Communications in
Analysis and Geometry
Volume 28, Number 7, 1603–1645, 2020

Integrability theorems and conformally constant Chern scalar curvature metrics in almost Hermitian geometry

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The various scalar curvatures on an almost Hermitian manifold are studied, in particular with respect to conformal variations. We show several integrability theorems, which state that two of these can only agree in the Kähler case. Our main question is the existence of almost Kähler metrics with conformally constant Chern scalar curvature. This problem is completely solved for ruled manifolds and in a complementary case where methods from the Chern–Yamabe problem are adapted to the non-integrable case. Also a moment map interpretation of the problem is given, leading to a Futaki invariant and the usual picture from geometric invariant theory.

†The second author was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – UP 85/2-1, UP 85/3-1.

The present paper is devoted to the conformal geometry of almost Hermitian structures, in particular to aspects relating to their scalar curvature.

The necessary background is briefly reviewed in §1. In particular, we recall the Chern connection and its torsion (see [28]) on almost Hermitian manifolds, which reflects also the almost complex structure. From it one derives three Ricci forms and two scalar curvatures: the Hermitian (or Chern) scalar curvature $s^H = 2s^C$ and the third scalar curvature s. From the Levi-Civita connection we also have the Riemannian scalar curvature s^g and all three evidently coincide in the Kähler case. In §2 their precise relationship in general is established by careful calculation in local coordinates (see Propositions 2.1, 2.2, 2.3). The formulas generalize those of Gauduchon in the integrable case [27].

These are applied in §3 to prove several new integrability theorems, which assert that when two scalar curvatures coincide we must already be in the Kähler case. These holds in any dimension when we have a nearly Kähler structure (Corollary 3.5). We also have results in any dimension in the almost Kähler case (see Corollary 3.1 and also Apostolov–Drăghici [7]). The completely general almost Hermitian case is restricted to dimension 4 (Theorem 3.2). We also obtain an interesting result (Corollary 3.7) on 6-dimensional compact non-Kähler, nearly Kähler manifolds: they all have $s^H = 0$.

We then compute in §4 the behaviour of our Ricci forms and scalar curvatures under conformal variations (see Corollaries 4.4 and 4.5). This allows us to prove another integrability theorem (Theorem 4.8) for conformally almost Kähler structures, relating the Hermitian and Riemannian scalar curvature.

In §5 we state the basic problem that will concern us for the rest of the paper: find almost Hermitian structures which have conformally constant Hermitian scalar curvature (ccHsc). We first extend the results of Angella–Calamai–Spotti [1] on the Chern–Yamabe problem to the non-integrable case and show some independent results of interest. In Corollary 5.10 we obtain that every almost Hermitian structure with non-positive fundamental constant (40) has ccHsc. The remaining case is much more difficult. It is not even known in general whether *one can find any ccHsc almost Hermitian structure*. This is our Existence Problem 5.2, where we restrict to the symplectic case.

In §6 we solve this problem for ruled manifolds given by the generalized Calabi construction (Theorem 6.4). Drawing on the fundamental work by

Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman [3, 5], the proof is carried out in §6.3 and quickly reduces to an ODE for a metric on the moment polytope, an interval in our case. The main difficulty is to show positivity of the solution and this is done by a careful asymptotic analysis in §6.4. The manifolds thus constructed are new examples of non-Kähler structures of constant Hermitian scalar curvature with positive fundamental constant.

Finally in §7 we give an interpretation of our existence problem in the framework of moment maps (Theorem 7.9). Here we assume a symmetry on the manifold, namely a Hamiltonian vector field which is Killing for some metric.

This leads to the usual existence and uniqueness conjectures in terms of geometric invariant theory, and also to a Futaki invariant (see §7.4). We end in §7.5 with concrete calculations in the toric case.

Acknowledgements

Both authors are grateful to Tedi Drăghici, Daniele Angella and Antonio Di Scala for their suggestions and interest in this work. Also we would like to thank a referee his for valuable comments.

1. Preliminaries

Let (M, J, g) be an almost Hermitian manifold of real dimension $m = 2n$. Thus $J\colon TM \to TM$ is an almost complex structure $J^2 = -1$ that is orthogonal for the Riemannian metric g. The associated fundamental form is $F = g(J\cdot, \cdot)$. We usually do not distinguish between the metric and the almost complex structure and write $g_J \coloneqq F(\cdot, J\cdot)$ for the metric corresponding to J. The volume form is $\mathrm{vol}_g = \frac{F^n}{n!}$. On the complexification $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ we consider the $\mathbb{C}$-bilinear extension of g, the Hermitian form $h(X \otimes z, Y \otimes w) = z\overline{w}g(X, Y)$, and the restriction of h to $T^{1,0}$, which we identify with TM using $X \mapsto X^{1,0} = \frac{X-iJX}{2}$

1.1. Complex notation

Let z_α denote a complex basis of $T^{1,0}$. Then $\bar{z}_{\bar{\alpha}}$ is the basis of $T^{0,1}$ obtained by conjugation. The dual basis is denoted $z^\alpha, \bar{z}^{\bar{\alpha}}$. The components of the Hermitian form are

$$h_{\alpha\bar{\beta}} = h(z_\alpha, z_\beta).$$

The transposed inverse of $h_{\alpha\bar{\beta}}$ is denoted $h^{\alpha\bar{\beta}}$. Thus $h_{\alpha\bar{\gamma}}h^{\beta\bar{\gamma}} = \delta_\alpha^\beta$ and $h_{\alpha\bar{\beta}} = \overline{h_{\beta\bar{\alpha}}}$. The fundamental form is then $F = ih_{\alpha\bar{\beta}}z^\alpha \wedge \bar{z}^{\bar{\beta}}$. We shall use the Hermitian form to raise and lower tensor indices. Note that for any tensor $(\cdot)^\alpha{}_\alpha = (\cdot)_{\bar{\alpha}}{}^{\bar{\alpha}}$.

We also get a J-adapted orthonormal frame $e_1, e_2 = Je_1, \ldots, e_{m-1}$, $e_m = Je_{m-1}$ of the real tangent bundle TM by decomposing z_α into the real and imaginary part:

$$z_\alpha = \frac{1}{2}(e_{2\alpha-1} - ie_{2\alpha}) \tag{1}$$

By convention $\alpha, \beta, \gamma, \ldots$ range over $1, \ldots, n$ while $i, j, k, \ldots$ range over $1, \ldots, m$.

The twisted exterior differential of a p-form ψ is defined as (see [24, (1.11.1)])

$$d^c\psi = JdJ^{-1}\psi,$$

where J acts on forms by $(J^{-1})^*$ (some authors have a different sign convention).

1.2. Type decomposition

Let E be a vector bundle on M with complex structure J^E. Unless $E = \mathbb{C}$ the space $\Omega^p(M; E)$ of E-valued differential p-forms has two different type decompositions.

Definition 1.1. A form $\psi \in \Omega^p(M; E)$ has *E-type* (r, s) when $p = r + s$ and

$$\sum_{k=1}^{p} \psi_{X_1 \cdots JX_k \cdots X_p} = (r - s)J^E\left(\psi_{X_1 \cdots X_p}\right) \qquad \forall X_i \in TM. \tag{2}$$

The subspace of forms of E-type (r, s) is denoted by $\Omega^{r,s}(M; E)$. We write $\psi^{r,s}$ for the projection with respect to this direct sum decomposition of $\Omega^p(M; E)$.

Hence ψ behaves like an ordinary (r, s)-form, except that it is vector-valued. For example, the Nijenhuis tensor $N \in \Omega^2(M; TM)$ has TM-type $(0, 2)$.

Lemma 1.2. *For a connection with $\nabla_X J = 0$, $\nabla_X\psi$ has the same E-type as ψ.*

To understand the E-type with respect to contractions, let us say that ψ has *ordinary type* (r,s) when in a local frame as in Section 1.1 we may write

$$\psi = \frac{1}{r!s!}\psi_{\alpha_1\cdots\alpha_r,\bar\beta_1\cdots\bar\beta_s} z^{\alpha_1}\cdots z^{\alpha_r}\wedge \bar z^{\bar\beta_1}\cdots \bar z^{\bar\beta_s}, \tag{3}$$

the coefficients being sections of E, anti-symmetric for $\alpha_1\cdots\alpha_r$ and for $\bar\beta_1\cdots\bar\beta_s$. Thus, the ordinary type behaves as expected under contraction with $(1,0)$ and $(0,1)$-vector fields. Concerning the E-type, we have the following observation:

Lemma 1.3. *A form ψ has E-type (r,s) precisely when it is the sum of an $E^{1,0}$-valued form of ordinary type (r,s) and an $E^{0,1}$-valued form of ordinary type (s,r).*

Finally, in case $E = TM$ we may use the metric g to identify $(p+1)$-forms with TM-valued p-forms. Using the musical isomorphism we get a map

$$i\colon \Omega^{p+1}(M)\hookrightarrow \Omega^p(M;TM),\quad i(\phi)_{X_1\cdots X_p} = \phi_{-,X_1\cdots X_p}{}^{\sharp_g}. \tag{4}$$

Note that $\phi\in\Omega^{p+1}(M)$ are *real* forms. From (4) we get a third type decomposition. This has been used by Gauduchon [28, (1.3.2)] in the case $p=2$.

Definition 1.4. A $(r+s)$-form ϕ has *real type* $(r,s)+(s,r)$ when the complexification of ϕ is a sum of a complex (r,s)-form and (its conjugate) (s,r)-form. We write $\Omega^{(r,s)+(s,r)}(M)$ for the space of real forms of real type $(r,s)+(s,r)$.

Lemma 1.5. *The map* (4) *identifies the real type decomposition*

$$\Omega^{(r,s+1)+(s+1,r)}(M) = \left[\Omega^{r,s}(M;TM)\oplus\Omega^{s+1,r-1}(M;TM)\right]\cap\Omega^{p+1}(M),$$

with the TM-grading. When $p=n$, we get $\Omega^{0,n}(M;TM)\cap\Omega^{n+1}(M)=\{0\}$.

Proof. Since both gradings decompose the entire space, it is enough to show an inclusion, which is a straightforward direct verification. □

1.3. Trace

Let ϕ be a real 2-tensor, which corresponds to an endomorphism $a \in \mathrm{End}_{\mathbb{R}}(TM)$ by letting

$$\phi_{XY} = g(a(X), Y), \qquad \forall X, Y \in TM.$$

Then ϕ is a 2-form $\iff$ a is skew-symmetric. In this case the *Lefschetz trace* is defined as

$$\Lambda(\phi) = g(\phi, F) = \mathrm{tr}(-J \circ a).$$

The complexification of ϕ will be denoted by the same letter so that the reals forms are characterized by $\overline{\phi_{ZW}} = \phi_{\bar{Z}\bar{W}}$ for $Z, W \in TM \otimes \mathbb{C}$. In the notation of 1.1

$$\Lambda(\phi) = -i{\phi_\alpha}^\alpha.$$

Recall that the definition of Λ is extended to forms of higher degrees by defining $\Lambda(\phi) = \iota_{F^\sharp}\phi = *(F \wedge *^{-1}\phi)$ where $F^\sharp$ is obtained by raising both indices.

1.4. Norms

Let E be a complex vector bundle on M with Hermitian form $\langle , \rangle$. Generalizing the case $E = \mathbb{C}$, the norm of an E-valued differential p-form is

$$(5) \qquad |\psi|^2_{\Omega^p(M;E)} = \frac{1}{p!}\langle \psi_{i_1\cdots i_p}, \psi^{i_1\cdots i_p}\rangle = \frac{1}{p!} g^{i_1 j_1} \cdots g^{i_p j_p} \langle \psi_{i_1\cdots i_p}, \psi_{j_1\cdots j_p}\rangle.$$

Unless $p = 1$ we shall not follow [28] in identifying TM-valued p-forms with $(0, p+1)$-tensors, since this leads to different conventions for the norm. We will only need (5) in the cases $E = \mathbb{C}$ and $E = TM$. When an E-valued p-form ψ is decomposed as a sum of elements (3) then

$$(6) \qquad |\psi|^2_{\Omega^p(M;E)} = \frac{1}{r!s!}\langle \psi_{\alpha_1\cdots\alpha_r,\bar{\beta}_1\cdots\bar{\beta}_s}, \psi^{\bar{\alpha}_1\cdots\bar{\alpha}_r,\beta_1\cdots\beta_s}\rangle.$$

In particular, the decomposition into E-type is orthogonal.

Lemma 1.6. *Let $\phi \in \Omega^{(r,r+1)+(r+1,r)}(M)$ be a real form with associated TM-valued form $\psi = i(\phi)$ using (4). For the $\Omega^{2r}(M;TM)$-norm of the projections we have*

$$\frac{1}{r!r!}|\psi^{r+1,r-1}|^2 = \frac{1}{(r-1)!(r+1)!}|\psi^{r,r}|^2. \tag{7}$$

In particular

$$|\phi|^2 = |\psi^{2,0}|^2 + |\phi^{(3,0)+(0,3)}|^2 \qquad \forall\phi \in \Omega^3(M) \tag{8}$$

When $n = 2$, we have $2|\psi^{2,0}| = |\psi^{1,1}|^2$ for every 3*-form ψ (see also [28, (1.3.9)]).*

1.5. Chern connection

The almost complex structure is parallel for the Levi-Civita connection D^g precisely when M is Kähler. Therefore one considers other metric connections that make J parallel.

Definition 1.7. The *Chern connection* ∇ is the unique Hermitian connection on TM whose $(0,1)$-part is the canonical Cauchy–Riemann operator

$$\bar\partial_X Z = [X^{0,1}, Z]^{1,0}, \qquad X \in TM, Z \in C^\infty(M, T^{1,0}). \tag{9}$$

(recall that a Hermitian connection is required to satisfy $\nabla g = 0, \nabla J = 0$.)

Equivalently, the Chern connection is the unique Hermitian connection whose torsion tensor $T_{XY} = \nabla_X Y - \nabla_Y X - [X,Y]$ is J-anti-invariant. The decomposition of $T \in \Omega^2(M;TM)$ into TM-type is then given by (see [28, p. 272])

$$T^{0,2} = N, \qquad T^{1,1} = 0, \qquad T^{2,0} = (d^cF)^{2,0}. \tag{10}$$

Here d^cF is a TM-valued 2-form via (4) and we take the $(2,0)$-part of its TM-type.

Remark 1.8. If $T = 0$ for the torsion of the Chern connection of an almost Hermitian manifold, then $\nabla = D^g$. Hence J is parallel for D^g and the structure is Kähler.

1.6. Torsion 1-form

Besides not being integrable, the difficulty in dealing with almost Hermitian manifolds is that the fundamental form is not closed ($dF = 0$ holds when M is *almost Kähler*). We thus consider the *torsion* 1*-form* $\theta = \Lambda(dF)$. Equivalently $dF = (dF)_0 + \frac{1}{n-1}\theta \wedge F$ for the trace-free part $(dF)_0$.

Lemma 1.9.

$$\theta = \Lambda(dF) = J\delta^g F. \tag{11}$$

Proof. Let $L\phi = F \wedge \phi$. Using [34, Corollary 1.2.28] we find

$$d(F^{n-1}) = (n-1)L^{n-2}dF = L^{n-1}\Lambda dF - \Lambda L^{n-1}dF = F^{n-1} \wedge \theta.$$

Combining this with $*F^k/k! = F^{n-k}/(n-k)!$ we compute

$$\begin{aligned} J\delta^g F = -J * d * F &= \frac{-1}{(n-1)!} J * d(F^{n-1}) = \frac{-1}{(n-1)!} J * (\theta \wedge F^{n-1}) \\ &= \frac{-1}{(n-1)!} J\iota_{\theta^\sharp}(*F^{n-1}) = -J\iota_{\theta^\sharp} F = \theta. \end{aligned}$$

□

The almost Hermitian structure is *Gauduchon* if $\delta^g\theta = 0$, and is *balanced* if $\theta = 0$. It is easy to check that the torsion 1-form $\theta_X = \mathrm{tr}(Z \mapsto T_{XZ})$ is the trace of the torsion tensor of the Chern connection. Thus

$$\theta = T_{\alpha\beta}{}^{\beta} z^{\alpha} + T_{\bar\alpha\bar\beta}{}^{\bar\beta} \bar z^{\bar\alpha}. \tag{12}$$

1.7. Ricci forms

Let R be a 2-form with values in skew-Hermitian endomorphisms of TM, for example the curvature tensor $R^{\nabla}_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ of the Chern connection. In the integrable case, the 2-form R^{∇} is J-invariant. In general, the complexification of R is not of type $(1,1)$ and has more components

$$R = \left(\frac{1}{2} R_{\alpha\beta\gamma}{}^{\delta}\, z^{\alpha} \wedge z^{\beta} + R_{\alpha\bar\beta\gamma}{}^{\delta}\, z^{\alpha} \wedge \bar z^{\bar\beta} + \frac{1}{2} R_{\bar\alpha\bar\beta\gamma}{}^{\delta}\, \bar z^{\bar\alpha} \wedge \bar z^{\bar\beta} \right) \otimes z^{\gamma} \otimes z_{\delta}.$$

Following [27], we consider three ways to contract the tensor R:

Definition 1.10. The *first* (or *Hermitian*) *Ricci form* ρ of R is the trace

$$\rho_{XY} = \mathrm{tr}_{\mathbb{C}}\,(J \circ R_{XY}) = -\Lambda(R_{XY}). \tag{13}$$

The complexification of ρ has components

$$\rho = \frac{i}{2} R_{\alpha\beta\gamma}{}^{\gamma} z^{\alpha} \wedge z^{\beta} + i R_{\alpha\bar\beta\gamma}{}^{\gamma} z^{\alpha} \wedge \bar z^{\bar\beta} + \frac{i}{2} R_{\bar\alpha\bar\beta\gamma}{}^{\gamma} \bar z^{\bar\alpha} \wedge \bar z^{\bar\beta}. \tag{14}$$

The first Ricci form is always closed and, when J is integrable, is of type $(1,1)$. Its cohomology class $2\pi c_1(TM, J)$ is the first Chern class of M.

Definition 1.11. The *second Ricci form* of R is $r_{XY} = -\Lambda(R_{\cdot,\cdot,XY})$, so

$$r = i R_{\gamma}{}^{\gamma}{}_{\lambda\bar\mu} z^{\lambda} \wedge \bar z^{\bar\mu}, \tag{15}$$

which is always a $(1,1)$-form, but not closed in general.

Definition 1.12. The *third Ricci form* is

$$\sigma = \frac{i}{2} R_{\mu\alpha}{}^{\alpha}{}_{\lambda} z^{\lambda} \wedge z^{\mu} + i R_{\bar\mu\alpha}{}^{\alpha}{}_{\lambda} z^{\lambda} \wedge \bar z^{\bar\mu} - \frac{i}{2} R_{\bar\mu}{}^{\alpha}{}_{\alpha\bar\lambda} \bar z^{\bar\lambda} \wedge \bar z^{\bar\mu} \tag{16}$$

1.8. Scalar curvatures

The Lefschetz traces of ρ and r agree. We thus define:

Definition 1.13. The *Chern scalar curvature* of R is

$$s^C = \Lambda(\rho) = \Lambda(r) = R_{\alpha}{}^{\alpha}{}_{\gamma}{}^{\gamma} = -\frac{1}{4} R_{e_i}{}^{Je_i}{}_{e_j}{}^{Je_j} \tag{17}$$

The *Hermitian scalar curvature* is $s^H = 2 \cdot s^C$ and coincides with the Riemannian scalar curvature in the Kähler case.

Definition 1.14. The *third scalar curvature* is the Lefschetz trace

$$s = \Lambda(\sigma) = R_{\alpha}{}^{\beta}{}_{\beta}{}^{\alpha} = \frac{1}{2} R_{i}{}^{j}{}_{j}{}^{i} = -\frac{1}{2} R_{ij}{}^{ij} \tag{18}$$

of the third Ricci form.Alternatively, s is the trace of the curvature operator.

We shall also consider the *Riemannian scalar curvature* s^g formed as usual from the Levi-Civita connection D^g.

Remark 1.15. Suppose J is integrable. Then the three Ricci forms coincide with those of [27, Section I.4]. Gauduchon uses the notation u for s^C and v for s. Liu–Yang define similar scalar curvatures from the Levi-Civita connection. The third scalar curvature s is like their 'Riemannian type scalar curvature' s_R in [43, 4.2].

2. Comparison of curvatures

When M is not Kähler, the three scalar curvatures defined above do not coincide. In this section, we quantify the differences. This is based on the following. Recall that the algebraic Bianchi identity for connections on TM with torsion asserts

$$\begin{aligned} R_{XY}Z + R_{ZX}Y + R_{YZ}X = {} & (\nabla_X T)_{YZ} + (\nabla_Z T)_{XY} + (\nabla_Y T)_{ZX} \\ & - T(X, T_{YZ}) - T(Z, T_{XY}) - T(Y, T_{ZX}). \end{aligned} \tag{19}$$

By [27, (2.1.4)] the difference between the Chern and Levi-Civita connection is

$$g(\nabla_X Y, Z) = g(D^g_X Y, Z) + \frac{3}{2} t_{XYZ} - g(X, T_{YZ}), \tag{20}$$

for the anti-symmetrization

$$t_{XYZ} = \frac{1}{3}\left(g(X, T_{YZ}) + g(Z, T_{XY}) + g(Y, T_{ZX})\right) \tag{21}$$

of the torsion tensor[1]. As a consequence of (10) we note (see also [28, (2.5.10)])

$$t = \frac{1}{3} d^c F. \tag{22}$$

Proposition 2.1. *We have* $s^C - s = \frac{1}{2}|\theta|^2 + \frac{1}{2}\delta^g\theta - \frac{9}{2}|t|^2 + \frac{1}{2}|T|^2$.

Here, $|T|^2$ is given by (5) as a TM-valued 2-form and $|\theta|^2$ and $|t|^2$ are the usual norms for differential 3-forms. The codifferential δ^g is taken with respect to g.

[1] In fact, as pointed out to us by the referee, (20) holds for any metric connection ∇ with torsion T and corresponding t defined by (21). The formula can be verified directly, using that the gauge potential $A_X = D^g_X - \nabla_X$ is then skew-symmetric and $T_{XY} = A_X Y - A_Y X$.

Proof. Choose an orthonormal frame $e_{2\alpha-1}, e_{2\alpha} = Je_{2\alpha-1}$ near $p \in M$ with $\nabla_X e_i = 0$ for all $X \in T_pM$. Such a frame may be constructed by extending parallelly an adapted orthonormal basis in p along geodesic rays. We work in the basis $z_\alpha = \frac{1}{2}(e_{2\alpha-1} - ie_{2\alpha})$ of $T^{1,0}$.

By evaluating (20) at p we get

$$(D_\alpha z^\gamma)_{\bar\beta} = -(D_\alpha z_{\bar\beta})^\gamma = -\frac{1}{2}T_{\bar\beta}{}^\gamma{}_\alpha, \quad (D_\alpha z^{\bar\gamma})_{\bar\beta} = -(D_\alpha z_{\bar\beta})^{\bar\gamma} = \frac{1}{2}T^{\bar\gamma}{}_{\alpha\bar\beta}.$$

We apply this to compute the codifferential at p:

$$\begin{aligned}
-\delta\theta &= h^{\alpha\bar\beta}(D_\alpha\theta)_{\bar\beta} + h^{\beta\bar\alpha}(D_{\bar\alpha}\theta)_\beta \\
&= h^{\alpha\bar\beta}\left(z_\alpha\theta_{\bar\beta} + \theta_\gamma(D_\alpha z^\gamma)_{\bar\beta} + \theta_{\bar\gamma}(D_\alpha z^{\bar\gamma})_{\bar\beta}\right) + \text{conjugate term} \\
&= z_\alpha\theta^\alpha - \frac{1}{2}\theta_\gamma T^{\alpha\gamma}{}_\alpha + \frac{1}{2}\theta_{\bar\gamma}T^{\bar\gamma}{}_\alpha{}^\alpha + \text{conjugate term} \\
&= z_\alpha\theta^\alpha + \frac{1}{2}|\theta|^2 + \text{conjugate term} \\
&= z_\alpha\theta^\alpha + z_{\bar\alpha}\theta^{\bar\alpha} + |\theta|^2
\end{aligned}$$

Since $\nabla J = 0$, the Chern connection preserves the type decomposition of TM-valued forms and so (19) reduces in our frame to

$$R_{\alpha\bar\beta\gamma}{}^\delta + R_{\bar\beta\gamma\alpha}{}^\delta = (\nabla_{\bar\beta}T)_{\gamma\alpha}{}^\delta - T_{\bar\beta,T(\gamma,\alpha)}{}^\delta.$$

Taking the double trace of this equation gives

$$s^C - s = R_\alpha{}^\alpha{}_\gamma{}^\gamma + R^\alpha{}_{\gamma\alpha}{}^\gamma = (\nabla_{\bar\alpha}T)_\gamma{}^{\bar\alpha\gamma} - T^\alpha{}_{T(\gamma,\alpha)}{}^\gamma = (\nabla_\alpha T)_{\bar\gamma}{}^{\alpha\bar\gamma} - T^{\bar\alpha}{}_{T(\bar\gamma,\bar\alpha)}{}^{\bar\gamma}.$$

(the last equation holds since $s^C - s$ is real.) In our frame at p

$$(\nabla_{\bar\alpha}T)_\gamma{}^{\bar\alpha\gamma} + (\nabla_\alpha T)_{\bar\gamma}{}^{\alpha\bar\gamma} = -z_\alpha\theta^\alpha - \bar z_{\bar\alpha}\theta^{\bar\alpha}$$

Putting the above together

$$s^C - s = \frac{1}{2}(|\theta|^2 + \delta\theta) - \frac{1}{2}\left(T^\alpha{}_{T(\gamma,\alpha)}{}^\gamma + T^{\bar\alpha}{}_{T(\bar\gamma,\bar\alpha)}{}^{\bar\gamma}\right).$$

Now apply the easy identities $T^\alpha{}_{T(\gamma,\alpha)}{}^\gamma = T^{\bar\alpha}{}_{T(\bar\gamma,\bar\alpha)}{}^{\bar\gamma} = T_{\alpha\beta\gamma}T^{\beta\gamma\alpha}$ and

$$9|t|^2 - |T|^2 = 2T_{\alpha\beta\gamma}T^{\beta\gamma\alpha} = T_{ijk}T^{jki}. \tag{23}$$

□

Proposition 2.2. *For the Riemannian scalar curvature s^g we have*

$$2s - s^g = |T|^2 - \frac{9}{2}|t|^2 - 2\delta\theta - |\theta|^2 \tag{24}$$

Proof. This is a similar computation, using normal coordinates e_i at $p \in M$. Thus $[e_i, e_j] = 0$ and $D^g_{e_i}(e_j)|_p = 0$. The Riemannian scalar curvature at p is then

$$s^g(p) = e_i g(D^g_{e_j} e_j, e_i) - e_j g(D^g_{e_i} e_j, e_i)$$

(we omit all summation signs) which, using

$$\nabla_{e_i} e_j = D^g_{e_i}(e_j) + \left(\frac{3}{2} t_{ij}{}^k - T_j{}^k{}_i\right) e_k \tag{25}$$

from (20), becomes

$$\begin{aligned} &e_i g(\nabla_{e_j} e_j, e_i) + e_i T_{jij} - e_j g(\nabla_{e_i} e_j, e_i) - e_j T_{jii} \\ &= 2s(p) + g(\nabla_{e_i} e_i, \nabla_{e_j} e_j) - g(\nabla_{e_i} e_j, \nabla_{e_j} e_i) - 2 e_j T_{jii}. \end{aligned}$$

Now $\delta\theta(p) = -e_j T_{jii}$ and inserting (25) gives

$$s^g(p) - 2s(p) = T_{iki} T_{jkj} + \frac{9}{4} t^2_{ijk} + 3 t_{ij}{}^k T_i{}^k{}_j - T_{jki} T_{ikj} + 2\delta\theta$$

Now apply (23) and $|\theta|^2 = T_{iki} T_{jkj}$ to get (24). □

Combining Propositions 2.1 and 2.2 gives (recall $s^H = 2s^C$):

Corollary 2.3. $s^H - s^g = -\delta\theta - \frac{27}{2}|t|^2 + 2|T|^2$.

Remark 2.4. By (22) and the orthogonal decomposition of 3-forms into real type

$$9|t|^2 = |dF|^2 = |dF^{(2,1)+(1,2)}|^2 + |dF^{(3,0)+(0,3)}|^2,$$

and by (8) and (10)

$$|T^{2,0}|^2 = |d^c F^{2,0}|^2_{\Omega^2} = |dF^{(2,1)+(1,2)}|^2, \qquad |T^{0,2}|^2 = |N|^2.$$

(for t and $dF^{(r,s)+(s,r)}$ we take the 3-form norm.) We conclude

$$\begin{aligned} &9|t|^2 - |T|^2 = |dF^{(3,0)+(0,3)}|^2 - |N|^2, \\ &\frac{9}{2}|t|^2 - |T|^2 = -\frac{1}{2}|dF^{(1,2)+(2,1)}|^2 + \frac{1}{2}|dF^{(3,0)+(0,3)}|^2 - |N|^2. \end{aligned}$$

When J is integrable, dF is of type $(2,1)+(1,2)$, so these equations reduce to $9|t|^2 = |T|^2$ and $\frac{9}{2}|t|^2 - |T|^2 = -\frac{1}{2}|dF^{(2,1)+(1,2)}|^2$. In this case (24) becomes [27, (32)] and Proposition 2.1 specializes to [27, Corollaire 2]:

$$\begin{aligned} 2s - s^g &= \frac{1}{2}|dF|^2 - 2\delta\theta - |\theta|^2, \\ s^H - 2s &= |\theta|^2 + \delta\theta, \\ s^H - s^g &= \frac{1}{2}|dF|^2 - \delta\theta. \end{aligned}$$

3. Integrability theorems

In the almost Kähler case, $dF = 0$, Propositions 2.1 and 2.2 immediate imply vanishing theorems. For in this case, $T = N$, $\theta = 0$, and $t = 0$ from (10), (11), and (22), respectively. The formulas then reduce to

$$2s - s^g = |N|^2, \qquad s^H - 2s = |N|^2. \tag{26}$$

Corollary 3.1. *[7] On an almost Kähler manifold we have $s^g \leq 2s \leq s^H$ with either equality precisely when (J, g, F) is Kähler.*

With some care in dimension four, these conclusions can be extended. Thus, assuming equality of various scalar curvatures on an almost Hermitian manifold will guarantee both the integrability af J and the Kähler condition $dF = 0$.

Theorem 3.2. *Let (M, J, g, F) be a closed almost Hermitian $4 = 2n$-manifold.*

i $\displaystyle\int_M (2s - s^g + |\theta|^2)\frac{F^n}{n!} \geq 0.$

ii $\displaystyle\int_M (s^H - s^g)\frac{F^n}{n!} \geq 0.$

iii $\displaystyle\int_M (s^C - s)\frac{F^n}{n!} \geq 0.$

In any case, equality holds if and only if the structure is Kähler.

Proof. Recall from (22) and (10) that

$$T = N + (d^cF)^{2,0}, \qquad t = \frac{1}{3}d^cF.$$

Since we are in dimension four, $t = t^{2,0} + t^{1,1}$ by Lemma 1.5. Also for the $\Omega^2(M;TM)$-norm defined in (5), Lemma 1.6 gives

$$|t|^2_{\Omega^2} = |t^{2,0}|^2 + |t^{1,1}|^2 = 3|t^{2,0}|^2 = \frac{1}{3}|(d^cF)^{2,0}|^2. \tag{27}$$

Combined with $|t|^2_{\Omega^3} = \frac{1}{3}|t|^2_{\Omega^2(M;TM)}$ we get

$$|T|^2_{\Omega^2} - \frac{9}{2}|t|^2_{\Omega^3} = |N|^2 + \frac{1}{2}|(d^cF)^{2,0}|^2. \tag{28}$$

Putting this into Proposition 2.2 and integrating gives

$$\int_M (2s - s^g + |\theta|^2) = \int_M \left(|N|^2 + \frac{1}{2}|(d^cF)^{2,0}|^2\right)\frac{F^n}{n!} \geq 0$$

where we use that the integral over $\delta\theta$ vanishes (since $\frac{F^n}{n!}$ is the Riemannian volume form). Of course, the left hand side can only vanish when $N = 0$ and $(d^cF)^{2,0} = 0$. Then $T = N + (d^cF)^{2,0} = 0$ so by Remark 1.8 we are in the Kähler case. Part ii) is a similar application of Corollary 2.3, while iii) uses Proposition 2.1. □

Remark 3.3. When M is a closed Hermitian manifold (the integrable case), one can deduce Theorem 3.2 in any dimension (see [27, 43] or apply the technique above to Remark 2.4). On the other hand, Dabkowski–Lock [17] have examples of *non-compact* Hermitian with $s^H = s^g$ which are not Kähler. Do higher-dimensional closed almost Hermitian non-Kähler manifolds with $s^H = s^g$ exist?

In the conformally almost Kähler case, we will extend ii) to higher dimensions in Theorem 4.8 below. We now proceed by proving an 'opposite' of Corollary 3.1 for nearly Kähler structures.

Definition 3.4. An almost Hermitian manifold (J, g, F) is *nearly Kähler* if

$$\left(D^g_X J\right)Y + \left(D^g_Y J\right)X = 0$$

where D^g is the Levi-Civita connection [30, 31].

It follows from the definition that $D^gF = \frac{1}{3}dF$. Moreover, dF is of type $(3,0)+(0,3)$ and $N = \frac{1}{3}d^cF$ is totally anti-symmetric. In particular, a nearly

Kähler manifold is balanced. Furthermore, a nearly Kähler manifold of dimension $2n = 4$ is Kähler. Also, if the nearly Kähler manifold is Hermitian then it is Kähler. Examples of nearly Kähler manifolds are S^6 with its standard almost-complex structure and metric, $S^3 \times S^3$ equipped with the bi-invariant almost complex structure and its 3-symmetric almost Hermitian structure and the twistor spaces over Einstein self-dual 4-manifolds, endowed with the anti-tautological almost complex structure (for more details about nearly Kähler manifolds see [13, 15, 45, 46, 51]). We deduce from Propositions 2.1 and 2.2 that for any nearly Kähler manifold

$$2s - s^g = -\frac{1}{6}|d^cF|^2_{\Omega^3}, \qquad s^H - 2s = -\frac{2}{3}|d^cF|^2_{\Omega^3}. \tag{29}$$

Corollary 3.5. *On a nearly Kähler manifold we have* $s^H \leq 2s \leq s^g$ *with either equality precisely when* (J, g, F) *is Kähler.*

Remark 3.6. This does not contradict Theorem 3.2 because in dimension 4 the notions Kähler and nearly Kähler agree.

A feature of nearly Kähler manifolds is that dF is of constant norm [14, 36]. Moreover, Gray proved [32, Theorem 5.2] that any non-Kähler nearly Kähler manifold of dimension 6 is Einstein of positive (constant) s^g. Furthermore, their first Chern class vanishes. Hence, the Hermitian scalar curvature s^H of any closed non-Kähler nearly Kähler manifold of dimension 6 vanishes.

Corollary 3.7. *On a closed non-Kähler, nearly Kähler manifold* (M, J, g) *of dimension* 6*, we have* $s^H = 0$.

Proof. From the discussion above and (29) we know already that s^H is constant. So we only need to prove that the total integral of s^H is zero. Now since $c_1(TM, J) = 0$, the Hermitian Ricci form is $\rho = da$ for some 1-form a. Thus

$$\begin{aligned}\int_M s^H \operatorname{vol}_g = 2\int_M \Lambda(\rho) \operatorname{vol}_g = 2\int_M g(da, F) \operatorname{vol}_g \\ = 2\int_M g(a, \delta F) \operatorname{vol}_g = -2\int_M g(a, J\theta) \operatorname{vol}_g = 0.\end{aligned}$$

since (M, J, g) is balanced. □

For more examples of almost Hermitian manifolds with vanishing Hermitian scalar curvature, we refer the reader to the work of Di Scala and Vezzoni [19, 20] (see also [52]).

4. Conformal variations

Let $\tilde{g} = e^{2f} g$ be a conformal variation of the metric along a smooth real-valued function f. Then $(M, J, \tilde{g}, \tilde{F})$ is again an almost Hermitian manifold and we shall be interested in how the associated Chern connection, Ricci forms, and scalar curvatures behave under this variation.

We begin by deriving an alternative expression for the Chern connection:

Lemma 4.1. *The Chern connection is given by*

$$h(W, \nabla_X Z) = X^{0,1} h(W, Z) + h(W, [X^{0,1}, Z]) + h([W, X^{0,1}], Z), \tag{30}$$

where $X \in TM$ and $W, Z \in C^\infty(M, T^{1,0})$.

Proof. (30) is easily seen to define a Hermitian connection whose $(0,1)$-part is given by (9). The result then follows from the uniqueness of such a connection. □

Lemma 4.2. $\tilde{\nabla}_X Z = \nabla_X Z + 2X^{1,0}(f) \cdot Z$ *for all* $X \in TM, Z \in T^{1,0}$.

Proof. This is an immediate consequence of Lemma 4.1. □

Proposition 4.3. *For the curvature tensors of the Chern connection we have*

$$\tilde{R}^{\tilde{\nabla}}(Z) = R^{\nabla}(Z) + i dd^c f \cdot Z. \tag{31}$$

Proof. Beginning with Lemma 4.2 a straightforward calculation gives

$$\tilde{R}^{\tilde{\nabla}}_{XY} Z = R^{\nabla}_{XY} Z + 2\left(X(Y^{1,0} f) - Y(X^{1,0} f) - [X, Y]^{1,0} f\right) \cdot Z. \qquad \square$$

Corollary 4.4. *The conformal variations of the three Ricci forms are given by*

$$\tilde{\rho} = \rho - n \cdot dd^c f, \tag{32}$$

$$\tilde{r} = r - \Lambda(dd^c f) \cdot F, \tag{33}$$

$$\tilde{\sigma} = \sigma - dd^c f. \tag{34}$$

Proof. Compute the three Ricci forms of the tensor $dd^c f \otimes F$. □

Corollary 4.5. *The conformal variations of the scalar curvatures are*

$$e^{2f}\tilde{s}^C = s^C - n\Lambda(dd^c f), \tag{35}$$

$$e^{2f}\tilde{s} = s - \Lambda(dd^c f). \tag{36}$$

Lemma 4.6. *For $f \in C^\infty(M)$ real we have*

$$-\Lambda(dd^c f) = \Delta^g(f) + g(\theta, df). \tag{37}$$

Here, Δ^g denotes the Hodge–de Rham operator $\Delta^g(f) = \delta^g df$.

Proof. Let $(e_i)_{i=1,\dots,m}$ by a J-adapted orthonormal frame as in (1). Then

$$\begin{aligned}\Lambda(dd^c f) &= \frac{1}{2}\sum_{i=1}^{m} (dd^c f)(e_i, Je_i) \\ &= \sum_{i=1}^{m} e_i\left(d^c f(Je_i)\right) - d^c f(JD^g_{e_i} e_i) - (d^c f)\left((D^g_{e_i} J)e_i\right) \\ &= -\Delta^g(f) - g(\theta, df).\end{aligned}$$

In the last equality we have used $\theta = J\delta^g F = -\sum_{i=1}^{m} g(J(D^g_{e_i} J)e_i, \cdot)$, which is a straightforward consequence of $D^g g = 0$ and $F = g(J\cdot, \cdot)$. □

Corollary 4.7. *For the Hermitian scalar curvature of $\tilde{g} = e^{2f} g$ we have*

$$e^{2f}\tilde{s}^H = s^H + m\Delta^g(f) + mg(\theta, df). \tag{38}$$

When J is integrable we recover [27, (23)].

Theorem 4.8. *Let (M, J, g, F) be a closed almost Hermitian manifold of real dimension $m = 2n$. Assume that g is conformally almost Kähler. Then*

$$\int_M (s^H - s^g)\frac{F^n}{n!} \geq 0.$$

Equality holds precisely when J is integrable and (J, g, F) is Kähler.

Proof. Suppose $(J, \tilde{g} = e^{2f}g)$ is almost Kähler. Then by (26) we have

$$\tilde{s}^H - \tilde{s}^{\tilde{g}} = 2|N|^2_{\tilde{g}} = 2e^{-2f}|N|^2_g$$

Since $\tilde{F} = e^{2f}F$ is closed, we have

$$0 = d(e^{2f}F) = e^{2f}\left((dF)_0 + \left(2df + \frac{\theta}{n-1}\right) \wedge F\right)$$

for the trace-free part $(dF)_0$. From this we read off the torsion 1-form

$$\theta = (2-m)df.$$

Putting this into (38) and combining with the formula for the conformal variation of the Riemannian scalar curvature (see Besse [12, Theorem 1.159]) we get

$$e^{2f}\left(\tilde{s}^H - \tilde{s}^{\tilde{g}}\right) = (s^H - s^g) + (2-m)\Delta f - (m-2)|df|^2. \tag{39}$$

Hence

$$\int_M 2|N|^2_g \frac{F^n}{n!} = \int_M (s^H - s^g)\frac{F^n}{n!} - (m-2)\int_M |df|^2_g \frac{F^n}{n!}.$$ □

5. Conformally constant Hermitian scalar curvature metrics

We shall be concerned with the existence of the following type of metrics:

Definition 5.1. An almost Hermitian metric (J, g, F) has *conformally constant Hermitian scalar curvature* (ccHsc) if for some $f \in C^\infty(M)$ the structure $(J, \tilde{g}, \tilde{F}) := (J, e^{2f}g, e^{2f}F)$ has $\tilde{s}^H = \text{const}$.

In Corollary 5.10 we prove a sufficient criterion for (J, g, F) to have conformally constant Hermitian scalar curvature (non-positive fundamental constant). This can be regarded as a generalization of the Chern–Yamabe problem [1] to the non-integrable case. Thus the problem is divided into the cases $C(J, [g]) \leq 0$ and $C(J, [g]) > 0$ according to the fundamental constant. The positive case in the Chern–Yamabe problem is difficult because (38) looses its nice analytic properties stemming from the maximum principle. The question remains open in this case. Restricting to symplectic manifolds, we shall consider instead the following more basic existence problem:

Existence Problem 5.2. *Let (M, ω) be a closed symplectic manifold. Does M admit* any *ω-compatible almost complex structure J such that (J, g, ω) has conformally constant Hermitian scalar curvature?*

Remark 5.3. We allow conformal variations because if we fix ω the existence of compatible metrics with $s^H = \text{const}$ is not guaranteed: sometimes one cannot find an *extremal* Kähler metric [16] and for instance on toric manifolds the existence of extremal Kähler metrics is conjecturally equivalent to the existence of extremal almost-Kähler metrics (see [22] and also for example [4, 5, 8, 35, 42, 47, 49]).

Chern–Yamabe Problem 5.4. *Given a closed almost Hermitian manifold M, find a conformal structure $(J, e^{2f} g, e^{2f} F)$ of constant Hermitian scalar curvature. In other words, does every (J, g, F) have ccHsc?*

As shown in (38), the Hermitian scalar curvature transforms by the same formula as in the integrable case. Here we show how to extend the main results of [1] to the non-integrable case, as well as some results of independent interest. We mention also the work [18] where a similar problem for the J-scalar curvature is studied, which is derived from the Riemannian curvature.

Recall that Gauduchon showed in [26] that every conformal class $[g]$ has a natural base-point $g_0 = e^{-2f_0} g$. It is characterized by having a co-closed torsion 1-form θ_0, once we normalize g_0 to unit volume. In terms of the *complex Laplacian*

$$L^g(f) := \Delta^g f + g(\theta, df),$$

this is equivalent to $(L^g)^* e^{(2-m)f_0} = 0$ and $\int_M e^{-mf_0} \frac{F^n}{n!} = 1$.

Definition 5.5. $(J, g_0 := e^{-2f_0} g, F_0 := e^{-2f_0} F)$ is the *Gauduchon metric* in the conformal class $[g]$. The *fundamental constant* is (see [1, 6, 10, 25])

$$C(M, J, [g]) := \int_M e^{(2-m)f_0} s^H \frac{F^n}{n!} \overset{(38)}{=} \int_M s_0^H \frac{F_0^n}{n!}. \tag{40}$$

In the Hermitian setting, the fundamental constant plays a central role in the Plurigenera Theorem [25] and is closely related to the Kodaira dimension. The different cases in the Chern–Yamabe problem are $C < 0$, $C = 0$, and $C > 0$.

Recall that the *Yamabe constant* is defined in terms of the Riemannian structure

$$Y[g] := \inf \left\{ \int_M s^{\tilde{g}} \operatorname{vol}_{\tilde{g}} \;\middle|\; \tilde{g} = e^{2f} g,\ \int_M \operatorname{vol}_{\tilde{g}} = 1 \right\}.$$

We remark that Yamabe, Trudinger, Aubin, and Schoen have shown that $[g]$ contains metrics of constant Riemannian scalar curvature $Y[g]$ (see [39] for a full account). From Theorem 3.2 we immediately get (see [6] in the integrable case):

Proposition 5.6. *In dimension $2n = 4$ we have the estimate*

$$Y[g] \leq C(M^4, J, [g])$$

with equality if and only if the Gauduchon metric (J, g_0, F_0) is Kähler of constant scalar curvature.

Proposition 5.7. *Let (M^m, J, g, F) be a closed almost Hermitian manifold. Then there exists a conformal metric $\tilde{g} \in [g]$ whose Hermitian scalar curvature has the same sign as C at every point (meaning zero when $C = 0$).*

Proof. The adjoint of the complex Laplacian L^{g_0} of the Gauduchon metric g_0 is $\Delta^{g_0} f - g_0(\theta_0, df)$, where we use $\delta^{g_0}\theta_0 = 0$. By the maximum principle $\ker(L^{g_0})^*$ are the constant functions (for more details, see [26]). Hence the equation

$$L^{g_0} f = C(J, [g]) - s^H_{g_0}$$

is solvable for f, since the right hand side is orthogonal to the constants. Defining $\tilde{g} = e^{2f} g_0$, equation (38) shows $\tilde{s}^H = e^{-2f} C(J, [g])$. □

Remark 5.8. This generalizes [1, Theorem 3.1] to the non-integrable case. It follows that the Chern–Yamabe problem is solvable when $C = 0$. The same conclusion (and same proof) holds for the third scalar curvature, where C is replaced by the integral of the third scalar curvature of the Gauduchon metric g_0.

A simple adaption of the argument given by Angella–Calamai–Spotti for [1, Theorem 4.1] gives the following statement:

Theorem 5.9 ([1]). *Let (M^m, J, g, F) be a closed almost Hermitian manifold, and let $S\colon M \to \mathbb{R}$ be any strictly negative smooth function (not necessarily the scalar curvature). Then the PDE*

$$mL^g(f) + S = \lambda e^{2f} \tag{41}$$

has a solution $(\lambda, f) \in \mathbb{R} \times C^\infty(M)$; in fact we must have $\lambda < 0$. The solution is unique up to replacing (λ, f) by $(\lambda e^{-2c}, f + c)$ for a constant c. Thus by scaling we may solve (41) for any given negative λ.

Proof. This is proven in [1, p. 11] by the continuity method. To apply their argument it is only important to see that for any solution f of (41) we have $\lambda < 0$. In our general setting, this follows since putting the formula $L^g(f) = e^{-2f_0} L^{g_0}(f)$ into (41) and integrating gives

$$\underbrace{m\int_M (\Delta^{g_0} f + g_0(\theta_0, df))\,\frac{F_0^n}{n!}}_{=0} + \underbrace{\int_M S e^{2f_0}\frac{F_0^n}{n!}}_{<0} = \underbrace{\lambda\int_M e^{2(f+f_0)}\frac{F_0^n}{n!}}_{>0}.$$

□

Combining this with Proposition 5.7 and (38) we thus obtain the following generalization of [1, Theorem 4.1]:

Corollary 5.10. *Every closed almost Hermitian manifold with $C(J, [g]) \leq 0$ has ccHsc (see also Remark 5.8).*

Remark 5.11. We refer also to [11] by Melvyn Berger for a related question. When (J, g) is Kähler (or more generally when $[g]$ is a balanced conformal class) he essentially constructs solutions of (41) where S is the Hermitian scalar curvature of (J, g) and λ is a given non-positive function.

6. Ruled manifolds

We begin our study of the Existence Problem 5.2 for positive fundamental constant with ruled manifolds. On complex manifolds, $C(J, [g]) > 0$ implies Kodaira dimension $-\infty$, by the Gauduchon Plurigenera Theorem [25]. The Kodaira dimension of ruled manifolds is $-\infty$ (conversely, this however does not imply $C(J, [g]) > 0$).

Angella–Calamai–Spotti [1, Section 5] have given first simple examples of Hermitian non-Kähler manifolds of positive constant Hermitian scalar curvature (for instance on the Hopf surface or abstractly by deformations using the implicit function theorem). In this section, we demonstrate the existence of almost Hermitian non-Kähler metrics of positive constant Hermitian scalar curvature on ruled manifolds (see [50] in the case of *extremal* Kähler metrics). We mention also Hong's work [33], which is different in that only the Kähler *class* is fixed.

6.1. The generalized Calabi construction

Let us briefly review the construction. The reader may consult [3–5] for more details and greater generality.

Let (S, ω_S) be a symplectic manifold. For a torus T with Lie algebra $\mathfrak{t}$ let $\pi\colon Q \to S$ be a principal T-bundle with connection $\theta \in \Omega^1(Q; \mathfrak{t})$. Assume

$$d\theta = p \cdot \pi^* \omega_S \tag{42}$$

for some fixed $p \in \mathfrak{t}$. Let (V, g_V, ω_V) be a toric almost Kähler manifold for the same torus and moment map $\mu\colon V \to \Delta \subset \mathfrak{t}^*$. Pick $c \in \mathbb{R}$ with (here $\langle , \rangle$ is evaluation)

$$P(w) \coloneqq \langle w, p\rangle + c > 0 \qquad \forall w \in \Delta. \tag{43}$$

Thus P is a positive function on the Delzant polytope Δ.

Given this data, the generalized Calabi construction determines a symplectic structure ω_M on the total space of the associated bundle

$$M \coloneqq Q \times_T V \xrightarrow{\pi} S. \tag{44}$$

On the free stratum $V^0 = \mu^{-1}(\Delta^0)$ over the interior of the Delzant polytope let $\alpha\colon TV^0 \to \mathfrak{t}$ be the g-orthogonal projection onto the orbits. The linear map

$$T_p Q \times T_v V^0 \to \mathfrak{t}, \quad (X, Y) \mapsto \theta(X) + \alpha(Y)$$

is invariant under the torus action and thus induces a 1-form θ^0 on $M^0 \coloneqq Q \times_T V^0$. The moment map factors over the projection to $\mu\colon M \to \Delta$. Set

$$\omega_M = P(\mu)\pi^*\omega_S + \langle d\mu \wedge \theta^0 \rangle. \tag{45}$$

Generally, the g-orthogonal projection α is a map $T_v V \to \mathfrak{t}/\mathfrak{t}_v$ up to the isotropy Lie algebra $\mathfrak{t}_v$. Since $d\mu_v^\xi$ vanishes for $\xi \in \mathfrak{t}_v$ the definition of $\langle d\mu \wedge \theta^0 \rangle$ naturally extends so that ω_M is also defined over all of M. (42) implies that ω_M is closed.

When S has an almost Kähler metric (J_S, g_S, ω_S) we get an almost Kähler metric on M as follows. Let $\mathbf{G}$ be the metric on $\Delta^0 \subset \mathfrak{t}^*$ that turns μ into a Riemannian submersion, let $\mathbf{H}$ be the dual metric on the cotangent bundle of Δ^0. Precomposing with μ we obtain pairings $\mathbf{G}_p \colon \mathfrak{t}^* \otimes \mathfrak{t}^* \to \mathbb{R}$, $\mathbf{H}_p \colon \mathfrak{t} \otimes \mathfrak{t} \to \mathbb{R}$ at each $p \in M$. Then

$$g_M \coloneqq P(\mu)\pi^* g_S + \mathbf{G}(d\mu \otimes d\mu) + \mathbf{H}(\theta^0 \otimes \theta^0). \tag{46}$$

Remark 6.1. The metric on V is determined by $\mathbf{G}$: recall that every metric $\mathbf{G}$ on Δ^0 subject to appropriate boundary conditions (see [3, Proposition 1] or (47), (48) below) compactifies to an ω_V-compatible almost complex structure g_V. Recall also that, up to symplectomorphism, any metric on V arises this way [3, Lemma 3].

6.2. Ruled manifolds

We now restrict to $T = S^1$. We shall say that a Hermitian line bundle L with connection has *degree* $p \in \mathbb{R}$ if $R^L = p\omega_S$ for the curvature.

Remark 6.2. Modifying ω_S slightly, such line bundles always exist for closed S. Indeed, an arbitrary small perturbation of ω_S is a symplectic form that represents a rational cohomology class, so some $q\omega_S$ with $q \in \mathbb{Q}$ represents an integer cohomology class (see [29, Observation 4.3]). The corresponding Kostant–Souriau line bundle has the required properties, with $p = 1/q$. Another important class of examples is when S is a Riemann surface. Here, holomorphic line bundles are determined by their degree $p \in \mathbb{Z}$ with $c_1(L) = p[\omega_S]$. Using the $\partial\bar{\partial}$-Lemma we find a Hermitian connection whose curvature 2-form is precisely $p\omega_S$.

Let $V = \mathbb{C}P^1$ with Fubini–Study symplectic form ω_{FS} and Delzant polytope $\Delta = [0, 1]$. As in (43) choose c with $P(x) \coloneqq px + c$ positive on $[0, 1]$.

Definition 6.3. The *ruled manifold* belonging to $(L \to S, c)$ is $M \coloneqq \mathbb{P}(L \oplus \mathbb{C})$ equipped with the symplectic form $\omega_{M,c}$ from (45).

Hence M is obtained by compactifying each fiber of L to a sphere. Following [5] we assume also that the base S has constant Hermitian scalar curvature.

6.3. Existence problem

Since the scalar curvature is S^1-invariant, it is our strategy to consider only conformal variations $u = \varphi \circ \mu$ for $\varphi\colon [0,1] \to \mathbb{R}^+$.

Theorem 6.4. *Let $(M^m = \mathbb{P}(L \oplus \mathbb{C}), \omega_{M,c})$ be a ruled manifold over a closed Kähler manifold (S^{m-2}, g_S, ω_S) of constant positive scalar curvature.*

Choose $a, b > 0$ and let $u = a\mu + b$. Rescaling the volume of S, for c sufficiently large there existsa compatible S^1-invariant Kähler metric g on M so that $\tilde{g} := u^{-2}g$ has (positive) constant Hermitian scalar curvature (for $\tilde{\omega} = u^{-2}\omega_{M,c}$).

This solves the Existence Problem 5.2 on ruled manifolds. For $n = 2$ they are examples of almost Hermitian manifolds of positive fundamental constant (see Proposition 6.6) which are not covered by the results of the previous section. Instead of rescaling the base one may also change the Fubini–Study form on the fibers by a fixed factor.

Remark 6.5. By Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman, ruled manifolds with c sufficiently large also admit an extremal metric [5, Theorem 4].

As recalled above in Remark 6.1, compatible S^1-invariant metrics on $(\mathbb{C}P^1, \omega_{\mathrm{FS}})$ correspond to smooth functions $\mathbf{H}\colon [0,1] \to \mathbb{R}$ satisfying the boundary conditions

$$\begin{aligned} \mathbf{H}(0) &= \mathbf{H}(1) = 0, \\ \mathbf{H}'(0) &= 2 = -\mathbf{H}'(1), \end{aligned} \tag{47}$$

$$\mathbf{H}(x) > 0 \quad (0 < x < 1). \tag{48}$$

Then $\mathbf{H}$ and g_S determine an ω_M-compatible almost Kähler metric g_M via (46). The metric g we seek will be of the form (46) and is hence determined by a function $\mathbf{G}^{-1} := \mathbf{H}$ satisfying (47), (48). Let $t\colon V^0 \to S^1$ be the angle coordinate on the round sphere $\mathbb{C}P^1 \setminus \{N, S\}$. Let (x^i) be local coordinates on S over which L is trivialized. The connection then corresponds to a local 1-form $A = A_i dx^i$ on the base. We have local coordinates (x^i, μ, t) on M^0 in which the induced 1-form can be written $\theta^0 = A + dt$. From (45) we then get the volume form

$$\mathrm{vol}_M = \frac{\omega_{M,c}^n}{n!} = \frac{1}{n} P(\mu)^{n-1} \,\mathrm{vol}_S \wedge d\mu \wedge dt. \tag{49}$$

According to (46), the local expression for the metric g is:

$$\big[P(\mu)g^S_{ij} + \mathbf{H}(\mu)A_i(\mu)A_j(\mu)\big]dx^i dx^j \\ + \mathbf{G}(\mu)d\mu d\mu + \mathbf{H}(\mu)dtdt + 2\mathbf{H}(\mu)A_i(x)dx^i dt$$

From this we see $d\mu^{\sharp_g} = \mathbf{H}(\mu)\frac{\partial}{\partial\mu}$. Putting this into the formula

$$d(i_{\mathrm{grad}_g f}\,\mathrm{vol}_M) = -\Delta^g(f)\,\mathrm{vol}_M$$

we get for the Hodge–de Rham Laplacian of the moment map

$$\Delta^g(\mu) = -\frac{(P^{n-1}\mathbf{H})'(\mu)}{P(\mu)^{n-1}}. \tag{50}$$

Note also the general formula $\Delta^g(\varphi\circ\mu) = \varphi'(\mu)\Delta^g\mu - \varphi''(\mu)|d\mu|^2_g$.

We shall refer the analytic part of the proof to the next subsection.

Proof of Theorem 6.4. According to Proposition 6.9 below with $\lambda := b/a$ we find unique $b_1, b_2 > 0$ and $f \in C^\infty([\lambda, 1+\lambda])$ strictly positive on the interior satisfying (56), (57). Rescaling the volume, we assume that b_2 is the scalar curvature of S.

By [5, Lemma 9] the Hermitian scalar curvature of $(M, g, \omega_{M,c})$ is (omitting the argument μ and where the derivatives are taken as functions of $x \in [0,1]$)

$$s^H = \frac{b_2}{P} - \frac{(P^{n-1}\mathbf{H})''}{P^{n-1}}. \tag{51}$$

Combining (38), (50), and (51) we get for $\tilde{g} = u^{-2}g$, where $u = \varphi\circ\mu$:

$$\tilde{s}^H = \varphi^2\frac{b_2}{P} - \varphi^2\frac{(P^{n-1}\mathbf{H})''}{P^{n-1}} + m\varphi\varphi'\frac{(P^{n-1}\mathbf{H})'}{P^{n-1}} + m\left(\varphi\varphi'' - (\varphi')^2\right)\mathbf{H}$$

If $\varphi(x) = ax + b$ and defining $f(x+\lambda) = P(x)^{n-1}\mathbf{H}(x)$ we see from (56) that we have found a solution $\mathbf{H}$ to this equation where $\tilde{s}^H = a^2 b_1$ which is clearly positive. Condition (47) is (57) and (48) is just the positivity of f. □

Proposition 6.6. *Let $M = P(L\oplus\mathbb{C})$ with the Kähler metric g from Theorem* 6.4*. When $n = \dim_{\mathbb{C}} M = 2$ the fundamental constant is*

$$C(M, J, [g]) = \frac{2s^H_S + 8c + 4p}{2c + p}. \tag{52}$$

Proof. The coarea formula applied to the submersion $\mu\colon M \to [0,1]$ and volume form $\omega_M^n/n!$ shows that for measurable $f\colon [0,1] \to \mathbb{R}$ we have (see also [5, p. 17])

$$\int_M (f\mu)\frac{\omega_M^n}{n!} = \int_0^1 f(x)\,\mathrm{vol}(\mu^{-1}x)dx. \tag{53}$$

On $\mu^{-1}(x)$ the form (45) restricts to $P(x)\pi^*\omega_S$, hence $\mathrm{vol}(\mu^{-1}x) = P(x)^{n-1}\,\mathrm{vol}(B)$. Without loss we may suppose $\mathrm{vol}(B) = 1$. The Kähler metric (ω_M, g) is Gauduchon and normalizing (40) to unit volume gives

$$C(M, J, [g]) = \frac{1}{\mathrm{vol}(M)} \int_M s^H \frac{\omega_M^n}{n!}$$

Recall $P(x) = px + c$, $n = 2$, and $P(x)\mathbf{H}(x) = f(x+\lambda)$. Evaluate using (53)

$$\begin{aligned}
\mathrm{vol}(M) &= \int_M \frac{\omega_M^2}{2!} = \int_0^1 P(x)dx = \frac{(p+c)^2 - c^2}{2p} \\
\int_M s^H \frac{\omega_M^2}{2!} &\overset{(51)}{=} \int_0^1 \frac{s_S^H - (P\mathbf{H})''(x)}{P(x)}\,\mathrm{vol}(\mu^{-1}x)dx \\
&= \int_0^1 \left[s_S^H - (P\mathbf{H})''(x)\right] dx \\
&= s_S^H - f'(x+\lambda)\big|_0^1 = s_S^H + 2(p+c) + 2c
\end{aligned}$$

using (47), which corresponds to (57) when $P(x)\mathbf{H}(x) = f(x+\lambda)$. □

6.4. Analytic part

We first describe the general method for obtaining the main result of this section. Let $a_n, \dots, a_0 \in C^\infty(I)$ and $Q_c^1, \dots, Q_c^m \in C^\infty(I)$ depending on a parameter $c \in \mathbb{R}$ be smooth functions on a fixed interval I. For any fixed c sufficiently large we wish to solve

$$\begin{aligned}
&a_n(x)f^{(n)}(x) + \cdots + a_1(x)f^{(1)}(x) + a_0(x)f(x) \\
&= b_1 Q_c^1(x) + \cdots + b_m Q_c^m(x)
\end{aligned} \tag{54}$$

with $n+m$ initial values (which may also depend on c) uniquely for $f \in C^\infty(I)$ and $b = (b_1, \dots, b_m) \in \mathbb{R}^m$ and to understand the asymptotic behaviour in c.

Suppose that for all b, c there exists a particular solution $f_{b,c}^\circ$ of (54). Since b appears linearly in (54) we may arrange also that $f_{b,c}^\circ$ is a linear function of b. Let $f_1, \dots, f_n$ be the n independent solutions of the homogeneous ODE corresponding to (54). Then the linear map

$$F_c \colon \mathbb{R}^{n+m} \to C^\infty(I), \quad (b,d) \mapsto f^\circ_{b,c} + d_1 f_1 + \dots + d_n f_n$$

parameterizes all solutions of (54) with right-hand side fixed by b, c.

Let the initial values be prescribed by a linear map $i \colon C^\infty(I) \to \mathbb{R}^{n+m}$ and a family of vectors $v_c \in \mathbb{R}^{n+m}$, $c \in \mathbb{R}$ (so that the initial value problem becomes (54) with $i(f) = v_c$). The problem above of finding solutions with given initial values can then be formulated as follows: prove that the endomorphism $M_c = i \circ F_c$ of $\mathbb{R}^{n+m}$ is invertible for c large and understand the asymptotics of $M_c^{-1}(v_c)$.

This is done with the following lemma. Here $O(c^n)$ stands for any Laurent polynomial in c of degree $\leq n$ with coefficients in $C^\infty(I)$, i.e. an expression

$$\varphi_n(x)c^n + \cdots \varphi_k(x)c^k, \qquad n \geq k \in \mathbb{Z}, \quad \varphi_j \in C^\infty(I). \tag{55}$$

Remark 6.7. We will use this notation to describe asymptotic behaviour. Note that it is very restrictive, since it requires the dependence on c to be a Laurent polynomial. Note also that the x-derivative of an expression (55) is of the same form. Thus $\frac{d}{dx}O(c^n) = O(c^n)$.

Lemma 6.8. *Let $M_c = (M_c^1, \dots, M_c^N) \in \mathbb{R}^{N \times N}$ be a matrix whose columns are functions of $c \in \mathbb{R}$ having the asymptotic behaviour $M_c^j = M_\infty^j c^{n_j} + O(c^{n_j - 1})$. Then for $M_\infty = (M_\infty^1, \dots, M_\infty^N)$*

$$\det(M_c) = \det(M_\infty) c^{n_1 + \cdots + n_N} + O(c^{n_1 + \cdots + n_N - 1}).$$

In particular, when M_∞ is invertible it follows that M_c is invertible for c sufficiently large. Assume this and $v_c = v_\infty c^m + O(c^{m-1}) \in \mathbb{R}^N$ and let x_c and x_∞ be the respective solutions of $M_c x_c = v_c$ and $M_\infty x_\infty = v_\infty$. Then for the j-th entry

$$x_c^j = x_\infty^j c^{m - n_j} + O(c^{m - n_j - 1}).$$

In particular, the unique solutions x_c are again Laurent polynomials in c.

This follows from elementary properties of the determinant (Cramer's rule). The main result of this section is as follows. It completes the proof of Theorem 6.4.

Proposition 6.9. *For any* $m = 2n \in \mathbb{N}$, $\lambda > 0$, *and* $P(x) = px + c$ *with* c *sufficiently large there exists a unique solution* $(b_1, b_2, f) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times C^\infty([\lambda, 1+\lambda])$ *of*

$$\begin{aligned}(56) \qquad & x^2 f''(x) - mxf'(x) + mf(x) \\ &= -b_1 \cdot P(x-\lambda)^{n-1} + b_2 \cdot x^2 \cdot P(x-\lambda)^{n-2}\end{aligned}$$

with initial values

$$(57) \quad f(\lambda) = f(1+\lambda) = 0 \quad f'(\lambda) = 2c^{n-1} \quad f'(1+\lambda) = -2(p+c)^{n-1}.$$

Moreover, f *is strictly positive on* $]\lambda, 1+\lambda[$.

Care must be taken to prove the positivity since near λ the solution functions f_c for $c \to \infty$ could oscillate around zero into the negative, even if the limiting function f_∞ is strictly positive on the interior.

Proof. We apply the method above with $I = [\lambda, 1+\lambda]$, $Q_c^1(x) = -(px - p\lambda + c)^{n-1}$, $Q_c^2(x) = x^2(px - p\lambda + c)^{n-2}$ and

$$i(f) = (f(\lambda), f(1+\lambda), f'(\lambda), f'(1+\lambda)), \quad v_c = (0, 0, 2c^{n-1}, -2(p+c)^{n-1})$$

First we need to determine the general solution for fixed b, c. The homogeneous equation corresponding to (56) has solutions x, x^m. Applying 'reduction of order' [48, p. 242] we obtain all solutions of (56)

$$\begin{aligned}(58) \qquad F_c(b,d)(x) = & \frac{b_1 x}{m-1} \int \frac{(px - p\lambda + c)^{n-1}}{x^2} \\ & - \frac{b_1 x^m}{m-1} \int \frac{(px - p\lambda + c)^{n-1}}{x^{m+1}} \\ & - \frac{b_2 x}{m-1} \int (px - p\lambda + c)^{n-2} \\ & + \frac{b_2 x^m}{m-1} \int \frac{(px - p\lambda + c)^{n-2}}{x^{m-1}} + d_1 x + d_2 x^m.\end{aligned}$$

where we fix choices of primitives (say by integrating from λ to x).

Let $M_c = i \circ F_c \colon \mathbb{R}^4 \to \mathbb{R}^4$, regarded as a matrix in the standard basis. The matrix entries are simple to write down from (58), but very long and not informative. Ultimately, we wish to show that M_c is invertible for c sufficiently large and according to Lemma 6.8 it suffices for this to understand the asymptotic behaviour. We have

$$f(x) = b_1\left(\frac{-c^{n-1}}{m} + O(c^{n-2})\right) + b_2\left(\frac{-c^{n-2}}{m-2} + O(c^{n-3})\right)x^2 + d_1x + d_2x^m \tag{59}$$

and using Remark 6.7 this implies

$$\begin{aligned} f'(x) &= b_1O(c^{n-2}) + b_2O(c^{n-3})2x \\ &\quad + b_2\left(\frac{-c^{n-2}}{m-2} + O(c^{n-3})\right)x^2 + d_1 + md_2x^{m-1} \\ &= b_1O(c^{n-2}) + b_2\left(\frac{-c^{n-2}}{m-2} + O(c^{n-3})\right) + d_1 + md_2x^{m-1} \end{aligned}$$

Hence

$$M_c = \begin{pmatrix} \frac{-c^{n-1}}{m} + O(c^{n-2}) & \frac{-c^{n-2}}{m-2}\lambda^2 + O(c^{n-3}) & \lambda & \lambda^m \\ \frac{-c^{n-1}}{m} + O(c^{n-2}) & \frac{-c^{n-2}}{m-2}(1+\lambda)^2 + O(c^{n-3}) & 1+\lambda & (1+\lambda)^m \\ O(c^{n-2}) & \frac{-c^{n-2}}{n-1}\lambda + O(c^{n-3}) & 1 & m\lambda^{m-1} \\ O(c^{n-2}) & \frac{-c^{n-2}}{n-1}(1+\lambda) + O(c^{n-3}) & 1 & m(1+\lambda)^{m-1} \end{pmatrix}.$$

Also $v_c = (0,0,2,-2)c^{n-1} + O(c^{n-2})$. The matrix

$$M_\infty = \begin{pmatrix} \frac{-1}{m} & \frac{-\lambda^2}{m-2} & \lambda & \lambda^m \\ \frac{-1}{m} & \frac{-(1+\lambda)^2}{m-2} & 1+\lambda & (1+\lambda)^m \\ 0 & \frac{-\lambda}{n-1} & 1 & m\lambda^{m-1} \\ 0 & \frac{-(1+\lambda)}{n-1} & 1 & m(1+\lambda)^{m-1} \end{pmatrix}$$

is invertible. The equation $M_\infty \cdot x_\infty = (0,0,2,-2)^{\mathrm{T}}$ has the solution

$$x_\infty = \big(2m\lambda(1+\lambda), 2(m-2), 2(1+2\lambda), 0\big).$$

Therefore Lemma 6.8 implies that M_c is invertible for c sufficiently large and also that we have the asymptotic behaviour

$$\begin{aligned} b_1 &= 2m\lambda(1+\lambda) + O(c^{-1}), & b_2 &= 2(m-2)c + O(c^0), \\ d_1 &= 2(1+2\lambda)c^{n-1} + O(c^{n-2}), & d_2 &= O(c^{n-2}). \end{aligned}$$

Putting this into (59) shows

$$f_c(x+\lambda) = 2c^{n-1}x(1-x) + O(c^{n-2}).$$

Hence

$$\frac{f_c'(x+\lambda)}{2c^{n-1}} = 1 - 2x + O(c^{-1})$$

uniformly in x.

It follows that for c sufficiently large f_c has precisely one extreme point on $[\lambda, 1+\lambda]$ (close to $1/2+\lambda$). From (57) we see that this must be a maximum and hence f_c is positive on $[\lambda, 1+\lambda]$. □

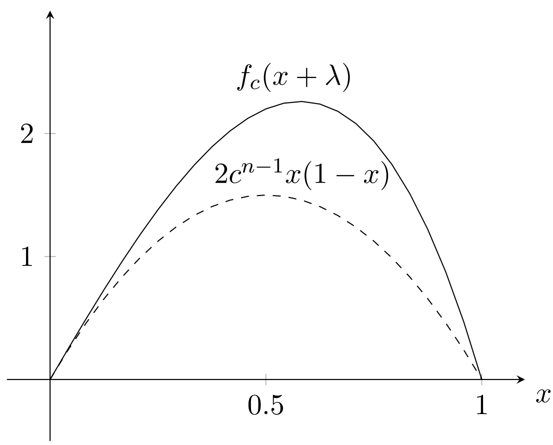

Figure 1: Plot of the solution f_c for $p=3, \lambda=\frac{1}{2}, m=4, c=3$ and of the 'ideal solution' at infinity

7. Moment map setup

In this section we give a moment map interpretation of the Existence Problem 5.2 inspired by the work of Apostolov and Maschler [9]. This leads to the familiar existence and uniqueness conjectures formulated in terms of geometric invariant theory, as well as to a version of the Futaki invariant. Our method applies to closed symplectic manifolds (M^m, ω) admitting a symmetry given by a Hamiltonian vector field, meaning we assume $\mathcal{AC}^f(\omega) \neq \emptyset$ (see Definition 7.2 below). An example is a manifold with Hamiltonian circle action, as above.

First we recall the general definition ($\mathfrak{g}^*$ gets the coadjoint action):

Definition 7.1. A symplectic action of a Lie group on a symplectic manifold (S, ω) is *Hamiltonian* if there exists a G-equivariant *moment map* $\mu\colon X \to \mathfrak{g}^*$ with

$$d\mu^\xi(X) = \omega(X, \xi^*) \qquad \forall \xi \in \mathfrak{g}, X \in TS. \tag{60}$$

(write $\mu^\xi = \mu(-)(\xi) \in C^\infty(S)$ and let $\xi^* \in \mathfrak{X}(S)$ be the infinitesimal action.)

7.1. The action

For fixed $f \in C^\infty(M)$ with $\int_M f \operatorname{vol} = 0$ let

$$u := e^{-nf}, \qquad K := \operatorname{grad}_\omega u. \tag{61}$$

Then $\mathfrak{L}_K\omega = \iota_K d\omega + d\iota_K\omega = 0 + ddu = 0$.

Definition 7.2. Let K be a Hamiltonian vector field on (M, ω) as in (61). By definition $\mathcal{AC}^f(\omega)$ is the Fréchet manifold of ω-compatible almost complex structures J satisfying $\mathfrak{L}_K J = 0$.

We equip $\mathcal{AC}^f(\omega)$ with the symplectic form [21, 23]

$$\mathbf{\Omega}_J(A, B) = \frac{1}{2}\int_M \operatorname{tr}(J \circ A \circ B)e^{nf} \operatorname{vol}, \qquad A, B \in T_J\mathcal{AC}^f(\omega). \tag{62}$$

Proposition 7.3. *When non-empty, $\mathcal{AC}^f(\omega)$ is contractible. This is the case if and only if K is a Killing vector field for some metric g on M.*

Proof. Letting $\mathfrak{Met}^f(M)$ denote the space of all K-invariant Riemannian metrics, the usual map restricts to a retraction

$$\mathfrak{Met}^f(M) \to \mathcal{AC}^f(\omega), \quad g \mapsto J_g, \text{ where } J_g := A_g|A_g|^{-1}, \ g(A_g-,-) := \omega$$

on the convex space of metrics g satisfying $\mathfrak{L}_K g = 0$.

To see that J_g is an almost complex structure, note that A_g is skew-symmetric and that A_g commutes with $|A_g| = (A_g^*A_g)^{1/2}$ by functional calculus. Thus

$$J_g^2 = A_g|A_g|^{-1}A_g|A_g|^{-1} = A_g^2|A_g|^{-2} = A_g^2(A_g^*A_g)^{-1} = A_g^2(-A_g^2)^{-1} = -1.$$

Conversely, when $\mathcal{AC}^f(\omega)$ is non-empty, let J be an element. Then K is a Killing field for $g = \omega(\cdot, J\cdot)$ since $\mathfrak{L}_K\omega = 0$ and $\mathfrak{L}_K J = 0$. ☐

Remark 7.4. As mentioned above, we make the *assumption* that $\mathcal{AC}^f(\omega)$ is non-empty. According to Bochner, there are no non-trivial Killing fields in the case of strictly negative Ricci curvature. We have $s^H = 2|N|^2 + s^g$ and so the results of this section mainly concern the case of positive fundamental constant.

Definition 7.5. $\mathrm{Ham}^f(\omega) \subset \mathrm{Symp}(\omega)$ is the subgroup of Hamiltonian symplectomorphisms ϕ satisfying $f \circ \phi = f$ (equivalently ϕ_* preserves $\mathrm{grad}_\omega f$).

We recall that by definition the Lie algebra of the Hamiltonian symplectomorphisms are the Hamiltonian vector fields $d\varphi = -\iota_X\omega$, where $\varphi \in C^\infty(M)$. The Lie algebra $\mathfrak{ham}^f(\omega)$ consists of Hamiltonian vector fields with $[\mathrm{grad}_\omega f, X] = 0$. Let $C^\infty_f(M)$ be the space of $\varphi \in C^\infty(M)$ with constant Poisson bracket $\{f, \varphi\}$. Then $\mathfrak{ham}^f(\omega)$ is canonically identified with $C^\infty_f(M)/\mathbb{R}$. The adjoint action of $\phi \in \mathrm{Ham}^f(\omega)$ on $\varphi \in C^\infty_f(M)/\mathbb{R}$ can be written $(\phi^{-1})^*\varphi$. On $C^\infty_f(M)$ consider

$$\langle h_1, h_2\rangle_{e^{(2+n)f}} = \int_M h_1 h_2 e^{(2+n)f}\,\mathrm{vol}. \tag{63}$$

It places $C^\infty_f(M)/\mathbb{R}$ in duality with

$$C^\infty_{0,f}(M) \coloneqq \{\varphi \in C^\infty_f(M) \mid \langle \varphi, 1\rangle_{e^{(2+n)f}} = 0\}.$$

We have an isomorphism to $C^\infty_{0,f}(M) \to C^\infty_f(M)/\mathbb{R}$ with inverse

$$C^\infty_f(M)/\mathbb{R} \to C^\infty_{0,f}(M), \quad \varphi \mapsto \mathring{\varphi} \coloneqq \varphi - \frac{\langle \varphi, 1\rangle_{e^{(2+n)f}}}{\langle 1, 1\rangle_{e^{(2+n)f}}} = \varphi - \frac{\int_M \varphi e^{(2+n)f}\,\mathrm{vol}}{\int_M e^{(2+n)f}\,\mathrm{vol}}.$$

For $\varphi, \psi \in C^\infty_f(M)$ note the formula

$$\langle \mathring{\varphi}, \psi\rangle_{e^{(2+n)f}} = \langle \varphi, \mathring{\psi}\rangle_{e^{(2+n)f}}. \tag{64}$$

Since $e^{nf}\,\mathrm{vol}$ is preserved by ϕ, the action of $\mathrm{Ham}^f(\omega)$ on $\mathcal{AC}^f(\omega)$ by $\phi_* \circ J \circ \phi_*^{-1}$ preserves the symplectic form (62). We will show that it is Hamiltonian.

7.2. Technical preparations

Lemma 7.6 (see [41, Lemma 1.3]). *Let (J, g, ω) be almost Kähler. Suppose the symplectic gradient $K = \mathrm{grad}_\omega u$ is a g-Killing field. Then the J-anti-invariant part $(D^g J du^\sharp)^{J,-}$ is anti-symmetric.*

Proof. Because $\mathfrak{L}_K\omega = 0$ is automatic, the field $K = Jdu^\sharp$ is Killing precisely when it is holomorphic. Therefore, combined with the fact that D^g is torsion-free,

$$0 = \mathfrak{L}_K J = D^g_K J - [D^g K, J].$$

So $(D^g K)^{J,-} = \frac{1}{2}J[D^g K, J] = \frac{1}{2}JD^g_K J = -\frac{1}{2}D^g_{JK}J$ which is anti-symmetric. □

Consider a path $J_t \in \mathcal{AC}(\omega)$ representing $\dot{J} = \frac{d}{dt}\big|_0 J_t$. Write $g_t = \omega(\cdot, J_t\cdot)$. The variation of the scalar curvature is given by the Mohsen formula:

Proposition 7.7 (see [24, 44]). $\frac{d}{dt}\big|_0 s^H_{g_t} = -\delta J(\delta\dot{J})^\flat$.

In this formula the codifferential of an endomorphism A is defined by

$$g(\delta A, X) = \delta\langle A, X\rangle + g(A, D^g X), \quad X \in \mathfrak{X}(M) \tag{65}$$

using the evaluation pairing $\langle , \rangle$. For 1-forms α, β we note also the simple formulas

$$g(\alpha^\sharp, A(X)) = g(A^*, \alpha \otimes X), \tag{66}$$

$$\frac{d}{dt}\bigg|_0 g_t(\alpha, \beta) = -g(\alpha, \dot{J}J\beta), \tag{67}$$

which are used in the proof of our main technical lemma:

Lemma 7.8. *For the metrics $\tilde{g}_t = e^{2f}\omega(\cdot, J_t\cdot)$ and any $h \in C^\infty(M)$ we have*

$$\frac{d}{dt}\bigg|_0 \int_M s^H_{\tilde{g}_t} h e^{(2+n)f}\,\mathrm{vol} = \int_M g(\dot{J}, D^g J dh^\sharp) e^{nf}\,\mathrm{vol}. \tag{68}$$

Proof. Recall $u := e^{-nf}$. By (38) the scalar curvature of the conformal variation is

$$s^H_{\tilde{g}_t} = e^{-2f}\left(s^H_{g_t} + m\Delta^{g_t}(f)\right) = u^{2/n}s^H_{g_t} - 2u^{2/n-1}\Delta^{g_t}(u) - 2u^{2/n-2}|du|^2_{g_t}.$$

Putting this and $e^{(2+n)f} = u^{-1-2/n}$ into the left hand side of (68) gives

$$\frac{d}{dt}\bigg|_0 \int_M s^H_{g_t} h u^{-1}\,\mathrm{vol} - 2\int_M \Delta^{g_t}(u) h u^{-2}\,\mathrm{vol} - 2\int_M g_t(du, du) h u^{-3}\,\mathrm{vol}.$$

Applying Proposition 7.7 and (67) to the second and third summand we get

$$-\int_M \delta J(\delta\dot{J})^\flat h u^{-1}\,\mathrm{vol} \\ +2\int_M g(du,\dot{J}Jd(hu^{-2}))\,\mathrm{vol}+2\int_M g\left(du,\dot{J}Jdu\right)hu^{-3}\,\mathrm{vol}$$

which, in view of (65) and (66) becomes

$$\int g\left(\dot{J}, D^g Jd(hu^{-1})^\sharp + 2du\otimes Jd(hu^{-2})^\sharp + 2hu^{-3}du\otimes Jdu^\sharp\right)\mathrm{vol}\,. \tag{69}$$

Now expand using the Leibniz rule:

$$D^g Jd(hu^{-1})^\sharp = 2u^{-3}hdu\otimes Jdu^\sharp - u^{-2}du\otimes Jdh^\sharp - u^{-2}dh\otimes Jdu^\sharp \\ + u^{-1}D^g(Jdh^\sharp) - u^{-2}hD^gJdu^\sharp \tag{70}$$

$$du\otimes Jd(hu^{-2})^\sharp = u^{-2}du\otimes Jdh^\sharp - 2u^{-3}hdu\otimes Jdu^\sharp \tag{71}$$

From (66) we see $g(\dot{J}, du\otimes Jdh^\sharp) = g(\dot{J}, dh\otimes Jdu^\sharp)$. Moreover, Lemma 7.6 implies $g(\dot{J}, D^gJdu^\sharp)=0$ since $\dot{J}$ is symmetric and J-anti-invariant, while the J-anti-invariant part of $D^gJdu^\sharp$ is anti-symmetric. Inserting (70) into (69) and applying these facts then gives the right hand side of (68). □

7.3. Proof of main theorem

We write $g_{f,J} := e^{2f}\omega(\cdot, J\cdot)$ and $g_J := g_{0,J}$.

Theorem 7.9. *Let (M,ω) be a closed symplectic manifold with Hamiltonian vector field $K = \mathrm{grad}_\omega u$, $u = e^{-nf}$, $\int_M f\,\mathrm{vol} = 0$, and $\mathcal{AC}^f(\omega)\neq\emptyset$. The action of $\mathrm{Ham}^f(\omega)$ on $\mathcal{AC}^f(\omega)$ with symplectic form (62) is Hamiltonian with moment map*

$$\mu\colon \mathcal{AC}^f(\omega)\times C^\infty_{0,f}(M)\to\mathbb{R},\quad \mu^{\mathring{h}}(J) = \int_M s^H_{g_{f,J}}\mathring{h}e^{(2+n)f}\,\mathrm{vol}\,. \tag{72}$$

Here the Hermitian scalar curvature of $g_{f,J}$ is viewed as a functional using (63).

Identifying $\mathfrak{ham}^f(\omega) = C^\infty_f(M)/\mathbb{R}$ and using (64) we may rewrite

$$\mu\colon \mathcal{AC}^f(\omega)\to (C^\infty_f(M)/\mathbb{R})^*,\quad \mu(J) = \int_M \mathring{s}^H_{g_{f,J}}he^{(2+n)f}\,\mathrm{vol}\,. \tag{73}$$

Proof. We must check that for any tangent vector $\dot{J} \in T_J\mathcal{AC}^f$ and $h \in C^\infty_{0,f}(M)$

$$\mathbf{\Omega}_J(h^*_J, \dot{J}) = d\mu^h(\dot{J}).$$

Here $h^*_J = -\mathfrak{L}_Z J \in T_J\mathcal{AC}^f$ for $Z = \mathrm{grad}_\omega h$ denotes the infinitesimal action of h at the point J. In terms of the adjoint of $D^g Z \in \mathrm{End}(TM)$ with respect to $g = g_J$, the infinitesimal action can be rewritten as $h^*J = -J \circ (D^g Z + (D^g Z)^*)$ and so

$$\begin{aligned}\mathbf{\Omega}_J(h^*_J, \dot{J}) &= \frac{1}{2}\int_M \left(\mathrm{tr}(D^g Z \circ \dot{J}) + \mathrm{tr}((D^g Z)^* \circ \dot{J})\right) e^{nf}\,\mathrm{vol} \\ &= \int_M \mathrm{tr}(D^g Z \circ \dot{J}) e^{nf}\,\mathrm{vol}\,.\end{aligned}$$

This is the right hand side of (68), as $Z = Jdh^\sharp$, while the left hand side of (68) is simply $d\mu^h(\dot{J})$. From $\phi^* g_{\phi\cdot J,f} = g_{J,\phi^* f}$ we get $\phi^* s^H_{\phi\cdot J,f} = s^H_{J,\phi^* f}$. Now $f \circ \phi = f$ by Definition 7.5 and so $\phi^*\mu(\phi\cdot J) = \mu(J)$, proving that (72) is also equivariant. □

The zeros of the moment map μ are $J \in \mathcal{AC}^f(\omega)$ such that the metric $g_{f,J}$ is of constant Hermitian scalar curvature. The geometric invariant theory formal picture suggests then the existence of a unique almost-Kähler metric in $\mathcal{AC}^f(\omega)$ conformal to a constant Hermitian scalar curvautre metric, modulo the action of $\mathrm{Ham}^f(\omega)$, in every "*stable*" "*complexified*" orbit of the action of $\mathrm{Ham}^f(\omega)$.

Remark 7.10. In [9, Remark 1], the zeros of the Apostolov–Maschler moment map are metrics $g_{f,J}$ with $s^g_{g_{f,J}} + |N|^2_{g_{f,J}}$ is constant, where $s^g_{g_{f,J}}$ is the Riemannian scalar curvature of $g_{f,J}$. Our choice of the weights in the volume form $\mathbf{\Omega}$ and in the inner product (63) is motivated by metrics $g_{f,J}$ of constant Hermitian scalar curvature. In [9], the chosen weights correspond to the study of conformally Kähler Einstein-Maxwell metrics. A very general setup was studied by Lahdili in [37, 38] (we also refer to [2]).

Corollary 7.11. *Minima of* $\|\mu\|^2$ *on* $\mathcal{AC}^f(\omega)$ *are ccHsc metrics.*

It may also be of interest to consider critical points of $\|\mu\|^2$.

7.4. Futaki invariant

Moment maps lead very generally to a Futaki invariant. In the context of Definition 7.1, this invariant is associated to any Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Letting $S^{\mathfrak{h}} = \{p \in S \mid \xi_p^* = 0 \ \forall \xi \in \mathfrak{h}\}$ be the $\mathfrak{h}$-fixed points, the restriction $\mu\colon S^{\mathfrak{h}} \to \mathfrak{h}^*$ is locally constant. Assuming $S^{\mathfrak{h}}$ is connected, the common value of μ is called the Futaki invariant $\mathcal{F}^{\mathfrak{h}} \in \mathfrak{h}^*$.

Applied to our situation $\mathfrak{h} = \mathbb{R} \cdot K \subset \mathfrak{ham}^f(\omega)$ corresponding to $u \in C_f^\infty(M)/\mathbb{R}$. The $\mathfrak{h}$-fixed points are all of $\mathcal{AC}^f(\omega)$. Identify $\mathfrak{h}^* = \mathbb{R}$ by evaluating at u.

Definition 7.12. For *any* $J \in \mathcal{AC}^f(\omega)$ the *Futaki invariant* is given by

$$\begin{aligned}\mathcal{F}^f(\omega) = \mu^u(J) = \langle \mathring{s}^H_{g_{f,J}}, u\rangle_{e^{(2+n)f}} &= \int_M \mathring{s}^H_{g_{f,J}} e^{2f} \operatorname{vol} \\ &\overset{(38)}{=} \int_M s^H_{g_J} \operatorname{vol} - \frac{\int_M s^H_{g_{f,J}} e^{(n+2)f} \operatorname{vol}}{\int_M e^{(n+2)f} \operatorname{vol}} \int_M e^{2f} \operatorname{vol}.\end{aligned}$$

The main point of the Futaki invariant, that it is independent of J, is a consequence of the general moment map setup and Proposition 7.3. Indeed, if we consider a path J_t in $\mathcal{AC}^f(\omega)$ then

$$\begin{aligned}\frac{d}{dt}\mu^u(J_t) &= \frac{d}{dt}\int_M s^H_{g_{J_t}} \operatorname{vol} - \frac{\int_M e^{2f}\operatorname{vol}}{\int_M e^{(n+2)f}\operatorname{vol}} \frac{d}{dt}\int_M s^H_{g_{f,J_t}} e^{(n+2)f} \operatorname{vol}, \\ &= -\int_M \delta_t J_t (\delta_t \dot{J}_t)^{\flat_t} \operatorname{vol} \\ &\quad - \frac{\int_M e^{2f}\operatorname{vol}}{\int_M e^{(n+2)f}\operatorname{vol}} \int_M g_t(\dot{J}_t, D^{g_t} J_t d\,(1)^\sharp)\, e^{nf} \operatorname{vol} = 0.\end{aligned}$$

In the second line, we used Proposition 7.7 and Equation (68) for $h \equiv 1$.

Corollary 7.13. *Suppose that* $\mathcal{AC}^f(\omega) \neq \emptyset$ *and assume the existence of* $J \in \mathcal{AC}^f(\omega)$ *such that* $g_{f,J} = e^{2f}\omega(\cdot, J\cdot)$ *has constant Hermitian scalar curvature. Then,* $\mathcal{F}^f(\omega) = 0$.

7.5. The toric case

Let (M^{2n}, ω) be a closed symplectic manifold equipped with an effective Hamiltonian action of a n-dimensional torus T. Let $z : M \to \Delta \subset \mathfrak{t}^*$ be the moment map, where Δ is the Delzant polytope in $\mathfrak{t}^*$ the dual of $\mathfrak{t} = \mathrm{Lie}(T)$. Denote by $\{u_1, \ldots, u_d\}$ the normals to the polytope Δ. The action of the torus T is generated by a family of Hamiltonian vector fields $\{K_1, \ldots, K_n\}$ linearly independent on an open set of the $2n$-dimensional symplectic manifold (M, ω) with $\omega(K_i, K_j) = 0$. The symplectic form ω and an ω-compatible

T-invariant almost Kähler metric g are given on z^{-1} of the interior of Δ by

$$\omega = \sum_{i=1}^{n} dz_i \wedge dt_i,$$
$$g = \sum_{i,j=1}^{n} G_{ij}(z) dz_i \otimes dz_j + H_{ij}(z) dt_i \otimes dt_j + P_{ij}(z) dz_i \odot dt_j,$$

where G, H are symmetric positive definite matrix-valued functions satisfying the compatibility conditions $GH - P^2 = Id$ and $HP = P^t H$ (P^t is the transpose of P). The coordinates z_i are the moment coordinates and t_i are the angle coordinates.

Denote by $H_{ij,k} = \frac{\partial H_{ij}}{\partial z_k}$ etc. It is shown [22] and [40, (4.6)] that the Hermitian scalar curvature is given by

$$s^H = - \sum_{i,j=1}^{n} H_{ij,ij}.$$

Let $u = a_1 z_1 + a_2 z_2 + \cdots + a_n z_n + a_{n+1}$ be a Hamiltonian Killing potential (a_i are real numbers). Then,

$$Jdu = \sum_{i,l=1}^{n} a_i P_{li} dz_i + a_i H_{il} dt_l.$$

Hence,

$$dJdu = \sum_{i,l=1}^{n} a_i P_{li,j} dz_j \wedge dz_i + a_i H_{il,j} dz_j \wedge dt_l.$$

Recall that $\Delta^g u = -g(dJdu, \omega)$. We obtain

$$\Delta^g u = - \sum_{i,j=1}^{n} a_i H_{ij,j}, \qquad |du|_g^2 = \sum_{i,j=1}^{n} a_i a_j H_{ij}.$$

Hence, the conformal change equation (38) becomes

$$s^H_{g_{f,J}} = -u^{\frac{2}{n}} \sum_{i,j=1}^{n} H_{ij,ij} + 2u^{\frac{2}{n}-1} \sum_{i,j=1}^{n} a_i H_{ij,j} - 2u^{\frac{2}{n}-2} \sum_{i,j=1}^{n} a_i a_j H_{ij}, \tag{74}$$

where $s^H_{g_{f,J}}$ is the Hermitian scalar curvature of $g_{f,J} = e^{2f}\omega(\cdot, J\cdot)$ with $e^{-nf} = u$. It is easy to check since H is symmetric that

$$\sum_{i,j=1}^{n} \left(e^{nf} H_{ij}\right)_{,ij} = \sum_{i,j=1}^{n} e^{nf} H_{ij,ij} - 2e^{2nf} a_i H_{ij,j} + 2e^{3nf} a_i a_j H_{ij}.$$

We conclude that (74) is equivalent to

$$e^{(n+2)f} s^H_{g_{f,J}} = - \sum_{i,j=1}^{n} \left(e^{nf} H_{ij}\right)_{,ij}. \tag{75}$$

Now, when the g-orthogonal distribution to the T-orbits is involutive (this is the case when $P = 0$), H has to satisfy the *boundary conditions* in [3, Proposition 1] and hence we can apply [9, Lemma 2] to get

Proposition 7.14. *For any H satisfying the boundary conditions [3, Proposition 1] and any affine function $\xi = \xi(z_1, \ldots, z_n)$,*

$$-\int_\Delta \left(\sum_{i,j=1}^{n} \left(e^{nf} H_{ij}\right)_{,ij} \right) \xi dv = 2 \int_{\partial\Delta} e^{nf} \xi d\mu,$$

where Δ is the polytope and $\partial\Delta$ its boundary, $dv = dz_1 \wedge \cdots \wedge dz_n$ and $d\mu$ is defined by $u_j \wedge d\mu = -dv$ for any codimension one face with inward normal u_j

If we suppose that $s^H_{g_{f,J}}$ is a constant, then (75) becomes using Proposition 7.14

$$2e^{(n+2)f} \frac{\int_{\partial\Delta} e^{nf} d\mu}{\int_\Delta e^{(n+2)f} dv} = - \sum_{i,j=1}^{n} \left(e^{nf} H_{ij}\right)_{,ij} \tag{76}$$

Define the Donaldson–Futaki invariant [22] for any smooth function ξ to be

$$\mathcal{F}_{\Delta,f}(\xi) = 2 \int_{\partial\Delta} e^{nf} \xi d\mu - 2 \frac{\int_{\partial\Delta} e^{nf} d\mu}{\int_\Delta e^{(n+2)f} dv} \int_\Delta \xi e^{(n+2)f} dv.$$

It is straightforward from Proposition 7.14 to conclude that if there exists a solution H (satisfying the boundary conditions) of (76), then $\mathcal{F}_{\Delta,f}(\xi) = 0$, for any affine function $\xi = \xi(z_1, \ldots, z_n)$. In fact, the existence of (J, g, ω) such that $g_{f,J}$ is of (positive) constant Hermitian scalar curvature can be related then to a notion of "stability" (see for instance [9, 22])

Remark 7.15. Proposition 7.14 implies that for any toric almost Kähler manifold (M, J, g, ω) with $P = 0$,

$$C(M, J, [g]) = 2\frac{\int_{\partial\Delta} e^{nf} d\mu}{\int_{\Delta} e^{(n+2)f} dv} > 0.$$

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Received June 29, 2017
Accepted March 11, 2018