Communications in
Analysis and Geometry
Volume 28, Number 7, 1577–1601, 2020

The isoperimetric problem of a complete Riemannian manifold with a finite number of C^0-asymptotically Schwarzschild ends

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We show existence and we give a geometric characterization of isoperimetric regions for large volumes, in C^2-locally asymptotically Euclidean Riemannian manifolds with a finite number of C^0-asymptotically Schwarzschild ends. This work extends previous results contained in [EM13b], [EM13a], and [BE13]. Moreover strengthening a little bit the speed of convergence to the Schwarzschild metric we obtain existence of isoperimetric regions for all volumes for a class of manifolds that we named C^0-strongly asymptotic Schwarzschild, extending results of [BE13]. Such results are of interest in the field of mathematical general relativity.

1. Introduction

1.1. Finite perimeter sets in Riemannian manifolds

We always assume that all the Riemannian manifolds M considered are smooth with smooth Riemannian metric g. We denote by V_g the canonical Riemannian measure induced on M by g, and by A_g the $(n-1)$-Hausdorff measure associated to the canonical Riemannian length space metric d of M, $U \subseteq M$ an open subset, $\mathcal{P}_g(E, U)$ the perimeter of Ω in U, whenever $U = M$ we will denote $\mathcal{P}_g(\Omega, M) =: \mathcal{P}_g(\Omega)$. We say that a sequence of finite perimeter sets (E_j) **converges in** $L^1_{loc}(M, g)$ to a finite perimeter set E, if $\chi_{E_j} \to \chi_E$ in $L^1_{loc}(M, g)$. Moreover, we say that a sequence of finite perimeter sets (E_j) **converges in the sense of finite perimeter sets** to another finite perimeter set E, if $E_j \to E$ in $L^1_{loc}(M, g)$, and $\lim_{j\to+\infty} \mathcal{P}_g(E_j) = \mathcal{P}_g(E)$. When it is already clear from the context, explicit mention of the metric g will be suppressed in what follows. For a more detailed discussion on locally

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finite perimeter sets and functions of bounded variation on a Riemannian manifold, one can consult [JPPP07].

1.2. Isoperimetric profile, compactness and existence of isoperimetric regions

Standard results of the theory of sets of finite perimeter, guarantee that $A(\partial^* E) = \mathcal{H}^{n-1}(\partial^* E) = \mathcal{P}(E)$ where $\partial^* E$ is the reduced boundary of E. In particular, if E has smooth boundary, then $\partial^* E = \partial E$, where ∂E is the topological boundary of E. More precisely we know that for every finite perimeter set E $spt(\mu_E) = \overline{\partial^* E}$, where μ_E is the vector valued Radon measure which is the distributional gradient of the caractecteristic function χ_E of E. Furthermore, up to modification on sets of measure zero we can always assume that $\overline{\partial^* E} = \partial E$ (see for instance (15.3) of [Mag12]). In the sequel we always make this technical non restricting assumption.

Definition 1.1. Let (M, g) be a Riemannian manifold of dimension n (possibly with infinite volume). We denote by $\tilde{\tau}_M$ the set of finite perimeter subsets of M. The function $I_M : [0, V(M)[\to [0, +\infty[$ defined by

$$I_M(v) := \inf\{\mathcal{P}(\Omega) = A(\partial^*\Omega) : \Omega \in \tilde{\tau}_M, V(\Omega) = v\}$$

is called the **isoperimetric profile function** (or shortly the **isoperimetric profile**) of the manifold M. If there exists a finite perimeter set $\Omega \in \tilde{\tau}_M$ satisfying $V(\Omega) = v$, $I_M(V(\Omega)) = A(\partial^*\Omega) = \mathcal{P}(\Omega)$ such an Ω will be called an **isoperimetric region**, and we say that $I_M(v)$ is **achieved**.

Compactness arguments involving finite perimeter sets implies always existence of isoperimetric regions if the ambient manifold is compact, but there are examples of manifolds without isoperimetric regions of some or every volumes. For further information about this point the reader could see the introduction of [Nar14a] or [MN16] and the discussion therein. So we cannot have always a compactness theorem if we stay in a non-compact ambient manifold. For completeness we remind the reader that if $n \leq 7$, then the reduced boundary $\partial^*\Omega$ of an isoperimetric region is smooth. More generally, the topological boundary of an isoperimetric region Ω (in fact of a good representative whose characteristic function is in the same BV-equivalence class) is the disjoint union of a regular part R and a singular part S. $R = \partial\Omega$ is smooth at each of its points and has constant mean curvature, while S has Hausdorff-codimension at least 8 in M. For more details on regularity theory for isoperimetric regions the reader can consult [Mor03] or [Mor09] Sect. 8.5, Theorem 12.2.

1.3. Main results

The main result of this paper is the following theorem which is a nontrivial consequence of the theory developed in [Nar14b], [Nar14a], [MnFN19], [FN20], combined with the work done in [EM13b]. This gives answers to some mathematical problems arising naturally in general relativity.

Theorem 1. *Let (M^n, g) be an $n \geq 3$ dimensional complete boundaryless Riemannian manifold. Assume that there exists an open relatively compact set $U \subset\subset M$ such that $M \setminus U = \mathring{\bigcup}_{i\in\mathcal{I}} E_i$, where $\mathcal{I} := \{1, \ldots, l\}$, $l \in \mathbb{N} \setminus \{0\}$, and each E_i is an end which is C^0-asymptotic to the Schwarzschild metric of mass $m > 0$ at rate γ, see Definition 2.5. Then there exists $V_0 = V_0(M, g) > 0$ such that for every $v \geq V_0$ there exists at least one isoperimetric region Ω_v enclosing volume v. Moreover there exists an end $E_i = E_{\Omega_v}$ such that $\Omega_v \cap E_{\Omega_v}$ is the region below a normal graph based on $\tilde{S}^i_r$ where $V_g(\Omega_v \cap E_{\Omega_v}) = V_g(\tilde{B}_r)$, i.e., $\Omega_v = x_i^{-1}(\varphi(B_r \setminus B_1)) \mathring{\cup} \Omega^*$, with $\Omega^* \subseteq B$ and $\varphi(B_r \setminus B_1) \subseteq \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, 1)$ is a suitable perturbation of $B_r \setminus B_1$. $\Omega_v \cap E_{\Omega_v}$ contains Σ_i and is an isoperimetric region as in Theorem 4.1 of [EM13b], $\Omega_v \setminus \mathring{E}_{\Omega_v}$ contains Σ_i and $\Omega_v \setminus \mathring{E}_{\Omega_v}$ has least relative perimeter with respect to all domains in $B \setminus \mathring{E}_{\Omega_v}$ containing Σ_i and having volume equal to $V(\Omega_v \setminus \mathring{E}_{\Omega_v})$.*

Remark 1.1. The characterization of isoperimetric regions enclosing large volumes in Theorem 1 is achieved using Theorem 4.1 of [EM13b] applied to the part of an isoperimetric region that have a sufficiently big volume far inside an end. The new part consists in proving that this can happen only in at most one end and that outside this end isoperimetric regions for large volumes cannot escape from a big but fixed big geodesic ball B that depends just on (M, g). The details and suitable modifications of the proofs are presented in Lemma 3.1.

Remark 1.2. Since the ends are like in [EM13b] it follows trivially that, if it happens that an end E is C^2-asymptotic to Schwarzschild, then the volume V_0 can be chosen in such a way that there exists a unique smooth isoperimetric (relatively to E) foliations of $E \setminus B$. Moreover, if E is asymptotically even (see Definition 2.1 of [EM13b]) then the centers of mass of $\partial\Omega_v$

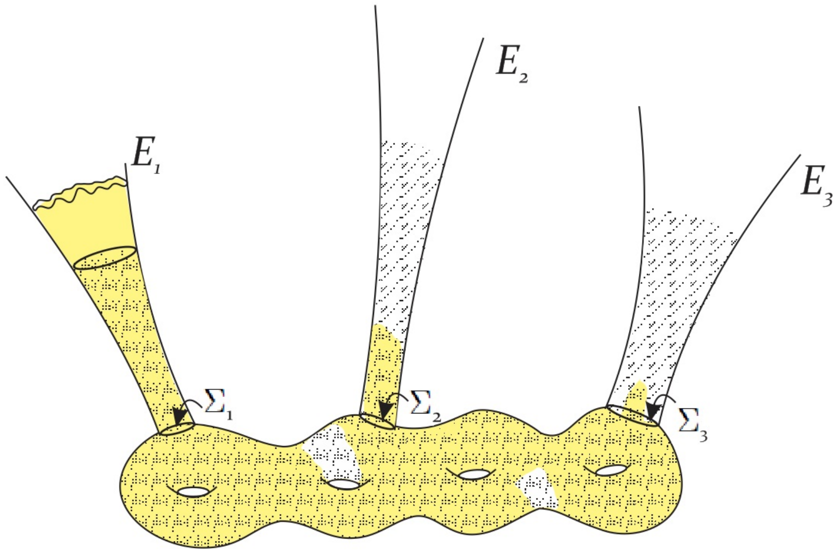

Figure 1: The isoperimetric region Ω is in yellow and B is dotted.

converge to the center of mass of E, as V goes to $+\infty$, compare Section 5 of [EM13b].

Corollary 1. *If we allow (M, g) in the preceding theorem to have each end E_i, with mass $m_i > 0$. Then there exists a volume $V_0 = V_0(M, g) > 0$ and a subset $\mathcal{I} \subseteq \{1, \ldots, l\}$, defined as $\mathcal{I} := \{i : m_i = \max\{m_1, \ldots, m_l\}\}$ such that for every volumes $v \in [V_0, +\infty[$ there exists an isoperimetric region Ω_v that satisfies the conclusion of Lemma 3.1 in which the preferred end $E_{\Omega_v} \in \{E_i\}_{i\in\mathcal{I}}$. In particular, if $m_i \neq m_j$ for all $i \neq j$, then $\mathcal{I} = \{i\}$ is reduced to a singleton and this means that there exists exactly one end E_i in which the isoperimetric regions for large volumes prefer to stay with a large amount of volume and this end corresponds to the end with the biggest mass.*

In the next theorem paying the price of strengthening the rate of convergence to the Schwarzschild metric inside each end, we can show existence of isoperimetric regions in every volumes. The proof uses the generalized existence theorem of [Nar14a] and a slight modification of the fine estimates for the area of balls that goes to infinity of Proposition 12 of [BE13].

Theorem 2. *Let (M^n, g) be an $n \geq 3$ dimensional complete boundaryless Riemannian manifold. Assume that there exists a relatively compact open* *set* $U \subset\subset M$ *such that* $M \setminus U = \mathring{\bigcup}_{i\in\mathcal{I}} E_i$*, where* $\mathcal{I} := \{1, \dots, l\}$*,* $l \in \mathbb{N} \setminus \{0\}$*, and each* E_i *is a* C^0*-strongly asymptotic to Schwarzschild of mass* $m > 0$ *end, see Definition 2.6. Then for every volume* $0 < v < V(M)$ *there exists at least one isoperimetric region* Ω_v *enclosing volume* v*.*

Corollary 2. *The conclusion of the preceding theorem still holds if we allow to the ends* E_i *of* M *to have different masses* $m_i > 0$*.*

2. Definition, notations and some basic facts

Theorem 2.1 (Generalized existence [Nar14a]). *Let* (M, g) *have* C^0*-locally asymptotically bounded geometry. Given a positive volume* $0 < v < V(M)$*, there are a finite number* N*, of limit manifolds at infinity such that their disjoint union with* M *contains an isoperimetric region of volume* v *and perimeter* $I_M(v)$*. Moreover, the number of limit manifolds is at worst linear in* v*. Indeed* $N \leq \left[\frac{v}{v^*}\right] + 1 = l(n, k, v_0, v)$*, where* v^* *is as in Lemma 3.2 of [Heb00] and* $\left[\frac{v}{v^*}\right]$ *denotes the integer part of the real number* $\frac{v}{v^*}$*.*

Remark 2.1. Observe that if $(M, g, p) \in \mathcal{M}^{m,\alpha}(n, Q, r)$ for any $m \geq 0$ and for every $p \in M$, then M have C^0-bounded geometry. So Theorem 2.1 applies to pointed manifolds in $\mathcal{M}^{m,\alpha}(n, Q, r)$. For the exact definitions see chapter 10 of [Pet06].

Now we come back to one of main interest in our theory of generalized existence and generalized compactness, i.e., to extend arguments valid for compact manifolds to noncompact ones. To this aim let us introduce the following definition suggested by Theorem 2.1.

Definition 2.1. We call $D_\infty = \bigcup_i D_{\infty,i}$ a finite perimeter set in $\tilde{M}$ a **generalized set of finite perimeter of** M and an isoperimetric region of $\tilde{M}$ a **generalized isoperimetric region**, where $\tilde{M}$ is the disjoint union of M with all the pointed limit manifolds obtained from diverging sequences of M.

Remark 2.2. We remark that D_∞ is a finite perimeter set of volume v in $\mathring{\bigcup}_i M_{\infty,i}$.

Remark 2.3. If D is a genuine isoperimetric region contained in M, then D is also a generalized isoperimetric region with $N = 1$ and

$$(M_{\infty,1}, g_{\infty,1}) = (M, g).$$

In general this does not prevent the existence of another generalized isoperimetric region of the same volume having more than one piece at infinity. If all the limit manifolds are constant curvature space forms, then obviously there is only one piece at infinity because from the point of view of isoperimetry in simply connected space forms two disjoints balls always do worst than one single ball.

Definition 2.2. Let $m \in \mathbb{N}$ and $\alpha \in [0,1]$ be given. We say that a complete Riemannian n-manifold (M,g) is $C^{m,\alpha}$**-locally asymptotically flat** or equivalently $C^{m,\alpha}$**-locally asymptotically Euclidean** if for every diverging sequence of points $(p_j)_{j\in\mathbb{N}}$ there exists a subsequence $(p_{j_l})_{l\in\mathbb{N}}$ such that the sequence of pointed manifolds
$(M, g, p_{j_l}) \to (\mathbb{R}^n, \delta, 0)$ in the pointed $C^{m,\alpha}$-topology, where δ is the canonical Euclidean metric of $\mathbb{R}^n$.

Definition 2.3. Let (M^n, g) be a complete Riemannian manifold. We say that M have **bounded geometry**, if there exist constants $k \in \mathbb{R}$ and $v_0 \in]0,+\infty[$ such that, the Ricci curvature tensor of the metric g, Ric_g satisfies $Ric_g \geq kg$ in the sense of quadratic forms and $V_g(B_{(M,g)}(p,1)) \geq v_0 > 0$ for every $p \in M$, where $B_{(M,g)}(p,r)$ denotes the geodesic ball of center p and radius $r > 0$.

Remark 2.4. Observe that a $C^{m,\alpha}$-locally asymptotically Euclidean manifold in the sense of Definition 2.2 is of bounded geometry in the sense of Definition 2.3.

Definition 2.4. An **initial data set** (M,g) is a connected complete boundaryless n-dimensional Riemannian manifold such that there exists a positive constant $C > 0$, a bounded open set $U \subset M$, a positive natural number $\tilde{N}$, such that $M \setminus U = \mathring{\bigcup}_{i=1}^{\tilde{N}} E_i$ each E_i is an end, and $E_i \cong_{x_i} \mathbb{R}^n \setminus B_1(0)$, in the coordinates induced by $x_i = (x_i^1, ..., x_i^n)$ satisfying

$$r|g_{ij} - \delta_{ij}| + r^2|\partial_k g_{ij}| + r^3|\partial^2_{kl} g_{ij}| \leq C, \tag{1}$$

for all $r \geq 2$, where $r := |x| = \sqrt{\delta_{ij}x^i x^j}$, (Einstein convention). We will use also the notations $B_r := \{x \in \mathbb{R}^n : |x| < r\}$, and $S_r := \{x \in \mathbb{R}^n : |x| = r\}$, for a **centered coordinate ball of radius** r and a **centered coordinate sphere of radius** r, respectively.

Remark 2.5. Observe that an initial data set in the sense of Definition 2.4 is C^2-locally asymptotically Euclidean in the sense of Definition 2.2.

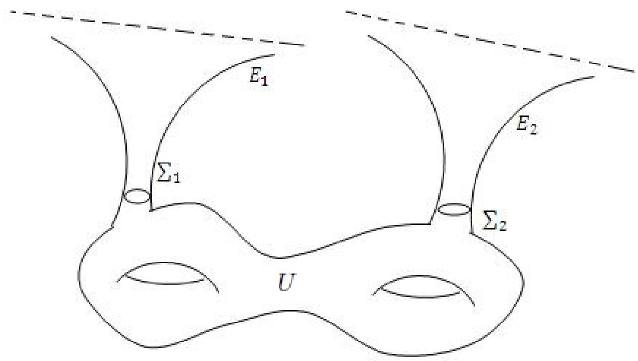

Figure 2: An example of a Schwarzschild Multiend Riemannian manifold, with $\Sigma_1 \mathring{\cup} \Sigma_2$ being the boundary of U and E_1 and E_2 being the ends of (M, g) with $\Sigma_1 \subset E_1$ and $\Sigma_2 \subset E_2$.

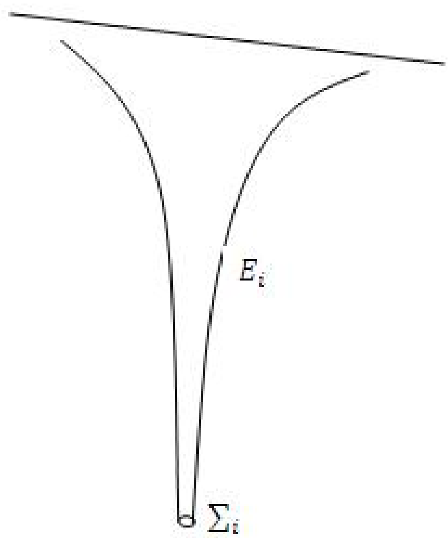

Figure 3: View of a single end. $\Sigma_i \subset E_i$ is the boundary of a typical end E_i.

In what follow we always assume that $n \geq 3$.

Definition 2.5 (see also [EM13b]). For any $m > 0$, $\gamma \in (0, 1]$, and $k \in N$, we say that an initial data set is C^k**-asymptotic to Schwarzschild of**

mass $m > 0$ **at rate** γ, if

$$\sum_{l=0}^{k} r^{n-2+\gamma+l}|\partial^l (g - g_m)_{ij}| \leq C, \tag{2}$$

for all $r \geq 2$, in each coordinate chart $x_i : E_i \cong \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,1)$, where $(g_m)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$ is the usual Schwarzschild metric on $(\mathbb{R}^n \setminus \{0\})$.

Definition 2.6 (see also [EM13b]). For any $m > 0$, $\gamma \in]0, +\infty[$, we say that an initial data set is C^0**-strongly asymptotic to Schwarzschild of mass** $m > 0$ **at rate** γ, if

$$r^{2n+\gamma}|\,(g - g_m)_{ij}\,| \leq C, \tag{3}$$

for all $r \geq 2$, in each coordinate chart $x_i : E_i \cong \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,1)$, where $(g_m)_{ij} = \left(1 + \frac{m}{2|x|^{n-2}}\right)^{\frac{4}{n-2}} \delta_{ij}$ is the usual Schwarzschild metric on $(\mathbb{R}^n \setminus \{0\})$.

3. Proof of Theorem 1 and of Corollary 1

In the following lemma we give a quite nice geometric description of the shape of an isoperimetric regions inside a Riemannian manifold satisfying the assumptions of Theorem 1. Let us denote by $\tilde{S}^i_r$ a coordinate sphere of radius r inside the end E_i and by $\tilde{B}^i_r$ a coordinate ball of radius r inside the end E_i.

Lemma 3.1. *Under the same assumptions of Theorem 1, there exists* $V_0 = V_0(M, g) > 0$, *and a large metric ball* $B \subseteq M$ *depending only on* (M, g) *such that if* $\Omega \subseteq M$ *is an isoperimetric region with* $V(\Omega) = v \geq V_0$, *then there exists an end* $E_i = E_\Omega$ *such that* $\Omega \cap E_\Omega$ *is the region below a normal graph based on* $\tilde{S}^i_r$ *where* $V_g(\Omega \cap E_\Omega) = V_g(\tilde{B}^i_r)$, *i.e.,* $\Omega = x_i^{-1}(\varphi(B_r \setminus B_1))\mathring{\cup}\Omega^*$, *with* $\Omega^* \subseteq B$ *and* $\varphi(B_r \setminus B_1) \subseteq \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0,1)$ *is a suitable perturbation of* $B_r \setminus B_1$. $\Omega \cap E_\Omega$ *contains* Σ_i *and is an isoperimetric region as in Theorem 4.1 of [EM13b],* $\Omega \setminus \mathring{E}_\Omega$ *contains* Σ_i *and* $\Omega \setminus \mathring{E}_\Omega$ *has least relative perimeter with respect to all domains in* $B \setminus \mathring{E}_\Omega$ *containing* Σ_i *and having volume equal to* $V(\Omega \setminus E_\Omega)$. *In particular there exists a constant* $c = c(M) > 0$ *such that the area outside a preferred end is less than c.*

Remark 3.1. In general B contains U and is much larger than U, see Figure 4. B could be chosen in such a way that $M \setminus B$ is a disjoint union of ends that are foliated by the boundary of isoperimetric regions of that end, provided this foliation exists. Furthermore B contain U and all the $\tilde{B}^i_r$, with r large enough to enclose a volume bigger than the volume V_0 given adapting the proof of by Theorem 4.1 of [EM13b] as shown in the Appendix.

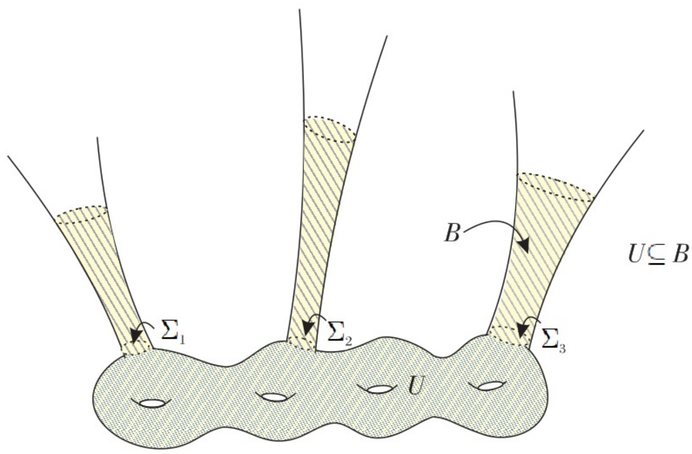

Figure 4: A schematic picture of a C^0-asymptotically Schwarzschild manifold (M, g) with $U \subseteq B$, $\Sigma_i = \partial E_i$ and E_1, E_2, E_3 ends.

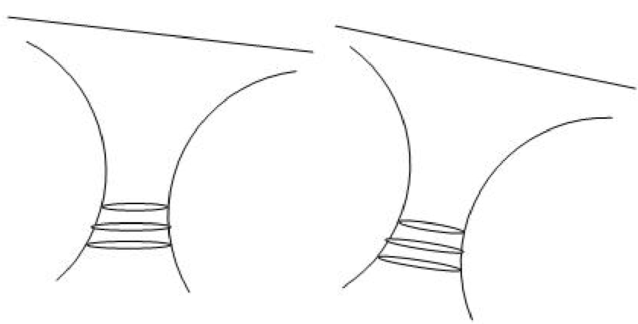

Figure 5: $M \setminus B$.

Remark 3.2. Corollary 16 of [BE13] is a particular instance of Lemma 3.1 when the number of ends is two. Of course, in Corollary 16 of [BE13] more accurate geometrical informations are given due to the very special features of the double Schwarzschild manifolds considered there. See Figure 6 in which the same notation of Corollary 16 of [BE13] are used.

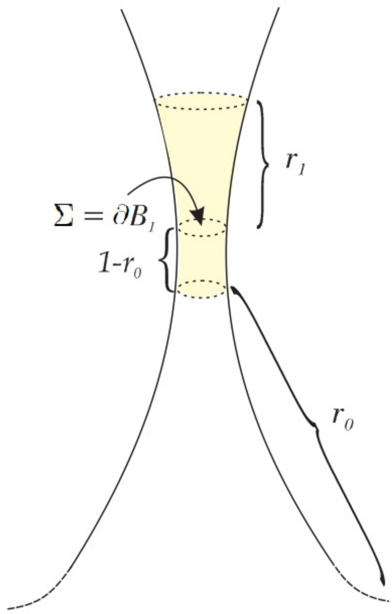

Figure 6: In Corollary 16, of [BE13], (M, g) is the double Schwarzschild manifold of positive mass $m > 0$, $(\mathbb{R}^n \setminus 0, g_m)$. The isoperimetric region Ω is in yellow, B is dotted, E_1 is the preferred end. Just in this example Σ is outermost minimal, w.r.t. any end.

Now we are ready to prove our lemma on the geometric characterization of the isoperimetric regions in our C^0-asymptotically Schwarzschild manifolds.

Proof of Lemma 3.1. Let $(\Omega_j)_j$ be a sequence of isoperimetric regions in M, such that $V(\Omega_j) \to +\infty$. It follows easily using the fact that the number of ends is finite that there exists at least one end $E_{i_j} =: E_j^*$, such that $V(\Omega_j \cap E_{i_j}) \to +\infty$. The crucial point is to show that this end E_{i_j} is unique. To show this we observe that the proof of Theorem 4.1 of [EM13b], applies exactly in the same way to our sequence $(\Omega_j)_{j\in\mathbb{N}}$ and our manifold (M, g). This application of the proof of Theorem 4.1 of [EM13b] to our ambient manifold (M, g), that we recalled here in the Appendix, gives us a volume

$V_0 > 0$ that depends only on the geometric data of the ends such that if Ω is an isoperimetric region of volume $v \geq V_0$ then there exists an end E such that $\Omega \cap E$ contains a large centered coordinate ball $\tilde{B}_{r/2}$ with $V(\tilde{B}_r) = V(\Omega \cap E)$. In particular, this discussion shows that for large values of the enclosed volume $v \geq V_0$ an isoperimetric region Ω is such that

$$\Sigma \subseteq \Omega, \tag{4}$$

and finally that $\Omega \cap E = x^{-1}(\varphi(B_r \setminus B_1))$, for larges values of $V(\Omega)$, where Σ is the boundary of E. Now we show that there is no infinite volume in more than one end. Roughly speaking, this follows quickly from the fact that the dominant term in the expansion of the area with respect to volume is the isoperimetric profile of the Euclidean space which shows that two big different coordinate balls each in one different end do worst than one coordinate ball in just one end enclosing a volume that is the sum of the two volumes of the others two balls. To show this rigorously we will argue by contradiction. Assume that for every j, there exist two distinct ends $E_{j1} \neq E_{j2}$ such that $V(\Omega_j \cap E_{j1}) =: v_{j1} \to +\infty$ and $V(\Omega_j \cap E_{j2}) =: v_{j2} \to +\infty$. Again an application of Theorem 4.1 of [EM13b] permits us to say that $\Omega_j \cap E_{j1}$ and $\Omega_j \cap E_{j2}$ are perturbations of large coordinates balls whose expansion of the area with respect to the enclosed volume is given by

$$A((\partial\Omega_j) \cap E_{ji}) = I_{\mathbb{R}^n}(v_{ji}) - m^*(v_{ji})^{\frac{1}{n}} + o(v_{ji}^{\frac{1}{n}}),\ \forall i \in \{1, 2\}. \tag{5}$$

Remark 3.3. A particular case of (5) can be found in (3) of [EM13a].

Now put $\Omega'_j := (\Omega_j \setminus E_{j2}) \cup \Omega''_j$, where $\Omega''_j \subset E_{j1}$ is such that $\Omega'_j \cap E_{j1}$ is a large coordinate ball in E_{j1} and $V(\Omega'_j) = V(\Omega_j) = v$, we have that

$$A(\partial\Omega'_j) = I_{\mathbb{R}^n}(v_{j1} + v_{j2}) - m^*(v_{j1} + v_{j2})^{\frac{1}{n}} + A(\Sigma_{j2}) + A_1, \tag{6}$$

furthermore by (5) we have

$$A(\partial\Omega_j) = I_{\mathbb{R}^n}(v_{j1}) + I_{\mathbb{R}^n}(v_{j2}) - m^* v_{j1}^{\frac{1}{n}} - m^* v_{j2}^{\frac{1}{n}} + A_2, \tag{7}$$

where the quantities A_1 and A_2 could go at infinity because of the contribution of the other ends, but in fact does not contributes to the asymptotic expansion of the difference of the areas Ω'_j and Ω_j because Ω'_j and Ω_j coincides on $M \setminus E_{j1} \mathring{\cup} E_{j2}$. Thus by direct elementary arguments we have

$$A(\partial\Omega'_j) - A(\partial\Omega_j) = I_{\mathbb{R}^n}(v_{j1} + v_{j2}) - I_{\mathbb{R}^n}(v_{j1}) - I_{\mathbb{R}^n}(v_{j2}) \tag{8}$$

$$- m^* (v_{j1} + v_{j2})^{\frac{1}{n}} + m^* v_{j1}^{\frac{1}{n}} + m^* v_{j2}^{\frac{1}{n}} + \cdots \tag{9}$$

$$\to -\infty, \tag{10}$$

when $v_{j1}, v_{j2} \to +\infty$. In particular we obtain that for large j that $\mathcal{P}(\Omega'_j) < \mathcal{P}(\Omega_j)$, which contradicts the hypothesis that Ω_j is a sequence of isoperimetric regions. This shows that we cannot have an infinite amount of volume of an isoperimetric region in more than one end at the same time which in turn implies that the volumes $V(\Omega_j \cap (M \setminus E_{i_j})) \leq K$ are uniformly bounded, by some fixed positive constant K independent of j. According to (4) we have $\Sigma \subset \Omega \setminus \mathring{E}_\Omega$, where we denoted by E_Ω the preferred end in which Ω have an amount of volume bigger than V_0. We prove that $\Omega \setminus \mathring{E}_\Omega \subseteq M \setminus \mathring{E}_\Omega$ is such that $A((\partial\Omega) \cap (M \setminus E_\Omega))$ is equal to

$$\inf \left\{ A(\partial D \cap (M \setminus E_\Omega)) : D \subseteq (M \setminus \mathring{E}_\Omega), \Sigma \subseteq D, V(D) = V(\Omega \setminus E_\Omega) \right\}. \tag{11}$$

Now we proceed to the detailed verification of (11). Indeed, if (11) was not true we can find a finite perimeter set Ω' (that can be chosen open and bounded with smooth boundary in $M \setminus E_\Omega$) inside $M \setminus \mathring{E}_\Omega$ such that $\Sigma \subseteq \Omega'$, $V(\Omega') = V(\Omega \setminus E_\Omega)$,

$$A(\partial\Omega' \cap (M \setminus E_\Omega)) < A(\partial\Omega \cap (M \setminus E_\Omega)). \tag{12}$$

Thus $V((\Omega \cap E_\Omega) \cup \Omega') = v$ and by the fact that $\Sigma \subset \Omega'$, $\Sigma \subset \Omega^{(1)}$, where $\Omega^{(1)}$ is the set of points with density 1 of Ω (that by regularity theory, see for example Theorem 1 of [GMT83] is an open set), from Inequality (12) we conclude

$$\begin{aligned} A\left(\partial^* \left[\left(\Omega \cap \mathring{E}_\Omega\right) \cup \Omega'\right]\right) &= A\left((\partial\Omega) \cap \mathring{E}_\Omega\right) + A((\partial\Omega') \cap (M \setminus E_\Omega)) \\ &< A\left((\partial\Omega) \cap \mathring{E}_\Omega\right) + A((\partial^*\Omega) \cap (M \setminus E_\Omega)) \\ &= A(\partial\Omega), \end{aligned}$$

which contradicts the fact that Ω is an isoperimetric region of volume v. At this point we have to show that the diameters of $\Omega_j \cap (M \setminus E_{i_j})$ are uniformly bounded with respect to j. We start this argument by noticing that as already mentioned the volumes $V(\Omega_j \cap (M \setminus E_{i_j})) \leq K$ are uniformly bounded, by the same fixed positive constant K. Analogously to what is done in Lemma 4.9 of [NOA20], in the spirit of the proof of Theorem 3 of [Nar14a] (these proofs were inspired by preceding works of Frank Morgan [Mor94] proving boundedness of isoperimetric regions in the Euclidean setting and Manuel Ritoré and Cesar Rosales in Euclidean cones, see Proposition 3.7 of [RR04]) and tacking into account the fact $H_{\partial\Omega_j} \to 0$ we can easily adapt those arguments to our isoperimetric problem with obstacle and conclude that

$$(13) \qquad diam(\Omega_j \cap (M \setminus E_{i_j})) \leq C(n,k,v_0)V(\Omega_j \cap (M \setminus E_{i_j}))^{\frac{1}{n}} \leq CK^{\frac{1}{n}},$$

which ensures the existence of our big geodesic ball B. For completeness sake we sketch the proof of (13) here. For more technical details the reader could consult the proof of Lemma 4.9 of [NOA20]. Let p_{Ω_j} be the center of the largest ball inscribed in $\hat{\Omega}_j := \Omega_j \cap (M \setminus E_{i_j})$ and denote by $V_{\hat{\Omega}_j}(r) := V_g(\hat{\Omega}_j \setminus B_g(p_{\Omega_j}, r))$, $\hat{B}_j := B_g(p_{\Omega_j}, r_K) \cap (M \setminus E_{i_j})$. Consider a fixed radius $r_K = C(n,k,v_0)K^{\frac{1}{n}}$ independent of j such that $V(B_g(p_{\Omega_j}, r_K) \cap (M \setminus E_{i_j})) > K$. Notice that this is always possible because our manifold (M, g) is of bounded geometry and in particular satisfies a noncollapsing condition on the volumes of balls of fixed radius. Set $l_{1j}(r) := \mathcal{H}^{n-1}_g((\partial^*\Omega_j) \cap B_g(p, r))$ and $l_{2j}(r) := \mathcal{H}^{n-1}_g((\partial^*\Omega_j) \cap (M \setminus \overline{B_g(p, r)})$. It is well known by classical slicing theory for finite perimeter sets that $\mathcal{P}_g(\Omega_j) = l_{1j}(r) + l_{2j}(r)$ a.e. r. For every $r \geq r_K$ consider the finite perimeter set $\hat{\Omega}'_j := (\hat{\Omega}_j \setminus B_g(p_{\Omega_j}, r))) \cup B_g(p_{\Omega_j}, r_j)))$ where r_j is chosen in such a way that $V(\hat{\Omega}'_j) = V(\hat{\Omega}_j)$. Having in mind the proof of Lemma 4.9 of [NOA20] it is not too hard to prove that

$$(14) \qquad (V_{\hat{\Omega}_j}^{1/n})'(r) \leq \frac{\tilde{C}_2^*}{2n} H_{\partial\hat{\Omega}_j} \left(V_{\hat{\Omega}_j}(r)\right)^{1/n} - \frac{C_{Heb}}{2n}, \ a.e.\ r \in [r_K, +\infty[,$$

where $\tilde{C}_2^* = \tilde{C}_2^*(n,k,v_0) > 0$ and $C_{Heb} = C_{Heb}(n,k,v_0)$. Recalling that $H_{\partial\hat{\Omega}_j} = H_{\partial\Omega_j} \sim H_{\partial\tilde{B}(v_j)} \sim \frac{C_1(n,k,v_0)}{v_j^{\frac{1}{n}}} > 0$ for large values of v_j, where $\partial\tilde{B}(v_j)$ is a large coordinate ball $C_1 > 0$ is independent of j, we obtain

$$(15) \qquad (V_{\hat{\Omega}_j}^{1/n})'(r) \leq \frac{\tilde{C}_2}{2n} \left(\frac{V_{\Omega_j}(r)}{v_j}\right)^{1/n} - \frac{C_{Heb}}{2n}, \ a.e.\ r \in [r_K, +\infty[.$$

Taking v_j large enough and using the fact that $V_{\Omega_j}(r) \leq V(\hat{\Omega}_j) \leq K$ we get

$$(16) \qquad (V_{\hat{\Omega}_j}^{1/n})'(r) \leq -\frac{C_{Heb}}{4n}, \ a.e.\ r \in [r_K, +\infty[.$$

Integrating the last inequality over the interval $[r_K, r]$ we obtain

$$r \leq r_K + \frac{4n}{C_{Heb}}(V_{\hat{\Omega}_j}^{1/n}(r_K) - V_{\hat{\Omega}_j}^{1/n}(r)) \leq r_K + \frac{4n}{C_{Heb}}K^{\frac{1}{n}} = CK^{\frac{1}{n}}.$$

Now that we have proved that the diameters of $\hat{\Omega}_j$ are uniformly bounded it is a trivial task to deduce the existence of our ball B. This follows easily from the fact that Σ_j is bounded and $\Sigma_j \subseteq \hat{\Omega}_j$ for every j. From this we conclude that we can replace M by B in (11) finishing the proof of the lemma. □

Now we prove Theorem 1.

Proof. Take a sequence of volumes $v_i \to +\infty$. Applying the generalized existence Theorem 1 of [Nar14a], we get that there exists $\Omega_i \subset M$ (Ω_i is eventually empty) isoperimetric region with $V(\Omega_i) = v_{i1}$ and $B_{\mathbb{R}^n}(0, r_i) \subset \mathbb{R}^n$ with

$$V(B_{\mathbb{R}^n}(0, r_i)) = v_{i2}, \tag{17}$$

satisfying $v_{i1} + v_{i2} = v_i$, and $I_M(v_i) = I_M(v_{i1}) + I_{\mathbb{R}^n}(v_{i2})$. We observe that $I_M(v_{i1}) = A(\partial\Omega_i)$ and that we have just one piece at infinity because two balls do worst than one in Euclidean space. Note that this argument was already used in the proof of Theorem 1 of [MN16]. If $v_{i2} = 0$ there is nothing to prove, the existence of isoperimetric regions follows immediately. If $v_{i2} > 0$ one can have three cases

1) $v_{i1} \to +\infty$,
2) there exists a constant $K > 0$ such that $0 < v_{i1} \leq K$ for every $i \in \mathbb{N}$,
3) $v_{i1} = 0$, for i large enough.

We will show, in first, that we can rule out cases 2) and 3). To do this, suppose by contradiction that $0 < v_{i1} \leq K < +\infty$ then remember that by Theorem 2 of [MnFN19] the isoperimetric profile function I_M is continuous (and actually by Theorem 2 of [MnFN19] I_M is local $\frac{n-1}{n}$-Hölder continuous) so $V(\Omega_i) + A(\partial\Omega_i) \leq K_1$ where $K_1 > 0$ is another positive constant. We can extract from the sequence of volumes v_{i1} a convergent subsequence named again $v_{i1} \to \bar{v} \geq 0$. By generalized existence we obtain a generalized isoperimetric region $D \subset \tilde{M}$ such that $V(D) = \bar{v}$, $I_M(\bar{v}) = A(\partial D)$. Again $D = D_1 \mathring{\cup} D_\infty$, with $D_1 \subset M$ and $D_\infty \subset \mathbb{R}^n$ isoperimetric regions in their respective volumes and in their respective ambient manifolds. Hence D_∞ is an Euclidean ball. But also by the continuity of I_M we get $I_M = I_{\tilde{M}}$ we have that $D \mathring{\cup} B_{\mathbb{R}^n}(0, r_i)$ is a generalized isoperimetric region of volume $\bar{v} + v_{i2}$, it follows that $H_{\partial D_\infty} = \frac{n-1}{r_i}$ for every $i \in \mathbb{N}$. As a consequence of the fact that

$v_i \to +\infty$ and (v_{i1}) is a bounded sequence we must have $v_{i2} \to +\infty$, hence by (17) $r_i \to +\infty$, and we get $H_{\partial D_\infty} = \lim_{r_i \to +\infty} \frac{n-1}{r_i} = 0$. As it is easy to see it is impossible to have an Euclidean ball with finite positive enclosed volume and zero mean curvature. This implies that $D_\infty = \emptyset$, for v_i large enough. As a consequence of the proof of Theorem 2.1 of [RR04] or Theorem 1 of [Nar14a] and the last fact we have $\Omega_i \to D_1$ in the sense of finite perimeter sets of M. This last assertion implies that $V(D_1) = \bar{v} = \lim_{i\to+\infty} v_{i1}$. By Lemma 2.7 of [Nar14a] we get $I_M \leq I_{\mathbb{R}^n}$. So

$$I_M(v_{i1}) + I_{\mathbb{R}^n}(v_{i2}) = I_M(v_i) \leq I_{\mathbb{R}^n}(v_i). \tag{18}$$

From (18) follows that

$$0 \leq I_M(v_{i1}) \leq I_{\mathbb{R}^n}(v_i) - I_{\mathbb{R}^n}(v_{i2}) \to 0, \tag{19}$$

because $v_i - v_{i2} \to \bar{v}$ and $I_{\mathbb{R}^n}$ is the function $v \mapsto v^{\frac{n-1}{n}}$, with fractional exponent $0 < \frac{n-1}{n} < 1$. By (19) we get immediately $\lim_{i\to+\infty} I_M(v_{i1}) = 0$. Since I_M is continuous we obtain

$$\lim_{i\to+\infty} I_M(v_{i1}) = I_M(\bar{v}) = 0 = A(\partial D_1),$$

which implies that $V(D_1) = \bar{v} = 0$.

Remark 3.4. As pointed out by the anonymous referee one can argue also in the following way

$$\lim_{i\to+\infty} I_M(v_{i1}) = I_M(\bar{v}) \leq I_{\mathbb{R}^n}(\bar{v}) \leq \lim_{i\to+\infty} I_{\mathbb{R}^n}(v_i) - I_{\mathbb{R}^n}(v_{i2}) = 0,$$

to get the desired contradiction.

Now for small nonzero volumes, isoperimetric regions are psedobubbles with small diameter and big mean curvature $H_{\partial\Omega_i} \to +\infty$, because M is C^2-locally asymptotically Euclidean, compare [Nar14b] (for earlier results in the compact case compare [Nar09]), but this is a contradiction because by first variation of area $H_{\partial\Omega_i} = H_{\partial B_{\mathbb{R}^n}(0,r_i)} = \frac{n-1}{r_i}$, with $r_i \to +\infty$. We have just showed that $v_{i1} = 0$ for i large enough provided v_{i1} is uniformly bounded with respect to the index i, that is case 2) is simply impossible to occur.

Remark 3.5. The argument just given here shows that v_{i1} uniformly bounded implies $v_{i1} = 0$ is a well formed formula valid in an arbitrary C^2-asymptotically Euclidean manifold, always for the same reason that in an Euclidean isoperimetric context two balls do worst then one.

Consider, now, the case 3), i.e., $v_{i1} = 0$ for i large enough. To rule out this case we compare a large Euclidean ball of enclosed volume v_{i2} with $\Omega_v := x_i^{-1}(B_r \setminus B_1)$ choosing r such that $V(\Omega_{v_{i2}}) = v_{i2}$, by (21), we get $A(\partial\Omega_{v_{i2}}) \leq c_n v_{i2}^{\frac{n-1}{n}}$. If $v_{i1} = 0$, for large i then we have that all the mass stays in a manifold at infinity and so if we want to have existence we need an isoperimetric comparison for large volumes between $I_M(v)$ and $I_{M_\infty}(v) = I_{\mathbb{R}^n}(v)$. This isoperimetric comparison is a consequence of (21) which gives that there exists a volume $v_0 = v_0(C, m)$ (where C is as in Definition 2.5) such that

$$I_M(v) < I_{\mathbb{R}^n}(v), \tag{20}$$

for every $v \geq v_0$. To see this we look for finite perimeter sets $\Omega'_v \subset M$ which are not necessarily isoperimetric regions, which have volume $V(\Omega'_v) = v$ and $A(\partial\Omega'_v) < I_{\mathbb{R}^n}(v)$. A candidate for this kind of domains are coordinate balls inside an end $\tilde{B}_r := x_i^{-1}(B_r \setminus B_1(0))$, with r such that $V(\tilde{B}_r) = v$, because after straightforward calculations

$$A(\partial\tilde{B}_{r(v)}) = I_{\mathbb{R}^n}(v) - m^* v^{\frac{1}{n}} + o(v^{\frac{1}{n}}) = c_n v^{\frac{n-1}{n}} - m^* v^{\frac{1}{n}} + o(v^{\frac{1}{n}}), \tag{21}$$

where $m^* > 0$ is the same coefficient that appears in the asymptotic expansion of

$$A_{g_m}(\partial\Omega_{v_m}) = I_{\mathbb{R}^n}(v_m) - m^* v_m^{\frac{1}{n}} + o(v^{\frac{1}{n}}),$$

$v_m := V_{g_m}(\Omega_v)$. Namely $m^* = c'_n m > 0$, where c'_n is a dimensional constant that depends only on the dimension n of M. The calculation of m^* is straightforward and we omit here the details, in the case of $n = 3$ it comes immediately from (3) of [EM13a]. It is worth to note here that the assumption (2) in Definition 2.5, is crucial to have the remainder in (21) of order of infinity strictly less than $v^{\frac{1}{n}}$. If the rate of convergence of g to g_m was of the order $r^{-\alpha}$ with $0 < \alpha \leq n-2$ then this could add some extra term to m^* in the asymptotic expansion (21) that we could not control necessarily. This discussion permits to exclude case 3).

Remark 3.6. As was pointed out to us by an anonymous referee (that we sincerely acknowledge here), there is a simpler way to show that when $v_i \to +\infty$ then $v_{i1} \to +\infty$ too. To see this we observe that asymptotically for large v_i it holds, by the asymptotic expansion of the area of a centered coordinate sphere with respect to the enclosed volume, that

$$I_{\mathbb{R}^n}(v_i - v_{i1}) \leq I_M(v_{i1}) + I_{\mathbb{R}^n}(v_i - v_{i1}) = I_M(v_i) \leq A(\partial\Omega_i^*) \\ = I_{\mathbb{R}^n}(v_i) - m^* v_i^{\frac{1}{n}} + o(v_i^{\frac{1}{n}}),$$

where Ω_i^* could be chosen as a domain $\Omega_i^* := U \cup \tilde{B}_{r_i}^1$ such that $V(\Omega_i^*) = v_i$. Now dividing by v_i the preceding inequality, the fact of assume $v_i \to +\infty$ and v_{i1} uniformly bounded with respect to i, leads to a contradiction. The advantage of the more sophisticated argument given prior to this remark is that it partially applies to more general Euclidean asymptotic metric than just to the Scwarzschild metric.

So we are reduced just to the case 1). We will show that the only possible phenomenon that can happen is $v_{i1} \to +\infty$ and $v_{i2} = 0$ for large i. With this aim in mind we will show first that it is not possible to have $v_{i1} \to +\infty$ and also $v_{i2} \to +\infty$ at the same time. A way to see this fact is to consider equation (21) and observe that the leading term is Euclidean, now we take all the mass v_{i2} and from infinity we add a volume v_{i2} to the part in the end E_i, in this way we construct a competitor set (as in the proof of Lemma 3.1) $\tilde{\Omega}_{v_i}$ which is isoperimetric in the preferred end E_i and such that $\tilde{\Omega}_{v_i} \setminus \Omega_{v_{i1}} = x_i^{-1}(\varphi(B_{\tilde{r}_i} \setminus B_{r_i})) \subseteq E_i$, where E_i is one fixed end with the property that $V(E_i \cap \Omega_{v_{i1}}) \to +\infty$, for suitable $\tilde{r}_i > r_i > 1$ and φ_i diffeomorphism satisfying $V(x_i^{-1}(\varphi_i(B_{\tilde{r}_i} \setminus B_{r_i}))) = v_{i2}$ in such a way that $\tilde{\Omega}_{v_i} \cap E$ is an isoperimetric region containing Σ_i of E_i, i.e., a pertubation of a large coordinate ball as prescribed by Theorem 4.1 of [EM13b], $\tilde{\Omega}_{v_i} \cap (B \setminus E) = \Omega_{v_i} \cap (B \setminus E) =: \bar{\Omega}_{\bar{v}_i}$ and $V(\tilde{\Omega}_{v_i}) = v_i$. Hence by virtue of (21) we get for large v_{i1}

$$I_M(v_{i1}) = A(\partial B_{r(v_{i1})}) + c(\bar{v}_{i1}) + \varepsilon(v_{i1}) = I_{\mathbb{R}^n}(v_{i1}) - m^* v_{i1}^{\frac{1}{n}} + o(v_{i1}^{\frac{1}{n}}), \tag{22}$$

where $\varepsilon(v_{i1}) \to 0$ when $v_{i1} \to +\infty$, $c(\bar{v}_{i1})$ is the relative area of the isoperimetric region $\bar{\Omega}_{\bar{v}_{i1}}$ of volume $\bar{v}_{i1}$ inside $B \setminus E$ where B is the fixed big ball of Lemma 3.1. Equation (22) also hold if we just require that $\Omega_{v_i} \cap E_i$ is a large coordinate ball such that $V(\Omega_{v_i}) = v_i$. It is easy to see that $\Omega_{v_{i1}} \cap (B \setminus E)$ could be characterized as the isoperimetric region for the relative isoperimetric problem in $B \setminus E$ which contain the boundary Σ of E. Such a relative isoperimetric region Ω' exists by standard compactness arguments of geometric measure theory, and regularity theory as in [EM13a], (compare also

Theorem 1.5 of [DS92]), in particular $A(\partial\Omega' \cap (B \setminus E))$ is equal to

$$\inf\left\{A(\partial D \cap (B \setminus E)) : D \subseteq B, \Sigma \subseteq D, V(D) = V(\Omega' \setminus E)\right\}. \tag{23}$$

Again by compactness arguments it is easy to show that the relative isoperimetric profile $I_{B\setminus E} : [0, V(B \setminus E)] \to [0, +\infty[$ is continuous (one can see this using the proof Theorem 2 of [MnFN19] that applies because we are in bounded geometry), and so $||I_{B\setminus E}||_\infty = c < +\infty$. If one prefer could rephrase this in terms of a relative Cheeger constant. This shows that there exists a positive constant $c = c(M) > 0$ (independent of v) such that the constant $c(v)$ appearing in (22) satisfies $c(v) \leq c$ for every $v \in [0, V(B \setminus E)]$. This last fact legitimates the second equality in equation (22). Thus by the strict subadditivity of the Euclidean isoperimetric profile, (22) readily follows

$$\begin{aligned} A(\partial\tilde{\Omega}_{v_i}) - I_M(v_i) &= A(\partial\tilde{\Omega}_{v_i}) - I_M(v_{i1}) - I_{\mathbb{R}^n}(v_{i2}) \\ &= A(\partial B_{r(v_{i1}+v_{i2})}) - A(\partial B_{r(v_{i1})}) - I_{\mathbb{R}^n}(v_{i2}) + o(v_{i1}^{\frac{1}{n}}) \\ &= (v_{i1} + v_{i2})^{\frac{n-1}{n}} - m^*(v_{i1} + v_{i2})^{\frac{1}{n}} \\ &\quad - v_{i1}^{\frac{n-1}{n}} + m^* v_{i1}^{\frac{1}{n}} - v_{i2}^{\frac{n-1}{n}} + o(v_{i1}^{\frac{1}{n}}) \\ &< -m^*(v_{i1} + v_{i2})^{\frac{1}{n}} + m^* v_{i1}^{\frac{1}{n}} + o(v_{i1}^{\frac{1}{n}}). \end{aligned}$$

Thus for large volumes $v_i \to +\infty$, we have

$$A(\partial\tilde{\Omega}_{v_i}) - I_M(v_i) < 0, \tag{24}$$

which is the desired contradiction. We remark that the use of Lemma 3.1 is crucial to have the right shape of $\Omega_{v_{i1}}$ inside the preferred end E. To finish the proof, the only case that remains to rule out is when $v_{i1} \to +\infty$ and $0 < v_{i2} \leq const.$ for every i. By the generalized compactness Theorem 1 of [FN20] there exists $v_2 \geq 0$ such that $v_{i2} \to v_2$. If $v_2 > 0$ then comparing the mean curvatures like already did in this proof, to avoid case 2) we obtain a contradiction, because the mean curvature of a large coordinate sphere tends to zero but the curvature of an Euclidean ball of positive volume v_2 is not zero. A simpler way to see this is again to look at formula (21), since the leading term is $I_{\mathbb{R}^n}$ that is strictly subadditive, we can consider again a competing domain $\tilde{\Omega}_{v_i}$ such that $\tilde{\Omega}_{v_i} \setminus \Omega_{v_{i1}} = x_i^{-1}(\varphi_i(B_{\tilde{r}_i} \setminus B_{r_i}))$, with E_i is such that $V(E_i \cap \Omega_{v_{i1}}) \to +\infty$, for suitable $\tilde{r}_i > r_i > 1$, φ_i diffeomorphism satisfying $V(x_i^{-1}(\varphi_i(B_{\tilde{r}_i} \setminus B_{r_i}))) = v_{i2}$. Now it is not too hard to see that (24) implies the claim. If $v_2 = 0$ the situation is even worst because the mean curvature of Euclidean balls of volumes going to zero goes to $+\infty$, again because isoperimetric regions for small volumes are nearly round balls, i.e., pseudobubbles as showed in [Nar14b], whose theorems apply here since M is C^2-locally asymptotically Euclidean. Hence we have necessarily that for v_i large enough $v_{i2} = 0$, which implies existence of isoperimetric regions of volume v_i, provided v_i is large enough. Since the sequence v_i is arbitrary the first part of the theorem is proved. Now that we have established existence of isoperimetric regions for large volumes the second claim in the statement of Theorem 1 follows readily from Lemma 3.1. □

Remark 3.7. If we allow each end E_i of M to have a mass $m_i > 0$ that possibly is different from the masses of the other ends, then we can guess in which end the isoperimetric regions for big volumes concentrates with "infinite volume". In fact the big volumes isoperimetric regions will prefer to stay in the end that for big volumes do better isoperimetrically and by (22) we conclude that the preferred end is to be found among the ones with bigger mass, because as it is easy to see an end with more positive mass does better than an end of less mass when we are considering large volumes. So from this perspective the worst case is the one considered in Theorem 1 in which all the masses m_i are equal to their common value m and we cannot say a priori which is the end that the isoperimetric regions for large volumes will prefer. However, Theorem 1 says that also in case of equal masses the number of ends in which the isoperimetric regions for large volumes concentrates is exactly one, but this end could vary from an isoperimetric region to another. An example of this behavior is given by Corollary 16 of [BE13], in which there are two ends and exactly two isoperimetric regions for the same large volume and they are obtained one from each other by reflection across the horizon, and each one of these isoperimetric regions chooses to have the biggest amount of mass in one end or in the other one.

After this informal presentation of the proof of Corollary 1, we are ready to go into its details.

Proof of Corollary 1. Here we treat the case in which the masses are not all equal, the case of equal masse being already treated in Theorem 1. Without loss of generality we can assume that $1 \in \mathcal{I}$, i.e.,

$$m_1 = \max\{m_1, \dots, m_{\tilde{N}}\}.$$

We will prove the corollary by contradiction. To this aim, suppose that the conclusion of Corollary 1 is false, then there exists a sequence of isoperimetric regions Ω_j such that $V(\Omega_j) = v_j \to +\infty$, and

$$E_{\Omega_j} \notin \{E_i\}_{i\in\mathcal{I}}.$$

Now we construct a competitor $\Omega'_j := (\Omega_j \setminus E_{\Omega_j})\mathring{\cup}\tilde{B}^1_{r_j}$, where $\tilde{B}^1_{r_j}$ is a large coordinate ball such that $V(\tilde{B}^1_{r_j}) = v'_j + v''_j$, with $v''_j := V(\Omega_j \cap E_1)$ and $v'_j := V(\Omega_j \setminus E_{\Omega_j})$. Roughly speaking we subtract the volume of Ω_j inside E_{Ω_j} and we put it inside the end E_1 in such a way $\Omega_j \cap E_1$ is a large coordinate ball and $V(\Omega'_j) = V(\Omega_j)$. As in the proof of Lemma 3.1, also in case of different masses we have that v''_j is uniformly bounded and $v'_j \to +\infty$. By construction $V(\Omega'_j) = V(\Omega_j) = v_j$. Furthermore, it is not too hard to prove that we have the following estimates

$$A(\partial\Omega'_j) - A(\partial\Omega_j) \leq -\left(m^*_1 - m^*_{E_{\Omega_j}}\right) {v'_j}^{\frac{1}{n}} + o({v'_j}^{\frac{1}{n}}). \tag{25}$$

This last estimate follows from an application of an analog of Lemma 3.1 in case of different masses which goes mutatis mutandis and uses in a crucial way Theorem 4.1 of [EM13b]. This cannot be avoided because again we need to control what happens to the area $A(\partial\Omega_j \cap E_{\Omega_j})$. The right hand side of (25), becomes strictly negative for $j \to +\infty$, since we have assumed $m^*_1 - m^*_{E_{\Omega_j}} > 0$. This yields to the desired contradiction. □

Here we prove Theorem 2.

Proof. By Proposition 12 of [BE13] and equation (3) we get by a direct calculation that for a given $0 < v < V(M)$ and any compact set $K \subseteq M$ there exists a smooth region $D \subset M \setminus K$ such that $V(D) = v$ and

$$A(\partial D) < c_n v^{\frac{n-1}{n}} = I_{\mathbb{R}^n}(v). \tag{26}$$

D is obtained by perturbing the closed ball $\bar{B} := \{x : |x - a| \leq r\}$, for bounded radius r and big $|a|$. The remaining part of the proof follows exactly the same scheme of Theorem 13 of [BE13], that was previously employed in another context in the proof of Theorem 1.1 of [MN16]. Now, using Theorem 1 of [Nar14a], reported here in Theorem 2.1 we get that there exists a generalized isoperimetric region $\Omega = \Omega_1\mathring{\cup}\Omega_\infty$, both $\Omega_1 \subseteq M$ and $\Omega_\infty \subseteq \mathbb{R}^n$ are isoperimetric regions in their own volumes in their respective ambient manifolds, with $V(\Omega) = v$, $V(\Omega_1) = v_1$, $V(\Omega_\infty) = v_\infty$, $v = v_1 + v_\infty$, moreover by Theorem 3 of [Nar14a] Ω_1 is bounded. If $\Omega_\infty = \emptyset$, the theorem follows promptly. Suppose, now that $\Omega_\infty \neq \emptyset$, one can chose as before a domain $D \subseteq M \setminus \Omega_1$ such that $V(D) = v_\infty$, $A(\partial D) < c_n v_\infty^{\frac{n-1}{n}} = I_{\mathbb{R}^n}(v_\infty)$.

This yields to the construction of a competitor $\Omega' := \Omega_1 \mathring{\cup} D \subseteq M$ such that $V(\Omega') = v$ and $A(\partial\Omega') = A(\partial\Omega_1) + A(\partial D) < I_M(v) = A(\partial\Omega)$, this leads to a contradiction, hence $D_\infty = \emptyset$ and the theorem follows. □

Remark 3.8. As a final remark we observe that the hypothesis of convergence of the metric tensor stated in (3) are necessary for the proof of Theorem 2, because a weaker rate of convergence could destroy the estimate (26), when passing from the model Schwarzschild metric to a C^0-asymptotically one.

To prove Corollary 2 it is enough to observe that the same proof of Theorem 2 applies mutatis mutandis.

4. Appendix

To make the paper self contained we will recall here the details of the arguments of Theorem 5.1 of [EM13a] and Theorem 4.1 of [EM13b] in our setting. We start with the following lemma that is the analog of Lemma 4.3 of [EM13a] in our context.

Lemma 4.1. *Let M a C^0-asymptotically Schwarzschild manifold having $N \geq 1$ ends, with each end E_i, with mass m_i. For every fixed $\Theta > 1$ there exists a volume $V_0 = V_0(\Theta, m_1, \dots, m_N, C, n, k, v_0) > 0$, such that for every isoperimetric region Ω of the entire M, having $V_g(\Omega \cap E) \geq V_0$ it holds*

$$A_g\left((\partial\Omega) \cap \tilde{B}_r\right) \leq \Theta r^{n-1}, \quad \forall r > 1, \tag{27}$$

$$A_g\left((\partial\Omega) \cap E\right) \leq \Theta V(\Omega \cap E)^{\frac{n-1}{n}}. \tag{28}$$

Proof. It is easily seen that,

$$\frac{A(\partial\Omega \cap E)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \leqslant \frac{A(\tilde{S}_r) + A(\Sigma_1)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \sim \frac{c_n v^{\frac{n-1}{n}} + \cdots + A(\Sigma_1)}{V(\Omega \cap E)^{\frac{n-1}{n}}} \leqslant \Theta.$$

□

We are ready to finish the explication of this appendix. With this aim in mind, take a sequence of isoperimetric regions $\Omega_i \subseteq M$ with $V_g(\Omega_i) \to +\infty$. We use the homothety $\mu_\lambda : \mathbb{R}^n \to \mathbb{R}^n$, $\mu_\lambda : x \mapsto \lambda x$, with scale factor $\lambda_i := \left(\frac{V_g(\Omega_i \cap E_i)}{\omega_n}\right)^{\frac{1}{n}}$ to obtain sets $\hat{\Omega}_i \subseteq \mathbb{R}^n \setminus B_{\mathbb{R}^n}(0, \lambda_i^{-1})$, $\hat{\Omega}_i := \mu_{\lambda_i}(x_i(\Omega_i \cap E_i))$ that are locally isoperimetric w.r.t. the metric $g_i := \frac{1}{\lambda_i^2}\mu_{\lambda_i}^* g$ and such that $V_{g_i}(\hat{\Omega}_i) = \omega_n$. As it is easy to check $\left(\mathbb{R}^n \setminus B(0, \frac{1}{\lambda_i}), g_i\right) \to (\mathbb{R}^n \setminus \{0\}, \delta)$ in the C^2_{loc} topology. Now we observe that $V_\delta(\hat{\Omega}_i) \sim V_{g_i}(\hat{\Omega}_i) = V_\delta(B_{\mathbb{R}^n}(0,1))$ and that for large volumes Lemma 4.1 implies

$$A_\delta(\partial\hat{\Omega}_i) \sim A_{g_i}(\partial\hat{\Omega}_i) \leq \Theta V_{g_i}(\hat{\Omega}_i)^{\frac{n-1}{n}} \leq K_3 = \Theta\omega_n^{\frac{n-1}{n}},$$

where K_3 is a constant. It follows that the sequence $\hat{\Omega}_i$ has volumes and boundaries uniformly bounded. This implies the existence of a finite perimeter set $\Omega \subset \mathbb{R}^n \setminus \{0\}$ such that $\chi_{\hat{\Omega}_i} \to \chi_\Omega$ in $L^1_{loc}(\mathbb{R}^n)$ topology. About this point the reader could consult the beggining of the proof of Theorem 2.1 of [RR04]. In particular $V_\delta(\Omega) \leq \omega_n = \lim_{i\to+\infty} V_\delta(\hat{\Omega}_i)$ and the inequality could be strict. If we show that

$$V_\delta(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0, 1+\theta)) = 0, \tag{29}$$

for every fixed $\theta > 0$, then it is straightforward to see that Ω must coincide with the unit ball of $\mathbb{R}^n$, i.e.,

$$\Omega = B_{\mathbb{R}^n}(0,1). \tag{30}$$

It is exactly in order to prove (29) that we need in a crucial way that our initial data set M is indeed C^0-asymptotically Schwarzschild. The arguments used here does not works in a general initial data set, but only in C^0-asymptotically Schwarzschild, because we can use the effective comparison Theorem 3.5 of [EM13b] that is a special feature of the Schwarzschild geometry and it is not a consequence of effective Euclidean isoperimetric inequality as explained very well in [EM13b]. We will prove (30) by contradiction. To this aim, assume that there exist $\frac{4\alpha_{n-1}}{c_n} > \varepsilon > 0$, $\theta > 0$, such that $V_{g_i}(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0, 1+\theta)) \geq \varepsilon > 0$, for every $i \in \mathbb{N}$. By a relative isoperimetric inequality

$$A_{g_i}((\partial\hat{\Omega}_i) \setminus B_{\mathbb{R}^n}(0, 1+\theta)) \geq c_n V_{g_i}(\hat{\Omega}_i \setminus B_{\mathbb{R}^n}(0, 1+\theta)) \tag{31}$$

$$\geq c_n\varepsilon \geq 2\eta A_{g_i}(\partial B_{\mathbb{R}^n}(0,1)). \tag{32}$$

We conclude that each $\Omega_i \cap E_i$ is $\left(1+\frac{\theta}{2}, \eta\right)$-off-center. At this point we can apply Theorem 3.5 of [EM13b] to $\Omega_i \cap E_i$ and deduce that $\Omega_i' := \tilde{B}_r \mathring{\cup} \Omega_i \setminus E_i$ satisfies

$$A_g(\partial\Omega_i') - A_g(\partial\Omega_i) \leq A_g(\Sigma_i) - c\eta m_i \left(\frac{\theta}{2+\theta}\right)^2 r(V_g(\Omega_i \cap E_i)), \tag{33}$$

where $\Sigma_i := \partial E_i$, $c = c(n) > 0$ is a dimensional constant, and $r(V_g(\Omega_i \cap E_i))$ is such that $V_g(\tilde{B}_r \setminus \tilde{B}_1) = V_g(\Omega_i \cap E_i)$. By the fact that $r(V_g(\Omega_i \cap E_i)) \to$

$+\infty$, when $i \to +\infty$, inequality (33) immediately shows that for large volumes Ω_i is not isoperimetric, which is the desired contradiction to our assumptions. To finish the proof at this point we follow a somewhat little bit different argument from the proof of Theorem 5.1 of [EM13a]. At this point we can use Theorem 1 of [Nar18] complemented with Remark 4.1 of [Nar18] to show that the boundary of Ω_i is the graph of a function based on a centered coordinate sphere. This is still not enough to guarantees that $\Sigma_i \subseteq \Omega_i$ for large volumes, but the arguments of the proof of Theorem 4.1 of [EM13b] show that $\tilde{B}_{r_i/2} \subseteq \Omega_i$ for large volumes. This finish easily the proof that $\Sigma_{j_i} \subseteq \Omega_i$ for large volumes.

References

[BE13] Simon Brendle and Michael Eichmair, *Isoperimetric and Weingarten surfaces in the Schwarzschild manifold*, J. Differential Geom. **94** (2013), no. 3, 387–407.

[DS92] Frank Duzaar and Klaus Steffen, *Area minimizing hypersurfaces with prescribed volume and boundary*, Math. Z. **209** (1992), no. 4, 581–618.

[EM13a] Michael Eichmair and Jan Metzger, *Large isoperimetric surfaces in initial data sets*, J. Differential Geom. **94** (2013), no. 1, 159–186.

[EM13b] Michael Eichmair and Jan Metzger, *Unique isoperimetric foliations of asymptotically flat manifolds in all dimensions*, Invent. Math. **194** (2013), no. 3, 591–630.

[FN20] Abraham Enrique Muñoz Flores and Stefano Nardulli, *Generalized compactness for finite perimeter sets and applications to the isoperimetric problem*, J. Dyn. Control Syst **1** (2020), no. 15, 1–11.

[GMT83] E. Gonzalez, U. Massari, and I. Tamanini, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J. **32** (1983), no. 1, 25–37.

[Heb00] Emmanuel Hebey, Non Linear Analysis on Manifolds: Sobolev Spaces and Inequalities, volume 5 of Lectures Notes, AMS-Courant Inst. Math. Sci., (2000).

[JPPP07] M. Miranda Jr., D. Pallara, F. Paronetto, and M. Preunkert, *Heat semigroup and functions of bounded variation on Riemannian manifolds*, J. reine angew. Math. **613** (2007), 99–119.

[Mag12] Francesco Maggi, Sets of Finite Perimeter and Geometric Variational Problems, volume 135 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, (2012). An introduction to geometric measure theory.

[MN16] Andrea Mondino and Stefano Nardulli, *Existence of isoperimetric regions in non-compact Riemannian manifolds under Ricci or scalar curvature conditions*, Comm. Anal. Geom. **24** (2016), no. 1, 115–138.

[MnFN19] Abraham Enrique Muñoz Flores and Stefano Nardulli, *Local Hölder continuity of the isoperimetric profile in complete noncompact Riemannian manifolds with bounded geometry*, Geom. Dedicata **201** (2019), 1–12.

[Mor94] Frank Morgan, *Clusters minimizing area plus length of singular curves*, Math. Ann. **299** (1994), no. 299, 697–714.

[Mor03] Frank Morgan, *Regularity of isoperimetric hypersurfaces in Riemannian manifolds*, Trans. Amer. Math. Soc. **355** (2003), no. 12, 5041–5052.

[Mor09] Frank Morgan, Geometric Measure Theory: A Beginner's Guide, Academic Press, fourth edition, (2009).

[Nar09] Stefano Nardulli, *The isoperimetric profile of a smooth Riemannian manifold for small volumes*, Ann. Glob. Anal. Geom. **36** (September 2009), no. 2, 111–131.

[Nar14a] Stefano Nardulli, *Generalized existence of isoperimetric regions in non-compact Riemannian manifolds and applications to the isoperimetric profile*, Asian J. Math. **18** (2014), no. 1, 1–28.

[Nar14b] Stefano Nardulli, *The isoperimetric profile of a noncompact Riemannian manifold for small volumes*, Calc. Var. Partial Differential Equations **49** (2014), no. 1-2, 173–195.

[Nar18] Stefano Nardulli, *Regularity of isoperimetric regions that are close to a smooth manifold*, Bull. Braz. Math. Soc. (N.S.) **49** (2018), no. 2, 199–260.

[NOA20] Stefano Nardulli and Luis Eduardo Osorio Acevedo, *Sharp isoperimetric inequalities for small volumes in complete noncompact Riemannian manifolds of bounded geometry involving the scalar curvature*, Int. Math. Res. Not. IMRN **2020** (2020), no. 15, 4667–4720.

[Pet06] Peter Petersen, Riemannian Geometry, volume 171 of Grad. Texts in Math., Springer Verlag, 2nd edition, (2006).

[RR04] Manuel Ritoré and César Rosales, *Existence and characterization of regions minimizing perimeter under a volume constraint inside Euclidean cones*, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4601–4622.

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Received July 24, 2016
Accepted April 22, 2018