$). This leads to a contradiction since, $N = J^5 > 16J + \frac{e^2\sigma_1 128J^4}{\sigma_0}$ if J is large enough and t_i is close enough to time T before scaling: we need t_i close to T to guarantee that the non-expanding and non-collapsing estimates hold for balls (after scaling) of radius $0 \leq r \leq 2J$.

This finishes the proof of the claim.

Note, that this estimate and (5.5) imply that

$$ {}^tB_{J/8}(p) \subseteq G = {}^{t=-1}B_{2J}(p) \subseteq {}^rB_{\frac{J^5}{10^4}}(p) \tag{5.9}$$

for all $t, r \in [-1, 0)$. Repeating the argument for $\frac{J^5}{10^4}$ instead of J, we get

$$\begin{aligned} {}^tB_{J/8}(p) &\subseteq G = {}^{t=-1}B_{2J}(p) \subseteq {}^tB_{\frac{1}{10^5}J^5}(p) \\ &\subseteq \tilde{G} := {}^{t=-1}B_{2J^5}(p) \subseteq {}^rB_{J^{25}}(p), \end{aligned} \tag{5.10}$$

for all $t, r \in [-1, 0)$, and $\mathrm{Sing}(M) \subseteq G$. This implies

$$\begin{aligned} \mathrm{Sing}(M) &= (\mathrm{Reg}(M))^c \subseteq (\mathrm{Reg}_{-1}(M))^c \\ &\subseteq {}^{t=-1}B_{2J}(p) \subseteq {}^tB_{\frac{1}{10^5}J^5}(p) \\ &\subseteq \tilde{G} = {}^{t=-1}B_{2J^5}(p) \subseteq {}^rB_{J^{25}}(p) \end{aligned} \tag{5.11}$$

for all $t, r \in [-1, 0)$.

The general case is as follows. We wish to cluster those points ${}^ip_k$ (the centre points of the balls appearing in the construction of $\mathcal{B}_R(g(-1))$) together if they satisfy the condition: $\mathrm{dist}(g(t = -1))({}^ip_k, {}^ip_l)$ remains bounded as $i \to \infty$. We assume that for each t_i, we obtained L balls (independent of i) in the construction of $\mathcal{B}_R(g(-1))$: if not, pass to a subsequence so that this is the case. Remember, that the solutions $(M, g(t) = g_i(t))_{t\in(-A_i,0)}$ are obtained by translating (in time) and scaling the original solution $(M, g(t))_{t\in[0,T)}$ at good times $t_i \nearrow T$ by $g_i(\cdot, \tilde{t}) := (T - t_i)g(\cdot, t_i + \frac{\tilde{t}}{T-t_i})$. So the points in the construction of $\mathcal{B}_R(g(-1))$ could depend on i. We can guarantee, after taking a subsequence in i if necessary, that there are $\tilde{L} \le L$ sets, (clusters of points) ${}^iT_j$, $j \in \{1, \dots, \tilde{L}\}$ and some large constant $\Lambda < \infty$ such that: for all i large enough, for all $k, l \in \{1, \dots, L\}$, exactly one of the following two statements is true:

- $\mathrm{dist}((g(t = -1))({}^ip_k, {}^ip_l) \le \Lambda$ if ${}^ip_k, {}^ip_l \in {}^iT_s$ for some $s \in \{1, \dots, \tilde{L}\}$, or
- $\mathrm{dist}(g(t = -1))({}^ip_k, {}^ip_l) \to \infty$ as $i \to \infty$ if ${}^ip_k \in {}^iT_s$ and ${}^ip_l \in {}^iT_v$ and $s \ne v, s, v \in \{1, \dots, \tilde{L}\}$.

We explain now how the sets ${}^iT_s$ are constructed. Fix $k, l \in \{1, \dots, L\}$. If there is a subsequence in i such that after taking this subsequence $\mathrm{dist}((g(t = -1))({}^ip_k, {}^i p_l) \to \infty$ as $i \to \infty$, then take this subsequence. Do this for all $k, l \in \{1, \dots, L\}$. As the index set $\{1, \dots, L\}$ is finite, after taking finitely many subsequences, we will arrive at the following situation: there exists a constant $\Lambda < \infty$ such that for all $k, l \in \{1, \dots, L\}$ one of the following two statements is true:

- $\mathrm{dist}((g(t = -1))({}^ip_k, {}^i p_l) \to \infty$ for all i *or*
- $\mathrm{dist}((g(t = -1))({}^ip_k, {}^i p_l) \le \Lambda$ for all i

Now we define ${}^iT_1$ as the set of all ${}^ip_k$, $k \in \{1, \dots, L\}$, such that $\mathrm{dist}((g(t = -1))({}^ip_k, {}^i p_1) \le \Lambda$ for all i. ${}^iT_2$ is the set of all ${}^ip_k$, $k \in \{1, \dots, L\}$, such that $\mathrm{dist}((g(t = -1))({}^ip_k, {}^i p_2) \le \Lambda$ for all i. And so on. This gives us sets ${}^iT_1, \dots, {}^i T_L$. Each set contains finitely many points, and for arbitrary $k, l \in \{1, \dots, L\}$ either ${}^iT_k \cap {}^iT_l = \emptyset$ for all $i \in \mathbb{N}$ or ${}^iT_k = {}^iT_l$ for all $i \in \mathbb{N}$. For fixed $i \in \mathbb{N}$: if a set appears more than once, we throw away all copies of the set except for one. This completes the construction of the sets $T_1, \dots, T_{\tilde{L}}$ (we drop the index i again for the moment).

Take one of these sets, for example T_1. $\mathcal{BB}_1$ will denote the union of the balls $B_{2R}(z)$ where $z \in T_1$. Let ${}^ip_1 \in \mathcal{BB}_1$ be arbitrary : we are rechoosing the points ${}^ip_j$ (we choose exactly one point ${}^ip_j$, arbitrarily, with ${}^ip_j \in \mathcal{BB}_j$, and we do this for each $j \in \{1, \dots, \tilde{L}\}$). Define $G_1 := {}^{t=-1}B_{2J}({}^ip_1)$, $H_1 := {}^{t=-1}B_J({}^ip_1)$ where $J \gg \max(\Lambda, R)$ is large but fixed (independent of i).

Arguing as in the case of one point as above, we see that (for i large enough)

$$\begin{aligned} G_1 &:= {}^{t=-1}B_{2J}(p) \subseteq {}^{t}B_{\frac{1}{10^5}J^5}(p) \\ &\subseteq \tilde{G}_1 = {}^{t=-1}B_{2J^5}(p) \subseteq {}^{r}B_{J^{25}}(p) \end{aligned} \tag{5.12}$$

for all $p \in \mathcal{BB}_1$ (the choice of ${}^ip_1 \in \mathcal{BB}_1$ was arbitrary), for all $t, r \in [-1, 0)$. Note that we need i large enough here, to guarantee that all other sets $T_2, \dots, T_{\tilde{L}}$ do not interfere with the arguments presented above: that is, we can guarantee that $\overline{H_1^c \cap G_1} \subseteq \mathrm{Reg}_{-1}(M)$ and $\overline{\tilde{H}_1^c \cap \tilde{G}_1} \subseteq \mathrm{Reg}_{-1}(M)$. Now do the same for the other sets $\mathcal{BB}_j$, $j \in \{1, \dots, \tilde{L}\}$.

We call the constant $\tilde{L}$ once again L. Hence,

$$\begin{aligned} \mathrm{Sing}(M) &\subseteq (\mathrm{Reg}_{-1}(M))^c \\ &\subseteq \cup_{j=1}^{L}({}^{t}B_{J_0}({}^ip_j)) \\ &\subseteq \tilde{G} = \cup_{j=1}^{L}({}^{-1}B_{J_1}({}^ip_j)) \\ &\subseteq \cup_{j=1}^{L}({}^{r}B_{J_2}({}^ip_j)) \end{aligned} \tag{5.13}$$

for all $t, r \in [-1, 0)$, where $J_0 := \frac{1}{10^5}J^5$, $J_1 := 2J^5$,$J_2 := J^{25}$.

Note, that by construction we have $d(-1)({}^{-1}B_{J_1}({}^ip_j), {}^{-1}B_{J_1}({}^ip_k)) \to \infty$ as $i \to \infty$ for $j \neq k$.

Scaling and translating back to the original solution, we get

$$\begin{aligned} \mathrm{Sing}(M) &\subseteq (\mathrm{Reg}_t(M))^c \\ &\subseteq \cup_{j=1}^{L}({}^{t}B_{J_0\sqrt{T-t_i}}({}^ip_j)) \\ &\subseteq G = \cup_{j=1}^{L}({}^{t=t_i}B_{J_1\sqrt{T-t_i}}({}^ip_j)) \\ &\subseteq \cup_{j=1}^{L}({}^{r}B_{J_2\sqrt{T-t_i}}({}^ip_j)) \end{aligned} \tag{5.14}$$

for all $t, r \in [t_i, T)$.

The proof of the claim of the theorem is as follows. Assume the conclusion of the theorem is false. Then for any constants J_0, J_1, J_2, we can find good times $t_i \in (T - w_i, T)$, where $w_i \to 0$, such that we cannot find points $p_1(t_i), \dots, p_L(t_i)$, with $L \leq \frac{K_0}{\varepsilon_0}$ for which (5.1) holds. Taking a subsequence, as above, and choosing $p_1(t_i) = {}^ip_1, \dots, p_L(t_i) = {}^ip_L$ leads to a contradiction if i is large enough. Note at first that it could be that $L = L(s) \leq L\frac{K_0}{\varepsilon_0}$ depends on s. But by adding regular points $p_{L(s)+1}(s), \dots, p_{\frac{K_0}{\varepsilon_0}}(s)$ which are in $\mathrm{Reg}_s(M)$, and satisfy $\mathrm{dist}(s)(p_i(s), p_j(s)) \geq \sigma_0 > 0$, for all $i \geq L(s) + 1$, for all $j \in \{1, \dots, \frac{K_0}{\varepsilon_0}\}$, the conclusion of the theorem is still correct, and the comments which follow this proof are still valid. $\square$

Remark 5.4. Note, that in the construction above,

$$d(-1)({}^{-1}B_{J_1}({}^ip_j), {}^{-1}B_{J_1}({}^ip_k)) \to \infty$$

as $i \to \infty$ for all $j \neq k$ (before scaling back). Hence, any smooth curve $\gamma : [0,1] \to M$ which lies in $(\cup_{j=1}^L({}^{-1}B_{J_1}({}^ip_j)))^c$ and has $\gamma(0) \in \partial({}^{-1}B_{J_1}({}^ip_j))$ and $\gamma(1) \in \partial({}^{-1}B_{J_1}({}^ip_k))$ must have $L_t(\gamma) \geq N(i)$ for all $t \in [-1,0)$ with $N(i) \to \infty$ as $i \to \infty$ (in $(\cup_{j=1}^L({}^{-1}B_{J_1}({}^ip_j)))^c$ we have $\frac{1}{10}g(t) \leq g(-1) \leq 10g(t)$ for all $t \in [-1,0)$. Hence $d(t)({}^{-1}B_{J_1}({}^ip_j), {}^{-1}B_{J_1}({}^ip_k)) \geq N(i)$ for all $t \in [-1,0)$ with $N(i) \to \infty$ as $i \to \infty$. Hence, without loss of generality we can assume that the $p_j(s)$ in the statement of the Theorem satisfy

$$d(t)({}^sB_{J_1\sqrt{T-s}}(p_j(s)), {}^sB_{J_1\sqrt{T-s}}(p_k(s))) \geq N(s)\sqrt{T-s} \tag{5.15}$$

for all $t \in (s,T)$ for all $j \neq k$, where $N(s) \to \infty$ as $s \nearrow T$. That is: the new claim is the claim of the Theorem 5.1, but with the extra claim 5.15. The proof is: repeat the contradiction argument at the end of the proof above for this new claim, using the information mentioned at the beginning of this remark.

Remark 5.5. Note that in the conclusion of the theorem, we may also assume, that

$$\begin{aligned} {}^tB_{J^5\sqrt{T-s}}(p_j(s)) &\subseteq {}^sB_{16J^5\sqrt{T-s}}(p_j(s)) \\ &\subseteq {}^rB_{J^{25}\sqrt{T-s}}(p_j(s)) \end{aligned} \tag{5.16}$$

for all $r,t \in [s,T)$, for all $j \in \{1,\dots,L\}$ holds (not just for the union of the balls). Repeating this part of the proof for larger J, but keeping the same $p_j(s)$, we see that in fact the following is also true:

$${}^tB_{K\sqrt{T-s}}(p_j(s)) \subseteq {}^sB_{16K\sqrt{T-s}}(p_j(s)) \subseteq {}^rB_{K^5\sqrt{T-s_2}}(p_j(s)) \tag{5.17}$$

for all $r,t \in [s,T)$, for all $j \in \{1,\dots,L\}$ for all $K \geq J^5 \in \mathbb{R}^+$ as long as $|T-s| \leq w(K)$ is small enough, for all good times s, in view of Remark 5.4 from above.

As a corollary we obtain that the distance is uniformly continuous in time. We explain this in the following.

Let $x,y \in M$ and $t \in [t_i,T)$ and $\gamma : [0,1] \to M$ be a distance minimising curve with respect to $g(t)$ from x to y, $d_t(x,y) = L_t(\gamma)$, t_i a good time close to T. We use the notation $L_t(\sigma) = L_{g(t)}(\sigma)$ here, to denote the length of a curve σ with respect to $g(t)$.

We modify the curve γ to obtain a new curve $\tilde{\gamma} : [0,1] \to M$ in the following way: if γ reaches the closure of the ball ${}^tB_{J_0\sqrt{T-t_i}}({}^ip_k)$ (here, ${}^ip_k = p_k(t_i)$, $k \in \{1, \ldots, L\}$ and J_0, J_1, J_2 are from the above construction) at a first point $\gamma(r)$ then let $\gamma(\tilde{r})$ be the last point which is in the closure of the ball ${}^tB_{J_0\sqrt{T-t_i}}({}^ip_k)$ (it could go out and come in a number of times). Remove $\gamma|_{(r,\tilde{r})}$ from the curve γ. In doing this we obtain the finite union of at most $L+4$ curves $\tilde{\gamma}_j$. Call this finite union $\tilde{\gamma}$ and consider it as a curve with finitely many discontinuities.

The new $\tilde{\gamma}$ has

$$L_t(\tilde{\gamma}) \leq L_t(\gamma) = d(x, y, t) \tag{5.18}$$

Now $(\cup_{k=1}^L {}^tB_{J_0\sqrt{T-t_i}}(p_k^i))^c \subseteq \mathrm{Reg}_{t_i}(M)$ (J_0 coming from (5.1) above), as we saw above, and the Riemannian metric is uniformly continuous (in time) on $\mathrm{Reg}_{t_i}(M)$ for good times t_i. That is, for all $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_i) > 0$ such that

$$(1-\varepsilon)g(y,t) \leq g(y,s) \leq (1+\varepsilon)g(y,t) \tag{5.19}$$

for all $y \in (\cup_{k=1}^L {}^tB_{J_0\sqrt{T-t_i}}(p_k^i))^c$ for all $t_i \leq t, s \leq T$, $|t-s| \leq \delta$ in view of (4.12) and the fact that $y \in (\cup_{k=1}^L {}^tB_{J_0\sqrt{T-t_i}}(p_k))^c \subseteq \mathrm{Reg}_{t_i}(M)$. Hence $L_t(\tilde{\gamma}) \geq L_s(\tilde{\gamma}) - c\varepsilon$ for all $T - \delta \leq t, s \leq T$ in view of the fact that the diameter of the manifold is bounded: more precisely, $L_t(\tilde{\gamma}) \geq \frac{1}{1+\varepsilon} L_s(\tilde{\gamma}) = (1 - \frac{\varepsilon}{1+\varepsilon}) L_s(\tilde{\gamma}) \geq L_s(\tilde{\gamma}) - \varepsilon L_s(\tilde{\gamma})$, and $L_s(\tilde{\gamma}) \leq (1+\varepsilon) L_t(\tilde{\gamma}) \leq (1+\varepsilon) d(x,y,t) \leq 2D$, in view of (5.18) and (5.19), and hence

$$L_t(\tilde{\gamma}) \geq L_s(\tilde{\gamma}) - 2D\varepsilon \tag{5.20}$$

for all $T - \delta \leq t, s \leq T$ as claimed. Putting (5.18) and (5.20) together we get

$$\begin{aligned} d(x,y,t) &\geq L_t(\tilde{\gamma}) \\ &\geq L_s(\tilde{\gamma}) - 2D\varepsilon \\ &\geq d(x,y,s) - 2L\varepsilon - 2D\varepsilon. \end{aligned} \tag{5.21}$$

The last inequality can be seen as follows: when $\tilde{\gamma}$ reaches a ball ${}^tB_{J_0\sqrt{T-t_i}}({}^ip_k)$, it must also be in ${}^sB_{J_2\sqrt{T-t_i}}({}^ip_k)$, by estimate (5.16). So the two points of discontinuity on $\tilde{\gamma}$ may be joined smoothly by a curve with length (with respect to $g(s)$) at most $2J_2\sqrt{T-t_i}$, which is without loss of generality less than ε. Doing this with all of the points of discontinuity (that is with all the balls), we obtain a new continuous curve $\hat{\gamma}$ from x to y with length $L_s(\hat{\gamma}) \leq 2L\varepsilon + L_s(\tilde{\gamma})$, which implies $L_s(\tilde{\gamma}) \geq L_s(\hat{\gamma}) - 2L\varepsilon \geq d(x, y, s) - 2L\varepsilon$ as claimed.

Swapping s and t in this argument gives us

$$|d(x, y, t) - d(x, y, s)| \leq C\varepsilon \tag{5.22}$$

for all $T - \delta \leq t, s \leq T$, where $x, y \in M$ are arbitrary, and the constant C appearing here does not depend on the choice of $x, y \in M$. Smoothness of the flow (and bounded diameter of M) for $t < T$ implies that:

Theorem 5.6. *Let $(M, g(t))_{t\in[0,T)}$ be a smooth solution on a compact manifold satisfying the basic assumptions. For all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|d(x, y, t) - d(x, y, s)| \leq \varepsilon \tag{5.23}$$

for all $x, y \in M$ for all $t, s \in [0, T)$ with $|t - s| \leq \delta$.

6. Convergence to a length space

The results of the previous sections imply that $(M, d(g(t))) \to (X, d_X)$ in the Gromov-Hausdorff sense as $t \nearrow T$, where (X, d_X) is a metric space, and that away from at most finitely many points $x_1, \ldots, x_L \in X$ we have that $X \backslash \{x_1, \ldots, x_L\}$ is a smooth Riemannian manifold with a natural metric and that the convergence is in the C^k Cheeger-Gromov sense. Furthermore, (X, d_X) is a length space (we explain all of this below).

In the paper [BZ], the authors also showed independently, with the help of estimates proved in their paper, a similar result to the result mentioned above (see Corollary 1.11 of their paper).

The previous sections of this paper give us lots of information on how well the limit (X, d) will be achieved and what the limit looks like, geometrically and topologically, near singular points. We will use the results of the previous sections combined with a method of G. Tian (in [Tian]) to show somewhat more than the result mentioned at the start of this section: namely, we will show that (X, d_X) is a C^0 **Riemannian orbifold**, smooth away from its singular points (this is shown in Section 8). In the last section of this paper we explain how it is possible to flow C^0 Riemannian orbifolds of this type using the orbifold Ricci flow and results from the paper [SimC0].

We construct the limit space (X, d_X) directly using the following Lemma, which relies on the uniform continuity of the distance function (in the sense of Theorem 5.6).

Lemma 6.1. *Let $(M, g(t))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the standard assumptions. Then*

$$X := \{[x] \mid x \in M\}$$
$$\text{where } [x] = [y] \text{ if and only if } d(x, y, t) \to 0 \text{ as } t \nearrow T. \tag{6.1}$$

X is well defined. Furthermore, the function $d_X : X \times X \to \mathbb{R}_0^+$,

$$d_X([x], [y]) := \lim_{t\nearrow T} d(x, y, t) \tag{6.2}$$

is well defined and defines a metric on X.

Proof. If $d(x, y, t_i) \to 0$ for some sequence $t_i \nearrow T$, then $d(x, y, s_i) \to 0$ for all sequences $s_i \nearrow T$, in view of Theorem 5.6. This means that $[x]$ is well defined, and hence X is well defined. Define $d_X([x], [y]) = \lim_{i\to\infty} d(x, y, t_i)$ where $t_i \nearrow T$ is any sequence of times approaching T. The limit on the right hand side is well defined in view of the theorem on the uniform continuity of distance (Theorem 5.6) and d_X is then also well defined, due to the theorem on the uniform continuity of distance (Theorem 5.6) and the triangle inequality on $d(\cdot, \cdot, t)$.

From the definition, we see that $d_X([x], [y]) = 0$ if and only if $[x] = [y]$. The triangle inequality of, and symmetry of d_X follows from the triangle inequality of, and symmetry of $d(\cdot, \cdot, t)$. □

This (X, d_X) is the limiting metric space of $(M, d(g(t)))_{t\in[0,T)}$ in view of the theorem on the uniform continuity of distance, as we now show.

Lemma 6.2. *Let everything be as in Lemma 6.1 above. The function $f : M \to X$ is defined by*

$$f(x) := [x]. \tag{6.3}$$

$f : (M, g(t)) \to (X, d_X)$ is a Gromov-Hausdorff approximation in the sense that

$$|d_X(f(x), f(y)) - d(g(t))(x, y)| \leq \varepsilon(|T - t|)$$
$$X := f(M), \tag{6.4}$$

where $\varepsilon(r) \to 0$ *as* $r \searrow 0$. f *is continuous and surjective and hence* (X, d_X) *is compact, precompact, connected and complete. In particular* $(M, d(g(t))) \to (X, d_X)$ *as* $t \nearrow T$.

Proof. The first claim of the theorem follows immediately from the theorem on the uniform continuity of distance and the definition of X. Now we show that f is continuous. Let U be open in X and ${}^{d_X}B_\varepsilon(p) \subseteq U$. Due to the uniform continuity of the distance function, we know the following: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $f({}^{d(t)}B_{\varepsilon/2}(q)) \subseteq {}^{d_X}B_\varepsilon(p)$ for all $|T - t| < \delta$ where q is an arbitrary point with $f(q) = p$ (there could be lots of such points). Hence ${}^{d(t)}B_{\varepsilon/2}(q) \subseteq f^{-1}(U)$. Since $p \in U$ was arbitrary, and q with $f(q) = p$ was arbitrary, we have shown the following: for any point $q \in f^{-1}(U)$ there exists an $\varepsilon(q)$ and a $t_{\varepsilon,q} < T$ such that ${}^{d(t_{\varepsilon,q})}B_{\varepsilon(q)}(q) \subseteq f^{-1}(U)$. So we can write

$$f^{-1}(U) = \cup_{q \in f^{-1}(U)} {}^{d(t_{\varepsilon,q})}B_{\varepsilon(q)}(q).$$

Each of the sets contained in the union is an open set in $(M, d(t))$ for **any** $t < T$ and hence, f is continuous. Hence (X, d_X) is compact, being the continuous image of a compact space, and hence complete and precompact as it is a metric space. $\square$

We have shown $f : M \to X$ is a continuous surjective map, where the topology on X comes from d_X and that on M is the initial topology of the manifold M, which agrees with that coming from the metric space $(M, d(g(t)))$ for any $t < T$. Note that the map $f : M \to X$ is **not** necessarily injective: it could be that a set Ω containing more than two points is all mapped onto one point in X by f.

Let ${}^ip_1, \ldots, {}^ip_L$ be the points constructed in the previous section (in the possibly singular region) for large i. Taking a subsequence we can assume that $f({}^ip_k) \to x_k$ as $i \to \infty$ for all $k \in \{1, \ldots, L\}$ for some fixed $x_1, \ldots, x_L$. We do not rule out the case $x_j = x_k$ for $j \neq k$. After renumbering the x_j's we have finitely many (we use the symbol L again) distinct points $x_1, \ldots, x_L$, and $x_i \neq x_j$ for all $i \neq j, i, j \in \{1, \ldots, L\}$.

Definition 6.3. Let $[x] \in X$, $x \in M$. We say $[x]$ is a *regular point in* X if $[x]$ contains only one point, and $[x]$ is a *singular point in* X, if $[x]$ contains more than one point.

Remark 6.4. Notice that the notion of *singular point* and *regular point* differs depending on whether the point is in X or M. The following theorem gathers together properties that we have already proved and shows that there is a connection between the different notions of *singular* and *regular*.

Theorem 6.5. *Let $(M, g(t))_{t\in[0,T)}$ be a solution to Ricci flow satisfying the basic assumptions, and (X, d_X), $x_1, x_2, \dots, x_L \in X$ as above. Then*

(i) $X\backslash\{x_1,\dots,x_L\} \subseteq f(\mathrm{Reg}(M)) \subseteq \mathrm{Reg}(X)$ *and* $f(\mathrm{Sing}(M)) \subseteq \{x_1,\dots,x_L\}$.

(ii) $V := f^{-1}(X\backslash\{x_1,\dots,x_L\}) \subseteq \mathrm{Reg}(M)$, *$V$ is open and $f|_V : V \to X$ is an open, continuous, bijective map, and hence $f|_V : V \to f(V) := X\backslash\{x_1,\dots,x_L\}$ is a homeomorphism.*

Proof. (i) Take a point $x \notin \{x_1,\dots,x_L\}$. Then $d_X(x, x_j) \geq \varepsilon > 0$ for all $j \in \{1,\dots,L\}$ for some $\varepsilon > 0$. Let $[z] = x$. Remembering that $[{}^ip_j] \to x_j$ as $i \to \infty$, we see that $d_X([z], [{}^ip_j]) \geq \frac{\varepsilon}{2} > 0$ for all $j \in \{1,\dots,L\}$ if i is large enough. Fix i large. Then we can find a $\tilde{t}(i)$ such that $d(z, {}^ip_j, t) \geq \varepsilon/4$ for all $\tilde{t}(i) \leq t < T$ near enough to T, for some $\tilde{t}(i) \geq t_i$, for all $j \in \{1,\dots,L\}$ in view of the definition of d_X. Scaling as in the proof of Theorem 5.1 (and using the notation of the proof), we see that $d(z, {}^ip_j, -\hat{s}_i) \to \infty$ for all $j \in \{1,\dots,L\}$, for some $0 \leq \hat{s}_i \leq 1$ as $i \to \infty$, and hence $z \in \mathrm{Reg}_{-1}(M) \subseteq \mathrm{Reg}(M)$ due to (5.13), and hence $x = [z] = f(z) \subseteq f(\mathrm{Reg}(M))$. This shows that $X\backslash\{x_1,\dots,x_L\} \subseteq f(\mathrm{Reg}(M))$ and hence we have shown the first inclusion of (i). Let $z \in \mathrm{Reg}(M)$ be arbitrary. Then $z \in \mathrm{Reg}_t(M)$ for t close enough to T by definition. Choose a good time t and scale the solution by $\frac{1}{T-t}$ and translate the the solution in time (as in the proof of Theorem 4.5 above). Then $z \in \mathrm{Reg}_{-1}(M)$. Hence $d(z, y, t) \geq \frac{1}{10} d(z, y, -1)$ for all $y \in \overline{{}^{-1}B_{\frac{R}{200}}(z)}$ for all $t \in (-1, 0)$ in view of (4.13), and $d(z, p, t) \geq \inf_{y\in\partial({}^{-1}B_{\frac{R}{200}}(z))} d(z, y, t) \geq \varepsilon_0 > 0$ for all $t \in (-1, 0)$, for all $p \in ({}^{-1}B_{R/200}(z))^c$ for the same reason. That is $f(z) = [z]$ is not singular, since $\lim_{t\nearrow 0} d(z, y, t) > 0$ for all $y \neq z$, $y \in M$. That is $f(\mathrm{Reg}(M)) \subseteq \mathrm{Reg}(X)$. This shows the second inclusion of (i). Now we prove the last statement of (i). Let $p \in \mathrm{Sing}(M)$. Assume $f(p) \in X\backslash\{x_1,\dots,x_L\}$. Then we know that there exists a $x \in \mathrm{Reg}(M)$ such that $f(x) = f(p)$ in view of the set inclusions just proved. But then $[x] = [p]$ and $x \neq p$ (since $\mathrm{Reg}(M)$ and $\mathrm{Sing}(M)$ are disjoint). Furthermore $[x] \in \mathrm{Reg}(X)$ due to the set inclusions just shown. This contradicts the definition of $\mathrm{Reg}(X)$. Hence, we must have $[p] = f(p) \in \{x_1,\dots,x_L\}$. This finishes the proof of (i).

(ii) Let $z \in f^{-1}(X\backslash\{x_1,\dots,x_L\})$. Then $f(z) \in X\backslash\{x_1,\dots,x_L\}$. If $z \in \mathrm{Sing}(M)$ were the case, then we would have $f(z) \in \{x_1,\dots,x_L\}$ from (i), which is a contradiction. Hence $z \in \mathrm{Reg}(M)$. That is

$$V := f^{-1}(X\backslash\{x_1,\dots,x_L\}) \subseteq \mathrm{Reg}(M).$$

V is open, since f is continuous, und $X\backslash\{x_1,\dots,x_L\}$ is open. From the above, $f|_V : V \to X$ is injective: assume there exists $x, y \in V$ with $f(x) = [x] = [y] = f(y)$. $x \in V$ implies $[x] \in X\backslash\{x_1,\dots,x_L\}$ and hence $[x] \in \mathrm{Reg}(X)$ from (i). Combining this with $[x] = [y]$, we see that $x = y$ in view of the definition of $\mathrm{Reg}(X)$, that is $f|_V : V \to X\backslash\{x_1,\dots,x_L\}$ is injective. Let $(f|_V)^{-1} : X\backslash\{x_1,\dots,x_L\} \to V$ be the inverse of $f|_V : V \to X\backslash\{x_1,\dots,x_L\}$. Then $(f|_V)^{-1} : N := X\backslash\{x_1,\dots,x_L\} \to M$ is continuous as we now show. Assume $[z_k] \to [z]$ in $f(V) = X\backslash\{x_1,\dots,x_L\}$ as $k \to \infty$. Using the fact that $f|_V : V \to X$ is injective, we see that there are unique points $z_k, z \in V$ such that $f(z_k) = [z_k]$ and $f(z) = [z]$. Furthermore, $z_k, z \in \mathrm{Reg}(M)$: if z_k respectively z were in $\mathrm{Sing}(M)$, then we would have $f(z_k)$ respectively $f(z) \in \{x_1,\dots,x_L\}$ which is a contradiction.

Assume z_k does not converge to z. $z \in \mathrm{Reg}(M)$ and hence we can find a good time t_i near T such that $z \in \mathrm{Reg}_{t_i}(M)$. Fix this t_i. z_k doesn't converge to z means: we can find a an $\varepsilon(i) = \varepsilon(t_i) > 0$ and a subsequence $(z_{k,i})_{k\in\mathbb{N}}$ of $(z_k)_{k\in\mathbb{N}}$ (depending possibly on i), such that $d(t_i)(z_{k,i}, z) \ge \varepsilon(i)$ for all $k \in \mathbb{N}$. Scale at a time t_i and translate as above to $t = -1$ (as in the proof of Theorem 4.5). Then we have $z \in \mathrm{Reg}_{-1}(M)$ and $d(-1)(z_{k,i}, z) \ge \tilde{\varepsilon}(i) > 0$ for all $k \in \mathbb{N}$.

Hence (arguing as above) $d(z, z_{k,i}, s) \ge \frac{1}{10} d(z, z_{k,i}, -1) \ge \varepsilon(i) > 0$ for all $z_{k,i} \in \overline{{}^{-1}B_{\frac{R}{200}}(z)}$ for all $s \in (-1, 0)$ in view of (4.13),and $d(z, z_{k,i}, s) \ge \inf_{y\in\partial({}^{-1}B_{\frac{R}{200}}(z))} d(z, y, s) \ge \varepsilon_0 > 0$ for all $s \in (-1, 0)$, for all $z_{i,k} \in ({}^{-1}B_{\frac{R}{200}}(z))^c$ for the same reason.

Taking a limit $s \nearrow 0$, we see $d_X([z_{k,i]}, [z]) \ge \hat{\varepsilon}(i) > 0$ for all $k \in \mathbb{N}$, which contradicts the fact that $[z_k] \to [z]$ as $k \to \infty$. □

These facts allows us to give $X\backslash\{x_1,\dots,x_L\}$ a natural manifold structure, as we now explain.

Proposition 6.6. *Let everything be as in Lemma 6.5 above. $N = X\backslash\{x_1,\dots,x_L\}$ has a natural manifold structure and with this structure $f|_V : V \to N$ is a diffeomorphism, $V := f^{-1}(N)$. There is a natural Riemannian metric l on N defined by $l := \lim_{t\nearrow T} f_* g(t)$.*

Proof. For $x \in N$, let $\tilde{x} \in V \subseteq M$ be the unique point in V with $f(\tilde{x}) = x$. Let $\psi : \tilde{U} \subset\subset V \subseteq M \to \mathbb{R}^4$ be a smooth chart on M with $\tilde{x} \in \tilde{U}$, and let $U := f(\tilde{U})$. U is open from the above. Define a coordinate chart $\varphi : U \subseteq N \to \mathbb{R}^4$ by $\varphi = \psi \circ (f|_V)^{-1}$. Clearly these maps define a C^∞ atlas on N (the topology induced by $f : V \to N$ on N is the same as that induced by d_X

on N). Using this atlas on N, $f|_V : V \to N$ is then a smooth diffeomorphism per definition.

Also, we can define a limit metric l on N in a natural way: let $l := \lim_{t \nearrow T} f_*(g(t))$. This metric is well defined. Let $[z] \in N$ and z be the corresponding point in V. $z \in \mathrm{Reg}(M)$ because of (ii) above. Hence $z \in \mathrm{Reg}_t M$ for all good times t near enough T and hence, after rescaling as in the proof of Theorem 4.5, $z \in \mathrm{Reg}_{-1}(M)$. Fix coordinates $\psi : \tilde{U} \subset\subset V \to \hat{U} \subseteq \mathbb{R}^4$ with $\tilde{U} \subseteq {}^{-1}B_{R/2}(z)$. Let $g_{ij}(\cdot, t)$ refer to the metric $g(\cdot, t)$ with respect to the coordinates ψ. Then $g_{ij}(t) \to l_{ij}$ as $t \nearrow 0$ for some smooth metric l on $\psi(\tilde{U})$, in view of the estimates in the statement of Theorem 4.5 (see for example the arguments in Section 8 of [HaForm]). Noting that $f_*(g(t))_{ij}(\cdot, t) = g_{ij}(\cdot, t)$ in the coordinates $\varphi = \psi \circ (f|_V)^{-1} : U \to \mathbb{R}^4$, we see that this limit is well defined. □

Notice that for each $x, y \in X$ we can find a z with $d_X(x, z) = d_X(z, y) = \frac{1}{2} d_X(x, y)$: this follows by using the Gromov-Hausdorff approximation f, and the fact that this is true for $d(g(t_i))$ (for a sequence of times $t_i \nearrow T$), and using the compactness of M and X. Hence, since (X, d_X) is complete, we have that (X, d_X) is also a length space. We include the statement of this fact and others, some of which appeared already in this section, in the following theorem.

Theorem 6.7. *Let everything be as in Proposition 6.6, and let $p_1, \ldots, p_L \in M$ be arbitrary points with $f(p_j) = x_j$ for all $j \in \{1, \ldots, L\}$. (X, d_X) is a compact length space, with length function L_X, and $(N, l) = (X \backslash \{x_1, \ldots, x_L\}, l)$ is a smooth Riemannian manifold with*

(a) $\sup_{x,y \in M} |d(g(t))(x, y) - d_X(f(x), f(y)| \to 0$ *as $t \nearrow T$, and hence*

$$\sup_{r \in [0,D]} d_{GH}({}^{d(g(t))}B_r(x_i), {}^{d_X}B_r(p_i)) \to 0 \text{ as } t \nearrow T, \tag{6.5}$$

for arbitrary $p_j \in M$ with $f(p_j) = x_j$.

(b) *Let $\hat{N}$ be a component of N and $d_{\hat{N},l}$ the metric induced by $(\hat{N}, l)$ on $\hat{N}$. Then, for all $x \in N$, there exists an open set $U \subset\subset N$ with $x \in U$, such that $d_X|_U = d_{\hat{N},l}|_U$ and $\mathrm{vol}_l(E \cap U) = d\mu_X(E \cap U)$ for all measurable $E \subseteq N$, where $d\mu_X$ refers to n-dimensional Hausdorff-measure with respect to the metric space (X, d_X), and vol_l is the volume form coming from l on N. Hence, $d\mu_X|_N = \mathrm{vol}_l$ if we restrict to measurable sets in N.*

(c) $L_X(\gamma) = L_l(\gamma)$, *in the case where* γ *is a piecewise smooth curve which lies completely in* $N = X \backslash \{x_1, \ldots, x_L\}$.

Proof. (a) follows directly from Lemmata 6.3, 6.5 and 6.6. As we mentioned above, for each $x, y \in X$ we can find a z with $d_X(x,z) = d_X(z,y) = \frac{1}{2}d_X(x,y)$: this follows by using the Gromov-Hausdorff approximation f, and the fact that this is true for $d(g(t_i))$, and using the compactness of M and X. Hence, since (X, d_X) is complete, we have that (X, d_X) is also a length space, see Chapter 2 and in particular Theorem 2.4.16 of [BBI]: in the proof of Theorem 2.4.6 in [BBI], it is shown, that one can construct a continuous curve $\gamma : [0, l := d_X(x,y)] \to X$ such that $d_X(\gamma(s), \gamma(t)) = |t - s|$ for all $0 < s, t \leq l$, and hence $d_X(x,y) = L_X(\gamma)$ where, for $\sigma : [a,b] \to \mathbb{R}$ a continuous curve, $L_X(\sigma)$ is the supremum of the sums $\Sigma(Y) = \sum_{i=1}^{N} d_X(\sigma(y_{i-1}), \sigma(y_i))$ over all finite partitions $Y = \{y_1, \ldots, y_N\}$, $N \in \mathbb{N}$ of $[a,b]$. Hence all points x, y can be joined by a continuous geodesic curve $\gamma : [0, s] \to X$ such that $L_X(\gamma|_{[a,b]}) = d_X(\gamma(b), \gamma(a))$ for all $0 \leq a, b \leq s$. We are using the notation of [BBI]: a *geodesic* in a length space is a continuous curve whose length realises the distance.

Let $q \in N = X \backslash \{x_1, \ldots, x_L\}$. Then $q \in \hat{N}$, the unique connected component of N containing q. For the proof of (b), d_l will refer to $d_{l,\hat{N}}$ the distance function associated to $(\hat{N}, l)$.

From the above (Lemmata 6.5 and 6.6), there exists a unique $\hat{q} \in M$ such that $f(\hat{q}) = q$, and we can find a neighbourhood $Z \subset\subset U \subset\subset N$ and coordinates $\varphi : U \to \mathbb{R}^4$, with $x \in U$, $\tilde{U} := \varphi(U)$, $\tilde{Z} := \varphi(Z)$, $\varphi(q) = p$. By choosing $\varepsilon > 0$ small enough, we can guarantee that ${}^{d_l}B_{100\varepsilon}(q)$ and ${}^{d_X}B_{100\varepsilon}(q)$ are compactly contained in Z. Using the fact that $g_{ij}(t) \to l_{ij}$ in the C^k norm on $\tilde{U} = \varphi(U)$, we see that every smooth, regular curve $\gamma : I \to \tilde{U}$ with $\gamma(0) \in \varphi({}^{d_l}B_{2\varepsilon}(q) \cap {}^{d_X}B_{2\varepsilon}(q))$ which leaves $\tilde{Z}$ must have length larger than 10ε with respect to $g_{ij}(t)$ if $|t - T| \leq \delta$ (and with respect to l_{ij}), in view of the fact that $(1 - \tilde{\varepsilon})l_{ij} \leq g_{ij}(t) \leq (1 + \tilde{\varepsilon})l_{ij}$ in $\tilde{U}$ if $|t - T| \leq \delta$, δ small enough. Hence, for $x, y \in {}^{d_l}B_{2\varepsilon}(q) \cap {}^{d_X}B_{2\varepsilon}(q)$, we have $d_l(x,y) = d_{\tilde{l},\tilde{U}}(\varphi(x), \varphi(y))$, where $\tilde{U} = \varphi(U)$, $\tilde{l} = \varphi_*(l) = (l_{ij})_{i,j \in \{1,\ldots,n\}}$ and $d_{\tilde{l},\tilde{U}}$ is the distance on the Riemannian manifold $(\tilde{U}, \tilde{l})$. Similarly, $d_{g(t)}(f^{-1}(x), f^{-1}(y)) = d_{\tilde{g}(t),\tilde{U}}(\varphi(x), \varphi(y))$, $\tilde{g}(t) = \psi_*(g(t)) = (g_{ij}(t))_{i,j \in \{1,\ldots,n\}}$ if $|T - t| \leq \tilde{\varepsilon}$, where we are using the coordinates $\varphi = \psi \circ f^{-1}$, introduced in Proposition 6.6 [Explanation. Without loss of generality, $|d_{g(t)}(f^{-1}(x), f^{-1}(y)) - d_X(x,y)| \leq \varepsilon$ for $|T - t| \leq \tilde{\varepsilon}$, and hence $d_{g(t)}(f^{-1}(x), f^{-1}(y)) \leq 3\varepsilon$. If γ is any curve in M between $f^{-1}(x)$ and $f^{-1}(y)$ whose length is less than 4ε, then γ must lie in $f^{-1}(Z)$: otherwise, pushing down to $\tilde{U}$ with the coordinates ψ, we would obtain a part of the curve having length larger than 10ε, which is a contradiction. End of the explanation]. This shows us

$$
\begin{aligned}
d_X(x,y) &= \lim_{t\nearrow T} d_{g(t)}(f^{-1}(x), f^{-1}(y)) \\
&= \lim_{t\nearrow T} d_{\tilde g(t),\tilde U}(\varphi(x),\varphi(y)) = d_{\tilde l,\tilde U}(\varphi(y),\varphi(y)) = d_l(x,y),
\end{aligned}
$$

as claimed. Furthermore, since l is smooth, we can assume that $\varepsilon > 0$ is so small, that $\mathrm{vol}_l\,|_{{}^{d_l}B_\varepsilon(q)} = \mathcal{H}^n_{d_l}|_{{}^{d_l}B_\varepsilon(q)}$, and hence $\mathrm{vol}_l\,|_{{}^{l}B_\varepsilon(q)} = \mathcal{H}^n_{d_X}|_{{}^{d_l}B_\varepsilon(q)}$, since $d_X = d_l$ on ${}^{d_l}B_\varepsilon(q)$, where here $\mathcal{H}^n_{d_l}$ is Hausdorff-measure on $(\hat N, d_l)$. This finishes the proof of (b).It follows, that $L_X(\sigma) = L_l(\sigma)$ for any piecewise smooth $\sigma : [0,1] \to X\backslash\{x_1,\dots,x_L\}$ curve: we cover the image by small balls for which on each of the balls $d_X = d_l$, and use the fact that locally, $L_l(\sigma)$ is the supremum of the sums $\Sigma(Y) = \sum_{i=1}^N d_l(\sigma(y_{i-1}),\sigma(y_i)) = \sum_{i=1}^N d_X(\sigma(y_{i-1}),\sigma(y_i))$ over all finite partitions Y,$Y = \{y_1,\dots,y_N\}$,$N \in \mathbb{N}$ of $[a,b]$ (without loss of generality, $\sigma[y_i,y_{i+1}]$ lies in a small ball on which $d_X = d_l$). This is (c). □

7. Curvature estimates on and near the limit space

Let $d\mu_X$ denote Hausdorff-measure on the metric space (X, d_X). This is an outer measure and defined for all sets in X. See for example Chapter 2 of [AT]. Let $d\mu_l = \mathrm{vol}_l$ refer to the measure on $N = X\backslash\{x_1,\dots,x_L\}$ coming from the Riemannian metric l. From (b) in Theorem 6.7 above, we saw that $d\mu_l = (d\mu_X)|_N$ when we restrict to measurable sets in N. Hence for any measurable set E in N, we have

(i) $d\mu_l(E) = d\mu_X(E) = \lim_{\varepsilon\searrow 0} d\mu_X(E\backslash^{d_X}B_\varepsilon(p))$
$= \lim_{\varepsilon\searrow 0} d\mu_l(E\backslash^{d_X}B_\varepsilon(p))$

(ii) By construction l is the limit of the pull back of the metrics $g(t)$ by f^{-1}, and hence, $c_0 r^4 \le d\mu_l({}^{d_X}B_{r/2,r}(x_i)) \le c_1 r^4$ for all $r \le diam(X)$, where c_0, c_1 are fixed constants. This can be seen as follows. Let $U := {}^{d_X}B_{r/2,r}(x_j)$. Then ${}^tB_{r/4}(p_j) \subseteq \hat U := f^{-1}(U) \subseteq {}^tB_{2r}(p_j)$ for all t with $|T - t| \le \delta$ small enough, in view of the definition of f, and the uniform continuity in time of the distance function (here p_j is an arbitrary point with $f(p_j) = x_j$), and hence $c_0 r^4 \le \mathrm{vol}_{g(t)}(\hat U) \le c_1 r^4$. Letting $t \nearrow T$ and using $\mathrm{vol}_{g(t)}(\hat U) = \mathrm{vol}_{f_*(g(t))}(U) \to \mathrm{vol}_l(U) = d\mu_l(U)$ implies the claimed estimate.

(iii) Hence the non-collapsing/non-expanding estimates $\tilde\sigma_0 r^4 \le d\mu_l({}^{d_X}B_r(z)) \le \tilde\sigma_1 r^4$ must also hold on X for some constants $0 < \tilde\sigma_0, \tilde\sigma_1 < \infty$, for all $r \le \mathrm{diam}(X)$ .We denote the constants $0 < \tilde\sigma_0, \tilde\sigma_1 < \infty$ once again by

$0 < \sigma_0, \sigma_1 < \infty$. That is, the non-collapsing / non-expanding estimates survive into the limit.

In view of the results of the previous sections we have

Theorem 7.1. *Let everything be as in the previous section (X, $x_1, \dots, x_L$ are defined in Lemma 6.5 and l is defined in Lemma 6.6). Then,*

(i)

$$\int_X |\operatorname{Riem}(l)(x)|^2 d\mu_X \leq K_0 := c_0(g_0, T) \tag{7.1}$$

where $c_0(g_0, T)$ is the constant appearing in (2.1), and we define $|\operatorname{Riem}(l)(x)| = 0$ for $x \in \{x_1, \dots, x_L\}$ (this is a measurable function, since $d\mu_X(S) = 0$ for any finite set $S \subseteq X$).

(ii) *The following flatness estimates are also true. Let $(a_i)_{i \in \mathbb{N}}$ be any sequence with $a_i \nearrow \infty$, and let $l_i = a_i^2 l$, $d_i = \sqrt{a_i} d_X$. Then for all $0 < \sigma < N < \infty$, $K \in N$, we have*

$$|\nabla^k \operatorname{Riem}(l_i)(x)| \qquad \leq \varepsilon(i, \sigma, N, K) \text{ on } {}^{d_i}B_{\sigma,N}(x_j) \tag{7.2}$$

where $\varepsilon(i, \sigma, N, K) \to 0$ as $i \to \infty$ for fixed N, σ, K, and $j \in \{1, \dots, L\}$.

Remark 7.2. Note that we obtain the result (7.2) *for all* sequences. It is not necessary to pass to a subsequence in order to obtain the result.

Remark 7.3. Compare the estimates with those stated in Corollary 1.11 in [BZ], which were obtained independently.

Proof. (i) Using the theorem on monotone convergence (see for example Theorem 2 Section 1.3 in [EG]) and the fact that $d\mu_X(\cup_{i=1}^L B_\varepsilon(x_i)) \to 0$ as $\varepsilon \searrow 0$, we see that

$$\begin{aligned}
&\int_X |\operatorname{Riem}(l)(x)|^2 d\mu_X(x) \\
&= \lim_{\varepsilon \searrow 0} \int_{X \setminus (\cup_{i=1}^L B_\varepsilon(x_i))} |\operatorname{Riem}(l)(x)|^2 d\mu_X \\
&= \lim_{\varepsilon \searrow 0} \lim_{t \nearrow T} \int_{f^{-1}(X \setminus (\cup_{i=1}^L B_\varepsilon(x_i)))} |\operatorname{Riem}(g(t))(x)|^2 d\mu_t \leq K_0 \qquad (7.3)
\end{aligned}$$

This finishes the proof of (i).

(ii) Let $c_i := \frac{1}{T-t_i}$ where t_i is a sequence of good times. Scale and translate in time $(M, g(t))_{t\in[0,T)}$ as in Theorem 4.5, we call the resulting solution also $(M, g(t))_{t\in(-A_i,0)}$, and scale d_X by $d_i = \sqrt{c_i} d_X$. Notice that $d_i(x_k, x_l) \to \infty$ as $i \to \infty$ and we will only be concerned with these blow ups near one point x_k: without loss of generality $x_k = x_1$. Assume $x_1 \in f(\mathrm{Sing}(M))$, and let $p_1 \in \mathrm{Sing}(M)$ be a point with $f(p_1) = x_1$. If $x_1 \in f(\mathrm{Reg}(M))$, then the theorem follows by blowing up the region around x_1, which has a Riemannian manifold structure. From the estimates of Theorem 5.1, we have $\mathrm{Sing}(M) \subseteq (\mathrm{Reg}_{-1}(M))^c \subseteq \cup_{k=1}^L ({}^{-1}B_{J_1}({}^ip_k))$ and hence $p_1 \in {}^{-1}B_{J_1}({}^ip_k)$ for some $k \in \{1,\dots,L\}$: renaming the $({}^ip_k)'s$ we can assume $p_1 \in {}^{-1}B_{J_1}({}^ip_1)$ and hence

$$|\nabla^j \mathrm{Riem}(g_i(\tilde t))| \le C_j \text{ on } (\cup_{k=1}^L ({}^{-1}B_{2J_1}({}^ip_k)))^c \text{ if } \tilde t \in \left(-\frac{1}{2}, 0\right), \tag{7.4}$$

in view of the estimates (4.17), where ${}^ip_1 = p_1$. From Remark 5.5, we see that, without loss of generality, ${}^tB_{2J_1}(p_1) \subseteq {}^{-1}B_{32J_1}(p_1) \subseteq {}^tB_{2^5J_1^5}(p_1)$ for all $t \in (-1, 0]$ and ${}^tB_N(p_1) \subseteq {}^{-1}B_{16N}(p_1) \subseteq {}^tB_{N^5}(p_1)$ for all $t \in (-1, 0]$ if i is large enough. Hence, using the fact that $d(-1)({}^ip_j, {}^ip_k) \to \infty$ as $i \to \infty$ (see Remark 5.4) for all $j \neq k$, we see that ${}^tB_N(p_1) \cap ({}^tB_{2^5J_1^5}(p_1))^c \subseteq {}^{-1}B_{16N}(p_1) \cap ({}^{-1}B_{32J_1}(p_1))^c$ and

$$\begin{aligned} &|\nabla^j \mathrm{Riem}(g_i(\tilde t))| \le C_j \\ &\qquad \text{on } {}^{-1}B_{16N}(p_1) \cap ({}^{-1}B_{32J}(p_1))^c \supseteq {}^tB_N(p_1) \cap ({}^tB_{2^5J_1^5}(p_1))^c \\ &\qquad \text{if } \tilde t \in \left(-\frac{1}{2}, 0\right), \end{aligned} \tag{7.5}$$

and hence, taking a limit $t \nearrow T$, we see that

$$|\nabla^j \mathrm{Riem}(l_i)| \le C_j \text{ on } {}^{d_i}B_N(x_1) \cap ({}^{d_i}B_{J_4}(x_1))^c, \tag{7.6}$$

where $l_i = c_i l$, $J_4 := 2^5 J_1^5$. Using that $\int_{{}^{d_X}B_r(x_1)} |\mathrm{Riem}(l)(x)|^2 d\mu_X(x) \to 0$ as $r \to 0$, we see that

$$\int_{{}^{d_i}B_{J_4,N}(x_1)} |\mathrm{Riem}(l_i)(x)|^2 d\mu(i)_X(x) \to 0 \text{ as } i \to \infty, \tag{7.7}$$

where $d\mu(i)_X$ is Hausdorff-measure on (X, d_i), and hence

$$|\mathrm{Riem}(l_i)(x)| \le \varepsilon(i) \to 0 \text{ as } i \to \infty \text{ on } {}^{d_i}B_{J_4+1,N-1}(x_1)$$

in view of the fact that $|\nabla^j \operatorname{Riem}(l_i)| \le C_j$for all $j \le K$ on the same set (C_j not depending on i). In fact we may assume smallness for all gradients up to a fixed order. This can be seen as follows. Introduce geodesic coordinates at a point $m_i \in {}^{d_i}B_{J^5+1,N-1}(x_1)$. The injectivity radius at m_i is larger that $\beta > 0$ for all metrics independent of i in view of the injectivity radius estimate of Cheeger-Gromov-Taylor, Theorem 4.3 in [CGT], and the non-collapsing/non-inflating estimates. Now using Theorem 4.11 of [HaComp], and writing l_i in these geodesic coordinates, we get $|D^k l_i|_{B_\beta(0} \le C(K)$ for all $k \in \{1, \dots, K\}$. Hence taking a subsequence, we get a limit metric in $C^{k-1}(B_\beta(0))$, which is equal to δ, by Theorem 4.10 of [HaComp].

That is, without loss of generality,

$$|\nabla^k \operatorname{Riem}(l_i)(x)| \le \varepsilon(i) \to 0 \text{ as } i \to \infty \text{ on } {}^{d_i}B_{J_4+1,N-1}(x_1) \tag{7.8}$$

for all $k \le K \in \mathbb{N}_0$, where K is fixed but as large as we like, for $l_i = c_i l$, $c_i = \frac{1}{(T-t_i)}$, where $t_i \nearrow T$ is a sequence of good times, where we took various subsequences to achieve this. In fact the equation (7.8) is true *for any sequence* $c_i \nearrow \infty$: it is not necessary to take a subsequence, and it is not necessary that c_i has the form $c_i = \frac{1}{(T-t_i)}$, where t_i are good times. We explain this now. First, the statement is true for any sequence of the form $c_i = \frac{1}{(T-t_i)}$: if not, then take a sequence for which it fails. Taking a subsequence, if necessary, in the proof above, we arrive at a contradiction.

Now let $c_i \to \infty$ be arbitrary. We can always write $c_i = \frac{\alpha_i}{(T-t_i)}$ for some sequence of good times $t_i \nearrow T$ and $\alpha_i \in (1/4, 4)$, in view of Lemma 4.2. Now (7.8) holds for the metrics $\tilde{l}_i = \frac{1}{(T-t_i)} l$, as we have just shown, and hence, for $l_i = \frac{\alpha_i}{(T-t_i)} l = \alpha_i \tilde{l}_i$, we get

$$|\nabla^k \operatorname{Riem}(l_i)(x)| \le \varepsilon(i) \to 0 \text{ as } i \to \infty \text{ on } {}^{d_i}B_{2(J_4+1),\frac{1}{2}(N-1)}(x_1) \tag{7.9}$$

Now let a_i be an arbitrary sequence going to infinity, and $\tilde{l}_i = a_i l$. Writing $l_i = c_i l$ with $c_i = \frac{4(J_4+1)}{\sigma^2} a_i$, we see that $\tilde{l}_i = \frac{\sigma^2}{4(J_4+1)} l_i$, and hence, using the fact that N was arbitrary (but large), we get,

$$|\nabla^k \operatorname{Riem}(l_i)(x)| \le \varepsilon(i) \to 0 \text{ as } i \to \infty \text{ on } {}^{d_i}B_{\sigma,N}(x_1) \tag{7.10}$$ □

So we see that the manifold is becoming very flat away from singular points, in the sense just described, after scaling. Using these flatness estimates we will show that X is a *generalised* C^0 *Riemannian orbifold*. We wish also to show that at each possible orbifold point there is only one component: that is, that X is actually a C^0 *Riemannian orbifold* with only finitely many orbifold points. To do this, it will be necessary to obtain approximations of the blow ups $({}^{d_i}B_{\sigma,N}(x_1))$ (constructed in the proof above) by Riemannian manifolds which have certain nice properties.

This is the content of the next theorem.

Theorem 7.4. (Approximation Theorem) *Let l and X, $x_1, \dots, x_L$, be as in Lemma 6.5 and Lemma 6.6. There exist smooth metrics g_i on M, and points $p_j \in M$ for $j \in \{1, \dots, L\}$ such that*

$$\begin{aligned} & d_{GH}({}^{g_i}B_{N(i)}(p_j), {}^{d_i}B_{N(i)}(x_j)) \le \alpha_i \\ & |\nabla^k \operatorname{Riem}(l_i)|^2 \le \alpha_i \text{ on } {}^{d_i}B_{\sigma(i),N(i)}, \text{ and} \\ & ({}^{d_i}B_{\sigma(i),N(i)}(x_j), l_i), \text{ is } \alpha_i \text{ close to } ({}^{g_i}B_{\sigma(i),N(i)}(p_j), g_i) \\ & \quad \text{in the } C^k \text{ sense, and} \end{aligned} \tag{7.11}$$

$$\int_M |\operatorname{Ricci}(g_i)|^4 d\mu_{g_i} \to 0, \text{ as } i \to \infty, \tag{7.12}$$

where $d_i(\cdot,\cdot) = a_i d_X(\cdot,\cdot)$, $l_i = a_i^2 l$ and $a_i, \sigma(i), a_i, N(i) \in \mathbb{R}^+$ are numbers satisfying $0 < \alpha_i, \sigma(i) \to 0$ as $i \to \infty$, $a_i, N(i) \nearrow \infty$ as $i \to \infty$.

The condition ε close in the C^k sense, *is made precise in the proof of the theorem, and the approximations are always achieved with f.*

Proof. Let x_j be fixed. If $x_j \in f(\operatorname{Reg}(M))$ then the theorem follows directly using the definition of C^k close and Theorem 4.5 (see below). So assume $x_j = x_1 \notin f(\operatorname{Reg}(M))$ and let $f(p_1) = x_1$.

Let t_i be a sequence of good times and scale by $a_i := \frac{1}{T-t_i}$ and translate as in the proof of (ii) Theorem 7.1.

First we use a similar argument to that given at the end of Section 5 to show that $|d_t(\cdot,\cdot) - d_s(\cdot,\cdot)| \le C(J_1)$ for all $t, s \in (-\delta(N, J_1), 0]$ for all $x, y \in {}^tB_{N/4}(p_1)$. We use the notation from the proof of (ii) Theorem 7.1 in this argument, and we take various subsequences when necessary.

Let $x, y \in {}^tB_{N/4}(p_1)$ be arbitrary in, and γ a distance minimising curve between these two points w.r.t to $g(t)$ (γ must lie in ${}^tB_N(p_1)$ and we have $L_t(\gamma) \le N$).

We modify the curve γ to obtain a new curve $\tilde\gamma : [0,1] \to M$ in the following way: if γ reaches the closure of the ball ${}^tB_{2J_1}(p_1)$ at a first point $\gamma(r)$ then let $\gamma(\tilde r)$ be the last point which is in the closure of the ball ${}^tB_{2J_1}(p_1)$ (it could go out and come in a number of times). Remove $\gamma|_{(r,\tilde r)}$ from the curve γ. In doing this we obtain the finite union of at most 2 curves $\tilde\gamma_1$ and $\tilde\gamma_2$. Call this finite union $\tilde\gamma$ and consider it as a curve with finitely many discontinuities.

The new $\tilde{\gamma}$ has

$$L_t(\tilde{\gamma}) \le L_t(\gamma) = d(x,y,t) \le N \tag{7.13}$$

From equation (7.5) in the proof above, we see that for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$(1-\varepsilon)g(y,t) \le g(y,s) \le (1+\varepsilon)g(y,t) \tag{7.14}$$

for all $y \in {}^tB_{J_4,N}(p_1) = {}^tB_{2^5 J_1^5,N}(p_1) \subseteq {}^{-1}B_{32J,16N}(p_1)$ for all $t,s \in (-\delta, 0]$ (δ independent of i: use (7.5) and the evolution equation $\frac{\partial}{\partial t} g = -2\mathrm{Ricci}(g)$). Hence $L_t(\tilde{\gamma}) \ge L_s(\tilde{\gamma}) - \varepsilon L_t(\tilde{\gamma}) \ge L_s(\tilde{\gamma}) - \varepsilon N$ for all $t,s \in (-\delta,0]$, which, when combined with (7.13), gives us

$$\begin{aligned} d(x,y,t) &\ge L_t(\tilde{\gamma}) \\ &\ge L_s(\tilde{\gamma}) - \varepsilon N \\ &\ge d(x,y,s) - \varepsilon N - J_4^6. \end{aligned} \tag{7.15}$$

The last inequality can be seen as follows: when $\tilde{\gamma}$ reaches the ball ${}^tB_{J_4}(p_1)$, it must also be in ${}^sB_{J_4^5}(p_1)$ in view of Remark 5.5. So the two points of discontinuity on $\tilde{\gamma}$ may be joined smoothly by a curve with length (with respect to $g(s)$) at most $2J_4^5$. Call this curve $\hat{\gamma}$. Hence $L_s(\hat{\gamma}) \le 2J_4^5 + L_s(\tilde{\gamma})$, which implies $L_s(\tilde{\gamma}) \ge L_s(\hat{\gamma}) - 2J_4^5 \ge d(x,y,s) - J_4^6$ as claimed. So we have $d(x,y,t) \ge d(x,y,s) - J_4^7$ if we choose $\varepsilon = \frac{1}{N}$. Swapping s and t, we see that

$$|d(x,y,t) - d(x,y,s)| \le J_4^7 \tag{7.16}$$

if $t,s \in (-\delta,0]$, and $x,y \in {}^tB_{N/4}(p_1)$ where $\delta = \delta(N,J)$ and may depend on the solution, but does not depend on i, as long as i is large enough.

In particular, ${}^tB_{J^{100},\frac{N}{8}}(p_1) \subseteq {}^sB_{J^{50},\frac{N}{4}}(p_1) \subseteq {}^tB_{J^5,N}(p_1)$ for all $N > J^{100}$ and i large enough, for all $t,s \in (-\delta,0]$ where $\delta = \delta(N,J)$ and may depend on the solution, but does not depend on i. Notice, by taking a limit $s \nearrow 0$, we see $f({}^tB_{J^{100},\frac{N}{8}}(p_1)) \subseteq {}^{d_i}B_{J^{50},\frac{N}{4}}(x_1)$ (*).

Using (7.5), and the evolution equation for the curvature as in Section 8 of [HaForm], we see that

$$|f_*(g(t)) - l_i|_{C^k({}^{d_i}B_{J^{50},N}(x_1),g(t))} \le \hat{\varepsilon} \tag{7.17}$$

if $t \in (-\delta(k,\hat{\varepsilon}),0]$. We explain now why Inequality (7.17) is true. To see this, work with fixed geodesic coordinates $\varphi : B_{i_0}(z) \to B_{i_0}(0)$ of radius larger $i_0 > 0$ at any point in $z \in {}^{d_i}B_{J^{50},N}(p_1)$ (these exist because of the curvature estimates of the previous theorem, Theorem 7.1, and the non-collapsing estimates). Writing l_i in these coordinates, (we drop the i in these coordinates and call $f_*(g(t))$ also $g(t)$ in the coordinates) we have $\frac{1}{C}\delta_{ij} \le l_{ij}(\cdot) \le C\delta_{ij}$, $\sum_{j=0}^{K} |D^j l|^2(\cdot) \le C$ on $B_{i_0}(0)$ for some C not depending on i, where here D is the standard euclidean derivative (the manifolds are non-collapsed and satisfy the curvature bounds of Theorem 7.1: see Corollary 4.12 in [HaComp] for example). Using the evolution equation for $g(t)$ and the curvature bounds, and the fact that $g(0) - l = 0$ we see, using arguments similar to those of Section 8 in [HaForm], $e^{-C|t|}l \le g(t) \le e^{C|t|}l$, $|D(g(t) - l)| \le C|t|$, $|D^2(g(t) - l)| \le Ct$ and so on. This implies $\sum_{j=0}^{k} |{}^{g(t)}\nabla^j(l - g(t))|^2_{g(t)} + |{}^{l}\nabla^j(l - g(t))|^2_l \le \varepsilon$ on $B_{i_0}(z)$ if $|t| \le \delta(C, \varepsilon)$, where δ is chosen near enough to 0. This finishes the explanation of why Inequality (7.17) is true. For a tensor T and a metric l defined on U, we have used the following notation:

$$|T|^2_{C^k(U,l)} := \sum_{j=0}^{k} \sup_{x\in U} |{}^{l}\nabla^j T|^2_l(x),$$

where ${}^{l}\nabla^j$ refers to the jth covariant derivative with respect to l, if $j \in \mathbb{N}$, and ${}^{l}\nabla^0 T := T$. Note that this δ doesn't depend on i. In fact, what we have shown, is $\sum_{j=0}^{k} |{}^{g(t)}\nabla^j(f^*(l_i) - g(t))|^2_{g_i(t)}(f^{-1}(z)) + |{}^{l_i}\nabla^j(l_i - f_* g(t))|^2_{l_i}(z) \le \varepsilon$ for all $t \in (-\delta, 0)$ if $z \in {}^{d_i}B_{J^{50},N}(p_1)$. Hence, using (*), we have also shown $\sum_{j=0}^{k} |{}^{g(t)}\nabla^j(f^*(l_i) - g(t))|^2_{g(t)}(w) + |{}^{d_i}\nabla^j(l_i - f_* g(t))|^2_{l_i}(f(w)) \le \varepsilon$ for all $t \in (-\delta, 0)$ if $w \in {}^{t}B_{J^{100},\frac{N}{8}}(p_1)$.

Scaling the solution by $(\frac{\sigma}{J^{100}})^2$, and assuming $N = \frac{8\tilde{N}J^{100}}{\sigma}$ we see that $|g(t) - f^*(l_i)|_{C^k({}^{t}B_{\sigma,\tilde{N}}(p_1),g(t))} \le \sigma$ and $|f_*(g(t)) - l_i|_{C^k({}^{d_i}B_{\sigma,\tilde{N}}(p_1),\tilde{l}_i)} \le \sigma$ if $t \in (-\hat{\delta}, 0]$ (the original $\hat{\varepsilon}$ is as small as we like) and $|d(x,y,t) - d(x,y,s)| \le \sigma$ if $t, s \in (-\hat{\delta}, 0]$ and $x, y \in {}^{t}B_{\tilde{N}}(p_1)$. Choosing i large enough, and a time $t_1 \in (-\hat{\delta}, -\frac{\hat{\delta}}{4})$ which corresponds to a good time of the original solution, we see that we may assume without loss of generality, that $g_1 := g(t_1)$ satisfies $\int_M |\mathrm{Ricci}(g_1)|^4 d\mu_{g_1} \le \sigma$. g_1 is our first metric. It satisfies

$$\begin{aligned}
&|g_1 - f^*(l_i)|_{C^k({}^{g_1}B_{\sigma,\tilde{N}}(p_1),l_i)} \le \alpha_1,\\
&|f_*(g_1) - l_i|_{C^k({}^{d_i}B_{\sigma,\tilde{N}}(x_1),l_i)} \le \alpha_1\\
&\text{and} \quad |d_i(f(x), f(y)) - d_{g_1}(x,y)| \le \alpha_1
\end{aligned}$$

on ${}^{g_1}B_{\tilde{N}}(p_1)$, where $\alpha_1 = \sigma$, and $\int_M |\mathrm{Ricci}(g_1)|^4 d\mu_{g_1} \le \alpha_1$.

Repeating the procedure, but scaling by $(\frac{(\sigma)^2}{J^{100}})^2$, at the end, with $N = \frac{2\times 8\tilde{N}J^{100}}{\sigma^2}$ leads to our second metric g_2, and g_2 satisfies (for a new larger i)

$$\begin{aligned}
&|g_2 - f^*(l_i)|_{C^k({}^{g_2}B_{\sigma^2,2\tilde{N}}(p_1),g_2)} \le \alpha_2, \\
&|f_*(g_2) - l_i|_{C^k({}^{d_i}B_{\sigma^2,2\tilde{N}}(x_1),l_i)} \le \alpha_2 \\
&\text{and} \quad |d_i(f(x), f(y)) - d_{g_2}(x,y)| \le \alpha_2
\end{aligned}$$

on ${}^{g_2}B_{2\tilde{N}}(p_1)$, where $\alpha_2 = \sigma^2$, and $\int_M |\mathrm{Ricci}(g_2)|^4 d\mu_{g_2} \le \alpha_2$.

And so on. Choosing σ_i to be an arbitrary sequence with $\sigma_i \gg \sigma^i$ and $\sigma_i \to 0$ as $i \to \infty$ completes the proof. □

For convenience we introduce some notation which will help us describe the phenomenon of metric annulli being C^k-close, as described in the theorem above. This phenomenon occurs at a number of points in the rest of the paper.

Definition 7.5. Let (X, d_X), (Y, d_Y) be complete, connected metric spaces. We assume also that these spaces have a given Riemannian structure with at most finitely many (possible) singularities in the following sense: $N := X\backslash\{x_1, \ldots, x_L\}$ and $V := Y\backslash\{y_1, \ldots, y_L\}$ are smooth manifolds, and l is a Riemannian metric on N and v on V. For $0 < r < R \le \infty$, $E \subseteq X$ an open set($E = X$ is allowed), and $x_0 \in \{1, \ldots, x_L\}$, $y_0 \in \{y_1, \ldots, y_L\}$ we say that

$$d_{C^k}(E \cap {}^{d_X}B_{r,R}(x_0), {}^{d_Y}B_{r,R}(y_0)) \le \varepsilon \tag{7.18}$$

(we always assume $\varepsilon \ll \min(r, R-r)$) if

- (i) $E \cap {}^{d_X}B_{r,R}(x_0) \subseteq N$ and ${}^{d_Y}B_{r,R}(y_0) \subseteq V$, and
- (ii) there exists a C^{k+1} map $f : E \cap {}^{d_X}B_{r,R}(x_0) \to V$, such that f is a C^{k+1} diffeomorphism onto its image, ${}^{d_Y}B_{r+\varepsilon,R-\varepsilon}(y_0) \subseteq f(E \cap {}^{d_X}B_{r,R}(x_0))$
- (iii) $|d_X(w, x_0) - d_Y(f(w), y_0)| \le \varepsilon$ for all $w \in E \cap {}^{d_X}B_{r,R}(x_0)$: in particular ${}^{d_Y}B_{s+\varepsilon,m-\varepsilon}(y_0) \subseteq f(E \cap {}^{d_X}B_{s,m}(x_0)) \subseteq {}^{d_Y}B_{s-\varepsilon,m+\varepsilon}(y_0)$ for all $0 < r \le s < m \le R$ with $s + \varepsilon < m - \varepsilon$.
- (iv) $|f^*(v) - l|^2_{C^k(E\cap {}^{d_X}B_{r,R}(x_0),l)}$
  $:= \sum_{j=0}^k \sup_{x\in E\cap {}^{d_X}B_{r,R}(x_0)} |{}^l\nabla^j(f^*(v) - l)|^2_l(x) \le \varepsilon$
  and $|v - f_*l|^2_{C^k({}^{d_Y}B_{r+\varepsilon,R-\varepsilon}(y_0),v)} \le \varepsilon$.

Remark 7.6. Note that in the Approximation Theorem above, Theorem 7.4, the f that occurs there is also a Gromov-Hausdorff approximation when considered as a map on the balls being considered (and $E = M$). Here we only require condition (iii), which is weaker.

Remark 7.7. The definition of C^k close is coordinate free. This allows us to compare elements of sequences of Annuli in a coordinate invariant way.

Remark 7.8. From the definition we see, that if (X, d_X), (Y, d_Y), are metric spaces of the type occurring in the theorem then

$$d_{C^k}({}^{d_X}B_{r,R}(x_0), {}^{d_Y}B_{r,R}(y_0)) \leq \varepsilon$$

implies

$$d_{C^k}({}^{d_Y}B_{r+4\varepsilon,R-4\varepsilon}(y_0), {}^{d_X}B_{r+4\varepsilon,R-4\varepsilon}(x_0)) \leq \varepsilon \quad (almostsymmetry)$$

8. Orbifold structure of the limit space

The flatness estimate (7.2) of the previous section, along with the non-collapsing and non-expanding estimates (which survive into the limit, as explained in (iii) at the beginning of Section 7) guarantee that X is actually a so called *generalised C^0 Riemannian orbifold* with only finitely many isolated *orbifold points* : points $q \in X$ for which there exists a neighbourhood $q \in U \subseteq X$ and a smooth diffeomorphism $\varphi : U \to \mathbb{R}^4$ are called *manifold points*, all other points in X are called *orbifold points*. These objects have been studied in [Tian], [And1], [BKN] . In the papers [HM, HM2], the authors also used generalised Riemannian orbifolds (they refer to them as *multifolds*: see section 3 of [HM2]) to prove an orbifold compactness result for solitons. They were introduced and used in the static (for example the Einstein) setting by M. Anderson [And1] (see also [BKN]), to describe non-collapsing limits of Einstein manifolds. The estimates required to show that X is a generalised C^0 Riemannian orbifold are contained in the previous section. *Generalised Riemannian orbifolds* can have a number of components at each orbifold type point. In our case we will see that there is exactly one component at each singular point. Before showing this, we state the general result which follows from the argument for example in [Tian] (see also [And1], [BKN]).

We use the following notation in the statement of the theorem and in the rest of the paper: $D_{r,R} \subseteq \mathbb{R}^4$ is the standard open annulus of inner radius $r \geq 0$ and outer radius $R \leq \infty$, $(r < R)$ centred at 0: $D_{r,R} = \{x \in \mathbb{R}^4 \mid |x| > r, |x| < R\}$. D_r represents the open disc of radius r centred at 0: $D_r := \{x \in \mathbb{R}^4 \mid |x| < r\}$. Note $D_{0,R} = \{x \in \mathbb{R}^4 \mid |x| > 0, |x| < R\} = D_R \backslash \{0\}$.

Theorem 8.1. *X is a generalised C^0 Riemannian orbifold in the following sense.*

(i) *$X\backslash\{x_1,\dots,x_L\}$ is a manifold, with the structure explained above in Lemmata 6.5 and 6.6.*

(ii) *There exists an $r_0 > 0$ small, and an $N < \infty$ such that the following is true. Let $x_i \in X$ be one of the singular points. Then the number of connected components $(E_{i,j}(r))_{j\in\{1,\dots,\tilde{N}_i\}}$ of ${}^{d_X}B_r(x_i)\backslash\{x_i\}$ in $X\backslash\{x_1,\dots,x_L\}$ is finite and bounded by N (that is $\tilde{N}_i \le N$) for $r \le r_0$, where $N = N(\sigma_0,\sigma_1) < \infty$.*

(iii) *Fix $i \in \{1,\dots,L\}, j \in \{1,2,\dots,\tilde{N}_i\}$, and let $E = E_{i,j}(r_0)$ be one of the components from (ii). Then there exists a $0 < \tilde{r} \le r_0$ and a diffeomorphism $k : D_{0,\tilde{r}} \to k(D_{0,\tilde{r}}) \subseteq \tilde{E}$ where $\tilde{E}$ is the universal covering space of $E \cap (\cup_{j=1}^{N_i} E_{i,j}(\tilde{r}(1+\varepsilon))$, such that the covering map $\pi_E : \tilde{E} \to E$ is finite and for $r \le \tilde{r}$ we have*

$$\sup_{D_{0,r}} |(\pi_E \circ k)^* l - \delta|_{C^0(D_{0,r})} \le \varepsilon_1(r) \tag{8.1}$$

where $\varepsilon_1(r) \ge 0$ is a decreasing function with $\lim_{r\searrow 0}\varepsilon_1(r) = 0$, δ is the standard euclidean metric on $\mathbb{R}^4$ or subsets thereof, $|\cdot|_{C^0(L)}$ is the standard euclidean norm on two tensors, $|v|^2_{C^0(L)} := \sup_{x\in L}\sum_{i,j=1}^n |v_{ij}(x)|^2$ for any set $L \subseteq \mathbb{R}^4$ and any two tensor $v = v_{ij}dx^i dx^j$.

Proof. (i) was shown above. (ii) follows from the non-expanding and non-collapsing estimates, exactly as in the proof of Lemma 3.4 in [Tian].

(iii) Follows as in the proof of Lemma 3.6 in [Tian] using the *flatness estimates*, (7.2), and the non-collapsing and non-expanding estimates. □

Remark 8.2. Some of the proofs of the Lemmata mentioned here (Lemma 3.6 and Lemma 3.4 of [Tian]) can be simplified at certain points, by using that $\mathrm{inj(B_r(p))} \ge \mathrm{c_0 r}$ for all balls $B_r(p)$ which are compactly contained in (D,h) where (D,h) is any smooth, open flat ($\mathrm{Riem}(h) = 0$) non-collapsed, non-inflated (on all scales) manifold without boundary: this follows from the injectivity radius estimate of Cheeger-Gromov-Taylor, Theorem 4.3 in [CGT], whose proof is local.

The construction of this k in [Tian] (see Lemma 3.6 in [Tian]) is achieved by pasting together maps $\varphi_i : D_{\frac{1}{2^{i+2}},\frac{1}{2^i}} \to \pi^{-1}(B_{\frac{1-\varepsilon}{2^{i+2}},\frac{1+\varepsilon}{2^i}})$ where $i \in \mathbb{N}$. That is $\varphi_1, \varphi_2, \dots$ are first constructed, and then φ_1 is pasted to φ_2 and φ_2 to φ_3

and so on. This leads to a map k with the properties given in the theorem above: see the proof of Lemma 3.6 in [Tian]. We construct a φ here using the method described in the proof of Lemma 3.6 in [Tian] with some minor modifications. The explicit construction we present here will be used in later sections. That is, in later sections we will use the maps φ_i used here to construct φ along with properties of the $\varphi_i's$. For this reason, we present details of this construction.

As shown in the proof of Lemma 3.6 of [Tian]: if we scale, $l_i = (2^{i+2})^2 l$, $d_i = 2^{i+2} d_X$, then

$$d_{C^k}((^{g(i)}B_{1/2,4}(0), g(i)), (^{d_i}B_{1/2,4}(x_1) \cap E, l_i)) \leq \varepsilon(i) \to 0 \text{ as } i \to \infty,$$

where $(^{g(i)}B_{1/4,4}(0), g(i)) \subseteq ((\mathbb{R}^4 \backslash \{0\})/\Gamma(i), g(i))$, and $g(i)$ is the standard metric on $(\mathbb{R}^4 \backslash \{0\})/\Gamma(i)$, and $\Gamma(i)$ is some finite subgroup of $O(4)$ with finitely many elements (less than or equal to N elements, N independent of i) acting freely on $\mathbb{R}^4 \backslash \{0\}$. Hence there exists a diffeomorphism

$$v_i : (^{g(i)}B_{1/2,4}(0), g(i)) \to (^{d_i}B_{1/2-\varepsilon(i),4+\varepsilon(i)}(0) \cap E, l_i) \subseteq (E, l_i), \tag{8.2}$$

such that

$$|v_i^* l_i - g(i)|_{C^k(^{g(i)}B_{1/2,4}(0), g(i))} + |(v_i)_* g(i) - l_i|_{C^k(^{d_i}B_{1/2+\varepsilon(i),4-\varepsilon(i)}(0) \cap E, l_i)} \leq \varepsilon(i)$$

In the following, $\varepsilon(i) > 0$ will refer to positive numbers with the property that $\varepsilon(i) \to 0$ as $i \to \infty$. As the notation suggests, in fact this $\Gamma(i)$ (and hence $g(i)$) could depend on the sequence we take, and could depend on $i \in \mathbb{N}$. However, $\mathrm{inj}((\mathbb{R}^4 \backslash \{0\})/\Gamma(\mathrm{i}), \mathrm{g}(\mathrm{i}))(\mathrm{x}) \geq |\mathrm{x}| \mathrm{i}_0$ for some fixed $i_0 > 0$, where $|x| = |\tilde{x}|_{\mathbb{R}^4}$ is the euclidean norm of the point x lifted to $\tilde{x} \in \mathbb{R}^4$ (any such $\tilde{x}$ has the same euclidean distance from the origin, regardless of which $\tilde{x}$, covering x, we choose) [Explanation 1: this follows in view of the construction: for any ball $^{d_i}B_r(x) \subseteq {}^{d_i}B_{1,4}(x_1) \cap E$ we have $r^4 \sigma_1 \geq \mathrm{vol}(^{d_i}B_r(x)) \geq r^4 \sigma_0$, and the norm of the curvature tensor on $^{d_i}B_{1,4}(x_1)$ goes to zero as $i \to \infty$. Hence $\mathrm{inj}(^{\mathrm{d_i}}\mathrm{B}_{1/100}(0), \mathrm{l_i})(\mathrm{x}) \geq \mathrm{i}_0$ for any $x \in {}^{d_i}B_{\frac{5}{4},3}(x_1)$, for some $i_0 > 0$, if i is large enough, in view of the injectivity radius estimate of Cheeger-Gromov-Taylor contained in Theorem 4.3 in [CGT]). Hence, using $d_{C^k}((^{g(i)}B_{1/2,4}(0), g(i)), (^{d_i}B_{1/2,4}(x_1) \cap E, l_i)) \leq \varepsilon(i)$, we see that we have $\mathrm{inj}(^{\mathrm{g(i)}}\mathrm{B}_{1/100}(\mathrm{x}), \mathrm{g(i)})(\mathrm{x}) \geq \mathrm{i}_0/2$ for some $i_0 > 0$ for any $x \in (^{g(i)}B_{\frac{3}{2},2}(0), g(i))$, if i is large enough.]

Let $\pi_i : \mathbb{R}^4 \backslash \{0\} \to (\mathbb{R}^4 \backslash \{0\})/\Gamma(i)$ be the standard projection, and $x \in (\mathbb{R}^4 \backslash \{0\})/\Gamma(i)$, $(\pi_i)^{-1}(x) = \{x_1, \ldots, x_N\}$. π_i is a covering map and a local isometry, and using the fact that $\mathrm{inj}(\mathbb{R}^4\backslash\{0\}/\Gamma(\mathrm{i}), \mathrm{g}(\mathrm{i}))(\mathrm{x}) \geq |\mathrm{x}|\mathrm{i}_0$, we see that $d_{\mathbb{R}^4}(x_k, x_l) \geq (i_0|x|)/20 > 0$ in $\mathbb{R}^4$ for $x_k, x_l \in (\pi_i)^{-1}(x)$, $k \neq l$.

Let $\psi_i : D_{1,4} \to E$ be the natural map $\psi_i = v_i \circ \pi_i|_{D_{1,4}}$ where $v_i : ({}^{g(i)}B_{1/2,4}(0), g(i)) \to ({}^{d_i}B_{1/2-\varepsilon(i),4+\varepsilon(i)}(0) \cap E, l_i) \subseteq (E, l_i)$ is the map defioned in (8.2) above, and $\pi_i : \mathbb{R}^4\backslash\{0\} \to (\mathbb{R}^4\backslash\{0\})/\Gamma(i)$, the standard projection, is as above. Define $\varphi_i(x) = \psi_i(2^{i+2}x)$ for $x \in D_{\frac{1}{2^{i+2}},\frac{1}{2^i}}$: this is the unscaled version of ψ_i. Later we will paste the φ_i's together. To do this, it is conveniant to work at the scaled level. We will require that neighbours φ_i and φ_{i+1} are close to one another for all $i \in \mathbb{N}$, in a C^k sense (to be described) on their common domain of definition, at least at the scaled level. To show this, we have to compare neighbours φ_i and φ_{i+1}, for all $i \in N$, on their common domain of definition $D_{\frac{1}{2^{i+2}},\frac{1}{2^{i+1}}}$. We do this at the scaled level: $\psi_i : D_{1,4} \to E$ is as defined above, $\psi_i(x) = \varphi_i(\frac{1}{2^{i+2}}x)$, and we define $\eta_{i+1} : D_{\frac{1}{2},2} \to E$ by $\eta_{i+1}(x) = \varphi_{i+1}(\frac{x}{2^{i+2}}) = \psi_{i+1}(2x)$

Notice that in defining the $\psi_i' s$, we have the freedom to change the coverings π_i by a deck transformation, that is by an element $A \in O(4)$. Also, in view of the definitions, and the notion of convergence introduced in Definition 7.5, we have $(\psi_i)^*(l_i)$ is C^k close to δ on $D_{1+\varepsilon(i),4-\varepsilon(i)}$ and $(\eta_{i+1})^*(l_i)$ is C^k close to δ on $D_{1/2+\varepsilon(i),2-\varepsilon(i)}$, in view of the fact that $(\eta_{i+1})^*(l_i)(x) = (\psi_{i+1})^*(l_{i+1})(2x)$

Step 1. For all $i \geq N \in \mathbb{N}$ the following is true: By changing the map π_{i+1} by an element $A \in O(4)$, if necessary, we can assume that the pair ψ_i and η_{i+1} are, for sufficiently large $i \in \mathbb{N}$, C^k close to one another on their common domain of definition, in a sense which we now describe: take any arbitrary ball ${}^{\delta}B_s(y) \subseteq D_{1+\delta/2,2-\delta/2}$ with some fixed $s > 0$ $s \leq \frac{i_0}{10}, s \leq \frac{\delta}{10}$ where $y \in D_{1+\delta,2-\delta}$, is in the common domain of definition of ψ_i and η_{i+1}, where $\delta > 0$ is some fixed small number. Then $d_i(\psi_i(x), \eta_{i+1}(x)) \leq \varepsilon(i)$ for all $x \in D_{1+\delta/2,2-\delta/2}$ and, $\psi_i(B_s(y)) \cup \eta_{i+1}(B_s(y)) \subseteq {}^{d_i}B_{2s}(\tilde{y})$, $|\theta \circ \psi_i - \theta \circ \eta_{i+1}|_{C^k(B_s(y),\mathbb{R}^4)} \leq \varepsilon(i)$, where $\theta : {}^{d_i}B_{2s}(\tilde{y}) \to {}^{\delta}B_s(0) \subseteq \mathbb{R}^4$ are geodesic coordinates on(M, l_i) centred at the point $\tilde{y} = \eta_{i+1}(y)$ (note these coordinates exist, in view of the fact that $d_{C^k}(({}^{d_i}B_{1,4}(x_1) \cap E, l_i), ({}^{g(i)}B_{1,4}(0), g(i))) \leq \varepsilon(i)$).

Proof of Step 1. Assume this is not the case. Then we find a sequence for which this is not true. Taking a subsequence (we denote the subsequence of the pairs ψ_i, η_{i+1} also by ψ_i, η_{i+1}), we see that $({}^{g(i)}B_{1,2}(0), g(i))$, and $({}^{g(i+1)}B_{1,2}(0), g(i+1))$ converge to the same limit space, $(B_{1,2}(0), g) \subseteq (\mathbb{R}^4\backslash\{0\}, \delta)/\Gamma$ (in the sense of C^k convergence described above in Definition 7.5), where Γ is a finite subgroup of $O(4)$ with finitely many (bounded by N) elements acting freely on $\mathbb{R}^4\backslash\{0\}$: the argument in the beginning of the proof of Lemma 3.6 in [Tian], for example, gives us this fact.

Let us denote by $Z_i : ({}^{g(i)}B_{1,2}(0), g(i)) \to (B_{1-\varepsilon(i),2+\varepsilon(i)}(0), g)$and Z_{i+1} : $({}^{g(i+1)}B_{1,2}(0), g(i+1)) \to (B_{1-\varepsilon(i),2+\varepsilon(i)}(0), g)$ the natural maps which are diffeomorphisms and almost C^k local isometries onto their images: these must exist in view of this convergence.

Let us denote by $R_i : (E \cap {}^{d_i}B_{1,2}(x_1), l_i) \to (B_{1-\varepsilon(i),2+\varepsilon(i)}(0), g)$ the natural map, which is also a diffeomorphism onto its image and almost an local isometry, that arises in this way: $R_i = Z_i \circ (v_i)^{-1}$ (if $\varepsilon(i)$ changes in the proof, but the new constant $\tilde{\varepsilon}(i) \to 0$ as $i \to \infty$, then we denote $\tilde{\varepsilon}(i)$ by $\varepsilon(i)$ again). Then $R_i \circ \psi_i$ converges (after taking a subsequence) to a map $\hat{\pi} : D_{1,2} \to (B_{1,2}(0), g) \subseteq ((\mathbb{R}^4\backslash\{0\}, \delta))/\Gamma$ which is a covering map, with $(\hat{\pi})^* g = \delta$ and $R_i \circ \eta_{i+1}$ converges (after taking a subsequence) to a map $\tilde{\pi} : D_{1,2} \to (B_{1,2}(0), g)$ which is a covering map with $\tilde{\pi}^* g = \delta$, and the convergence is in the usual C^k sense of convergence of maps between fixed smooth Riemannian manifolds[1]. Hence the two maps differ only by a deck transformation, which is an element A in $O(4)$: $\tilde{\pi} = \hat{\pi} \circ A$. Before taking a limit, we can change η_{i+1} by this element, $\hat{\eta}_{i+1} := \eta_{i+1} \circ A$. Remembering the definitions of η_{i+1} and ψ_{i+1}, we see that we have $\hat{\eta}_{i+1}(x) = (\eta_{i+1} \circ A)(x) = \psi_{i+1}(A(2x)) = (\psi_{i+1} \circ A)(2x) = ((v_{i+1}) \circ (\pi_{i+1}) \circ A)(2x)$. That is we change the covering map π_{i+1} to the covering map $\hat{\pi}_{i+1} = \pi_{i+1} \circ A$, and then define $\hat{\eta}_{i+1} := (v_{i+1}) \circ \hat{\pi}_{i+1}(2x)$: we have this freedom in the choice of our π_{i+1}'s. Now both $R_i \circ \psi_i$ and $R_i \circ \hat{\eta}_{i+1} = R_i \circ \eta_{i+1} \circ A$ converge to $\tilde{\pi}$ in the sense explained above. In particular, returning to $({}^{d_i}B_{1,2}(x_1), l_i)$ with $(R_i)^{-1}$ and writing things in geodesic coordinates, we see that $\hat{\eta}_{i+1}$ is arbitrarily close to ψ_i, which leads to a contradiction. Here we used the following fact. In geodesic coordinates $\beta : B_s(p) \subseteq (B_{1+\varepsilon(i),2-\varepsilon(i)}(0), g) \to B_s(0) \subseteq \mathbb{R}^4$,

---

[1]Explanation: $R_i \circ \psi_i, R_i \circ \eta_{i+1} : D_{1+\varepsilon(i),2-\varepsilon(i)} \to (B_{1-2\varepsilon(i),2+2\varepsilon(i)}(0), g)$, have $(R_i \circ \psi_i)^* g$ and $(R_i \circ \eta_{i+1})^*(g)$ are $\varepsilon(i)$ close in the C^k norm to δ, and hence, taking a subsequence, we obtain maps $\hat{\pi}, \tilde{\pi} : D_{1,2} \to (B_{1,2}(0), g)$ with $\hat{\pi}^*(g) = \tilde{\pi}^*(g) = \delta$. We work now with $\hat{\pi}$: the same argument works for $\tilde{\pi}$. For any $x \in D_{1,2}$ we can find a small neighbourhood $U \subset\subset D_{1,2}$ with $x \in U$ such that $\hat{\pi}(U) \subseteq {}^gB_s(p)$ where ${}^gB_s(p) \subseteq ({}^gB_{1,2}(0), g)$ is a geodesic ball and there exist geodesic coordinates $\beta : {}^gB_s(p) \to {}^\delta B_s(0)$ (s small enough). Then $\beta \circ \hat{\pi} : U \to \mathbb{R}^4$ is well defined, and has $\det(D(\beta \circ \hat{\pi})) = 1$ and hence $\hat{\pi} : D_{1,2} \to {}^gB_{1,2}(0)$ is a local diffeomorphism. The map is, per construction, surjective (here the definition of the convergence of annuli from Definition 7.5 is used). It is also proper, since by construction, $D_{r,s} \subseteq D_{1,2}$ is mapped onto $(B_{r,s}(0), g) \subseteq (B_{1,2}(0), g(0))$ (here the definition of the convergence of annuli from Definition 7.5 is used). Hence, $\hat{\pi}$ is a covering map (see, for example, Proposition 2.19 in [Lee]).

the metric is δ. Hence for geodesic coordinates $\gamma : {}^{d_i}B_{s/2}(z) \subseteq {}^{d_i}B_{1,2}(x_1) \to B_{s/2}(0) \subseteq \mathbb{R}^4$ with $R_i(z) = p$, we see $\beta \circ R_i \circ \gamma^{-1} : B_{s/2}(0) \to \mathbb{R}^4$ is C^k close to an element in $O(4)$, in view of, for example, Corollary 4.12 in [HaComp]. End of the Explanation]. We assume in the following, that we have made the necessary modifications to the $\varphi_i's$ (note, that in changing π_i by a deck transformation, we are also changing the $\varphi_i's$ and hence the ψ_i's), so that the above C^k *closeness* of neighbours ψ_i, η_{i+1} for all $i \in \mathbb{N}$ large enough is guaranteed. These modifications are made inductively: for $i \in \mathbb{N}$ sufficiently large, first change π_{i+1} by a deck transformation if necessary, then π_{i+2} by a deck transformation if necessary, then π_{i+3} by a deck transformation if necessary, and so on. **End of Step 1.**

Now, **Step 2**, we explain how to join φ_i and φ_{i+1}, assuming we have made the necessary modifications to the $\varphi_i's$, as explained in Step 1. The resulting map, at the unscaled level will be φ.

For large $i \in \mathbb{N}$, we know that $(v_i^{-1} \circ \psi_i) : D_{1+\varepsilon(i),4-\varepsilon(i)} \to (B_{1,4}(0), g(i))$ and $(v_i^{-1} \circ \eta_{i+1}) : D_{1/2+\varepsilon(i),2-\varepsilon(i)} \to (B_{1,4}(0), g(i))$ are well defined smooth maps which are C^k close to one another on the common domain of definition $D_{1+\varepsilon(i),2-\varepsilon(i)}$ and C^k close to $\pi_i : D_{1+\varepsilon(i),2-\varepsilon(i)} \to (B_{1,2}(0), g(i))$ on $D_{1+\varepsilon(i),2-\varepsilon(i)}$ in the sense just described. Lifting these maps to $D_{0,4}(0) \subseteq \mathbb{R}^4$ with respect to the covering $\pi_i : D_{0,4}(0) \to (B_{0,4}(0), g(i))$, we see that we obtain maps $\tilde{\psi}_i : D_{1+\varepsilon(i),4-\varepsilon(i)} \to D_{1,4}(0)$ and $\tilde{\eta}_{i+1} : D_{1/2+\varepsilon(i),2-\varepsilon(i)} \to D_{1,2}(0)$ (these maps are lifts with respect to π_i, that is $\pi_i \circ \tilde{\psi}_i = (v_i^{-1} \circ \psi_i)$, $\pi_i \circ \tilde{\eta}_{i+1} = (v_i^{-1} \circ \eta_{i+1})$, and these lifts exist, since the domain of the maps we are lifting are simply connected: see Corollary 11.19 in [Lee2]) which are C^k close to the same element in $O(4)$ on $D_{1+\varepsilon(i),2-\varepsilon(i)}$, which is without loss of generality the identity (transform the lifts $\tilde{\psi}_i, \tilde{\eta}_{i+1}$ by the inverse of this element in the target space: the resulting maps are still lifts). Also $\tilde{\psi}_i^*(\delta)$ and $\tilde{\eta}_{i+1}^*(\delta)$ are C^k close to δ, on their domains of definition, and hence $\tilde{\psi}_i$ is C^k close to an element in $O(4)$ on $D_{1+\varepsilon(i),4-\varepsilon(i)}$ and $\tilde{\eta}_{i+1}$ is C^k close to an element in $O(4)$ on $D_{1/2+\varepsilon(i),2-\varepsilon(i)}$, and using the information in the previous line, this element is the identity in each case.

Defining

$$\begin{aligned} &\tilde{\varphi}_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to {}^{d_i}B_{1/2,4}(x_1), \\ &\tilde{\varphi}_i := v_i \circ \pi_i \circ (\eta\tilde{\psi}_i + (1-\eta)\tilde{\eta}_{i+1}) \end{aligned} \tag{8.3}$$

where $\eta : \mathbb{R}^4 \to \mathbb{R}_0^+$ is a smooth cutoff function, with $0 \leq \eta \leq 1$, $\eta = 0$ on $D_{0,2-2\delta}$, $\eta = 1$ on $D_{2-\delta,\infty}$, (**) we obtain a smooth map, which is equal to

η_{i+1} on $D_{1/2+\varepsilon(i),2-2\delta}$ and equal to ψ_i on $D_{2-\delta,4-\varepsilon(i)}$, and for which $(v_i)^{-1} \circ \tilde{\varphi}_i : D_{1/2+2\varepsilon(i),4-2\varepsilon(i)} \to {}^{g(i)}B_{1/2,4}(0)$ is C^k close to π_i. The map $\tilde{\varphi}_i$ satisfies

$$(1-\varepsilon(i))|x| \leq d_i(\tilde{\varphi}_i(x), x_1) \leq (1+\varepsilon(i))|x| \tag{8.4}$$

on $D_{1/2+\varepsilon(i),4-\varepsilon(i)}$, by construction. We can now define $\varphi : D_{0,\varepsilon} \to X\backslash\{x_1\}$. For $x \in [\frac{1-7\delta}{2^{i+1}}, \frac{1-4\delta}{2^i}]$ and $i \in \mathbb{N}$ large, we define $\varphi(x) := (\tilde{\varphi}_i)(2^{i+2}x)$. This map is smooth and well defined: fix $i \in \mathbb{N}$, and let $x \in [\frac{1-7\delta}{2^{i+1}}, \frac{1-2\delta}{2^{i+1}}]$. Then $\varphi(x) = \tilde{\varphi}_i(2^{i+2}x) = \eta_{i+1}(2^{i+2}x) = \varphi_{i+1}(x)$, and if $x \in [\frac{1}{2^{i+1}}, \frac{1-4\delta}{2^i}]$, then $\varphi(x) = \tilde{\varphi}_i(2^{i+2}x) = \psi_i(2^{i+2}x) = \varphi_i(x)$. **This finishes Step 2**.

We examine, in the following, various properties of φ.

By construction, $\varphi : D_{0,\varepsilon} \to X$ satisfies: $|d_X(\varphi(x), x_1) - |x|| \leq \varepsilon(|x|)|x|$, where $\varepsilon(|x|) \to 0$ as $|x| \to 0$: this follows from (8.4) and the definition of φ. We consider $\tilde{V} := \varphi^{-1}(\varphi(D_{0,\varepsilon}))$ and $V := \varphi(D_{0,\varepsilon})$. We claim that $\varphi|_{\tilde{V}} : \tilde{V} \to V$ is a covering map if $\varepsilon > 0$ is small enough. Note: we do **not** claim that V or $\tilde{V}$ have smooth boundary. We first note, that the cardinality of $(\varphi|_{\tilde{V}})^{-1}(x)$ for $x \in \tilde{V}$ is bounded if ε is small enough. Assume there are points $z_1, \ldots, z_K$, $z_s \neq z_j$ for all $s \neq j \in \{1, \ldots K\}$, with $\varphi(z_1) = \varphi(z_2) = \cdots \varphi(z_K) = m$. We can always find an $i \in \mathbb{N}$ with $z_1 \in [\frac{1-5\delta}{2^{i+1}}, \frac{1-5\delta}{2^i}]$, and hence

$$(1-\varepsilon(i))|z_1| \leq d_X(m, x_1) \leq (1+\varepsilon(i))|z_1|$$

implies

$$(1-\varepsilon(i)\frac{1-5\delta}{2^{i+1}} \leq d_X(m, x_1) \leq (1+\varepsilon(i))\frac{1-5\delta}{2^i}$$

and hence

$$\frac{(1-\varepsilon(i))}{(1+\varepsilon(i))}\frac{(1-5\delta)}{2^{i+1}} \leq |z_j| \leq \frac{(1-5\delta)}{2^i}\frac{(1+\varepsilon(i))}{(1-\varepsilon(i))} \quad \text{for } j = 1, \ldots, K.$$

Hence, after scaling by 2^{i+2}, we have $\tilde{z}_1, \ldots, \tilde{z}_K \in [2-11\delta, 4-19\delta]$ with $\tilde{\varphi}_i(\tilde{z}_1) = \tilde{\varphi}_i(\tilde{z}_2) = \cdots \tilde{\varphi}_i(\tilde{z}_K)$. At the scaled level, we know that, $(v_i)^{-1} \circ \tilde{\varphi}_i : D_{1/2+2\varepsilon(i),4-2\varepsilon(i)} \to (B_{1/2,4}, g_i(0))$ is C^k close to π_i, the standard projection, and the pull back of $g(i)$ with this map is C^k close to δ on $D_{1/2+2\varepsilon(i),4-2\varepsilon(i)}$. In fact $(v_i)^{-1} \circ \tilde{\varphi}_i = \pi_i \circ h_i$ where $h_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to \mathbb{R}^4$ is C^k close to the identity. In particular, $(v_i)^{-1} \circ \tilde{\varphi}_i(B_{s/2}(z)) \subseteq B_s((v_i)^{-1} \circ \tilde{\varphi}_i(z))$ for any $z \in [2-11\delta, 4-19\delta]$ for $0 < s \leq \frac{i_0}{10}$ fixed and small. Let $\psi : {}^{g(i)}B_s(\hat{z}_j) \to B_s(0) \subseteq \mathbb{R}^4$ be geodesic coordinates in $(B_{1/2,4}, g_i(0))$, where $\hat{z}_j = (v_i)^{-1} \circ \tilde{\varphi}_i(\tilde{z}_j)$. The map $\psi \circ (v_i)^{-1} \circ \tilde{\varphi}_i : B_{s/2}(\tilde{z}_j) \to \mathbb{R}^4$ is C^k close to an isometry $B(i,j) = A(i,j) + \tau_{\tilde{z}_j}$of $\mathbb{R}^4$, where $A(i,j) \in O(4)$ and $\tau_{\tilde{z}_j}$ is $\tau_{\tilde{z}_j}(x) =$

$x - \tilde{z}_j$, and hence after a rotation in the geodesic coordinates and a translation, C^k close to the identity. In particular, this map is a diffeomorphism when restricted to $B_{s/2}(\tilde{z}_j)$, and hence $\tilde{z}_i \notin B_{s/2}(\tilde{z}_j)$ for all $j \neq i$. Hence, $\mathrm{vol}(D_{1/2,4}) \geq \sum_{j=1}^{K} \mathrm{vol}(B_{s/2}(\tilde{z}_j)) \geq K\omega_4(s/2)^4$ which leads to a contradiction if K is too large.

If we scale the map $\varphi : D_{[\frac{1-7\delta}{2^{i+1}}, \frac{1-4\delta}{2^i}]} \to X$ by 2^{i+2}, that is let $\hat{\varphi} : D_{[2-14\delta, 4-16\delta]} \to X$ be defined by $\hat{\varphi}(x) = \varphi(\frac{x}{2^{i+2}})$, then we obtain the map $\tilde{\varphi}_i$: $\hat{\varphi} = \tilde{\varphi}_i|_{[2-14\delta,4-16\delta]}$. The argument above, shows that $(v_i)^{-1} \circ \tilde{\varphi}_i|_{B_{s/2}(z)} : B_{s/2}(z) \to \mathbb{R}^4$ is a diffeomorphism for all $|z| \in [2-10\delta, 4-18\delta]$ if $s \ll \delta$,$s < \frac{i_0}{100}$, i sufficiently large. That is $\hat{\varphi}|_{B_{s/2}(z)} = \tilde{\varphi}_i|_{B_{s/2}(z)} : B_{s/2}(z) \to X$ is a diffeomorphism for all $|z| \in [2-10\delta, 4-18\delta]$ and hence $\hat{\varphi}|_{D_{(2-10\delta,4-18\delta)}}$ is a local diffeomorphism, which tells us, scaling back, that $\varphi : D_{(\frac{1-5\delta}{2^{i+1}}, \frac{1-(9/2)\delta}{2^i})} \to X$ is a local diffeomorphism.

That is, $\varphi : D_{0,\varepsilon} \to X$ is a local diffeomorphism, if $\varepsilon > 0$ is small enough.

Hence $V := \varphi(D_{0,\varepsilon})$ is open if $\varepsilon > 0$ is small enough (this corresponds to i being sufficiently large), and $\varphi : \tilde{V} := \varphi^{-1}(V) \to V$ is a local diffeomorphism and an open map. V is connected, as it is the image under a continuous map of a connected region. In fact $\tilde{V}$ is also connected: this will be shown below.

$\varphi : \tilde{V} \to V$ is proper: Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $K \subseteq V$, where K is compact in V. This means, there is a subsequence of x_i (also denoted x_i) such that $x_i \to x \in K \subseteq V$, $x = \varphi(m)$ for some $m \in D_{0,\varepsilon}$. Let $z_1, z_2, \ldots, z_N \in \varphi^{-1}(x)$ be the finitely many points in $\tilde{V}$ with $\varphi(z_j) = m$. We can choose a small neighbourhood U_j of each one, such that $U_i \subset\subset \tilde{V}$ and $\varphi|_{U_j} : U_j \to \varphi(U_j)$ is a diffeomorphism, and without loss of generality $\varphi(U_j) = U \subset\subset V$ for all j, and $\varphi(m) \in U$. Hence *any* sequence $y_k \in \varphi^{-1}(K)$ with $\varphi(y_k) = x_k$ has a convergent subsequence, $y_k \to z_i$ as $k \to \infty$ for some $z_i \in \{z_1, \ldots, z_N\}$. Hence $(\varphi|_{\tilde{V}})^{-1}(K)$ is sequentially compact in $\tilde{V}$. That is $\varphi : \tilde{V} \to V$ is proper. That is, $\varphi : \tilde{V} \to V$ is a proper, surjective, local diffeomorphism. In particular lifts $\tilde{\gamma} : I \to \tilde{V}$ of curves $\gamma : I \to V$, $I = [a,b] \subseteq \mathbb{R}$, always exist and are uniquely determined by their starting points $\tilde{\gamma}(0)$ which is an arbitrary point in $\varphi^{-1}(\gamma(0))$.

$\tilde{V}$ is also connected. Let $\hat{x}$ and $\hat{y}$ be points in $\tilde{V}$ and $x = \varphi(\hat{x}) \in V$, $y = \varphi(\hat{y}) \in V$. $x = \varphi(\hat{x}) \in \varphi(D_{0,\varepsilon})$ implies $x = \varphi(x_0)$ for an $x_0 \in D_{0,\varepsilon}$. Let x_1 be the point $x_1 = x_0/4$. Then $x_1 \in D_{0,\varepsilon/4}$ and $\varphi^{-1}(\varphi(x_1)) \in D_{0,\varepsilon/3}$ if i is sufficiently large, in view of the construction of φ (see the above).

Joining x_0 to x_1 with a ray $\alpha : I \to D_{0,\varepsilon}$ (w.r.t to the euclidean metric) which points into 0 and pushing this down to V again with φ, we obtain a continuous map $\sigma = \varphi \circ \alpha : I \to V$ with $\sigma(0) = \varphi(x_0) = \varphi(\hat{x})$ and $\sigma(1) = \varphi(x_1)$. Taking the lift of this map, and using the starting point $\hat{x}$, we obtain a continuous curve $\tilde{\sigma} : I \to \tilde{V}$ with $\tilde{\sigma}(0) = \hat{x}$ and $\tilde{\sigma}(1) \in \varphi^{-1}(\varphi(x_1)) \in D_{0,\varepsilon/3}$. We may perform the same procedure with y to get a continuous curve $\tilde{\beta} : I \to \tilde{V}$ with $\tilde{\beta}(0) = \hat{y}$ and $\tilde{\beta}(1) \in D_{0,\varepsilon/3}$. We may join $\tilde{\beta}(1)$ to $\tilde{\sigma}(1)$ in $D_{0,\varepsilon/3} \subseteq \tilde{V}$ with a curve $T : I \to D_{0,\varepsilon/3}$, as this space is connected. Hence, following the curve $\tilde{\sigma}$ from $\tilde{\sigma}(0) = \hat{x}$ to $\tilde{\sigma}(1)$ in $\tilde{V}$ and then from $\tilde{\sigma}(1)$ to $\tilde{\beta}(1)$ with T and then from $\tilde{\beta}(1)$ to $\tilde{\beta}(0)$ by going backwards along the curve $\tilde{\beta}$, we see that we have constructed a continuous curve in $\tilde{V}$ from $\hat{x}$ to $\hat{y}$ as required.

Hence $\tilde{V}$ is also connected.

That is, $\varphi : \tilde{V} \to V$ is a proper, surjective, local diffeomorphism, between two path connected spaces, and hence $\varphi : \tilde{V} \to V$ is a covering map (see Proposition 2.19 in [Lee]).

In fact, $\tilde{V}$ is simply connected if $\varepsilon > 0$ is sufficiently small, and hence $\tilde{V}$ is the universal covering space of V. We explain this now. Let i be sufficiently large, and we consider the map $\tilde{\varphi}_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to X$ from above. $v_i \circ \tilde{\varphi}_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to (B_{1/2,4}, g(i))$ is C^k close to $\pi(i)$ as shown above. In particular, $\tilde{\varphi}_i|_{B_s(x)} : B_s(x) \to X$ is a diffeomorphism onto its image and $\tilde{\varphi}_i(B_{s/8}(x)) \subseteq B_{s/4}(z)$ and $B_{s/4}(z) \subseteq \tilde{\varphi}_i(B_s(x))$ for all $x \in D_{2,5/2}(0)$ for all z with $z = \tilde{\varphi}_i(x)$, for a fixed $s > 0$, s independent of i, and $(\tilde{\varphi}_i)^*(l_i)$ is C^k close to δ, as shown above. Let $p \in D_{2,5/2}(0)$ and let $p = p_1, p_2, \ldots, p_N \in D_{2-\varepsilon(i),(5/2)+\varepsilon(i)}$ be the distinct points with $\tilde{\varphi}_i(p_j) = \tilde{\varphi}_i(p)$ for all $j = 1, \ldots, N$. $\theta_j := (\tilde{\varphi}_i)^{-1}_{B_s(p)} \circ \tilde{\varphi}_i : B_{s/8}(p_j) \to B_s(p)$ is C^k close to an element in $O(4)$, and has $\theta_j(p_j) = p$. θ_j is C^k close to an element in $O(4)$ means $\theta_j(x) = A_j \cdot x + \beta_{i,j}(x)$ for all $x \in B_{s/8}(p_j)$, where $|\beta_{i,j}|_{C^1(B_{s/8}(p_j))} \le \varepsilon(i)$ and $A_j \in O(4)$, and hence $A_j(p_j) = p + \beta_{ij}(p_j)$ where $|\beta_{ij}(p_j)| \le \varepsilon(i)$. In particular,

$$\begin{aligned}
\partial_r(\theta_j((1-r)p_j)) &= -D\theta_j((1-r)p_j) \cdot p_j \\
&= -A_j \cdot p_j - D\beta_{i,j}((1-r)p_j) \cdot p_j \\
&= -p + v_j(r)
\end{aligned} \tag{8.5}$$

where $|v_j(r)| \le \varepsilon(i)$. That is, using $\theta_j(p_j) = p \in D_{2,5/2}(0)$, we see that $\theta_j((1-r)p_j) \in D_{2-s,5/2}(0)$ for all $r \in [0, s/100]$.

That is $(1-r)p_j \in (\theta_j)^{-1}(D_{2-s,5/2}(0)) \subseteq (\tilde{\varphi}_i)^{-1}(\tilde{\varphi}_i(D_{2-s,5/2}))$ for all $r \in [0, s/100]$:

$$\begin{aligned}
\theta_j((1-r_0)p_j) - p &= \theta_j((1-r_0)p_j) - \theta_j(p_j) \\
&= \int_0^{r_0} \partial_r(\theta_j((1-r)p_j))dr = -r_0 p + r_0 \tilde{v}_j,
\end{aligned}$$

with $|\tilde{v}_j| \leq \varepsilon(i)$ implies $\theta_j((1-r_0)p_j) = (1-r_0)p + r_0\tilde{v}_j$ and hence $|\theta_j((1-r)p_j)| = |(1-r_0)p + r_0\tilde{v}_j| \leq (5/2)(1-r_0) + \varepsilon(i)r_0 < (5/2)$ (respectively $\geq (1-r_0)2 - r_0\varepsilon(i) \geq 2-s$)

As $p \in D_{2,5/2}$ was arbitrary, we see $(1-r)q \in (\tilde{\varphi}_i)^{-1}(\tilde{\varphi}_i(D_{2-s,5/2}))$ for all $r \in [0, s/100]$ for all $q \in (\tilde{\varphi}_i)^{-1}(\tilde{\varphi}_i(D_{2,5/2}))$ for large enough i. Furthermore $(1-s/100)q \in D_{2-s,(5/2)-(s/200)} \subseteq D_{2-s,5/2}$ for large enough i. We assume that this i corresponds to ε: that is $\varepsilon = (\frac{5}{2}) \cdot (\frac{1}{2^{i+2}})$. Then, we have just shown that $(1-r)\tilde{q} \in \varphi^{-1}(\varphi(D_{0,\varepsilon}))$ for all $r \in [0, s/100]$, for all $\tilde{q} \in \varphi^{-1}(\varphi(D_{0,\varepsilon}))$, and we also know that $(1-s/100)q \in D_{0,\varepsilon} \subseteq \tilde{V} := \varphi^{-1}(\varphi(D_{0,\varepsilon}))$. That is, there exists a smooth map $c : [0, s/100] \times \tilde{V} \to \tilde{V}$, $c(r,x) = (1-r)x\eta(x) + (1-\eta(x))x$, where η is a rotationally symmetric cut off function on $D_{0,\varepsilon}$ with, $0 \leq \eta \leq 1$, $\eta = 1$ on $D_{\varepsilon/2,\varepsilon}$ and $\eta = 0$ on $D_{0,\varepsilon/4}$, such that $c(0,\cdot) = Id$ and $(c(s/100,\cdot))(\tilde{V}) \subseteq D_{0,\varepsilon}$, und $D_{0,\varepsilon}$ is a simply connected space. Hence $\tilde{V}$ is itself simply connected.

Notice also, that $E \cap B_r(x_1)$ is contained in V for r small enough (***). We explain this now. There is some $x \in E \cap B_r(x_1)$ with $x \in V$ by construction. Let $\gamma : [0,1] \to {}^lB_r(x_1)\backslash\{x_1\}$ be a smooth path of finite length in ${}^lB_r(x_1)\backslash\{x_1\}$ with $\gamma(0) = x \in E \cap V \cap B_r(x_1)$, and $|\gamma'(\cdot)|_l \leq C$. Let s be a value for which $\gamma(t) \in E \cap V$ for all $t < s$ and $\gamma(s) \in E \cap (V)^c$. Lifting $\gamma : [0,s) \to X$ with φ, we get a curve $\tilde{\gamma} : [0,s) \to D_{0,\frac{3}{2}r}$ with $r \ll \varepsilon$. Clearly, $\tilde{\gamma}(t) \to d \in D_{0,2r}$ as $t \nearrow s$. Hence $\gamma(t) = \varphi(\tilde{\gamma}(t)) \to \varphi(d) \in V$. On the other hand, $\gamma(t) \to \gamma(s)$ as $t \to s$. Hence $\gamma(s) = \varphi(d) \in V$, per definition of V, which is a contradiction. Hence γ is also a curve in V, that is $E \cap B_r(x_1)$ is contained in V.

Also, $V \subseteq E$ if $\varepsilon > 0$ is small enough in the definition of $V := \varphi(D_{0,\varepsilon})$ [Explanation. As we noted above, V is connected. Furthermore, $V \cap E \neq \emptyset$ by definition of V, and E is a connected component of $B_{r_0}(x_1)\backslash\{x_1\}$, and, without loss of generality, $\varepsilon \ll r_0$. This means that we have: V is connected, $V \subseteq B_{r_0}(x_1)\backslash\{x_1\}$, and E is a connected component of $B_{r_0}(x_1)\backslash\{x_1\}$, and $V \cap E \neq \emptyset$. Hence V is contained in E].

We will see that for r_0 small enough in the above theorem, that in fact ${}^{d_X}B_r(x_i)\backslash\{x_i\} \subseteq X$ has exactly one component for all $r \leq r_0$. This will follow by considering the manifolds (M, g_i, p_1), which approximate a blow up $(X, d_i := \sqrt{c_i}d_X, x_1)$ in the sense explained above in the Approximation Theorem, Theorem 7.4.

The approximations and the blow ups of X itself will converge to a metric cone of the form $\mathbb{R}^4\backslash\{0\}/\Gamma$ for some Γ, where Γ is a finite subgroup of $O(4)$ acting freely on $R^4\backslash\{0\}$, and the number of elements in Γ is bounded by $C(\sigma_0, \sigma_1) < \infty$. That is, each blow up near a singular point consists of exactly *one* cone. This will show us that for each i, $B_{r_0}(x_i)\backslash\{x_i\} \subseteq X$ has exactly one component. These facts are collected in the following theorem

Theorem 8.3. *X is a C^0 Riemannian orbifold in the following sense.*

(i) *$X\backslash\{x_1,\dots,x_L\}$ is a manifold, with the structure explained above in Lemmata 6.5 and 6.6.*

(ii) *There exists an $r_0 > 0$ small such that the following is true. Let $x_i \in X$ be one of the singular points. Then $B_r(x_i)\backslash\{x_i\}$ is connected for all $r \le r_0$.*

(iii) *There exists a $0 < \tilde{r} \le r_0$ and a smooth map $\varphi : D_{0,\tilde{r}} \to X\backslash\{x_1,\dots,x_L\}$ such that $\varphi : \tilde{V} \to V$ is a covering map, V and $\tilde{V}$ are connected sets, $\tilde{V}$ is simply connected, and, for all $r \le \tilde{r}$, we have*

$$\varphi(S^3_r(0)) \subseteq {}^{d_X}B_{r(1-\varepsilon_1(r)),r(1+\varepsilon_1(r))},$$

and

$$\sup_{D_{0,r}} |(\varphi)^* l - \delta|_\delta \le \varepsilon_1(r) \tag{8.6}$$

where $\varepsilon_1(r) \le \frac{\tilde{r}}{100}$ is a decreasing function with $\lim_{r\searrow 0}\varepsilon_1(r) = 0$, and $V := \varphi(D_{0,\frac{\tilde{r}}{2}})$, $\tilde{V} := \varphi^{-1}(V) \subseteq D_{0,\tilde{r}}$, and $S^3_r(0) := \{x \in \mathbb{R}^4 \mid |x| = r\}$, and here δ is the standard euclidean metric on $\mathbb{R}^4$ or subsets thereof.

Remark 8.4. Using the facts *** mentioned at the end of the construction of φ, we see that $B_r(x_1) \subseteq V \cup \{x_1\}$ for all $r \le r_0$ small enough, and hence $V \cup \{x_1\}$ is an open neighbourhood of x_1 in X.

Proof. Fix $x_1 \in \{x_1,\dots,x_L\}$ and assume that $B_{r_0}(x_1)\backslash\{x_1\}$, r_0 as above, contains more than one component: $B_{r_0}(x_1)\backslash\{x_1\} = \cup_{i=1}^N E_i$ with $E_i \cap E_j = \emptyset$ for all $i,j \in \{1,\dots,N\}$, $i \ne j$, and $N \ge 2$. Let E, G denote two distinct components, $E := E_1 \ne E_2 =: G$. We use the following notation: for $p \in E \cap B_{r_0/4}(x_1)$ and $q \in G \cap B_{r_0/4}(x_1)$, $\hat{q}, \hat{p}$ will denote the unique points in M with $f(\hat{q}) = q$, $f(\hat{p}) = p$: these points are unique since p, q are not singular in X.

Our proof is essentially a modified version of the *Neck Lemma*, Lemma 1.2 of [AnCh2], of M. Anderson and J. Cheeger adapted to our situation. Note that we do not have Ricci bounded from below (as they do) for our approximating sequences, but we do know that they all satisfy

$$\int_M |\mathrm{Rc}|^4(g_i)d\mu_{g_i} \to 0 \quad \text{as } i \to \infty.$$

Hence we can use the volume estimates of P. Petersen G.-F. Wei, [PeWe], in place of the Bishop-Gromov volume estimates. The estimates we require do not appear in [PeWe], although they follow after making minor modifications to the proof of their estimates. See Appendix C in [SiArxiv]. for example, for more details.

Let (M, g_i) and (X, d_i) be as in the Approximation Theorem, Theorem 7.4. Let E be as above, and let $z_i \in E \cap {}^{d_i}B_{1/4,10}(x_1)$ satisfy $d_i(x_1, z_i) = 1$ and $v_i \in T_{z_i}E$ be a vector such that there is a length minimising geodesic $\gamma_i : [0,1] \to X$ on (X, d_i) with $\gamma_i(0) = z_i$, $\gamma_i(1) = x_1$, $\gamma_i'(0) = v_i$, and $|\gamma_i'(t)|_{l_i} = 1$ for all $t \in [0,1)$ (γ_i' makes sense on E, since $x_i \notin E$ for all $i \in \{x_1, \ldots, x_L\}$, and $(X \backslash \{x_1, \ldots, x_L\}, l_i)$ is a smooth Riemannian manifold). We define $\hat{z}_i := f^{-1}(z_i) \in M$, the corresponding point in M, and $\hat{v}_i := f^* v_i$, the corresponding vector in $T_{\hat{z}_i}M$, where (M, g_i) are as in the Approximation Theorem.(TT)

We remember, that $\mathrm{inj(b)} > \mathrm{i}_0/1000$ for all $b \in E \cap {}^{d_i}B_{1/4,10}(x_1)$ due to the injectivity radius estimate of Cheeger-Gromov-Taylor (Theorem 4.3 in [CGT]) and the non-inflating/non-collapsing estimates. For any i, $(T_{\hat{z}_i}M, g_i(\hat{z}_i) = g(i))$ is isometric (as a vector space) to $(\mathbb{R}^4, \delta)$. We will make this identification in the following, sometimes without further mention.

Let $S_i \subseteq S^3_1(0)$ denote the set of vectors $\hat{w}$ in $S^3_1(0) \subseteq (\mathbb{R}^4, \delta) = (T_{\hat{z}_i}M, g(\hat{z}_i))$ (using the isometry above) which satisfy $\angle(\hat{v}_i, \hat{w}) \leq \alpha$ with respect to the euclidean metric, where $\alpha > 0$ is a small but positive angle. We claim

Claim 1. There exists a small $\tilde{\varepsilon}(\alpha) > 0$ such that any geodesic $\exp(g_i)_{\hat{z}_i}(\cdot\, m) : [0, 100] \to M$ does not go through ${}^{g_i}B_\varepsilon(p_1)$ if $m \in S^3_1(0) \cap (S_i)^c$ and $0 < \varepsilon \leq \tilde{\varepsilon}(\alpha)$ is small enough, and $i \geq N$ large enough.

Proof of Claim 1. Let $\alpha > 0$ be fixed. We assume we can find $\hat{w}_i \in (S_i)^c \cap S^3_1(0) \subseteq \mathbb{R}^4 = T_{\hat{z}_i}M$ and $r_i \in (0, 100]$ such that $g_i(\hat{z}_i)(\hat{w}_i, \hat{v}_i) > \alpha$, and $\exp(g_i)(r_i\hat{w}_i) \in \partial({}^{g_i}B_\varepsilon(p_1))$ but $\exp(g_i)(s\hat{w}_i) \in ({}^{g_i}B_\varepsilon(p_1))^c$ for all $0 \leq s < r_i$ for i arbitrarily large. We shall see, that this leads to a contradiction, if $\varepsilon \leq \tilde{\varepsilon}(\alpha)$ is chosen small enough. Let w_i be the push forward back to X, $w_i := f_*(\hat{w}_i)$.

For any $\delta > 0$, we know that $E \cap {}^{d_i}B_{\delta,1/\delta}(x_1)$ converges, after taking a subsequence if necessary, to $({}^gB_{\delta,1/\delta}(0), g) \subseteq \mathbb{R}^4 \backslash \{0\}/\Gamma$ in the sense of convergence given in Definition 7.5, in view of Lemma 3.6 of [Tian]: there exist diffeomorphisms $F_i : {}^{d_i}B_{\delta,1/\delta}(x_1) \cap E \to (\mathbb{R}^4\backslash\{0\})/\Gamma$ for i large enough, such that $(F_i)_* l_i \to g$ in the C^k sense, where g is the Riemannian metric on $(\mathbb{R}^4\backslash\{0\})/\Gamma$ **and** $|d_i(F_i^{-1}([x]), x_1) - |x|| \leq \varepsilon(i) \to 0$ as $i \to \infty$ for all $[x] \in B_{2\delta,\frac{1}{2\delta}}(0)/\Gamma$, where $|x| = d([x],[0])$ here refers to the standard norm in $\mathbb{R}^4$ of x, and $[x] = \{\Gamma_i(x) \mid \Gamma_i \in \Gamma\}$. In particular, the curves $F_i \circ f \circ \exp(\cdot\hat{v}_i) : [0, 1-3\delta] \to (\mathbb{R}^4\backslash\{0\})/\Gamma$ and $F_i \circ f \circ \exp(\cdot\hat{w}_i) : [0, r_i] \to (\mathbb{R}^4\backslash\{0\})/\Gamma$, are well defined and converge smoothly to geodesic curves $\gamma : [0, 1-3\delta] \to (\mathbb{R}^4\backslash\{0\})/\Gamma$ respectively $\tilde{\gamma} : [0, r] \to (\mathbb{R}^4\backslash\{0\})/\Gamma$, $r \leq 100$, with $\gamma(0) = \tilde{\gamma}(0) = z$ with $d(z,[0]) = 1$, and $g(\gamma'(0), \tilde{\gamma}'(0)) \geq \alpha$ and $\gamma(1-3\delta) \in {}^gB_{0,3\delta}(0)$ and $\tilde{\gamma}(r) \in {}^gB_{0,3\varepsilon}(0)$. Here, we can choose $\delta > 0$ arbitrarily small. By considering the lift of the curve γ to to $\mathbb{R}^4\backslash\{0\}$ (which must be a straight line in $\mathbb{R}^4\backslash\{0\}$), and using that δ is arbitrarily small, we see that $\gamma : [0, 1-3\delta] \to (\mathbb{R}^4\backslash\{0\})/\Gamma$ is arbitrarily close to the projection of a ray coming out of 0 (in $\mathbb{R}^4$) on $[0, 1-\sigma]$ for $\sigma > 0$ as small as we like (choose $\delta \ll \sigma$). Now lifting $\tilde{\gamma}$ to a curve in $\mathbb{R}^4\backslash\{0\}$ (which is also a straight line in $\mathbb{R}^4\backslash\{0\}$), and using the fact that $g(\gamma'(0), \tilde{\gamma}'(0)) \geq \alpha$ (which is also true for the lift), we see that $\tilde{\gamma}(r) \in (B_{0,\tilde{\varepsilon}(\alpha)}(0))^c$, for some $\varepsilon(\alpha) > 0$. This leads to a contradiction to the fact that $\tilde{\gamma}(r) \in {}^gB_{0,3\varepsilon}(0)$ if $\varepsilon > 0$ is chosen smaller than say $\frac{\tilde{\varepsilon}(\alpha)}{6}$. *End of the proof of Claim 1.*

Claim 2. For all $z \in f^{-1}(E \cap {}^{d_i}B_{\frac{1}{4},2}(x_1))$ and $w \in f^{-1}(G \cap {}^{d_i}B_{\frac{1}{4},2}(x_1))$, any length minimising geodesic from z to w must go through ${}^{g_i}B_{\varepsilon(i)}(p_1)$, where $\varepsilon(i) \to 0$ as $i \to \infty$.

Proof of Claim 2. Assume we can find i arbitrarily large, and points $\hat{z}_i \in f^{-1}(E \cap {}^{d_i}B_{\frac{1}{4},2}(x_1))$ and $\hat{w}_i \in f^{-1}(G \cap {}^{d_i}B_{\frac{1}{4},2}(x_1))$ and a length minimising geodesic $\hat{\gamma}_i : [0, r_i] \to M$ (w.r.t. g_i), parameterised by arclength, such that $\hat{\gamma}_i(0) = \hat{z}_i$ and $\hat{\gamma}_i(r_i) = \hat{w}_i$, for which $\hat{\gamma}_i$ doesn't go through ${}^{d_i}B_\sigma(p_1)$ for some $\sigma > 0$ (*) .

Note, the Approximation Theorem, Theorem 7.4, guarantees that $\hat{z}_i, \hat{w}_i \in f^{-1}({}^{d_i}B_{\frac{1}{4},2}(x_1)) \subseteq {}^{g_i}B_{10}(p_1))$. Hence $\hat{\gamma}_i([0, r_i]) \subseteq {}^{g_i}B_{40}(p_1))$, and hence, once again using the Approximation Theorem, $f(\hat{\gamma}_i([0, r_i])) \subseteq {}^{d_i}B_{41}(x_1)$.

Let $\gamma_i : [0, r_i] \to X$ be the curve $\gamma_i := f \circ \hat{\gamma}_i$. The Approximation Theorem guarantees that $\gamma_i([0, r_i]) \subseteq {}^{d_i}B_{41}(x_1)$ as we just noted (*).

There must be a first value $r_0(i) \in [0, r_i]$ with $\gamma_i(r_0(i)) = x_1$: the curve is continuous and goes from E to G, and so there must be some point $r_0(i)$ with $\gamma(r_0(i)) \in \partial E$. $\gamma(r_0(i))$ must be equal to x_1, since $d_i(\partial E\backslash\{x_1\}, x_1) \to \infty$ as $i \to \infty$ and $\gamma_i([0, r_i]) \subseteq {}^{d_i}B_{41}(x_1)$.

By assumption, $\hat\gamma_i(r) \notin {}^{g_i}B_\sigma(p_1)$ for all $r \in [0, r_i]$. But then, once again by the Approximation Theorem, $f \circ \hat\gamma_i([0, r_i]) \cap {}^{d_i}B_{\sigma/2}(x_1) = \emptyset$, which contradicts the fact that $f \circ \hat\gamma_i(r_0(i)) = x_1$. *End of the proof of Claim 2.*

Let $z_i \in E \cap ({}^{d_i}B_{\frac{1}{4},2}(x_1))$, $\hat z_i$, S_i v_i $\hat v_i$ be as above (see (TT) above): $S_i \subseteq S^3_1(0)$ denotes the set of vectors $\hat w$ in $S^3_1(0) \subseteq \mathbb{R}^4 = T_{\hat z_i}M$ which satisfy $\angle(\hat v_i, \hat w) \leq \alpha$, where we have identified vectors $T_{\hat z_i}M$ and vectors in $\mathbb{R}^4$ using the isometry between $(T_{\hat z_i}M, g_i(0))$ and $(\mathbb{R}^4, \delta)$ explained above.

Let $W_r := \{\exp(g_i)_{\hat z_i}(t\hat w) \mid t \in [0, r], \hat w \in S_i\}$, $V_r := \{\exp(g_i)_{\hat z_i}(s\hat w) \mid s \in [0, r], \hat w \in S_i$ and $\exp(g_i)_{\hat z_i}(\cdot \hat w) : [0, s] \to M$ is a minimising geodesic $\}$. E_r is the set in Euclidean space which corresponds to W_r: $E_r := \{t\beta \mid \beta \in S_i, \angle(\beta, e_1) \leq \alpha, t \leq r\}$

Claim 3. Let $\hat Z := f^{-1}(G \cap {}^{d_i}B_{1/2,1}(x_1))$. Then $\hat Z \subseteq V_3$, if i is large enough.

Proof of Claim 3. Let $\gamma(\cdot) := \exp(g_i)_{\hat z_i}(\cdot m_i) : [0, r_i] \to M$ be a length minimising geodesic from $\hat z_i$ to a point $\hat a_i \in f^{-1}(G \cap {}^{d_i}B_{1/2,1}(x_1))$ parameterised by arclength. Using the Approximation Theorem, Theorem 7.4, we must have $\hat a_i, \hat z_i \in {}^{g_i}B_{1+\varepsilon(i)}(p_1)$, since $d_i(z_i, x_1) = 1$ and hence we must have $r_i = d(g_i)(\hat a_i, \hat z_i) \leq 5/2$. Assume $m_i \in (S_i)^c$. Claim 1 tells us that the curve does not go through $B_{\varepsilon(\alpha)}(p_1)$ for some $\varepsilon(\alpha) > 0$ if i is large enough. But this contradicts Claim 2, if i is large enough. Hence $m_i \in S_i$ and hence $\hat Z \subseteq V_3$ in view of the definition of these two sets. *End of the proof of Claim 3.*

Note for later, that $\mathrm{vol}(g_i)(\hat Z) \geq \theta > 0$ for i large enough, where this θ is independent of α, i, and independent of which subsequence we take, in view of the fact that $(\hat Z, g_i)$ converges to $(B_{1,1/2}(0)/\Gamma)$ in the sense of C^k manifold convergence given in Definition 7.5 (this follows from the Approximation Theorem 7.4 and Lemma 3.6 of [Tian]), and we have bounds on the number of elements of Γ, and this gives as a non-collapsing estimate.

The volume comparison of Peterson/Wei shows (see Appendix C in [SiArxiv]) that

$$\left(\frac{\mathrm{vol}\, V_3}{\mathrm{vol}(E_3)}\right)^{1/8} - \left(\frac{\mathrm{vol}\, V_{1/2}}{\mathrm{vol}(E_{1/2})}\right)^{1/8} \leq \frac{c}{\alpha^{3/8}}\left(\int_M |\mathrm{Rc}|^4\right)^{1/8} \tag{8.7}$$

where we have used that the volume of $S_i \subseteq S^3_1(0)$ on the sphere $S^3_1(0)$ with respect to the the metric on the sphere $d\theta$ is $\alpha^3 c$ where c is a universal constant. Multiplying everything by $\mathrm{vol}(E_3)^{1/8}(\leq (\omega_4 3^4)^{1/8})$ we get

$$(8.8)\qquad \begin{aligned} &(\mathrm{vol}\, V_3)^{1/8} - (\mathrm{vol}\, V_{1/2})^{1/8}\left(\frac{\mathrm{vol}(E_3)}{\mathrm{vol}(E_{1/2})}\right)^{1/8} \\ &\leq \frac{c}{\alpha^{3/8}}(\mathrm{vol}(E_3)^{1/8})\left(\int_M |\mathrm{Rc}|^4\right)^{1/8}. \end{aligned}$$

The quantities $\mathrm{vol}(E_3)$ and $\mathrm{vol}(E_{1/2})$ are fixed and positive and depend on α (they are uniformly bounded above by the volume of $B_3(0)$ for every α). The quantity $\frac{\mathrm{vol}(E_3)}{\mathrm{vol}(E_{1/2})}$ is a fixed positive constant which don't depend on α, so we may write $c_* = (\frac{\mathrm{vol}(E_3)}{\mathrm{vol}(E_{1/2})})^{1/8}$, where c_* is independent of α and i. Using this in equation (8.8) we get

$$(8.9)\qquad (\mathrm{vol}\, V_3)^{1/8} \leq c_*(\mathrm{vol}\, V_{1/2})^{1/8} + \frac{c}{\alpha^{3/8}}\left(\int_M |\mathrm{Rc}|^4\right)^{1/8}.$$

From Claim 3 above, we see that

$$\mathrm{vol}(V_3) \geq \mathrm{vol}(\hat{Z}) \geq \theta$$

for some fixed $\theta > 0$ since on each component the metric approaches the euclidean metric divided out by a finite subgroup of $O(4)$ acting freely on $\mathbb{R}^4 \backslash \{0\}$. Recall that ${}^{d_i}B_{\delta,\frac{1}{\delta}} \cap E$ converges to $({}^{g}B_{\delta,\frac{1}{\delta}}, g) \subseteq (\mathbb{R}^4\backslash\{0\})/\Gamma$ in the sense of Definition 7.5 using a map $F_i : {}^{d_i}B_{\delta,\frac{1}{\delta}} \cap E \to {}^{g}B_{\delta,\frac{1}{\delta}}$, and ${}^{g_i}B_{\delta,\frac{1}{\delta}}(p_1)$ is $\varepsilon(i)$ C^k close to ${}^{d_i}B_{\delta,\frac{1}{\delta}}$ in the sense of Definition 7.5, using the map f, in view of the Approximation Theorem, Theorem 7.4. Since $V_r \subseteq W_r$, we have $\mathrm{vol}(V_{1/2}) \leq \mathrm{vol}\, W_{1/2} \leq c\alpha^3$ which goes to zero as $\alpha \to 0$ [Explanation. Let $F_i \circ f(\hat{z}_i) =: x_i$. x_i is at a distance $1 \pm \varepsilon(i)$ away from 0. We use the fact that $f(W_{1/2}) \subseteq E$ in the following without further mention: this follows from the fact that $f(W_{1/2}) \cap \{x_1, \ldots, x_L\} = \emptyset$, which follows from the Approximation Theorem.

Using the fact that $(F_i \circ f)_*(g_i) \to g$ on ${}^{g}B_{\delta-\varepsilon(i),\frac{1}{\delta}+\varepsilon(i)}$ as $i \to \infty$, we see that $(F_i \circ f)_*(S_i) \subseteq \tilde{S}_i$, where $\tilde{S}_i := \{v \in T_{x_i}(\mathbb{R}^4\backslash\{0\}/\Gamma) \mid g(x_i)(n_i, v) \leq \alpha + \varepsilon(i), |v|_g \in (1-\varepsilon(i), 1+\varepsilon(i))\}$ and $n_i := (F_i \circ f)_*(\hat{v}_i) = (F_i) * (v_i)$ is a vector of length almost one. Hence, using a compactness argument,

$$\begin{aligned} F_i \circ f(W_{1/2}) \subseteq \{\, &\exp(x_i)(rm) \mid r \in [0, 1/2+\delta], m \in T_{x_i}(\mathbb{R}^4\backslash\{0\}/\Gamma), \\ &\angle(m, n_i) \leq \alpha + \delta, ||m|_g - 1| \leq \varepsilon(i)\} \end{aligned}$$

for all $\delta > 0$, for $i \geq I(\delta) \in \mathbb{N}$ large enough, and hence

$$\begin{aligned} \operatorname{vol} W_{1/2} \leq (1+\delta(i)) \operatorname{vol}\Big(\{\exp(x_i)(rm) \mid r \in [0, 1/2+\delta(i)], \\ m \in T_{x_i}(\mathbb{R}^4\backslash\{0\}/\Gamma),\ ||m|_g - 1| \leq \varepsilon(i), \\ \angle(m, n_i) \leq \alpha + \delta(i)\}, g\Big) \\ \to \operatorname{vol}(\tilde{W}_{1/2}) \text{ as } i \to \infty \\ \leq \operatorname{vol}(\pi^{-1}(\tilde{W}_{1/2})), \end{aligned}$$

where $\tilde{W}_{1,2} = \{\exp(rm) \mid r \in [0, 1/2],\quad m \in T_x(\mathbb{R}^4\backslash\{0\}/\Gamma),\quad \angle(m, n) \leq \alpha,$ $|m|_g = 1\}$ and π is the standard projection from $\mathbb{R}^4\backslash\{0\}$ to $(\mathbb{R}^4\backslash\{0\})/\Gamma$. That is$\operatorname{vol}(V_{1/2}) \leq \operatorname{vol}(W_{1/2}) \leq \operatorname{vol}(\pi^{-1}(\tilde{W}_{1/2})) + \delta(i) \leq c\alpha^3$ since geodesics in $\mathbb{R}^4$ are straight lines, and $\pi^{-1}(W_{1/2})$ is a cone of angle α and length $1/2$ in $\mathbb{R}^4\backslash\{0\}$. End of the Explanation].

Using these two facts in (8.9), gives us

$$(\theta)^{1/8} \leq (\operatorname{vol} V_3)^{1/8} \leq c_* c^{1/8}\alpha^{3/8} + \frac{c}{\alpha^{3/8}}\left(\int_M |\mathrm{Rc}|^4\right)^{1/8}. \tag{8.10}$$

This leads to a contradiction if α is chosen small enough and then i is chosen large enough, since $(\int_M |\mathrm{Rc}|^4)^{1/2}$ goes to zero as $i \to \infty$.

That is, there cannot be two distinct components E and G as described above. □

9. Extending the flow

Since (X, d_X) is a C^0 Riemannian orbifold, it is possible to extend the flow past the singularity using the orbifold Ricci flow. We have

Theorem 9.1. *Let everything be as above. Then there exists a smooth orbifold, $\tilde{X}$, with finitely many orbifold points, $v_1, \ldots, v_L$, and a smooth solution to the orbifold Ricci flow, $(\tilde{X}, h(t))_{t\in(0,S)}$ for some $S > 0$, such that $(\tilde{X}, d(h(t))) \to (X, d_X)$ in the Gromov-Hausdorff sense as $t \searrow 0$.*

Proof. Fix $x_i \in \{x_1, \ldots, x_L\} \subseteq X$, where $\{x_1, \ldots, x_L\}$ are defined in Theorem 6.5. On ${}^{d_X}B_\varepsilon(x_1)$ we have a potentially non-smooth orbifold structure given by the map φ: the non-smoothness may also be present without considering the Riemannian metric, as we now explain. As explained above, if we consider $\tilde{V} := \varphi^{-1}(\varphi(D_{0,\varepsilon}))$ and $V := \varphi(D_{0,\varepsilon})$, then $\varphi|_{\tilde{V}} : \tilde{V} \to V$ is a covering map, $\tilde{V}$,V are connected, and $\tilde{V}$ is simply connected, if $\varepsilon > 0$ is small enough.

Let $x \in \tilde{V}$ be fixed, and $G_1, \ldots, G_N : \tilde{V} \to \tilde{V}$ the deck transformations, which are uniquely determined by $G_i(x) = x_i$, where $x_1, x_2, \ldots, x_N \in \tilde{V}$ are the distinct points with $\varphi(x_i) = \varphi(x_j)$ for all $i, j \in \{1, \ldots, N\}$.

We can extend $G_1, \ldots, G_N$ to maps $G_1, \ldots, G_N : \tilde{V} \cup \{0\} \to \tilde{V} \cup \{0\}$ by defining $G_i(0) = 0$ for all $i \in \{1, \ldots, N\}$. Then the maps $G_i : \tilde{V} \cup \{0\} \to \tilde{V} \cup \{0\}$ are homeomorphisms, but not necessarily smooth at 0. In this sense, the structure of the orbifold may not be smooth. Also, as we saw above, we can extend the metric to a continuous metric on $\tilde{V} \cup \{0\}$ by defining $g_{ij}(0) = \delta_{ij}$, but this extension is not necessarily smooth. In order to do Ricci flow of this C^0 orbifold, we will proceed as follows: **Step 1.** modify the metric g and the maps $G_1, \ldots, G_L : \tilde{V} \to \tilde{V}$ inside $D_{0, \frac{1}{2^i}}$ to obtain a new metric $\tilde{g}$ on $\tilde{V}$ and new maps $\tilde{G}_1, \ldots, \tilde{G}_L : \tilde{V} \to \tilde{V}$ which are isometries of $\tilde{V}$ with respect to $\tilde{g}$, and such that these new objects can be smoothly extended to 0. We do this in a way, so that the metric and maps are only slightly changed (see below for details). With the help of $\tilde{g}$ and $\tilde{G}_1, \ldots, \tilde{G}_L$ we will define a new smooth Riemannian orbifold: essentially this construction smooths out the $G_i's$ near the cone tips (the points $x_1, \ldots, x_L \in X$) in such a way, that a group structure is preserved, and the rest of the orbifold is not changed. For $i \in \mathbb{N}, i \to \infty$, we denote the smooth Riemannian orbifolds which we obtain in this way by (X_i, d_i). The construction will guarantee that $(X_i, d_i) \to (X, d)$ in the Gromov-Hausdorff sense, actually in the Riemannian C^0 sense: see below. In **Step 2**, we flow each of these spaces (X_i, d_i) by Ricci flow, and we will see, that the solution exists on a time interval $[0, T)$ with $T > 0$ being independent of i, and that each of the solutions satisfies estimates, independent of i. In **Step 3**, we take an orbifold limit of a subsequence of the solutions constructed in Step 2 to obtain a limiting smooth orbifold solution to Ricci flow $(\tilde{X}, h(t))_{t \in (0,T)}$ which satisfies $(\tilde{X}, d(h(t))) \to (X, d_X)$ as $t \searrow 0$, in the Gromov-Hausdorff sense.

Step 1. Let $G_1, \ldots, G_N : \tilde{V} \to \tilde{V}$ be the deck transformations of $\varphi : \tilde{V} \to V$, let $g := \varphi^*(l)$, and $\hat{\varphi} : \hat{V} \to V$ be $\hat{\varphi}(\hat{x}) = \varphi(\frac{\hat{x}}{c})$, where $\hat{V} := c\tilde{V}$. We use, in the following, the notation $\hat{x} = cx$. Then $\hat{\varphi}$ is a covering map, with deck transformations $H_1, \ldots, H_N : \hat{V} \to \hat{V}$, $H_i(\hat{x}) = cG_i(\frac{\hat{x}}{c})$. We know that $G_1, \ldots, G_N : \tilde{V} \to \tilde{V}$ are isometries with respect to g. Let $\hat{l} := c^2 l$ and $\hat{g} := (\hat{\varphi})^*(\hat{l})$. Then $\hat{g}_{ij}(\hat{x}) = g_{ij}(x)$, and $H_1, \ldots, H_N : \hat{V} \to \hat{V}$ are local isometries w.r.t. $\hat{g}$, and hence global isometries w.r.t.

$$\begin{aligned} \hat{g} : \hat{g}(\hat{x})(DH_i(\hat{x})(v), DH_i(\hat{x})(w)) &= g(x)(DG_i(x)(v), DG_i(x)(w)) \\ &= g(x)(v, w). \end{aligned}$$

Scaling with $c = 2^{i+2}$ we see $\hat{\varphi}|_{[2-14\delta,4-16\delta]} = \tilde{\varphi}_i|_{[2-14\delta,4-16\delta]}$, as shown above.

We go back to the construction of the map $\tilde{\varphi}_i$. Remember that $\tilde{\varphi}_i : D_{1/2+\varepsilon(i)\delta,4-\varepsilon(i)} \to X\backslash\{x_1\}$ was defined by $\tilde{\varphi}_i := v_i \circ \pi_i \circ (\eta\tilde{\psi}_i + (1-\eta)\tilde{\eta}_{i+1}) : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to {}^{d_i}B_{1/2,4}(x_1)$, where $\eta : \mathbb{R}^4 \to \mathbb{R}^+_0$ is a smooth cutoff function, with $\eta = 1$ on $D_{2-\delta,\infty}$ and $\eta = 0$ on $D_{0,2-2\delta}$ and $\eta\tilde{\psi}_i + (1-\eta)\tilde{\eta}_{i+1}$ is C^k close to the identity on $D_{1/2+\varepsilon(i),4-\varepsilon(i)}$ (see (8.3)). As we pointed out during the construction of $\tilde{\varphi}_i$, this means that $(v_i)^{-1} \circ \tilde{\varphi}_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to ({}^{g(i)}B_{1/2,4}(0), g(i))$ is C^k close to

$$\pi_i : D_{1/2+\varepsilon(i),4-\varepsilon(i)} \to ({}^{g(i)}B_{1/2+\varepsilon(i),4-\varepsilon(i)}(0), g(i)).$$

We define

$$\begin{aligned} &\alpha_i : D_{0,4-\varepsilon(i)} \to ({}^{g(i)}B_{0,4}(0), g(i)) \\ &\alpha_i := \pi_i \circ (\eta\tilde{\psi}_i + (1-\eta)Id) \end{aligned} \tag{9.1}$$

Then α_i is C^k close to $\pi_i : D_{0,4-\varepsilon(i)} \to ({}^{g(i)}B_{0,4-\varepsilon(i)}(0), g(i))$, and equal π_i on $D_{0,2-2\delta}$. Hence, using the same argument we used above to show that $\varphi : \tilde{V} \to V$ was a covering map, and $\tilde{V}$ is simply connected, we have $\alpha_i : \hat{Z} := (\alpha_i)^{-1}(\alpha_i(D_{0,4-\tilde{\varepsilon}})) \to Z := \alpha_i(D_{0,4-\tilde{\varepsilon}})$ is a covering map, if i is large enough ($\tilde{\varepsilon} > 0$ fixed and small), $\hat{Z}$ is simply connected, and $Z, \hat{Z}$ are connected. We also have $v_i \circ \alpha_i = \tilde{\varphi}_i$ on the set $D_{2-\delta,4-\tilde{\varepsilon}}$. In particular, α_i has the same number of deck transformations as $\tilde{\varphi}_i$ and hence as φ[2].

The Riemannian metric l_i on X can be pulled back to $(B_{1/2+\varepsilon(i),4-\varepsilon(i)}, g(i))$ with v_i: $h_i := (v_i)^*(l_i)$. This metric h_i is C^k close to $g(i)$. We interpolate between $h(i)$ and $g(i)$ on $(B_{1+\delta,2-4\delta}, g(i))$ by

$$\beta(i) := \hat{\eta}h_i + (1-\hat{\eta})g(i)$$

where $\hat{\eta} \geq 0$ is a smooth cut-off function on $(B_{1,4}, g(i))$ with $\hat{\eta} = 0$ on $B_{0,1+2\delta}$ and $\hat{\eta} = 1$ on $(B_{1+4\delta,\infty}, g(i))$. Note that $\beta(i) = h_i$ on $D_{2-\delta,4-\tilde{\varepsilon}}$.

Let $\hat{H}_1, \ldots, \hat{H}_N : \hat{Z} \to \hat{Z}$ be the deck transformations of the covering map $\alpha_i : \hat{Z} \to Z$. These maps are isometries w.r.t. $\hat{k}(i) := (\alpha_i)^*(\beta(i))$ on $\hat{Z}$. Scaling these maps leads to maps $H_k : \tilde{Z} \to \tilde{Z}$, $H_k(x) := \frac{1}{2^{i+2}}\hat{H}_k(x2^{i+2})$

[2]Explanation: $\hat{\alpha}_i := v_i \circ \alpha_i : \hat{Z} \to v_i(Z)$ is a covering map. Choose $w \in D_{5/2,3}$ and let $w = w_1, w_2, \ldots, w_{\tilde{N}}$ be the distinct points in $\hat{Z}$ with $\alpha_i(w_j) = \alpha_i(w)$ for all $j = 1, \ldots, \tilde{N}$. Then $w_1, \ldots, w_{\tilde{N}} \in D_{2,7/2}$ and furthermore $\hat{\alpha}_i(w_j) = \hat{\alpha}_i(w)$ for all $j = 1, \ldots, \tilde{N}$, and hence $\tilde{\varphi}_i(w_j) = \tilde{\varphi}_i(w)$ for all $j = 1, \ldots, \tilde{N}$. Hence $\tilde{N} \leq N$. Similarly, by considering the distinct points $w = \tilde{w}_1, \ldots, \tilde{w}_N \in D_{5/2,3}$ such that $\tilde{\varphi}_i(w) = \tilde{\varphi}_i(\tilde{w}_j)$ for all $j = 1, \ldots, \tilde{N}$, we see $\tilde{N} \geq N$.

for $k \in \{1, \ldots, N\}$, $\tilde{Z} := \{\frac{x}{2^{i+2}} \mid x \in \hat{Z}\}$. These maps are isometries w.r.t. $k(x) := k(i)(x) := \hat{k}(i)(\hat{x})$ on $\tilde{Z}$ (see the beginning of the proof).

Note that $k(x) := k(i)(x) = \hat{k}(i)(\hat{x}) = (\alpha_i)^*(\beta(i))(\hat{x}) = (\alpha_i)^*(h_i)(\hat{x}) = (v_i \circ \alpha_i)^*(l_i)(\hat{x}) = (\tilde{\varphi}_i)^*(l_i)(\hat{x}) = \varphi^*(l)(x) = g(x)$ on $\tilde{Z} \cap D_{\frac{2-\delta}{2^{i+2}}, \frac{4-\tilde{\varepsilon}}{2^{i+2}}}$. Where we used the fact that $v_i \circ \alpha_i$ is equal to $\tilde{\varphi}_i$ on the set $D_{2-\delta,4-\tilde{\varepsilon}}$. Hence the Riemannian metric $\tilde{g}$, which is defined to be the metric k on $D_{0,\frac{1}{2^{i+1}}}$ and g on $D_{\frac{1}{2^{i+1}},\infty} \cap \tilde{V}$, is smooth and well defined. It satisfies: $\tilde{g}(x) = k(x) = \hat{k}(\hat{x}) = \delta$ for $|x| \leq c(i)$ small enough. Furthermore, $|\tilde{g} - \delta|_{C^0(\tilde{V})} \leq \sigma$ where $\sigma > 0$, can be made as small as we like, by choosing $\varepsilon > 0$ (in the definition of $\tilde{V}$) small.

Using the fact that $v_i \circ \alpha_i$ is equal to $\tilde{\varphi}_i$ on the set $D_{2-\delta,4-\tilde{\varepsilon}}$ again, we see that $\hat{G}_1, \ldots, \hat{G}_N$ are the same as $\hat{H}_1, \ldots, \hat{H}_N$ when all of these transformations are restricted to $D_{2-\delta+4\tilde{\varepsilon},4-4\tilde{\varepsilon}}$ (we assume $\tilde{\varepsilon} \ll \delta$). Let $w \in D_{2-\delta+4\tilde{\varepsilon},4-4\tilde{\varepsilon}}$ and $w = w_1, w_2, \ldots, w_N \in D_{2-\delta+2\tilde{\varepsilon},4-2\tilde{\varepsilon}}$ be the distinct points with $\tilde{\varphi}_i(w_1) = \cdots = \tilde{\varphi}_i(w_N)$. Let $0 < s \ll \min(\tilde{\varepsilon}, i_0/100)$ be a fixed small number and i large enough. Then we have

$$\begin{aligned}
\hat{G}_k|_{B_s(w)} &= ((\tilde{\varphi}_i)|_{B_s(w_k)})^{-1} \circ (\tilde{\varphi}_i)_{B_s(w)} \\
&= (\tilde{\varphi}_i|_{B_s(w_k)})^{-1} \circ (v_i)^{-1} \circ v_i \circ (\tilde{\varphi}_i)_{B_s(w)} \\
&= (v_i \circ \tilde{\varphi}_i|_{B_s(w_k)})^{-1} \circ (v_i \circ (\tilde{\varphi}_i)_{B_s(w)}) \\
&= (\alpha_i|_{B_s(w_k)})^{-1} \circ (\alpha_i)_{B_s(w)}) \\
&= \hat{H}_k|_{B_s(w)}
\end{aligned}$$

This means the maps H_i can be extended smoothly to all of $\tilde{V} \cup \{0\}$, by defining $H_i = G_i$ on $\tilde{V} \cap (\tilde{Z})^c$ and $H_i(0) = 0$: call these new maps $\tilde{G}_i$. Note that these maps are now smooth. Near 0, $k(x) = \delta$, and $H_j(D_{0,s}) \subseteq D_{0,2s}$, $H_j : \tilde{Z} \to \tilde{Z}$ are isometries, and hence $H_j|_{D_{0,s}} \in O(4)$ for s small enough.

Note also, that for $x \in \tilde{Z}$, we always have $\tilde{G}_j(x) = H_j(x) \in \tilde{Z}$, and for $y \in \tilde{V} \cap (\tilde{Z})^c$, we have $\tilde{G}_j(y) \in \tilde{V} \cap (\tilde{Z})^c$. To see that the last statement is true, assume that $\tilde{G}_j(y) \in \tilde{Z}$ holds for some $y \in \tilde{V} \cap (\tilde{Z})^c$. Then we must have $y = (\tilde{G}_j)^{-1}(\tilde{G}_j)(y)) \in \tilde{Z}$ in view of the fact that $(\tilde{G}_j)^{-1}(\tilde{Z}) \subseteq \tilde{Z}$, and this is a contradiction to the fact that $y \in \tilde{V} \cap (\tilde{Z})^c$. This shows also that the $\tilde{G}_j' s$ are diffeomorphisms, with $(\tilde{G}_i)|_{\tilde{Z}} = H_i$ and $(\tilde{G}_i)|_{(\tilde{Z})^c \cap \tilde{V}} = G_i|_{(\tilde{Z})^c \cap \tilde{V}}$ for all $i \in \{1, \ldots, N\}$. In particular, $\{\tilde{G}_1, \ldots \tilde{G}_N\}$ forms a subgroup of the family of diffeomorphisms on $\tilde{V} \cup \{0\}$. The metric $\tilde{g}$ agrees with k on $\tilde{Z}$ and agree with g on $(\tilde{Z})^c \cap \tilde{V}$ Also, the $\tilde{G}_i$'s are isometries on $(\tilde{Z}, k) = (\tilde{Z}, \tilde{g})$, since $\tilde{G}_i = H_i$ on $\tilde{Z}$, and the $\tilde{G}_i$'s are isometries on $((\tilde{Z})^c \cap \tilde{V}, g) = ((\tilde{Z})^c \cap \tilde{V}, \tilde{g})$ since $\tilde{G}_i = G_i$ on $(\tilde{Z})^c \cap \tilde{V}$. Hence $\{\tilde{G}_1, \ldots \tilde{G}_N\}$ are global isometries on $\tilde{V} \cup \{0\}$, each with one fixed point, 0. The orbifold structure can now be defined as follows: let $\tilde{W} := \tilde{V} \cup \{0\}$. $(\tilde{W}, \tilde{G}_1, \ldots, \tilde{G}_N)$ determines one orbifold chart

$\psi : \tilde{W} \to \tilde{W}/\{\tilde{G}_1, \dots, \tilde{G}_L\}$, where $\psi(x) := [x] = \{\tilde{G}_i(x) \mid i = 1, \dots, N\}$. On $X \backslash ({}^{d_X}B_{\varepsilon/100}(x_1) \cup {}^{d_X}B_{\varepsilon/100}(x_2) \cup \dots \cup {}^{d_X}B_{\varepsilon/100}(x_L))$, we take a covering by the inverse of K manifold charts, for example, geodesic coordinates:

$$(\theta_\alpha) : {}^{l}B_{\tilde{\varepsilon}_0}(0) \to {}^{l}B_{\tilde{\varepsilon}_0}(y_\alpha) \subseteq (X \backslash ({}^{d_X}B_{\varepsilon/1000}(x_1) \cup {}^{d_X}B_{\varepsilon/1000}(x_2) \cup \cdots \cup {}^{d_X}B_{\varepsilon/1000}(x_L))), \quad \alpha \in \{1, \dots, K\}$$

(for orbifold charts the maps always go from an open set in $\mathbb{R}^4$ to an open set in the orbifold). These are fixed for this construction and don't depend on i. Since we don't change anything on $X \backslash ({}^{d_X}B_{\varepsilon/1000}(x_1) \cup {}^{d_X}B_{\varepsilon/1000}(x_2) \cup \dots \cup {}^{d_X}B_{\varepsilon/1000}(x_L))$, these charts, along with $\tilde{g}$, define an Riemannian orbifold $(\hat{X}, \tilde{g})$. To be a bit more specific: define

$$\hat{X} = X \backslash ({}^{d_X}B_{\varepsilon/100}(x_1) \cup {}^{d_X}B_{\varepsilon/100}(x_2) \cup \cdots \cup {}^{d_X}B_{\varepsilon/100}(x_L)) \cup \tilde{W}/\{\tilde{G}_1, \dots, \tilde{G}_L\}$$

where we identify points

$$z \in X \backslash ({}^{d_X}B_{\varepsilon/100}(x_1) \cup {}^{d_X}B_{\varepsilon/100}(x_2) \cup \cdots \cup {}^{d_X}B_{\varepsilon/100}(x_L))$$

with points $[v] \in \tilde{W}/\{\tilde{G}_1, \dots, \tilde{G}_L\}$ if $z \in \varphi(\tilde{V})$ and $[\varphi^{-1}(z)] = [v]$. The topology is defined by saying $x_i \to x \in \hat{X}$ if and only if $x, x_i \in \tilde{W}/\{\tilde{G}_1, \dots, \tilde{G}_L\}$ for all $i \geq N(x) \in \mathbb{N}$ and $x_i \to x$ in $\tilde{W}/\{\tilde{G}_1, \dots, \tilde{G}_L\}$, or $x, x_i \in X \backslash (\overline{{}^{d_X}B_{\varepsilon/100}(x_1)} \cup \overline{{}^{d_X}B_{\varepsilon/100}(x_2)} \cup \dots \cup \overline{{}^{d_X}B_{\varepsilon/100}(x_L)})$ for all $i \geq N(x) \in \mathbb{N}$ and $x_i \to x$ in $X \backslash (\overline{{}^{d_X}B_{\varepsilon/100}(x_1)} \cup \overline{{}^{d_X}B_{\varepsilon/100}(x_2)} \cup \dots \cup \overline{{}^{d_X}B_{\varepsilon/100}(x_L)})$. The charts are given above.

Call the resulting orbifold space $(X_i, \tilde{g}_i)$.

This finishes the construction of the modified orbifolds and metrics.

Step 2. Now we have a smooth orbifold and a smooth metric, so we may evolve it with the orbifold Ricci flow, to obtain a smooth solution $(X_i, Z_i(t))_{t \in (0, T_i)}$ to the orbifold Ricci flow: see Section 2 of [HaThreeO] and Section 5 [KLThree]. The new metric $g_i(0)$ at time zero on D_σ is ε away from δ, and smooth. In particular,

$$|g_i(0) - g_j(0)|_{C^0(D_\sigma, g_j(0))} \leq 2\varepsilon \text{ for all } i, j \in \mathbb{N}. \tag{9.2}$$

if $\sigma > 0$ is small enough. One method to construct a solution to the orbifold Ricci flow is using the so called *DeTurk trick* ([DeT]). We can use any valid smooth background metric h to do this: taking $h = g_j(0)$ for a fixed $j \in \mathbb{N}$, we have $|g_i(0) - h|_{C^0(D_\sigma,h)} \le \varepsilon$ on the whole of $(X_i, g_i(0))$. Now we use the h-flow in place of the Ricci-flow, that is locally the equation looks like,

$$\begin{aligned} \frac{\partial}{\partial t} g_i &= (g_i)^{\alpha\beta} \nabla^2_{\alpha\beta} g_i + \mathrm{Riem}(h) * (g_i) * (g_i)^{-1} * (h)^{-1} \\ &\quad + (g_i)^{-1} * (g_i)^{-1} * (\nabla g_i) * (\nabla g_i), \end{aligned} \tag{9.3}$$

where here, $\nabla = {}^h\nabla$. Using the estimates contained in the proof of Theorem 5.2 in [SimC0], we see that the solution $g_i(t)_{t\in[0,T_i)}$ can be extended to $g_i(t)_{t\in[0,S)}$ for some fixed $S = S(h) > 0$ and that the solution satisfies

$$\begin{aligned} &|g_i(t) - h|_{C^0(X_i,h)} \le 2\varepsilon \\ &|{}^h\nabla^k g_i(t)|^2_{C^0(X_i,h)} \le \frac{c(K,h)}{t^k} \text{ for all } 0 \le t \le S \end{aligned} \tag{9.4}$$

for all $k \le K \in \mathbb{N}$, as long as $t \le S$, where $c(K,h)$ doesn't depend on $i \in \mathbb{N}$. We also have

$$|g_i(t) - g_i(0)|_{C^0(X,h)} \le c(h,t) \le 2\varepsilon \text{ for all } 0 \le t \le S \tag{9.5}$$

where $c(h,t) \to 0$ as $t \searrow 0$, and $c(h,t)$ doesn't depend on $i \in \mathbb{N}$, in view of the inequalities (5.5) and (5.6) in [SimC0] (the $\varepsilon > 0$ appearing in (5.5) and (5.6) there is arbitrary: see the proof of Theorem 5.2 in [SimC0]). In particular,

$$d_{GH}((X_i, d(g_i(t))), (X_i, d(g_i(0)))) \le c(t) \tag{9.6}$$

with $c(t) \to 0$ as $t \searrow 0$. Using the smooth time dependent orbifold vector fields $V^k(\cdot,t) = -g_i(\cdot,t)^{sm}(\Gamma^k_{sm}(g_i)(\cdot,t) - \Gamma^k_{sm}(h)(\cdot))$ and the orbifold diffeomorphisms $\varphi_t : X_i \to X_i$ with $\frac{\partial}{\partial t}\varphi_t = V$, $\varphi_0 = Id$ we obtain a solution to the orbifold Ricci flow, $Z_i(t) := \varphi_t^* g_i(t)$ which satisfies

$$\begin{aligned} &d_{GH}((X_i, d(Z_i(t))), (X_i, d(Z_i(0)))) \le c(t) \\ &|\nabla^j \mathrm{Riem}(Z_i)|(\cdot,t) \le \frac{c(j,h)}{t^{1+(j/2)}} \text{ for all } 0 \le t \le S, \end{aligned} \tag{9.7}$$

see for example [Shi] for details. This finishes Step 2. In Step 2 we obtained various estimates which are necessary for Step 3.

Step 3. Using the Ricci flow orbifold compactness theorem, see [Lu] and Section 5.3 in [KLThree], we can now take a limit in $i \to \infty$ for $t \in (0,S)$, and we obtain an orbifold solution $(\tilde{X}, Z(t))_{t\in(0,S)}$ to the Ricci flow with

$$\begin{aligned} &d_{GH}((X,d_X),(\tilde{X},d_{Z(t)})) \le c(t) \\ &|\nabla^j \operatorname{Riem}(Z)|(\cdot,t) \le \frac{c(j,h)}{t^{1+(j/2)}} \text{ for all } 0 < t \le S \end{aligned} \tag{9.8}$$

where $c(t) \to 0$ as $t \searrow 0$. Here we used, that $(X_i, d(Z_i(0))) \to (X, d_X)$ in the Gromov-Hausdorff sense, which follows by the construction of the spaces $(X_i, d(Z_i(0))$. Hence we have a found a solution $(\tilde{X}, Z(t))_{t\in(0,S)}$ to the orbifold Ricci flow, with initial value $(X, d_X(0))$ in the sense that

$$d_{GH}((X,d_X),(\tilde{X},d_{Z(t)})) \to 0 \quad \text{as } t \searrow 0.$$

In this sense we have extended the flow $(M, g(t))_{t\in(0,T)}$ through the singular limit (X, d_X). □

Remark 9.2. Some of the estimates above can be obtained using Perelman's first pseudolocality theorem and Shi's estimates. However, the estimate on the Gromov-Hausdorff distance, which we require when showing that the initial value of the limit solution is (X, d_X), does not immediately follow from the pseudolocality theorem. We use the estimates given in [SimC0] to show that the initial value of the solution is (X, d_X).

References

[AT] L. Ambrosio and P. Tilli, Topics on Analysis in Metric Spaces, Oxford Lecture Series in Math. and Its Applications, (2004).

[And1] M. Anderson, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math. **102** (1990), 429–445.

[And2] M. Anderson, *Ricci curvature bounds and Einstein metrics on compact manifolds*, Journal American Math. Soc. **2** (1989), no. 3, 455–490.

[AnCh] M. Anderson and J. Cheeger, *C^α compactness for manifolds with Ricci curvature and injectivity radius bounded below*, J. Differential Geom. **35** (1992), no. 2, 265–281.

[AnCh2] M. Anderson and J. Cheeger, *Diffeomorphism finiteness for manifolds with Ricci curvature and $L^{n/2}$-norm of curvature bounded*, Geom. Funct. Anal. **1** (1991), no. 3, 231–252.

[ACK] S. Angenent, C. Caputo, and D. Knopf, *Minimally invasive surgery for Ricci flow singularities*, J. Reine Angew. Math. (Crelle) **672** (2012), 39–87.

[BZ] R. Bamler and Q. Zhang, *Heat kernel and curvature bounds in Ricci flows with bounded scalar curvature*, `arXiv:1501.01291`, (2015).

[BBI] D. Burago, Y. Burago, and S. Ivanov, A Course in Metric Geometry, Graduate Studies in Mathematics, Vol. 33. A.M.S., Providence, RI, (2001).

[CaoX] X.-D. Cao, *Curvature pinching estimate and singularities of the Ricci flow*, Comm. Anal. Geom. **19** (2011), no. 5, 975–990.

[BKN] S. Bando, A. Kasue, and H. Nakajima, *On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth*, Inv. Math. **97** (1989), 313–349.

[Chav] I. Chavel, Riemannian Geometry: A Modern Introduction, second edition, Cambridge Studies in Advanced Mathematics **98** (2006).

[Ch1] J. Cheeger,*Comparison and finiteness theorems for Riemannian manifolds*, Ph.D. Thesis, Princeton University, (1967).

[Ch2] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math. **92** (1970), 61–74.

[ChowII] B. Chow, *On the entropy estimate for the Ricci flow on compact* 2*-orbifolds*, J. Diff. Geom. **33** (1991), 597–600.

[ChowWu] B. Chow and L.-F. Wu, *The Ricci flow on compact* 2*-orbifolds with curvature negative somewhere*, Comm. Pure Appl. Math. **44** (1991), 275–286.

[CGT] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds*, Journal Differential Geom. **17** (1982), no. 1, 15–53.

[CTZ] B.-L. Chen, S.-H. Tang, and X.-P. Zhu, *Complete classification of compact four manifolds with positive isotropic curvature*, J. Differential Geom. **91** (2012), no. 1, 41–80.

[ChenYWangI] X. Chen and Y. Wang, *Bessel functions, heat kernel and the conical Kaehler-Ricci flow*, `arXiv:1305.0255`, (2013).

[ChenYWangII] X. Chen and Y. Wang, *On the long time behavior of the conical Kaehler-Ricci flows*, `arXiv:1402.6689`, (2014).

[ChenWang] X. Chen and B. Wang, *On the conditions to extend the Ricci flow III*, Int. Math. Res. Notices **2013** (2013), no. 10, 2349–2367.

[CLN] B. Chow, P. Lu, and L. Ni, Hamilton's Ricci Flow, Graduate Studies in Mathematics, Vol. 77, AMS Science Press, (2000).

[DeT] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors*, J. Differential Geom. **18** (1983), no. 1, 157–162.

[DeTK] D. DeTurck and J. Kazdan, *Some regularity theorems in Riemannian geometry*, Annales Sci.de l'École Norm. Sup. **14** (1981), no. 3, 249–260.

[Evans] L. Evans, Partial Differential Equations, second edition, Graduate Studies, Vol. 19, AMS, (2010).

[EG] L. Evans and R. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, (1992).

[EGZ] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Weak solutions to degenerate complex Monge-Ampère flows I*, Math. Ann. **362** (2015), 931–963.

[EGZII] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Weak solutions to degenerate complex Monge-Ampère flows II*, Adv. Math. **293** (2016), 37–80.

[FIK] M. Feldman, T. Ilmanen, and D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kaehler-Ricci solitons*, J. Differential Geom. **65** (2003), no. 2, 169–209.

[GT] D. Gilbarg and N. Trudinger, Elliptic Partial Differential Equations of Second Order, third revised edition, Springer, (2000).

[HM] R. Haslhofer and R. Mueller, *A compactness theorem for complete Ricci shrinkers*, Geom. Funct. Anal. **21** (2011), no. 5, 1091–1116.

[HM2] R. Haslhofer and R. Mueller, *A note on the compactness theorem for 4d Ricci shrinkers*, arXiv:1407.1683, (2014).

[HaComp] R. S. Hamilton, *A compactness property of the Ricci flow*, American Journal of Mathematics **117** (1995), 545–572.

[HaThree] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[HaThreeO] R. S. Hamilton, *Three-orbifolds with positive Ricci curvature*, in: Collected Papers on Ricci Flow, Ser. Geom. Top. **37**, International Press, (2003).

[HaFour] R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179.

[HaForm] R. S. Hamilton, *The formation of singularities in the Ricci flow*, in: Collection: Surveys in Differential Geometry, Vol. II (Cambridge, MA), 7–136, (1995).

[JK] J. Jost and H. Karcher, *Geometrische methoden zur Gewinnung von a-priori Schranken für harmonische Abbildungen*, Manuscripta Math. **40** (1982), 27–77.

[Ka] A. Kasue, *A convergence theorem for Riemannian manifolds and some applications*, Nagoya Math. J. **114** (1989), 21–51.

[KlLo] B. Kleiner and J. Lott, *Singular Ricci flows I*, arXiv:1408.2271, (2014).

[KLThree] B. Kleiner and J. Lott, *Geometrization of three-dimensional orbifolds via Ricci flow*, arXiv:1101.3733, (2011).

[KL] H. Koch and T. Lamm, *Geometric flows with rough initial data*, Asian J. Math. **16** (2012), no. 2, 209–235.

[Lee] J. Lee, Introduction to Smooth Manifolds, Graduate Texts in Math., Springer, (2002).

[Lee2] J. Lee, *Introduction to Topological Manifolds*, 2nd. edition, Springer, (2010).

[Li] Y. Li, *Smoothing Riemannian metrics with bounded Ricci curvature in dimension four. II*, Ann. Glob. Anal. Geom. **41** (2012), 407–421.

[LiuZhang] J. Liu and X. Zhang, *The conical Kaehler-Ricci fow on Fano manifolds*, arXiv:1402.1832, (2014).

[Lu] P. Lu, *A compactness property for solutions of the Ricci flow on orbifolds*, American Journal of Mathematics **123** (2001), no. 6, 1103–1134.

[MT] R. McCann and P. Topping, *Ricci flow, entropy and optimal transportation*, American Journal of Math. **132** (2010), 711–730.

[MRS] R. Mazzeo, Y. Rubinstein, and N. Sesum, *Ricci flow on surfaces with conic singularities*, arXiv:1306.6688, (2013).

[On] B. O'neill, Semi-Riemannian Geometry with Applications to Relativity, Academic Press, (1983).

[Petersen] P. Petersen, Convergence Theorems in Riemannian Geometry, Comparison Geometry, MSRI Publications, Vol. 30, (1997).

[PeWe] P. Petersen and G.-F. Wei, *Relative volume comparison with integral curvature bounds*, GAFA **7** (1997), 1031–1045.

[Pe1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159, (2002).

[SS] I. Sabitov and S. Sefel, *Connections between the order of smoothness of a surface and that of its metric*, Akademija Nauk SSSR. Sibirskoe Otdelenie. Sibirskii Matematiceskii Zurnal **17**, no. 4, 916–925.

[Sesum] N. Sesum, *Curvature tensor under the Ricci flow*, American Journal of Mathematics, **127** (2005), no. 6, 1315–1324.

[SesumLe] N. Sesum and N. Q. Le, *Remarks on the curvature behaviour at the first singular time of the Ricci flow*, Pacific Journal of Mathematics **255** (2012), no. 1, 155–175.

[Shi] W.-X. Shi, *Ricci deformation of the metric on complete noncompact Riemannian manifolds*, J. Differential Geom. **30** (1989), 303–394.

[Si1] M. Simon, *Some integral curvature estimates for the Ricci flow in four dimensions*, arXiv:1504.02623.

[SiArxiv] M. Simon, *Extending four dimensional Ricci flows with bounded scalar curvature*, arXiv:1504.02910.

[SimC0] M. Simon, *Deformation of C^0 Riemannian metrics in the direction of their Ricci curvature*, Comm. Anal. Geom. **10** (2002), no. 5, 1033–1074.

[SimSmoo] M. Simon, *Local smoothing results for the Ricci flow in dimensions two and three*, Miles Simon. Geometry and Topology **17** (2013), 2263–2287.

[SongWeinkove1] J. Song and B. Weinkove, *Contracting exceptional divisors by the Kähler-Ricci flow*, Duke Math. J. **162** (2013), no. 2, 367–415.

[SongWeinkove2] J. Song and B. Weinkove, *Contracting exceptional divisors by the Kähler-Ricci flow II*, Proc. Lond. Math. Soc. (3) **108** (2014), no. 6, 1529–1561.

[Tian] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Inventiones Mathematicae **101** (1990), no. 1, 101–172.

[TZ] G. Tian and Z. Zheng, *Regularity of Kähler Ricci flows on manifolds*, `arXiv:1310.5897`.

[TME] P. Topping, R. Mueller, and J. Enders, *On type I singularities in Ricci flow*, Communications in Analysis and Geometry **19** (2011), 905–922.

[Topping] P. Topping, *Diameter Control under Ricci flow*, Communications in Analysis and Geometry **13** (2005), 1039–1055.

[Wang1] B. Wang, *On the conditions to extend the Ricci flow*, Int. Math. Res. Notices **2008** (2008), rnn012, 30 pages.

[Wang2] B. Wang, *On the conditions to extend the Ricci flow (II)*, Int. Math. Res. Notices **2012** (2012), no. 14, 3192–3223.

[WangY] Y. Wang, *Smooth approximations of the conical Kähler-Ricci flows*, `arXiv:1401.5040`, (2014).

[Wei] G. Wei, *Manifolds with a lower Ricci curvature bound*, Surveys in Differential Geometry **XI**, eds. J. Cheeger and K. Grove, International Press, Somerville, MA, pp. 203–228, (2007).

[WuLF] , L.-F. Wu, *The Ricci flow on* 2*-orbifolds with positive curvature*, J. Diff. Geom. **33** (1991), 575–596.

[YangD] D. Yang, *Convergence of Riemannian manifolds with integral bounds on curvature I*, Ann. Sci. de l'ENS, 4, tome 25, no. 1 (1992), 77–105.

[Ye] R. Ye, *The logarithmic Sobolev inequality along the Ricci flow*, `arXiv:0707.2424`.

[Yin] H. Yin, *Ricci flow on surfaces with conical singularities*, J. Geom. Anal. **20** (2010), 970–995.

[YinII] H. Yin, *Ricci flow on surfaces with conical singularities II*, `arXiv:1305.4355`, (2013).

[Zhang] Z. Zhang, *Scalar curvature behaviour for finite-time singularity of Kahler Ricci flow*, Michigan Math. J. **59** (2010), no. 2, 419–433.

[Zhang1] Q. S. Zhang, *A uniform Sobolev inequality under Ricci flow*, Int. Math. Research Notices **2007** (2007), rnm056, 17 pages.

[Zhang2] Q. S. Zhang, *Erratum to: A uniform Sobolev inequality under Ricci flow*, Int. Math. Research Notices **2007** (2007), rnm096, 4 pages.

[Zhang3] Q. S. Zhang, *Bounds on volume growth of geodesic balls under Ricci flow*, Mathematical Research Letters **19** (2012), no. 1, 245–253.

Otto von Guericke University, Magdeburg, IAN
Universitätsplatz 2, Magdeburg 39104, Germany
E-mail address: `miles.simon@ovgu.de`

Received July 18, 2016
Accepted February 4, 2018