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Total p-powered curvature of closed curves and flat-core closed p-curves in $\mathrm{S}^2(G)$

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We consider a variational problem of p-elastic curves in two-dimensional sphere. We give its first variation formula, and in two-dimensional sphere, we give a realization of a solution which satisfies that the first variation formula is zero. We also show the existence of a flat-core, closed p-elastic curve.

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1. Introduction

An elastica is a curve which appears as a critical point, i.e., a stationary curve of a variational problem for the total squared curvature of curves under certain constraints. In the two-dimensional Euclidean space case, in 1691, James Bernoulli proposed a problem of finding the possible shapes of an inextensible rod such that its bottom end is fixed perpendicular to

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the ground and it is bent by a load sufficient to make its top end horizontal. After that, Daniel Bernoulli discovered a functional which is suitable for the problem, and Euler solved the variational problem; see [12, 13, 19]. It is known such variational problem has broad connections with various fields, such as elliptic function theory, differential geometry, soliton theory, etc. For example, it can be seen that stationary wave solutions (cn-wave and dn-wave solutions) of the mKdV equations are essentially planar elasticae; see [18, Chapter 5]. As for the relation with differential geometry, Langer-Singer [11] and Bryant-Griffiths [7] studied the problem in Riemannian manifolds with nonzero constant scalar curvature, in particular, in two-dimensional spheres and two-dimensional hyperbolic spaces. Arroyo-Garay-Mencía [1–6], Huang [9], Jurdjevic [10] and others developed the problem in Riemannian manifolds by considering various functionals; see also [16]. In [1], for a C^∞ function $P(t)$, Arroyo-Garay-Mencía considered the functional

$$J(c) = \int_0^L P(\kappa)\, ds \tag{1.1}$$

for each $c : [0, L] \to \mathbf{S}^2(G)$ of class C^4 such that $dc/ds \neq 0$, where L is the length of c, κ is the curvature of c and $\mathbf{S}^2(G)$ is a two-dimensional sphere with constant Gaussian curvature G. They showed a first variation formula for the functional, a closedness condition for a stationary curve, a second variation formula, etc. They considered the particular case $P(t) = \sqrt{t^2 + \lambda}$ with $\lambda > 0$. In [3] and [6], they also studied the case $P(t) = (t - \kappa_0)^2$ in two dimensional spheres and in two dimensional hyperbolic spaces, respectively. We note that Langer-Singer [11] and Bryant-Griffiths [7] studied the case $P(t) = t^2 + \lambda$ with $\lambda \in \mathbb{R}$.

In this paper, we consider the case $P(t) = |t|^p + \lambda$ with $p > 2$ and $\lambda > 0$. In [1, Proposition 8], Arroyo-Garay-Mencía showed that if $P(t) = t^p$, $p > 2$ and $p \in \mathbb{N}$, then C^4-stationary curves of J are only geodesics. (The assumption $p \in \mathbb{N}$ comes from that they assumed $P(t)$ is of class C^∞.) Instead of C^4-curves, we consider functional (1.1) for $P(t) = |t|^p + \lambda$ with $p > 2$ and $\lambda > 0$ on

$$\mathcal{D} = \big\{ c \in C^2([0,1], \mathbf{S}^2(G)) \mid c_t(t) \neq 0 \text{ for each } t \in [0,1] \big\}.$$

By a formal calculation based on the argument in [11, p.3] (see also Remark 1 below), we can see that if $c \in \mathcal{D}$ is a stationary curve, then it seems that the curvature κ of c satisfies

$$(1.2)\quad p(p-1)|\kappa|^{p-2}\kappa_{ss} + p(p-1)(p-2)|\kappa|^{p-4}\kappa\kappa_s^2 + (p-1)|\kappa|^p\kappa + Gp|\kappa|^{p-2}\kappa - \lambda\kappa = 0.$$

We note that in the case $p = 2$, (1.2) is identical with [11, (1.2)]. However, since $c \in \mathcal{D}$ has only C^2 regularity, κ is only continuous in general. So it seems to be difficult to consider its derivatives κ_s, κ_{ss} as in (1.2). In addition, in the case $2 < p < 3$, since (1.2) has a negative exponent of κ, it seems to be difficult to find a stationary curve of J whose curvature has a zero point, and hence it seems to be hopeless to find C^2 stationary curves in the case $p \neq 2$. Moreover, even if we obtain a solution of (1.2), it does not directly imply the existence of a stationary curve of J. In the case $p = 2$, it was done with the aid of Killing vector field or Noether's theorem; see [1, 7, 11] and others. However, since a solution κ of (1.2) may not be differentiable in our setting, it seems to be difficult to apply them. Overcoming these difficulties, we show the existence of C^2 stationary curves of J other than geodesics whose curvatures may have zero points. Moreover, we will show the existence of rather curious stationary curves of J, which we call *flat-core* stationary curves of J. In [20], the second author considered a similar problem in $\mathbb{R}^2$, he showed the existence of flat-core stationary curves for a corresponding functional, like J. We note that the concept of flat-core solution itself was introduced by Guedda-Veron [8] and recently developed by Takeuchi [17] for 1-dimensional nonlinear eigenvalue problems (not for elasticae).

This paper is organized as follows. In the next section, we give a formulation of our problem and a first variation formula to the problem. In Section 3, we will show that if κ satisfies that the first variation formula is zero, like (1.2), then there is a stationary curve in $\mathbf{S}^2(G)$ whose curvature is κ. In Section 4, we give some classifications of the stationary curves. In particular, we define minimal period crossing and non minimal period crossing stationary curves. In the final section, we show the existence of closed, flat-core stationary curves and we also show some numerical results of closed stationary curves for various $p > 2$ and $\lambda > 0$. Although numerical computations indicate the existence of various types of stationary closed curves in any case of $p > 2$ and $\lambda > 0$, it does not seem to be easy to give rigorous proofs of their existence in general. We give some pictures of them. In our future work, we will prove their existence in general.

2. A local coordinate and first variation formula

Let $\mathbf{S}^2(G)$ be a compact two-dimensional submanifold in $\mathbb{R}^3$ with constant Gaussian curvature G. Since any compact two-dimensional submanifold in $\mathbb{R}^3$ of constant Gaussian curvature must be a sphere, which is Liebmann's theorem ([15, Theorem 3.7]), we consider that $\mathbf{S}^2(G)$ is a sphere whose radius is $r = 1/\sqrt{G}$. We note that we use the relation

$$G = \frac{1}{r^2}$$

throughout this paper. Since we consider that $\mathbf{S}^2(G)$ is a sphere in $\mathbb{R}^3$ with radius r, we represent a point (x, y, z) in $\mathbf{S}^2(G)$ with the polar coordinate defined by

$$\begin{aligned}(2.1)\qquad &(x, y, z) = (r \sin v \cos u, r \sin v \sin u, -r \cos v),\\ &(0 \leq u < 2\pi, 0 \leq v \leq \pi),\end{aligned}$$

and we consider that $\mathbf{S}^2(G)$ has the standard Riemannian metric which is induced from the embedding from $\mathbf{S}^2(G)$ into $\mathbb{R}^3$. That is, we consider that

$$g_{uu} = r^2 \sin^2 v, \quad g_{uv} = g_{vu} = 0, \quad g_{vv} = r^2$$

is our Riemannian metric tensor. We say $c = (u, v)$ is a *curve in* $\mathbf{S}^2(G)$ if it is represented as

$$c(t) = (r \sin v(t) \cos u(t), r \sin v(t) \sin u(t), -r \cos v(t)), \quad t \in \mathbb{R}.$$

Let $\mathcal{D}$ be a set of C^2 curves in $\mathbf{S}^2(G)$ defined by

$$\begin{aligned}\mathcal{D} = \big\{c = (u, v) \in C^2([0, 1], \mathbf{S}^2(G)) \mid\ &\\ &c(t) = (r \sin v(t) \cos u(t), r \sin v(t) \sin u(t), -r \cos v(t)),\\ &c_t(t) \neq (0, 0, 0) \text{ and } 0 < v(t) < \pi \text{ for each } t \in [0, 1]\big\}.\end{aligned}$$

Let $c = (u, v) \in \mathcal{D}$ whose length is L, and let s be its arclength parameter. Then it satisfies

$$(2.2)\qquad r^2 \sin^2 v(s) u_s(s)^2 + r^2 v_s(s)^2 = 1.$$

Such an arclength parameter can be defined as follows. Let $s(t)$ be the function from $[0,1]$ into $[0,L]$ defined by

$$\begin{aligned} s(t) &= \int_0^t \sqrt{\langle c_t(t), c_t(t)\rangle_{\mathbb{R}^3}}\, dt \\ &= \int_0^t \sqrt{r^2 \sin^2 v(t) u_t(t)^2 + r^2 v_t(t)^2}\, dt, \quad t \in [0,1], \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$ is the standard inner product in $\mathbb{R}^3$. Since

$$ds/dt = \sqrt{\langle c_t(t), c_t(t)\rangle_{\mathbb{R}^3}} > 0,$$

by the inverse function theorem, we can define its inverse function $t(s)$ from $[0,L]$ into $[0,1]$. We note $t_s(s) > 0$, and we write $c(s) = (u(s), v(s))$ instead of $c(t(s)) = (u(t(s)), v(t(s)))$. Let $\mathbf{e}_1(s)$ be the unit tangent vector $c_s(s)$ and $\mathbf{e}_2(s)$ its $\pi/2$-rad anti-clockwise rotation at $c(s)$. Then, by the Frenet-Serret formula, we have

$$\nabla_{\mathbf{e}_1(s)} \mathbf{e}_1(s) = \kappa(s)\mathbf{e}_2(s), \qquad \big(\mathbf{e}_1(s) = c_s(s)\big).$$

Here, $\nabla_{\mathbf{e}_1(s)}$ is the covariant derivative to the direction $\mathbf{e}_1(s)$. We obtain the expression of the curvature of a curve in $\mathcal{D}$ as follows.

Lemma 1. *Let $c = (u, v) \in \mathcal{D}$ and let s be its arclength parameter. Then the curvature κ of c is expressed as*

$$\begin{aligned} \kappa(s) = r^2 \Big(&- u_{ss}(s) v_s(s) \sin v(s) + u_s(s) v_{ss}(s) \sin v(s) \\ &- 2u_s(s) v_s(s)^2 \cos v(s) - u_s(s)^3 \sin^2 v(s) \cos v(s) \Big). \end{aligned} \tag{2.3}$$

If a parameter t of c does not represent its arclength, then κ is expressed as

$$\kappa(t) = \frac{r^2 \Big(-u_{tt} v_t \sin v + u_t v_{tt} \sin v - 2u_t v_t^2 \cos v - u_t^3 \sin^2 v \cos v \Big)}{\Big(r^2 \sin^2 v\, u_t^2 + r^2 v_t^2 \Big)^{\frac{3}{2}}}. \tag{2.4}$$

Proof. By direct calculations, we have

$$c_s(s) = r\begin{pmatrix} v_s(s)\cos u(s)\cos v(s) - u_s(s)\sin u(s)\sin v(s) \\ u_s(s)\cos u(s)\sin v(s) + v_s(s)\sin u(s)\cos v(s) \\ v_s(s)\sin v(s)\end{pmatrix}^T,$$

$$c_{ss}(s) = r\begin{pmatrix} \cos v\left(v_{ss}\cos u - 2u_s v_s \sin u\right) - \sin v\left(u_{ss}\sin u + \cos u\left(u_s^2+v_s^2\right)\right) \\ \cos v\left(2u_s v_s\cos u + v_{ss}\sin u\right) + \sin v\left(u_{ss}\cos u - \sin u\left(u_s^2+v_s^2\right)\right) \\ v_{ss}\sin v + v_s^2\cos v\end{pmatrix}^T.$$

Using (2.2), we can see

$$\begin{aligned} &\left\langle c_{ss}(s), c(s)\right\rangle_{\mathbb{R}^3} = -r^2(u_s(s)^2\sin^2 v(s) + v_s(s)^2) = -1, \\ &\langle c(s)\times c_s(s), c(s)\times c_s(s)\rangle_{\mathbb{R}^3} = r^4(u_s(s)^2\sin^2 v(s) + v_s(s)^2) = r^2, \end{aligned}$$

where $\times$ is the outer product, and hence we get

$$\begin{aligned} \nabla_{\mathbf{e}_1(s)}\mathbf{e}_1(s) &= c_{ss}(s) - \frac{\left\langle c_{ss}(s), c(s)\right\rangle_{\mathbb{R}^3}}{r^2}c(s) \\ &= r\begin{pmatrix} -u_s^2\cos u\cos^2 v\sin v - 2u_s v_s\cos v\sin u - u_{ss}\sin u\sin v + v_{ss}\cos u\cos v \\ -u_s^2\sin u\cos^2 v\sin v + 2u_s v_s\cos u\cos v + u_{ss}\cos u\sin v + v_{ss}\sin u\cos v \\ -u_s^2\cos v\sin^2 v + v_{ss}\sin v\end{pmatrix}^T, \\ \mathbf{e}_2(s) &= \frac{c(s)\times c_s(s)}{\langle c(s)\times c_s(s), c(s)\times c_s(s)\rangle_{\mathbb{R}^3}^{\frac{1}{2}}} = r\begin{pmatrix} u_s\cos u\cos v\sin v + v_s\sin u \\ u_s\sin u\cos v\sin v - v_s\cos u \\ u_s\sin^2 v\end{pmatrix}^T. \end{aligned}$$

From $\kappa(s) = \left\langle \nabla_{\mathbf{e}_1(s)}\mathbf{e}_1(s), \mathbf{e}_2(s)\right\rangle_{\mathbb{R}^3}$, we obtain (2.3). In the case when t is not an arclength parameter, by changing variables, we can show (2.4). □

Lemma 2. *Let $(u,v)\in\mathcal{D}$ and let s be its arclength parameter. Then it holds that*

$$\begin{cases} u_{ss}(s)\sin v(s) = -v_s(s)\big(2u_s(s)\cos v(s) + \kappa(s)\big), \\ v_{ss}(s) = u_s(s)\sin v(s)\big(u_s(s)\cos v(s) + \kappa(s)\big). \end{cases}$$

Proof. From (2.2) and (2.3), we have

$$\begin{aligned} &\begin{pmatrix} u_s(s)\sin v(s) & v_s(s) \\ -v_s(s) & u_s(s)\sin v(s)\end{pmatrix}\begin{pmatrix} u_{ss}(s)\sin v(s) \\ v_{ss}(s)\end{pmatrix} \\ &= \begin{pmatrix} -u_s(s)^2 v_s(s)\sin v(s)\cos v(s) \\ \kappa(s)/r^2 + 2u_s(s)v_s(s)^2\cos v(s) + u_s(s)^3\sin^2 v(s)\cos v(s)\end{pmatrix}. \end{aligned}$$

Using (2.2), we obtain

$$\begin{aligned}\begin{pmatrix} u_{ss}(s)\sin v(s) \\ v_{ss}(s)\end{pmatrix} &= r^2 \begin{pmatrix} u_s(s)\sin v(s) & -v_s(s) \\ v_s(s) & u_s(s)\sin v(s)\end{pmatrix} \\ &\quad\cdot \begin{pmatrix} -u_s(s)^2 v_s(s)\sin v(s)\cos v(s) \\ \kappa(s)/r^2 + 2u_s(s)v_s(s)^2\cos v(s) + u_s(s)^3\sin^2 v(s)\cos v(s)\end{pmatrix} \\ &= \begin{pmatrix} -v_s(s)\big(2u_s(s)\cos v(s) + \kappa(s)\big) \\ u_s(s)\sin v(s)\big(u_s(s)\cos v(s) + \kappa(s)\big)\end{pmatrix}.\end{aligned}$$

□

In the rest of this paper, we always assume $p > 2$ and $\lambda > 0$. We consider the functional

$$J(c) = \int_0^L \left(\left|\kappa(s)\right|^p + \lambda\right) ds = \int_0^1 \left(\left|\kappa(t)\right|^p + \lambda\right) \langle c_t, c_t\rangle_{\mathbb{R}^3}^{\frac{1}{2}} dt, \qquad c \in \mathcal{D},$$

where s represents an arclength parameter of c, κ is the curvature of c and L is the total length of c. In the case $p = 2$, it coincides with the one treated in [11]. We say a mapping $c(w,t) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ with $\varepsilon > 0$ is a *variation* of $c \in \mathcal{D}$ if it satisfies

(i) $c(0,t) = c(t)$ for each $t \in [0,1]$,

(ii) for each $w \in (-\varepsilon, \varepsilon)$, $c(w, \cdot)$ is an element of $\mathcal{D}$,

(iii) for each $t \in [0,1]$, $c(\cdot, t)$ is smooth.

An example of such a mapping can be obtained by

$$c(w,t) = (u(w,t), v(w,t)) = \exp_{(u(0,t),v(0,t))}\big(w(u_w(0,t), v_w(0,t))\big)$$

for $(w,t) \in (-\varepsilon, \varepsilon) \times [0,1]$ with some $\varepsilon > 0$. We say $c \in \mathcal{D}$ is a *p-elastic curve* if

$$\left.\frac{dJ(c(w,\cdot))}{dw}\right|_{w=0} = 0$$

for each variation $c(w,t) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ of c satisfying

$$\text{(2.5)} \qquad \begin{aligned} c(w,0) = c(0), &\quad c_t(w,0) = c_t(0), \\ c(w,1) = c(1) \quad \text{and} &\quad c_t(w,1) = c_t(1)\end{aligned}$$

for each $w \in (-\varepsilon, \varepsilon)$. We denote by $\mathcal{S}$ the set of closed curves in $\mathcal{D}$, i.e.,

$$\mathcal{S} = \big\{c \in \mathcal{D} \mid c(0) = c(1),\ c_t(0) = c_t(1),\ c_{tt}(0) = c_{tt}(1)\big\}.$$

We also say $c \in \mathcal{S}$ is a *closed p-elastic curve* if

$$\left.\frac{dJ(c(w,\cdot))}{dw}\right|_{w=0} = 0$$

for each variation $c(w,t) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ of c satisfying $c(w,\cdot) \in \mathcal{S}$ for each $w \in (-\varepsilon, \varepsilon)$.

Now, we give a first variation formula of J.

Theorem 1. *Let $(u,v) \in \mathcal{D}$ such that $|\kappa|^{p-2}\kappa$ is of class C^2, where κ is the curvature of (u,v). Then for each variation $c(w,t) = (u(w,t), v(w,t)) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ of (u,v) with $\varepsilon > 0$, there holds*

$$\begin{aligned} \frac{1}{r^2}\left.\frac{dJ(c(w,\cdot))}{dw}\right|_{w=0} &= \int_0^L X(s)\sin v(s)\big(-v_s(s)u_w(0,s) + u_s(s)v_w(0,s)\big)\,ds \\ &\quad + \Big[Y_1(s)u_w(0,s) + Y_2(s)v_w(0,s) + Y_3(s)u_{ws}(0,s) + Y_4(s)v_{ws}(0,s)\Big]_0^L \end{aligned}$$

in the coordinate (2.1)*, where s is an arclength parameter of $(u(0,\cdot), v(0,\cdot))$, L is the total length of $(u(0,\cdot), v(0,\cdot))$ and*

$$\begin{cases} X(s) = p\big(|\kappa(s)|^{p-2}\kappa(s)\big)_{ss} + (p-1)|\kappa(s)|^p\kappa(s) + Gp|\kappa(s)|^{p-2}\kappa(s) - \lambda\kappa(s), \\ Y_1(s) = \sin v(s)\left(\lambda u_s(s)\sin v(s) + pv_s(s)(|\kappa(s)|^{p-2}\kappa(s))_s\right) \\ \qquad\qquad - \dfrac{p}{r^2}|\kappa(s)|^{p-2}\kappa(s)\cos v(s) - (p-1)|\kappa(s)|^p u_s(s)\sin^2 v(s), \\ Y_2(s) = v_s(s)\left(\lambda - p|\kappa(s)|^{p-2}\kappa(s)u_s(s)\cos v(s) - (p-1)|\kappa(s)|^p\right) \\ \qquad\qquad - pu_s(s)(|\kappa(s)|^{p-2}\kappa(s))_s\sin v(s), \\ Y_3(s) = -p|\kappa(s)|^{p-2}\kappa(s)v_s(s)\sin v(s), \\ Y_4(s) = p|\kappa(s)|^{p-2}\kappa(s)u_s(s)\sin v(s). \end{cases}$$

Remark 1. If κ is of class C^2, $X(s) \equiv 0$ is equivalent to (1.2). We note that if a curve in $\mathbf{S}^2(G)$ is of class C^4, its curvature κ is of class C^2.

Before giving the proof of Theorem 1, we give a direct consequence of the theorem.

Corollary 1. *Let $(u,v) \in \mathcal{D}$ such that $|\kappa|^{p-2}\kappa$ is of class C^2, and let s and $X(s)$ be as in Theorem 1, and assume $X(s) \equiv 0$. Then (u,v) is a p-elastic curve in $\mathbf{S}^2(G)$. Moreover, if $(u,v) \in \mathcal{S}$ then (u,v) is a closed p-elastic curve in $\mathbf{S}^2(G)$.*

Proof. Let $c(w,t) = (u(w,t), v(w,t)) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ be a variation of (u,v) which satisfies (2.5) for each $w \in (-\varepsilon, \varepsilon)$. Let L be the total length of $(u(0,\cdot), v(0,\cdot))$. We consider that s is the arclength parameter of $(u(0,\cdot), v(0,\cdot))$ defined as in the beginning of this section. As we stated, we write $u(w,s)$ and $v(w,s)$ instead of $u(w,t(s))$ and $v(w,t(s))$, respectively. For the sake of completeness, we note that s is generally not an arclength parameter of $(u(w,\cdot), v(w,\cdot))$ for $w \neq 0$, and that the meanings of $u_{ws}(w,s)$ and $v_{ws}(w,s)$ are $u_{wt}(w,t(s))t_s(s)$ and $v_{wt}(w,t(s))t_s(s)$, respectively. From (2.5), we have

$$\begin{aligned} &u_w(0,0) = u_w(0,L) = v_w(0,0) = v_w(0,L) = 0, \\ &u_{ws}(0,0) = v_{ws}(0,0) = u_{ws}(0,L) = v_{ws}(0,L) = 0, \end{aligned}$$

and hence

$$\Big[Y_1(s)u_w(0,s) + Y_2(s)v_w(0,s) + Y_3(s)u_{ws}(0,s) + Y_4(s)v_{ws}(0,s)\Big]_0^L = 0.$$

So, by Theorem 1, we can find that (u,v) is a p-elastic curve in $\mathbf{S}^2(G)$.

Next, let $(u,v) \in \mathcal{S}$ and assume $c(w,\cdot) \in \mathcal{S}$ for each $w \in (-\varepsilon, \varepsilon)$. From

$$Y_i(0) = Y_i(L) \quad \text{for each } i = 1,2,3,4,$$

where Y_1, Y_2, Y_3 and Y_4 are as in Theorem 1, and

$$\begin{aligned} u_w(0,0) &= u_w(0,L), & v_w(0,0) &= v_w(0,L), \\ u_{ws}(0,0) &= v_{ws}(0,0), & u_{ws}(0,L) &= v_{ws}(0,L), \end{aligned}$$

we can see that $(u,v) \in \mathcal{S}$ is a closed p-elastic curve in $\mathbf{S}^2(G)$. $\square$

Now, we give the proof of Theorem 1.

Proof of Theorem 1. Let $c(w,t) = (u(w,t), v(w,t)) : (-\varepsilon, \varepsilon) \times [0,1] \to \mathbf{S}^2(G)$ be a variation of (u,v). We put

$$\begin{aligned} l(w,t) &= \Big(r^2 \sin^2 v(w,t) u_t(w,t)^2 + r^2 v_t(w,t)^2\Big)^{\frac{1}{2}}, \\ m(w,t) &= r^2 \Big[-u_{tt}(w,t)v_t(w,t)\sin v(w,t) + u_t(w,t)v_{tt}(w,t)\sin v(w,t) \\ &\qquad - 2u_t(w,t)v_t(w,t)^2 \cos v(w,t) - u_t(w,t)^3 \sin^2 v(w,t) \cos v(w,t)\Big]. \end{aligned}$$

Let s be the arclength parameter of $(u(0,\cdot), v(0,\cdot))$ defined as in the beginning of this section. We write $u(w,s)$ and $v(w,s)$ instead of $u(w,t(s))$ and

$v(w,t(s))$, respectively. We define $\tilde{l}(w,s)$, $\tilde{m}(w,s)$ by

$$\begin{aligned}\tilde{l}(w,s) &= \left(r^2 \sin^2 v(w,s) u_s(w,s)^2 + r^2 v_s(w,s)^2\right)^{\frac{1}{2}},\\ \tilde{m}(w,s) &= r^2 \Big[-u_{ss}(w,s)v_s(w,s)\sin v(w,s) + u_s(w,s)v_{ss}(w,s)\sin v(w,s)\\ &\quad - 2u_s(w,s)v_s(w,s)^2 \cos v(w,s) - u_s(w,s)^3 \sin^2 v(w,s)\cos v(w,s)\Big].\end{aligned}$$

From

$$\begin{aligned}u_s(w,s) &= u_t(w,t(s))t_s(s),\\ u_{ss}(w,s) &= u_{tt}(w,t(s))(t_s(s))^2 + u_t(w,t(s))t_{ss}(s),\\ v_s(w,s) &= v_t(w,t(s))t_s(s),\\ v_{ss}(w,s) &= v_{tt}(w,t(s))(t_s(s))^2 + v_t(w,t(s))t_{ss}(s),\end{aligned}$$

we can see

$$\begin{aligned}&\tilde{l}(w,s) = l(w,t(s))t_s(s), \quad \tilde{m}(w,s) = m(w,t(s))\,(t_s(s))^3,\\ &\tilde{l}(0,s) = 1, \quad \tilde{m}(0,s) = \kappa(s).\end{aligned}$$

Using (2.2) and Lemma 2, we have

$$\begin{aligned}\frac{\partial \tilde{l}}{\partial w}(0,s) = r^2 \Big[&u_s(s)^2 \sin v(s)\cos v(s) v_w(0,s)\\ &+ u_s(s)\sin^2 v(s) u_{ws}(0,s) + v_s(s)v_{ws}(0,s)\Big]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{m}}{\partial w}(0,s) &= r^2\Big(-u_{ss}(0,s)v_s(0,s)\cos v(0,s) + u_s(0,s)v_{ss}(0,s)\cos v(0,s)\\ &\quad + 2u_s(0,s)v_s(0,s)^2 \sin v(0,s) - 2u_s(0,s)^3 \sin v(0,s)\cos^2 v(0,s)\\ &\quad + u_s(0,s)^3 \sin^3 v(0,s)\Big)v_w(0,s) + r^2\big(v_{ss}(0,s)\sin v(0,s)\\ &\quad - \cos v(0,s)(2v_s(0,s)^2 + 3u_s(0,s)^2\sin^2 v(0,s))\big)u_{sw}(0,s)\\ &\quad - r^2(u_{ss}(0,s)\sin v(0,s) + 4u_s(0,s)v_s(0,s)\cos v(0,s))v_{sw}(0,s)\\ &\quad - r^2 v_s(0,s)\sin v(0,s)u_{ssw}(0,s) + r^2 u_s(0,s)\sin v(0,s)v_{ssw}(0,s)\\ &= r^2\Big(v_s(s)^2(2u_s(s)\cos v(s) + \kappa(s))\cos v(s)\\ &\quad + u_s(s)^2 \sin v(s)(u_s(s)\cos v(s) + \kappa(s))\cos v(s)\\ &\quad + 2u_s(s)v_s(s)^2 \sin v(s) - 2u_s(s)^3 \sin v(s)\cos^2 v(s)\\ &\quad + u_s(s)^3\sin^3 v(s)\Big)v_w(0,s) + r^2\Big(u_s(s)\sin^2 v(s)(u_s(s)\cos v(s)\\ &\quad + \kappa(s)) - \cos v(s)(2v_s(s)^2 + 3u_s(s)^2\sin^2 v(s))\Big)u_{sw}(0,s)\end{aligned}$$

$$
\begin{aligned}
&+ r^2(v_s(s)(2u_s(s)\cos v(s) + \kappa(s)) - 4u_s(s)v_s(s)\cos v(s))v_{sw}(0,s) \\
&- r^2 v_s(s)\sin v(s) u_{ssw}(0,s) + r^2 u_s(s)\sin v(s) v_{ssw}(0,s) \\
= {}& r^2\Big[\frac{\kappa(s)}{r^2}\cot v(s) + 2u_s(s)v_s(s)^2\csc v(s) \\
&- u_s(s)^3\sin v(s)\cos 2v(s)\Big] v_w(0,s) \\
&+ r^2\Big[\kappa(s)u_s(s)\sin^2 v(s) - \frac{2}{r^2}\cos v(s)\Big]u_{ws}(0,s) \\
&+ r^2 v_s(s)\big[\kappa(s) - 2u_s(s)\cos v(s)\big]v_{ws}(0,s) \\
&- r^2 v_s(s)\sin v(s) u_{wss}(0,s) + r^2 u_s(s)\sin v(s) v_{wss}(0,s).
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
\frac{dJ(c(w,\cdot))}{dw}\bigg|_{w=0} &= \frac{d}{dw}\int_0^1 \big(l(w,t)^{1-3p}|m(w,t)|^p + \lambda l(w,t)\big)\,dt\Big|_{w=0} \\
&= \int_0^1 \bigg((1-3p)l(0,t)^{-3p}\frac{\partial l}{\partial w}(0,t)|m(0,t)|^p \\
&\quad + p l(0,t)^{1-3p}|m(0,t)|^{p-2}m(0,t)\frac{\partial m}{\partial w}(0,t) + \lambda\frac{\partial l}{\partial w}(0,t)\bigg)dt \\
&= \int_0^L \bigg((1-3p)\frac{\partial \tilde{l}}{\partial w}(0,s)|\kappa(s)|^p \\
&\quad + p|\kappa(s)|^{p-2}\kappa(s)\frac{\partial \tilde{m}}{\partial w}(0,s) + \lambda\frac{\partial \tilde{l}}{\partial w}(0,s)\bigg)ds \\
&= r^2\int_0^L \bigg(((1-3p)|\kappa(s)|^p + \lambda)\Big[u_s(s)^2\sin v(s)\cos v(s) v_w(0,s) \\
&\quad + u_s(s)\sin^2 v(s) u_{ws}(0,s) + v_s(s)v_{ws}(0,s)\Big] + \Big[\frac{\kappa(s)}{r^2}\cot v(s) \\
&\quad + 2u_s(s)v_s(s)^2\csc v(s) - u_s(s)^3\sin v(s)\cos 2v(s)\Big]v_w(0,s) \\
&\quad + \Big[\kappa(s)u_s(s)\sin^2 v(s) - \frac{2}{r^2}\cos v(s)\Big]u_{ws}(0,s) \\
&\quad + v_s(s)\big[\kappa(s) - 2u_s(s)\cos v(s)\big]v_{ws}(0,s) \\
&\quad - v_s(s)\sin v(s)u_{wss}(0,s) + u_s(s)\sin v(s)v_{wss}(0,s)\bigg)ds \\
&= r^2\int_0^L \Big(A(s)v_w(0,s) + B(s)u_{ws}(0,s) + C(s)v_{ws}(0,s) \\
&\quad + D(s)u_{wss}(0,s) + E(s)v_{wss}(0,s)\Big)ds,
\end{aligned}
$$

where

$$
\begin{aligned}
A(s) =&|\kappa(s)|^p\Big[(1-3p)u_s(s)^2\sin v(s)\cos v(s)+\frac{p}{r^2}\cot v(s)\Big]\\
&+|\kappa(s)|^{p-2}\kappa(s)\Big[2pu_s(s)v_s(s)^2\csc v(s)-pu_s(s)^3\sin v(s)\cos 2v(s)\Big]\\
&+\lambda u_s(s)^2\sin v(s)\cos v(s),\\
B(s) =&-(2p-1)|\kappa(s)|^p u_s(s)\sin^2 v(s)-\frac{2}{r^2}p|\kappa(s)|^{p-2}\kappa(s)\cos v(s)\\
&+\lambda u_s(s)\sin^2 v(s),\\
C(s) =&-(2p-1)|\kappa(s)|^p v_s(s)-2p|\kappa(s)|^{p-2}\kappa(s)u_s(s)v_s(s)\cos v(s)+\lambda v_s(s),\\
D(s) =&-p|\kappa(s)|^{p-2}\kappa(s)v_s(s)\sin v(s)=Y_3(s),\\
E(s) =&p|\kappa(s)|^{p-2}\kappa(s)u_s(s)\sin v(s)=Y_4(s).
\end{aligned}
$$

Since $|\kappa(s)|^{p-2}\kappa(s)$ is of class C^2, $|\kappa(s)|^p$ is of class C^1. Hence, $A(s)$, $B(s)$, $C(s)$ are of class C^1. From Lemma 2, we have

$$
\begin{cases}
u_{ss}(s)|\kappa(s)|^{p-2}\kappa(s)=-v_s(s)\csc v(s)\big(2u_s(s)\cos v(s)|\kappa(s)|^{p-2}\kappa(s)+|\kappa(s)|^p\big),\\
v_{ss}(s)|\kappa(s)|^{p-2}\kappa(s)=u_s(s)\sin v(s)\big(u_s(s)\cos v(s)|\kappa(s)|^{p-2}\kappa(s)+|\kappa(s)|^p\big).
\end{cases}
$$

So, we can find that $u_{ss}(s)|\kappa(s)|^{p-2}\kappa(s)$ and $v_{ss}(s)|\kappa(s)|^{p-2}\kappa(s)$ are of class C^1, and hence $D(s)$, $E(s)$ are of class C^2. Then, we obtain

$$
\begin{aligned}
\frac{1}{r^2}\frac{dJ(c(w,\cdot))}{dw}\bigg|_{w=0}=&\int_0^L\Big(\big(-B_s(s)+D_{ss}(s)\big)u_w(0,s)\\
&\quad+\big(A(s)-C_s(s)+E_{ss}(s)\big)v_w(0,s)\Big)ds\\
&+\Big[\big(B(s)-D_s(s)\big)u_w(0,s)+\big(C(s)-E_s(s)\big)v_w(0,s)\Big]_0^L\\
&\quad+\Big[D(s)u_{ws}(0,s)+E(s)v_{ws}(0,s)\Big]_0^L.
\end{aligned}
$$

Using Lemma 2, we have

$$
\begin{aligned}
D_s(s)=&-p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)\sin v(s)\\
&-p|\kappa(s)|^{p-2}\kappa(s)u_s(s)\sin^2 v(s)(u_s(s)\cos v(s)+\kappa(s))\\
&-p|\kappa(s)|^{p-2}\kappa(s)v_s(s)^2\cos v(s)\\
=&-p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)\sin v(s)\\
&-\frac{p}{r^2}|\kappa(s)|^{p-2}\kappa(s)\cos v(s)-p|\kappa(s)|^p u_s(s)\sin^2 v(s),
\end{aligned}
$$

and hence we obtain

$$\begin{aligned}B(s)-D_s(s)&=-(p-1)|\kappa(s)|^p u_s(s)\sin^2 v(s)-\frac{p}{r^2}|\kappa(s)|^{p-2}\kappa(s)\cos v(s)\\&\quad+p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)\sin v(s)+\lambda u_s(s)\sin^2 v(s)=Y_1(s).\end{aligned}$$

From Lemma 2 and $(p-1)(|\kappa(s)|^p)_s=p\kappa(s)(|\kappa(s)|^{p-2}\kappa(s))_s$, we can see

$$\begin{aligned}-B_s(s)+D_{ss}(s)&=p\kappa(s)(|\kappa(s)|^{p-2}\kappa(s))_s u_s(s)\sin^2 v(s)\\&\quad-(p-1)|\kappa(s)|^p v_s(s)(2u_s(s)\cos v(s)+\kappa(s))\sin v(s)\\&\quad+2(p-1)|\kappa(s)|^p u_s(s)v_s(s)\sin v(s)\cos v(s)\\&\quad+\frac{p}{r^2}(|\kappa(s)|^{p-2}\kappa(s))_s\cos v(s)\\&\quad-\frac{p}{r^2}|\kappa(s)|^{p-2}\kappa(s)v_s(s)\sin v(s)\\&\quad-p(|\kappa(s)|^{p-2}\kappa(s))_{ss}v_s(s)\sin v(s)\\&\quad-p(|\kappa(s)|^{p-2}\kappa(s))_s u_s(s)\sin^2 v(s)(u_s(s)\cos v(s)+\kappa(s))\\&\quad-p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)^2\cos v(s)\\&\quad+\lambda v_s(s)(2u_s(s)\cos v(s)+\kappa(s))\sin v(s)\\&\quad-2\lambda u_s(s)v_s(s)\sin v(s)\cos v(s)\\&=-X(s)v_s(s)\sin v(s).\end{aligned}$$

Similarly, we have

$$\begin{aligned}C(s)-E_s(s)&=-(p-1)|\kappa(s)|^p v_s(s)-p|\kappa(s)|^{p-2}\kappa(s)u_s(s)v_s(s)\cos v(s)\\&\quad-p(|\kappa(s)|^{p-2}\kappa(s))_s u_s(s)\sin v(s)+\lambda v_s(s)=Y_2(s),\\-C_s(s)+E_{ss}(s)&=p\kappa(s)(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)\\&\quad+(p-1)|\kappa(s)|^p u_s(s)\sin v(s)(u_s(s)\cos v(s)+\kappa(s))\\&\quad+p(|\kappa(s)|^{p-2}\kappa(s))_s u_s(s)v_s(s)\cos v(s)\\&\quad-p|\kappa(s)|^{p-2}\kappa(s)v_s(s)^2(2u_s(s)\cos v(s)+\kappa(s))\cot v(s)\\&\quad+p|\kappa(s)|^{p-2}\kappa(s)u_s(s)^2\sin v(s)(u_s(s)\cos v(s)\\&\quad+\kappa(s))\cos v(s)-p|\kappa(s)|^{p-2}\kappa(s)u_s(s)v_s(s)^2\sin v(s)\\&\quad+p(|\kappa(s)|^{p-2}\kappa(s))_{ss}u_s(s)\sin v(s)\\&\quad-p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)(2u_s(s)\cos v(s)+\kappa(s))\sin v(s)\\&\quad+p(|\kappa(s)|^{p-2}\kappa(s))_s u_s(s)v_s(s)\cos v(s)\\&\quad-\lambda u_s(s)\sin v(s)(u_s(s)\cos v(s)+\kappa(s))\end{aligned}$$

$$\begin{aligned}
&= p(|\kappa(s)|^{p-2}\kappa(s))_{ss}u_s(s)\sin v(s) + (p-1)|\kappa(s)|^p\kappa(s)u_s(s)\sin v(s)\\
&\quad + |\kappa(s)|^p\big((2p-1)u_s(s)^2\sin v(s)\cos v(s) - pv_s(s)^2\cot v(s)\big)\\
&\quad + |\kappa(s)|^{p-2}\kappa(s)\big(-2pu_s(s)v_s(s)^2\cos v(s)\cot v(s)\\
&\quad + pu_s(s)^3\sin v(s)\cos^2 v(s) - pu_s(s)v_s(s)^2\sin v(s)\big)\\
&\quad - \lambda\kappa u_s(s)\sin v(s) - \lambda u_s(s)^2\sin v(s)\cos v(s),
\end{aligned}$$

and hence we obtain

$$\begin{aligned}
&A(s) - C_s(s) + E_{ss}(s)\\
&\quad= p(|\kappa(s)|^{p-2}\kappa(s))_{ss}u_s(s)\sin v(s) + (p-1)|\kappa(s)|^p\kappa(s)u_s(s)\sin v(s)\\
&\qquad + |\kappa(s)|^p\Big[(1-3p)u_s(s)^2\sin v(s)\cos v(s) + \frac{p}{r^2}\cot v(s)\\
&\qquad + (2p-1)u_s(s)^2\sin v(s)\cos v(s) - pv_s(s)^2\cot v(s)\Big]\\
&\qquad + |\kappa(s)|^{p-2}\kappa(s)\Big[2pu_s(s)v_s(s)^2\csc v(s) - pu_s(s)^3\sin v(s)\cos 2v(s)\\
&\qquad - 2pu_s(s)v_s(s)^2\cos v(s)\cot v(s) + pu_s(s)^3\sin v(s)\cos^2 v(s)\\
&\qquad - pu_s(s)v_s(s)^2\sin v(s)\Big] - \lambda\kappa u_s(s)\sin v(s)\\
&\quad= X(s)u_s(s)\sin v(s).
\end{aligned}$$

Therefore, we have shown our assertion. □

3. Realization theorem of $X(s) \equiv 0$ in $\mathrm{S}^2(G)$

In this section, we will find a condition that $(u, v) \in \mathcal{D}$ satisfies $X(s) \equiv 0$, where X is the function given in Theorem 1. That is, for a given $\kappa(s)$ which satisfies $X(s) \equiv 0$, we give a p-elastic curve in $\mathbf{S}^2(G)$ whose curvature is κ. If we set $\kappa(s) = |\omega(s)|^{\frac{2-p}{p-1}}\omega(s)$, then $X(s) = 0$ is transformed to

$$p\omega_{ss}(s) + (p-1)|\omega(s)|^{\frac{2}{p-1}}\omega(s) + Gp\omega(s) - \lambda|\omega(s)|^{\frac{2-p}{p-1}}\omega(s) = 0. \tag{3.1}$$

We study the solutions of (3.1). Multiplying $2p\omega_s(s)$ to (3.1) and integrating it, we have

$$p^2\omega_s(s)^2 + (p-1)^2|\omega(s)|^{\frac{2p}{p-1}} + Gp^2\omega(s)^2 - 2\lambda(p-1)|\omega(s)|^{\frac{p}{p-1}} = d$$

with some constant d. We define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(\omega) = (p-1)^2|\omega|^{\frac{2p}{p-1}} + Gp^2\omega^2 - 2\lambda(p-1)|\omega|^{\frac{p}{p-1}} \quad \text{for } \omega \in \mathbb{R}.$$

We note that F is even and

$$F_\omega(\omega) = 2p(p-1)|\omega|^{\frac{2}{p-1}}\omega + 2Gp^2\omega - 2\lambda p|\omega|^{\frac{2-p}{p-1}}\omega \quad \text{for } \omega \in \mathbb{R}.$$

Here, we consider a differential equation

$$\begin{cases} 2p^2\omega_{ss}(s) + F_\omega(\omega(s)) = 0, & s \in \mathbb{R}, \\ \omega_s(0) = 0, \quad \omega(0) = \omega_0 \in \mathbb{R}. \end{cases} \tag{3.2}$$

We note that the first line in (3.2) is the same as (3.1).

In order to study (3.2), we investigate the behavior of F. We define

$$H(\omega) = (p-1)|\omega|^{\frac{p}{p-1}} + Gp|\omega|^{\frac{p-2}{p-1}} - \lambda \qquad \text{for } \omega \in \mathbb{R}. \tag{3.3}$$

We note it holds

$$F_\omega(\omega) = 2p|\omega|^{\frac{2-p}{p-1}}\omega H(\omega) \qquad \text{for } \omega \in \mathbb{R} \tag{3.4}$$

and both (3.1) and the first line in (3.2) are the same as

$$p\omega_{ss}(s) + |\omega(s)|^{\frac{2-p}{p-1}}\omega(s)H(\omega(s)) = 0, \quad s \in \mathbb{R}.$$

Recall that we always assume $p > 2$ and $\lambda > 0$. Hence, by (3.3), we can easily see that $H(\omega) = 0$ has exactly one positive root, and we put it $\omega_{1;\lambda}$. For the sake of completeness, we note that

$$\{\omega \in \mathbb{R} \mid F_\omega(\omega) = 0\} = \{0, \pm\omega_{1;\lambda}\},$$

and the following holds.

Lemma 3. *F has local extremes as follows: 0 is a local maximizer, $\pm\omega_{1;\lambda}$ are global minimizers, and F does not have any other local extreme.*

Proof. We note it holds for $\omega > 0$,

$$F_\omega(\omega) = 2p\omega^{\frac{1}{p-1}}H(\omega), \qquad H_\omega(\omega) = p\omega^{\frac{1}{p-1}} + G\frac{p(p-2)}{p-1}\omega^{-\frac{1}{p-1}},$$

and $H(0) = -\lambda < 0$. Since H is monotone increasing on $(0,\infty)$, F_ω changes its sign only once from minus to plus at $\omega = \omega_{1;\lambda}$ on the interval $(0,\infty)$. Thus, F takes its minimum at $\omega = \pm\omega_{1;\lambda}$. Hence, we can easily see that our assertion holds. □

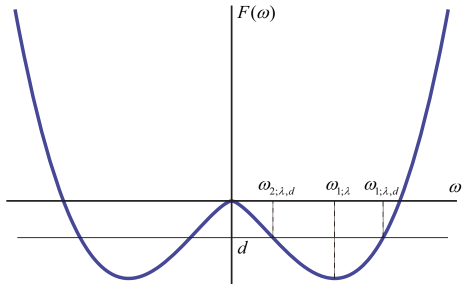

Figure 1: The Graph of F in the case $p = 3$, $\lambda = 7$ and $G = 1$.

For reader's convenience, we give a figure of the function F. The definitions of $\omega_{1;\lambda,d}$, $\omega_{2;\lambda,d}$ in the figure are given as follows. We set

$$\min F = \min_{\omega \in \mathbb{R}} F(\omega)(= F(\omega_{1;\lambda})).$$

From Lemma 3, we see that for each $d \geq \min F$, the equation $F(\omega) = d$ has at most two positive roots and at least one positive root. We put these roots $\omega_{1;\lambda,d} > \omega_{2;\lambda,d}$ as long as they exist. In other words, we set

$$\{\omega > 0 \mid F(\omega) = d\} = \begin{cases} \{\omega_{1;\lambda,d}\} & \text{in the case } d > 0 \text{ or } d = \min F, \\ \{\omega_{1;\lambda,d}, \omega_{2;\lambda,d}\} & \text{in the case } \min F < d < 0. \end{cases}$$

Now, we study the solutions of (3.2) in Lemmas 4 and 5. In particular, we will show that problem (3.2) has multiple solutions in the case $F(\omega_0) = 0$.

Lemma 4. *Let $\omega_0 \in \mathbb{R}$ such that*

$$F(\omega_0) \neq 0 \quad \textit{and} \quad F(\omega_0) \neq F(\omega_{1;\lambda}). \tag{3.5}$$

Then, (3.2) has a unique solution defined on $\mathbb{R}$, and it is non constant and periodic.

Proof. From the evenness of F and $\omega_0 \neq 0$, without loss of generality, we may assume $\omega_0 > 0$. From

$$F_{\omega\omega}(\omega) = 2p\left((p+1)|\omega|^{\frac{2}{p-1}} + Gp - \frac{\lambda}{p-1}|\omega|^{\frac{2-p}{p-1}}\right) \qquad \text{for } \omega \neq 0,$$

we can find the local Lipschitz property of F_ω as follows:

(L) F_ω is locally Lipschitz on $\mathbb{R} \setminus \{0\}$ and it is not locally Lipschitz at 0.

Since (3.2) is equivalent to the initial value problem of the system of first order differential equations

$$\begin{pmatrix} \omega_s(s) \\ \phi_s(s) \end{pmatrix} = \begin{pmatrix} \phi(s) \\ -F_\omega(\omega(s))/(2p^2) \end{pmatrix} \quad \text{for } s \in \mathbb{R} \quad \text{and} \quad \begin{pmatrix} \omega(0) \\ \phi(0) \end{pmatrix} = \begin{pmatrix} \omega_0 \\ 0 \end{pmatrix},$$

from (L) and $\omega_0 > 0$, we can see that (3.2) has a unique solution in a neighborhood of $s = 0$. By a similar argument in the beginning of this section, we see that each solution of (3.2) satisfies

$$p^2 \omega_s(s)^2 = F(\omega_0) - F(\omega(s)) \tag{3.6}$$

as long as it exists. Let I be the maximal interval such that (3.2) has a unique solution on I. We will show $I = (-\infty, \infty)$. To the contrary, without loss of generality, we may assume $0 < s_1 < \infty$, where $s_1 = \sup I$. From (3.6), we have

$$F(\omega_0) \geq F(\omega(s)) \quad \text{for each } s \in I. \tag{3.7}$$

In the case $F(\omega_0) < 0$, from (L), we can easily see that the solution is uniquely extendable on an interval $I \cup [s_1, s_1 + \varepsilon_1)$ with some $\varepsilon_1 > 0$. This is a contradiction. So, we consider the case $F(\omega_0) > 0$. If $\omega(s_1) \neq 0$, then from (L), we can easily see the solution is uniquely extendable on an interval $I \cup [s_1, s_1 + \varepsilon_2)$ with some $\varepsilon_2 > 0$. Hence, without loss of generality, we may assume $\omega(s) > 0$ for each $s \in [0, s_1)$ and $\omega(s_1) = 0$. From (3.6) and (3.7), we have $\omega_s(s) < 0$ for each $s \in (0, s_1]$. We note that in a small open interval containing s_1 on which ω_s does not vanish, the differential equations $2p^2\omega_{ss}(s) + F_\omega(\omega(s)) = 0$ and $p^2\omega_s(s)^2 = F(\omega_0) - F(\omega(s))$ are equivalent. Thus we have

$$\omega_s(s) = -\frac{1}{p}\sqrt{F(\omega_0) - F(\omega(s))} \quad \text{for } s \in [0, s_1].$$

From $F(\omega_0) > 0$, the mapping $\omega \mapsto \sqrt{F(\omega_0) - F(\omega)}$ is locally Lipschitz at $\omega = 0$, Hence, we can see that the solution is uniquely extendable on an interval $I \cup [s_1, s_1 + \varepsilon_3)$ with some $\varepsilon_3 > 0$. This is also a contradiction. Therefore, we can see $I = (-\infty, \infty)$, i.e., under assumption (3.5), problem (3.2) has a unique solution.

Next, we will show that under assumption (3.5), each solution of (3.2) is non constant and periodic. From (3.5), we have $F_\omega(\omega_0) \neq 0$, and hence the unique solution of (3.2) is non constant. We recall we assumed $\omega_0 > 0$. We set ω_1 by

$$\omega_1 = \begin{cases} -\omega_{1;\lambda,F(\omega_0)} & \text{if } F(\omega_0) > 0; \text{ (we note } \omega_0 = \omega_{1;\lambda,F(\omega_0)} \text{ in this case),} \\ \omega_{2;\lambda,F(\omega_0)} & \text{if } F(\omega_0) < 0 \text{ and } \omega_0 = \omega_{1;\lambda,F(\omega_0)}, \\ \omega_{1;\lambda,F(\omega_0)} & \text{if } F(\omega_0) < 0 \text{ and } \omega_0 = \omega_{2;\lambda,F(\omega_0)}. \end{cases}$$

We will show that there is $s_0 \in (0, \infty]$ such that $\omega(s) \neq \omega_1$ for each $s \in (0, s_0)$ and $\omega(s) \to \omega_1$ as $s \to s_0$. If such $s_0 \in (0, \infty]$ does not exist, from (3.6), we can see $\omega_s(s) \neq 0$ for each $s > 0$, and hence $\inf\{|\omega_s(s)| : s \in (0, \infty)\} > 0$. Since ω is bounded by (3.6), it is a contradiction. Hence such s_0 exists. We will show that s_0 is finite. Once $0 < s_0 < \infty$ was shown, we can see that ω is periodic and $2s_0$ is the minimal period of ω. In the case $\omega_1 < \omega_0$, i.e., in the case $\omega_0 = \omega_{1;\lambda,F(\omega_0)}$, we have $F_\omega(\omega_0) > 0$, $F_\omega(\omega_1) < 0$,

$$\frac{1}{\sqrt{F(\omega_0) - F(\omega)}} = \frac{1}{\sqrt{F_\omega(\omega_0)}} \big(1 + O(\omega_0 - \omega)\big)(\omega_0 - \omega)^{-\frac{1}{2}} \quad \text{as } \omega \to \omega_0 - 0,$$

and

$$\begin{aligned} \frac{1}{\sqrt{F(\omega_0) - F(\omega)}} &= \frac{1}{\sqrt{F(\omega_1) - F(\omega)}} \\ &= \frac{1}{\sqrt{-F_\omega(\omega_1)}} \big(1 + O(\omega - \omega_1)\big)(\omega - \omega_1)^{-\frac{1}{2}} \end{aligned}$$

as $\omega \to \omega_1 + 0$. Using (3.6), we see that

$$s_0 = \int_0^{s_0} ds = \int_{\omega_0}^{\omega_1} \frac{ds}{d\omega}\, d\omega = \int_{\omega_0}^{\omega_1} \frac{p}{\sqrt{F(\omega_0) - F(\omega)}}\, d\omega$$

is finite. In the case $\omega_0 < \omega_1$, i.e., in the case $\omega_0 = \omega_{2;\lambda,F(\omega_0)}$, we can show $s_0 < \infty$ similarly. Hence, we can see that the unique solution of (3.2) oscillates between ω_0 and ω_1 and it is periodic. □

Lemma 5. *The following hold.*

(i) *If* $F(\omega_0) = 0$*, then problem* (3.2) *has multiple solutions defined on* $\mathbb{R}$.

(ii) *If* $F(\omega_0) = F(\omega_{1;\lambda})$*, then problem* (3.2) *has a unique solution defined on* $\mathbb{R}$ *and it is constant.*

Proof. We can easily see (ii) by Lemma 3 and (3.6). We will show (i). Let $\omega_0 \in \mathbb{R}$ with $F(\omega_0) = 0$. Without loss of generality, we may assume $\omega_0 > 0$.

By similar arguments as in the proof of the previous lemma, we can see that there is $s_0 \in (0, \infty]$ such that $\omega(s) > 0$ for each $s \in [0, s_0)$ and $\omega(s) \to 0$ as $s \to s_0$. We will show that s_0 is finite. Noting $F_\omega(\omega_0) > 0$, we have

$$\begin{aligned}\frac{1}{\sqrt{-F(\omega)}} &= \frac{1}{\sqrt{F(\omega_0) - F(\omega)}} \\ &= \frac{1}{\sqrt{F_\omega(\omega_0)}}(\omega_0 - \omega)^{-\frac{1}{2}} + O(\omega_0 - \omega) \quad \text{as } \omega \to \omega_0 - 0.\end{aligned}$$

Since $p/(2(p-1)) < 1$ and

$$\frac{1}{\sqrt{-F(\omega)}} = \frac{1}{\omega^{\frac{p}{2(p-1)}}\sqrt{2\lambda(p-1) - (p-1)^2\omega^{\frac{p}{p-1}} - Gp^2\omega^{\frac{p-2}{p-1}}}} \quad \text{for } \omega \in (0, \omega_0),$$

we can see that

$$s_0 = p\int_0^{\omega_0} \frac{d\omega}{\sqrt{-F(\omega)}} \tag{3.8}$$

is finite. Hence, for example, both $\hat{\omega}(s)$ and $\tilde{\omega}(s)$ defined by

$$\hat{\omega}(s) = \begin{cases}\omega(s - 2ns_0) & \text{for } 2ns_0 \le s \le (2n+1)s_0 \text{ and } n \in \mathbb{Z}, \\ \omega(2ns_0 - s) & \text{for } (2n-1)s_0 \le s \le 2ns_0 \text{ and } n \in \mathbb{Z}\end{cases}$$

and

$$\tilde{\omega}(s) = \begin{cases}0 & \text{for } s \le -s_0, \\ \omega(-s) & \text{for } -s_0 \le s \le 0, \\ \omega(s) & \text{for } 0 \le s \le s_0, \\ 0 & \text{for } s \ge s_0\end{cases}$$

are solutions of (3.2). Thus, we have shown (i). □

The next lemma classifies the solutions ω of (3.2) with $\omega_0 \in \mathbb{R}$ and $d = F(\omega_0)$ into four types (I)–(IV) below. We note that such solutions satisfy

$$F(\omega_0) = \max_{s\in\mathbb{R}} F(\omega(s))$$

and that type (I) solutions exist only when $d = 0$ or $d = F(\omega_{1;\lambda})$, and types (II), (III) and (IV) solutions exist only when $d < 0$, $d > 0$ and $d = 0$, respectively.

(I) ($d = 0$ or $d = F(\omega_{1;\lambda})$) Constant solution $\omega \equiv \omega_0$. (ω_0 is one of 0 and $\pm\omega_{1;\lambda}$.)

(II) ($d < 0$) Positive or negative periodic solution which oscillates between $\omega_{1;\lambda,d}$ and $\omega_{2;\lambda,d}$ or $-\omega_{1;\lambda,d}$ and $-\omega_{2;\lambda,d}$. (For the sake of completeness, if ω oscillates between $\omega_{1;\lambda,0}$ and 0 or $-\omega_{1;\lambda,0}$ and 0, we do not call it type (II) solution, but we call it type (IV) solution; see (IV) below.)

(III) ($d > 0$) Sign changing periodic solution which oscillates between $-\omega_{1;\lambda,d}$ and $\omega_{1;\lambda,d}$.

(IV) ($d = 0$) Solution constructed along the following rule.
- It consists of the following (IV-i), (IV-ii) and (IV-iii).
- It includes at least one of (IV-ii) or (IV-iii).
- It is obtained by gluing (IV-i), (IV-ii) and (IV-iii) in arbitrary order.

(IV-i) Constant zero solution on $[s_0, s_1]$, where s_0, s_1 are any elements in $[-\infty, \infty]$ with $s_0 < s_1$. In the case $s_0 = -\infty$, $[-\infty, s_1]$ is considered as $(-\infty, s_1]$ and in the case $s_1 = \infty$, $[s_0, \infty]$ is considered as $[s_0, \infty)$.

(IV-ii) The solution of

$$\begin{cases} 2p^2\omega_{ss}(s) + F_\omega(\omega(s)) = 0, & s \in [s_0, s_0 + T], \\ \omega_s(s_0) = \omega(s_0) = 0, \\ \omega(s) > 0, & s \in (s_0, s_0 + T), \end{cases}$$

where s_0 is any real number and $T > 0$ is the constant given by

$$T = \int_0^{\omega_{1;\lambda,0}} \frac{2p}{\sqrt{-F(\omega)}} d\omega. \tag{3.9}$$

(IV-iii) The solution of

$$\begin{cases} 2p^2\omega_{ss}(s) + F_\omega(\omega(s)) = 0, & s \in [s_0, s_0 + T], \\ \omega_s(s_0) = \omega(s_0) = 0, \\ \omega(s) < 0, & s \in (s_0, s_0 + T), \end{cases}$$

where s_0 is any real number and $T > 0$ is the constant given in (3.9).

We sometimes call type (IV) solution of (3.2) a *flat-core solution.* Glued solutions such as (IV-ii), (IV-i)-(IV-ii)-(IV-iii)-(IV-i), (IV-ii)-(IV-iii), (IV-ii)-(IV-i)-(IV-iii) are examples of flat-core solutions. For the last one, we give a graph of a flat-core solution; see Figure 2. The following is a direct conse-

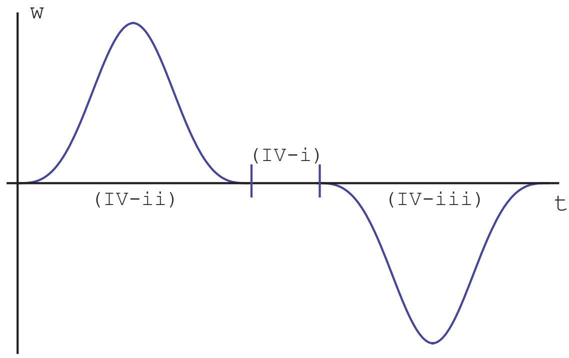

Figure 2: A graph of a flat-core solution of equation (3.2) composed by (IV-ii), (IV-i), (IV-iii); $p = 3$, $\lambda = 7$ and $G = 1$.

quence of Lemmas 4 and 5. We note that we can see the existence of flat-core solutions of (3.2) by Lemma 5 (i).

Lemma 6. *Let $d \geq \min F$. Then all the solutions ω of (3.2) with $d = F(\omega_0)$ are classified as in the following table.*

	d	solution type
(i)	$d = F(\omega_{1;\lambda})$	(I)×2 $(\omega = \pm\omega_{1;\lambda})$
(ii)	$F(\omega_{1;\lambda}) < d < 0$	(II)×2 $(-\omega_{1;\lambda,d} \leq \omega \leq -\omega_{2;\lambda,d},\ \omega_{2;\lambda,d} \leq \omega \leq \omega_{1;\lambda,d})$
(iii)	$d = 0$	(I) $(\omega = 0)$, (IV)
(iv)	$d > 0$	(III) $(-\omega_{1;\lambda,d} \leq \omega \leq \omega_{1;\lambda,d})$

Table 1: Table of solution types.

Figure 3 gives graphs of prototypical solutions of (3.2).

In order to realize a solution of $X(s) \equiv 0$ in $\mathbf{S}^2(G)$, we define d_λ by

$$d_\lambda = F\left(\left(\frac{\lambda}{p-1}\right)^{\frac{p-1}{p}}\right) = Gp^2\left(\frac{\lambda}{p-1}\right)^{\frac{2(p-1)}{p}} - \lambda^2\,(\geq \min F = F(\omega_{1;\lambda})).$$

The following lemma is crucial to realize solutions of $X(s) \equiv 0$ in $\mathbf{S}^2(G)$; see Theorem 3. Moreover, it gives information of the shapes of the graphs of the realized solutions in $\mathbf{S}^2(G)$; see Section 4.

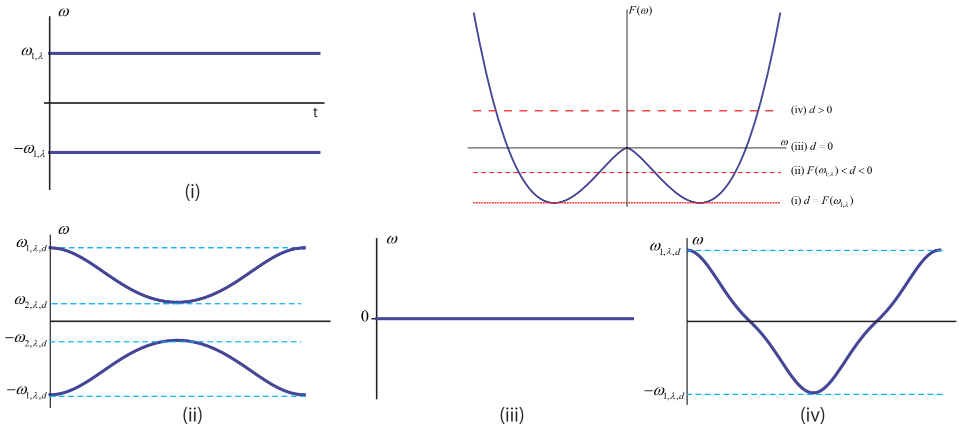

Figure 3: The graph of F in the case $p = 3$, $\lambda = 7$ and $G = 1$, and graphs of solutions (3.2) for (I)–(III).

Lemma 7. *Let* $d \geq \min F$*. Then* $d + \lambda^2 > 0$ *and*

$$
\begin{cases} \omega_{1;\lambda,d} < \sqrt{\dfrac{d+\lambda^2}{Gp^2}} < \left(\dfrac{\lambda}{p-1}\right)^{\frac{p-1}{p}} & \text{in the case } d < d_\lambda, \\ \omega_{1;\lambda,d} = \sqrt{\dfrac{d+\lambda^2}{Gp^2}} = \left(\dfrac{\lambda}{p-1}\right)^{\frac{p-1}{p}} & \text{in the case } d = d_\lambda, \\ \omega_{1;\lambda} < \left(\dfrac{\lambda}{p-1}\right)^{\frac{p-1}{p}} < \omega_{1;\lambda,d} < \sqrt{\dfrac{d+\lambda^2}{Gp^2}} & \text{in the case } d > d_\lambda. \end{cases} \tag{3.10}
$$

Proof. From $d \geq \min F$ and

$$
F(\omega) = \left((p-1)|\omega|^{\frac{p}{p-1}} - \lambda\right)^2 + Gp^2\omega^2 - \lambda^2, \tag{3.11}
$$

we have $d + \lambda^2 > 0$. From (3.11),

$$
F_\omega\left(\left(\frac{\lambda}{p-1}\right)^{\frac{p-1}{p}}\right) = 2Gp^2\left(\frac{\lambda}{p-1}\right)^{\frac{p-1}{p}} > 0, \tag{3.12}
$$

and the definitions of $\omega_{1;\lambda,d}$ and d_λ, we have

$$
\omega_{1;\lambda,d} = \left(\frac{\lambda}{p-1}\right)^{\frac{p-1}{p}} \Leftrightarrow d = d_\lambda.
$$

Hence, if $d \neq d_\lambda$, then $d+\lambda^2 = F(\omega_{1;\lambda,d}) + \lambda^2 > Gp^2\omega_{1;\lambda,d}^2$. So we have

$$\sqrt{\frac{d+\lambda^2}{Gp^2}} > \omega_{1;\lambda,d} \quad \text{if } d \neq d_\lambda, \qquad \text{and} \qquad \sqrt{\frac{d_\lambda+\lambda^2}{Gp^2}} = \omega_{1;\lambda,d_\lambda}. \tag{3.13}$$

On the other hand, since

$$\sqrt{\frac{d+\lambda^2}{Gp^2}} = \Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} \Leftrightarrow d = Gp^2\Big(\frac{\lambda}{p-1}\Big)^{\frac{2(p-1)}{p}} - \lambda^2 (= d_\lambda),$$

we can easily see

$$\begin{cases} \sqrt{\frac{d+\lambda^2}{Gp^2}} < \Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} & \text{if } d < d_\lambda, \\ \sqrt{\frac{d+\lambda^2}{Gp^2}} = \Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} & \text{if } d = d_\lambda, \\ \Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} < \sqrt{\frac{d+\lambda^2}{Gp^2}} & \text{if } d > d_\lambda. \end{cases} \tag{3.14}$$

Hence, from (3.13) and (3.14), we can see (3.10) except for the case $d > d_\lambda$. We consider the case $d > d_\lambda$. Since we have $\omega_{1;\lambda,d} \geq \omega_{1;\lambda}$, $F(\omega_{1;\lambda,d}) = d > d_\lambda = F((\lambda/(p-1))^{(p-1)/p})$ and (3.12), from Lemma 3, we obtain

$$\omega_{1;\lambda} < \Big(\frac{\lambda}{p-1}\Big)^{\frac{p-1}{p}} < \omega_{1;\lambda,d}.$$

Noting (3.13), we have shown (3.10) in the case $d > d_\lambda$. $\square$

Now, we give our realization theorems of $X(s) \equiv 0$ in $\mathbf{S}^2(G)$.

Theorem 2. *Let $\omega_0 \in \{0, \pm\omega_{1;\lambda}\}$, and let*

$$v_0 = \operatorname{arccot}\Big(-r|\omega_0|^{\frac{2-p}{p-1}}\omega_0\Big). \tag{3.15}$$

Then s represents an arclength parameter of the curve

$$\Big(\frac{s}{r\sin v_0}, v_0\Big) \tag{3.16}$$

in $\mathbf{S}^2(G)$, (3.16) *has a constant curvature* $|\omega_0|^{\frac{2-p}{p-1}}\omega_0$, *and* (3.16) *satisfies* $X(s) \equiv 0$. *In particular, for each* $s_0, s_1 \in \mathbb{R}$ *with* $s_0 \neq s_1$,

$$\left(\frac{s_0 + (s_1 - s_0)t}{r \sin v_0}, v_0\right), \qquad t \in [0,1] \tag{3.17}$$

is a p-elastic curve in $\mathbf{S}^2(G)$.

Proof. Let (u, v) has form (3.16). From (2.2), we can find that s represents its arclength parameter, and from (2.3), we can see that it has constant curvature $\kappa_0 = -(\cot v_0)/r$. By $\kappa_0 = -(\cot v_0)/r$ and (3.15), we have $\kappa_0 = |\omega_0|^{\frac{2-p}{p-1}}\omega_0$. From $\omega_0 \in \{0, \pm\omega_{1;\lambda}\}$, we can see

$$X(s) = p(|\kappa_0|^{p-2}\kappa_0)_{ss} + \kappa_0 H(|\kappa_0|^{p-2}\kappa_0) = |\omega_0|^{\frac{2-p}{p-1}}\omega_0 H(\omega_0) = 0.$$

Therefore, we have shown that our assertion holds. □

Theorem 3. *Let* $d \geq \min F$ *such that* $d \neq d_\lambda$. *Let* ω *be a solution of* (3.2) *with* $\omega_0 \in \mathbb{R}$ *and* $d = F(\omega_0)$, *and let* (u, v) *be a curve in* $\mathbf{S}^2(G)$ *satisfying*

$$v(s) = \arccos\left(-\sqrt{\frac{G}{d+\lambda^2}}p\omega(s)\right), \tag{3.18}$$

$$u_s(s) = \sqrt{\frac{G}{d+\lambda^2}} \cdot \frac{\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}}{1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2} \tag{3.19}$$

for $s \in \mathbb{R}$. *Then* s *represents an arclength parameter of the curve* (u, v), *i.e., it satisfies* (2.2), *the curvature* $\kappa(s)$ *of* (u, v) *satisfies*

$$\kappa(s) = |\omega(s)|^{\frac{2-p}{p-1}}\omega(s) \qquad (\textit{which is equivalent to } \omega(s) = |\kappa(s)|^{p-2}\kappa(s)), \tag{3.20}$$

and there hold

$$X(s) \equiv 0 \quad \textit{for each } s \in \mathbb{R}$$

and

$$0 < v(s) < \pi \quad \textit{for each } s \in \mathbb{R}. \tag{3.21}$$

In particular, for each $s_0, s_1 \in \mathbb{R}$ *with* $s_0 \neq s_1$, *the curve in* $\mathbf{S}^2(G)$ *defined by*

$$c(t) = (u(s_0 + (s_1 - s_0)t), v(s_0 + (s_1 - s_0)t)), \qquad t \in [0,1], \tag{3.22}$$

which belongs to $\mathcal{D}$, is a p-elastic curve in $\mathbf{S}^2(G)$, and it passes neither the north pole nor the south pole in the coordinate (2.1)*.*

Remark 2. In Theorem 3, if ω is constant, a curve (u, v) satisfying (3.18) and (3.19) is essentially the same as (3.16). We will show it. Assume that ω in the theorem is constant. From $X(s) \equiv 0$, we have $\omega(0) = 0$, or $\omega(0) \neq 0$ and $H(\omega(0)) = 0$. In the case $\omega(0) \neq 0$ and $H(\omega(0)) = 0$, we can see $\lambda - (p-1)|\omega(0)|^{\frac{p}{p-1}} = Gp|\omega(0)|^{\frac{p-2}{p-1}}$, v is a constant function, $\cos v = -\sqrt{G/(d+\lambda^2)}\, p\omega(0)$, and

$$\begin{aligned} u_s(s) &= \sqrt{\frac{G}{d+\lambda^2}} \cdot \frac{\lambda - (p-1)|\omega(0)|^{\frac{p}{p-1}}}{1 - \frac{Gp^2}{d+\lambda^2}\omega(0)^2} \\ &= -\frac{\cos v}{p\omega(0)} \cdot \frac{Gp|\omega(0)|^{\frac{p-2}{p-1}}}{\sin^2 v} = -\frac{1}{r\sin v} \cdot \frac{\cot v}{r|\omega(0)|^{\frac{2-p}{p-1}}\omega(0)}. \end{aligned}$$

From $\cos v = -\sqrt{G/(d+\lambda^2)}\, p\omega(0)$ and (3.10), we have

$$\begin{cases} 0 < v < \pi/2 & \text{in the case } \omega(0) < 0, \\ \pi/2 < v < \pi & \text{in the case } \omega(0) > 0. \end{cases}$$

So we have $u_s(s) > 0$. Since s represents an arclength parameter of (u, v) from Theorem 3, we obtain

$$1 = (r\sin v)u_s(s) = -\frac{\cot v}{r|\omega(0)|^{\frac{2-p}{p-1}}\omega(0)},$$

which yields (3.15) and $(u(s), v(s)) = (s/(r\sin v_0), v_0)$. In the case $\omega(0) = 0$, we have $u_s(s) \equiv \sqrt{G/(d+\lambda^2)}\,\lambda > 0$ and $v(s) \equiv \pi/2$, which implies (3.15) and $(u(s), v(s)) = (s/(r\sin v_0), v_0)$. Therefore, in both cases, we can see that our assertions hold.

Proof of Theorem 3. We note that the denominator $d + \lambda^2$ in (3.18) is positive by Lemma 7. Since $d \geq \min F$, $d \neq d_\lambda$ and $|\omega(s)| \leq \omega_{1;\lambda,d}$ for each $s \in \mathbb{R}$, from Lemma 7, we have

$$\sup_{s\in\mathbb{R}} \left| \sqrt{\frac{G}{d+\lambda^2}}\, p\,\omega(s) \right| < 1.$$

Thus the denominator in (3.19) is also positive and (3.21) is proved. We will show that s is an arclength parameter of (u, v). From (3.18), we have

$$\begin{aligned} &\sin v(s) v_s(s) = \sqrt{\frac{G}{d+\lambda^2}}\, p\, \omega_s(s), \\ &\sin^2 v(s) = 1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2. \end{aligned} \tag{3.23}$$

Using (3.6) and (3.11), we can see

$$\begin{aligned} &\big(\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}\big)^2 + p^2\omega_s(s)^2 \\ &\quad = \big(\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}\big)^2 + d - F(\omega(s)) \\ &\quad = d + \lambda^2 - Gp^2\omega(s)^2. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &r^2\big(\sin v(s)\big)^2 u_s(s)^2 + r^2 v_s(s)^2 \\ &\quad = r^2\left(1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2\right)\frac{G}{d+\lambda^2}\frac{\big(\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}\big)^2}{\big(1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2\big)^2} \\ &\qquad + r^2\frac{G}{d+\lambda^2}p^2\omega_s(s)^2\frac{1}{1 - \frac{Gp^2}{d+\lambda^2}\omega(s)^2} \\ &\quad = \frac{\big(\lambda - (p-1)|\omega(s)|^{\frac{p}{p-1}}\big)^2 + p^2\omega_s(s)^2}{d + \lambda^2 - Gp^2\omega(s)^2} = 1. \end{aligned}$$

Thus, we have shown that s is an arclength parameter of (u, v). Next, we will show (3.20). From Lemma 2, we have

$$\kappa(s) = \frac{-u_s(s)^2\cos v(s)\sin^2 v(s) + v_{ss}(s)\sin v(s)}{u_s(s)\big(\sin v(s)\big)^2}. \tag{3.24}$$

By differentiating the first equation in (3.23), we obtain

$$v_{ss}(s)\sin v(s) = -v_s(s)^2\cos v(s) + \sqrt{\frac{G}{d+\lambda^2}}\, p\, \omega_{ss}(s),$$

which, (2.2) and (3.1) yield

$$\begin{aligned}
&-u_s(s)^2\cos v(s)\sin^2 v(s)+v_{ss}(s)\sin v(s)\\
&=-u_s(s)^2\cos v(s)\sin^2 v(s)-v_s(s)^2\cos v(s)+\sqrt{\frac{G}{d+\lambda^2}}\,p\omega_{ss}(s)\\
&=-G\cos v(s)-\sqrt{\frac{G}{d+\lambda^2}}\Big((p-1)|\omega(s)|^{\frac{2}{p-1}}\omega(s)\\
&\qquad+Gp\omega(s)-\lambda|\omega(s)|^{\frac{2-p}{p-1}}\omega(s)\Big)\\
&=\sqrt{\frac{G}{d+\lambda^2}}\big(\lambda-(p-1)|\omega(s)|^{\frac{p}{p-1}}\big)|\omega(s)|^{\frac{2-p}{p-1}}\omega(s).
\end{aligned}$$

From (3.19) and the second equation in (3.23), we have

$$u_s(s)\sin^2 v(s)=\sqrt{\frac{G}{d+\lambda^2}}\big(\lambda-(p-1)|\omega(s)|^{\frac{p}{p-1}}\big),$$

which and (3.24) yield (3.20). Since $\omega(s)$ is of class C^2, so is $|\kappa(s)|^{p-2}\kappa(s)$. Thus the assumption of class C^2 of $|\kappa(s)|^{p-2}\kappa(s)$ in Corollary 1 (Theorem 1) is satisfied. From (3.6) and (3.20), we have

$$\begin{aligned}
p^2((|\kappa(s)|^{p-2}\kappa(s))_s)^2&=d-(p-1)^2|\kappa(s)|^{2p}\\
&\quad-Gp^2|\kappa(s)|^{2(p-1)}+2\lambda(p-1)|\kappa(s)|^p,
\end{aligned}\tag{3.25}$$

which yields $X(s)\equiv 0$. Hence from Corollary 1, we can see that (3.22) is a p-elastic curve in $\mathbf{S}^2(G)$. □

Remark 3. We will show how to find (3.18) and (3.19). Assume that v takes the form

$$\cos v(s)=C_1|\kappa(s)|^{p-2}\kappa(s).\tag{3.26}$$

In Theorem 1, we have shown that if s is an arclength parameter of the curve $(u,v)\in\mathcal{D}$ then $X(s)=0$ is sufficient to be that (u,v) is a p-elastic curve. Moreover, in the proof, we have shown

$$\frac{dY_1}{ds}(s)=B_s(s)-D_{ss}(s)=v_s(s)\sin v(s)X(s).\tag{3.27}$$

So it is sufficient that s is an arclength parameter of (u,v) and $Y_1(s) = C_2$ with some constant C_2. We assume $Y_1(s) = C_2$, i.e.,

$$\begin{aligned}(3.28)\qquad &\big((p-1)|\kappa(s)|^p - \lambda\big)\big(\sin v(s)\big)^2 u_s(s)\\ &= -C_2 + p(|\kappa(s)|^{p-2}\kappa(s))_s v_s(s)\sin v(s)\\ &\quad - Gp|\kappa(s)|^{p-2}\kappa(s)\cos v(s).\end{aligned}$$

From (3.27) and $Y_1(s) = C_2$, we have $X(s) = 0$, and hence (3.25) holds with some $d \geq \min F$. From (3.26), $v_s(s)\sin v(s) = -C_1(|\kappa(s)|^{p-2}\kappa(s))_s$ and (3.28), we have

$$\begin{aligned}u_s(s) &= \frac{-C_2 - \frac{C_1}{p}\big(p^2((|\kappa(s)|^{p-2}(\kappa(s))_s)^2 + Gp^2|\kappa(s)|^{2(p-1)}\big)}{\big((p-1)|\kappa(s)|^p - \lambda\big)\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}\\ &= \frac{-C_2 - \frac{C_1}{p}\big(d - (p-1)^2|\kappa(s)|^{2p} + 2\lambda(p-1)|\kappa(s)|^p\big)}{\big((p-1)|\kappa(s)|^p - \lambda\big)\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}\\ &= \frac{\frac{C_1}{p}\big((p-1)^2|\kappa(s)|^{2p} - 2\lambda(p-1)|\kappa(s)|^p - \frac{pC_2}{C_1} - d\big)}{\big((p-1)|\kappa(s)|^p - \lambda\big)\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}.\end{aligned}$$

We choose C_2 to satisfy $-pC_2/C_1 - d = \lambda^2$. Then we have

$$(3.29)\qquad u_s(s) = \frac{C_1\big((p-1)|\kappa(s)|^p - \lambda\big)}{p\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}.$$

From (3.25), (3.26), $v_s(s)\sin v(s) = -C_1(|\kappa(s)|^{p-2}\kappa(s))_s$ and (3.29), we have

$$\begin{aligned}u_s(s)^2\sin^2 v(s) + v_s(s)^2 &= \frac{C_1^2\big((p-1)|\kappa(s)|^p - \lambda\big)^2}{p^2\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)^2}\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)^2\\ &\quad + \frac{C_1^2\big((|\kappa(s)|^{p-2}\kappa(s))_s\big)^2}{p^2\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}\\ &= \frac{C_1^2\big(d + \lambda^2 - Gp^2|\kappa(s)|^{2(p-1)}\big)}{p^2\big(1 - C_1^2|\kappa(s)|^{2(p-1)}\big)}.\end{aligned}$$

Hence, if $C_1 = -\sqrt{G/(d+\lambda^2)}\,p$, then we can see (2.2) and $C_2 = \sqrt{G(d+\lambda^2)}$. Therefore, we have deduced the expressions (3.18) and (3.19).

4. Classifications of p-elastic curves in $\mathbf{S}^2(G)$

In this section, we give some classifications of our p-elastic curves in $\mathbf{S}^2(G)$. Let $d \geq \min F$. We say a p-elastic curve in $\mathbf{S}^2(G)$ is of type (I) if it is defined as in (3.17), and we say a p-elastic curve in $\mathbf{S}^2(G)$ is of type (II), (III) or (IV) if it is defined by (3.18) and (3.19) with a solution ω of (3.2), and ω is of type (II), (III) or (IV), respectively. We also say a p-elastic curve in $\mathbf{S}^2(G)$ is *flat-core* (resp. *normal*) if it is of type (IV) (resp. if it is of one of type (I), (II) or (III)). Following [11, 16], we say a p-elastic curve in $\mathbf{S}^2(G)$ is *orbitlike* if its curvature does not change sign and it is *wavelike* if its curvature changes sign. Thus type (II), p-elastic curves in $\mathbf{S}^2(G)$ are normal, orbitlike, and type (III), p-elastic curves in $\mathbf{S}^2(G)$ are normal, wavelike. We also note that if a p-elastic curve in $\mathbf{S}^2(G)$ is of type (IV) with a solution ω of (3.2), and ω does not include (IV-ii) or (IV-iii) (resp. ω includes both (IV-ii) and (IV-iii)), then it is orbitlike (resp. wavelike).

Remark 4. Let $d \geq \min F$ such that $d \neq d_\lambda$, and let (u, v) be a curve in $\mathbf{S}^2(G)$ which satisfies (3.18) and (3.19) with some solution ω of (3.2). From (3.20), we can see that (u, v) is wavelike (resp. orbitlike) if and only if $\omega(s)$ changes sign (resp. $\omega(s)$ does not change sign). Moreover, from (3.18), we can see that (u, v) is wavelike (resp. orbitlike) if and only if the range of $v(s)$ intersects both of the intervals $(0, \pi/2)$ and $(\pi/2, \pi)$ (resp. the range of $v(s)$ is included in one of the interval $(0, \pi/2]$ or $[\pi/2, \pi)$).

For each $d \geq \min F$ and $(\mathfrak{I}) \in \{(\mathrm{II}), (\mathrm{III})\}$, if there is a solution ω of (3.2) such that $d = F(\omega(0))$ and ω is a type $(\mathfrak{I})$ solution of (3.2), we define

$$(4.1)\qquad \begin{aligned} T^{(\mathfrak{I})}_{p;\lambda,d} &= \inf\{s > 0 \mid \omega(s) = \omega(0),\, \omega_s(s) = \omega_s(0) = 0\}, \\ \Lambda^{(\mathfrak{I})}_{p;\lambda,d} &= \int_0^{T^{(\mathfrak{I})}_{p;\lambda,d}} u_s(s)ds \,(= u(T^{(\mathfrak{I})}_{p;\lambda,d}) - u(0)), \end{aligned}$$

where u is defined by (3.19). We can see $T^{(\mathfrak{I})}_{p;\lambda,d}$ and $\Lambda^{(\mathfrak{I})}_{p;\lambda,d}$ are given by

$$T^{(\mathrm{II})}_{p;\lambda,d} = \int_{\omega_{2;\lambda,d}}^{\omega_{1;\lambda,d}} \frac{2p}{\sqrt{d - F(\omega)}} d\omega, \qquad T^{(\mathrm{III})}_{p;\lambda,d} = \int_0^{\omega_{1;\lambda,d}} \frac{4p}{\sqrt{d - F(\omega)}} d\omega,$$

and

$$\Lambda^{(\mathrm{II})}_{p;\lambda,d} = 2p\sqrt{\frac{G}{d+\lambda^2}}\int_{\omega_{2;\lambda,d}}^{\omega_{1;\lambda,d}} \frac{\lambda-(p-1)\omega^{\frac{p}{p-1}}}{\left(1-\frac{Gp^2}{d+\lambda^2}\omega^2\right)\sqrt{d-F(\omega)}}\,d\omega,$$
$$\Lambda^{(\mathrm{III})}_{p;\lambda,d} = 4p\sqrt{\frac{G}{d+\lambda^2}}\int_{0}^{\omega_{1;\lambda,d}} \frac{\lambda-(p-1)\omega^{\frac{p}{p-1}}}{\left(1-\frac{Gp^2}{d+\lambda^2}\omega^2\right)\sqrt{d-F(\omega)}}\,d\omega.$$

In the case $d=0$, although each non constant solution ω of (3.2) may not be periodic, we define $T^{(\mathrm{IV})}_{p;\lambda,0}$ and $\Lambda^{(\mathrm{IV})}_{p;\lambda,0}$ by (3.9) and (4.1) with $(\mathfrak{I})=(\mathrm{IV})$, respectively. More precisely, we define $T^{(\mathrm{IV})}_{p;\lambda,0}$ and $\Lambda^{(\mathrm{IV})}_{p;\lambda,0}$ by

$$T^{(\mathrm{IV})}_{p;\lambda,0} = \int_0^{\omega_{1;\lambda,0}} \frac{2p}{\sqrt{-F(\omega)}}d\omega,$$
$$\Lambda^{(\mathrm{IV})}_{p;\lambda,0} = \frac{2p\sqrt{G}}{\lambda}\int_0^{\omega_{1;\lambda,0}} \frac{\lambda-(p-1)\omega^{\frac{p}{p-1}}}{\left(1-\frac{Gp^2}{\lambda^2}\omega^2\right)\sqrt{-F(\omega)}}\,d\omega,$$

respectively. From Corollary 1 and Theorems 2 and 3, we can obtain the following closedness condition for p-elastic curves in $\mathbf{S}^2(G)$.

Theorem 4. *Let $d\geq \min F$ and let ω be a solution of (3.2) with $\omega_0\in\mathbb{R}$ and $d=F(\omega_0)$. Then the following hold.*

(i) *If ω is of type (I), then*

$$c(t)=(\pm 2\pi t, v_0), \qquad t\in[0,1]$$

is a closed p-elastic curve in the coordinate (2.1), where $\omega(0)\in\{0,\pm\omega_{1;\lambda}\}$ and

$$v_0=\operatorname{arccot}(-r|\omega(0)|^{\frac{2-p}{p-1}}\omega(0)).$$

(ii) *If $d\neq d_\lambda$, ω is of type (II) (resp. ω is of type (III)), (u,v) is a curve in $\mathbf{S}^2(G)$ which satisfies (3.18) and (3.19), and $m\Lambda^{(\mathrm{II})}_{p;\lambda,d}=2n\pi$ (resp. $m\Lambda^{(\mathrm{III})}_{p;\lambda,d}=2n\pi$) with some $(n,m)\in\mathbb{Z}\times\mathbb{N}$ satisfying $n=0$, or $n\neq 0$ and $\gcd(|n|,m)=1$, then $c(t)$ given by*

$$c(t)=(u(\pm mT^{(\mathrm{II})}_{p;\lambda,d}t), v(\pm mT^{(\mathrm{II})}_{p;\lambda,d}t)), \qquad t\in[0,1],$$
$$(\textit{resp.}\quad c(t)=(u(\pm mT^{(\mathrm{III})}_{p;\lambda,d}t), v(\pm mT^{(\mathrm{III})}_{p;\lambda,d}t)), \qquad t\in[0,1],)$$

which belongs to $\mathcal{S}$, is a closed p-elastic curve in $\mathbf{S}^2(G)$ and it passes neither the north pole nor the south pole in the coordinate (2.1).

(iii) *If* ω *is of type* (IV) *and there is* $s_0 > 0$ *satisfying* $\omega(s_0) = \omega(0)$ *and* $\omega_s(s_0) = \omega_s(0) = 0$, *and* (u, v) *is a curve satisfying* (3.18), (3.19) *and*

$$u(s_0) - u(0) = 2n\pi$$

with some $n \in \mathbb{Z}$, *then* (u, v) *given by*

$$c(t) = (u(\pm s_0 t), v(\pm s_0 t)), \qquad t \in [0, 1]$$

is a closed p*-elastic curve in* $\mathbf{S}^2(G)$ *and it passes neither the north pole nor the south pole in the coordinate* (2.1).

In order to study the shapes of the graphs of the realized solutions in $\mathbf{S}^2(G)$, we define the regions $\mathcal{N}$ and $\mathcal{C}$ in $(0, \infty) \times \mathbb{R}$ by

$$\begin{aligned}\mathcal{N} &= \Big\{(\lambda, d) \in (0, \infty) \times \mathbb{R} \mid \min F \leq d < d_\lambda\Big\},\\ \mathcal{C} &= \Big\{(\lambda, d) \in (0, \infty) \times \mathbb{R} \mid d > d_\lambda\Big\},\end{aligned}$$

respectively. We say a p-elastic curve (u, v) in $\mathbf{S}^2(G)$ satisfying (3.18) and (3.19) with a solution ω of (3.2) is *minimal period crossing* (resp. *non minimal period crossing*) if $(\lambda, d) \in \mathcal{C}$ (resp. $(\lambda, d) \in \mathcal{N}$ or (u, v) is of type (I)). For the sake of simplicity, we abbreviate minimal period crossing (resp. non minimal period crossing) to *MP–crossing* (resp. *non MP–crossing*). The reason why we define p-elastic curves in $\mathbf{S}^2(G)$ are MP–crossing or non MP–crossing is that the shapes of the curves in the case $(\lambda, d) \in \mathcal{C}$ and those in the case $(\lambda, d) \in \mathcal{N}$ are drastically different. In fact, if $(\lambda, d) \in \mathcal{N}$ or (u, v) is of type (I), from (3.10) and (3.19), we see that $u(s)$ is monotone increasing on $(0, T^{(\mathfrak{I})}_{p;\lambda,d})$, while if $(\lambda, d) \in \mathcal{C}$, $u_s(s)$ changes its sign on $(0, T^{(\mathfrak{I})}_{p;\lambda,d})$, where $(\mathfrak{I}) \in \{(\mathrm{II}), (\mathrm{III}), (\mathrm{IV})\}$. So, if $(\lambda, d) \in \mathcal{C}$, each p-elastic curve $\{(u(s), v(s)) : 0 \leq s \leq T^{(\mathfrak{I})}_{p;\lambda,d}\}$ in $\mathbf{S}^2(G)$ satisfying (3.18) and (3.19) with a solution ω of (3.2) and $\omega_0 \in \mathbb{R}$ satisfying $d = F(\omega_0)$ has a self-intersection point; even in the case when ω is of type (IV), the curve $(u(s), v(s))$ corresponding to each of the parts (IV-ii) and (IV-iii) has a self-intersection point. See the figures of MP–crossing, closed p-elastic curves and non MP–crossing, closed p-elastic curves in the next section. Figure 4 shows the regions $\mathcal{N}$ and $\mathcal{C}$ in $\mathbb{R}^2$. For the figure, we note that from the definition of d_λ, it holds

$$d_\lambda < 0 \Leftrightarrow \lambda > \frac{G^{\frac{p}{2}} p^p}{(p-1)^{p-1}} \quad \text{and} \quad d_\lambda > 0 \Leftrightarrow 0 < \lambda < \frac{G^{\frac{p}{2}} p^p}{(p-1)^{p-1}}. \tag{4.2}$$

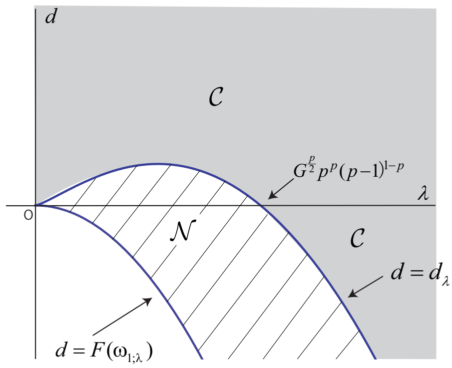

Figure 4: The regions $\mathcal{N}$ and $\mathcal{C}$.

5. Existence of flat-core, closed *p*-elastic curves and numerical examples of closed *p*-elastic curves

In this section, we show the existence of a closed, flat-core p-elastic curves in $\mathbf{S}^2(G)$. For closed, normal p-elastic curves, it is not easy to give complete conditions for p, λ and d when such curves exist. We consider that it is our future work. Instead of giving rigorous proofs of the existence of p-elastic curves of type (II) or those of type (III), we give some numerical examples of them. However, we can easily show the existence of closed flat-core, p-elastic curves as follows.

Theorem 5. *There exist infinitely many closed flat-core, p-elastic curves in* $\mathbf{S}^2(G)$.

Proof. A flat-core, p-elastic curve in $\mathbf{S}^2(G)$ can be obtained through a type (IV) solution ω of (3.2). Since ω can have (IV-i) as its part, we can find that there exist infinitely many closed, type (IV) solutions of (3.2). In fact, if $m\Lambda^{(\mathrm{IV})}_{p;\lambda,0} \neq 2n\pi$ with $(m,n) \in \mathbb{N} \times \mathbb{Z}$, we can get a closed, flat-core solution by gluing (IV-i). Even if $m_0\Lambda^{(\mathrm{IV})}_{p;\lambda,0} = 2n_0\pi$ with some $(m_0,n_0) \in \mathbb{N} \times \mathbb{Z}$, we can get a closed, flat-core solution by gluing (IV-i), and we can glue (IV-i), (IV-ii) and (IV-iii) in arbitrary order. Hence, there exist infinitely many closed flat-core, p-elastic curves in $\mathbf{S}^2(G)$. □

From (4.2), we can see that in the case $0 < \lambda < G^{\frac{p}{2}}p^p(p-1)^{1-p}$, flat-core, p-elastic curves are non MP–crossing, and in the case $\lambda > G^{\frac{p}{2}}p^p(p-1)^{1-p}$,

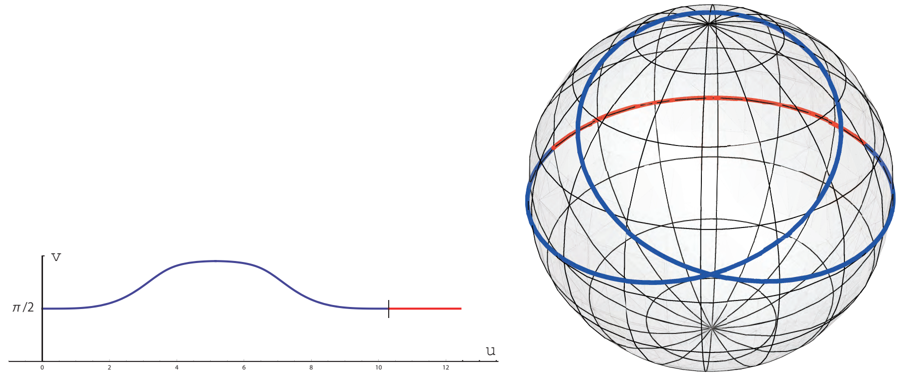

Figure 5: A non MP–crossing, flat-core, orbitlike, closed 3-elastic curve with $\lambda = 5, G = 1$.

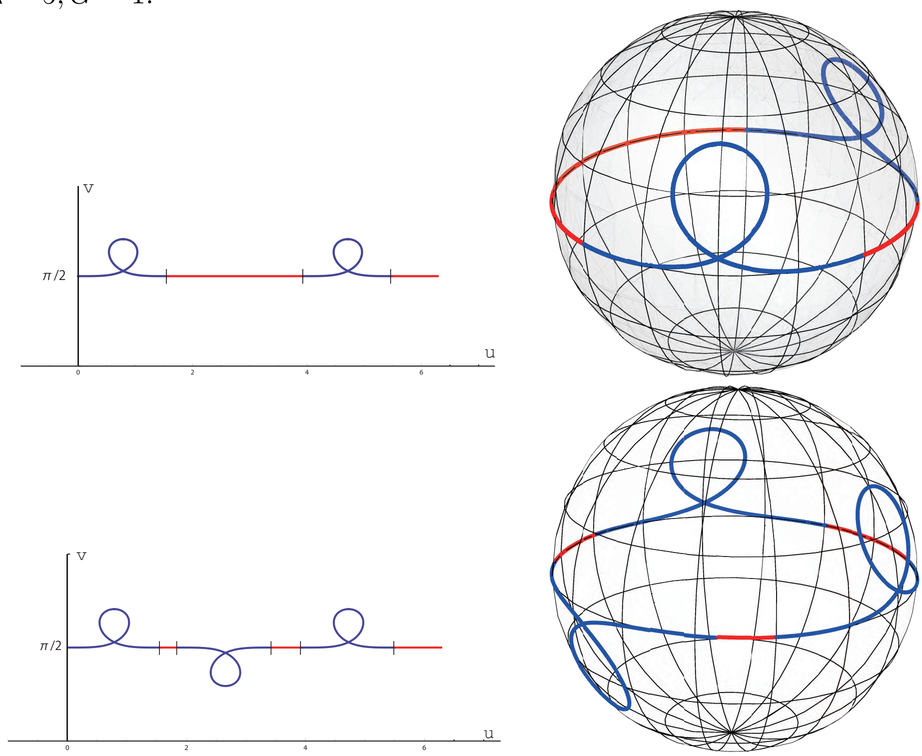

Figure 6: MP–crossing, flat-core, orbitlike and wavelike, closed 3-elastic curves with $\lambda = 100, G = 1$.

they are MP–crossing. Figure 5 shows an example of a non MP–crossing, flat-core, closed p-elastic curve, and Figure 6 shows examples of MP–crossing, flat-core, closed p-elastic curves. In these figures, red lines represent (IV-i).In the rest of this section, we show some numerical examples of normal, closed p-elastic curves. For each $(n, m) \in \mathbb{Z} \times \mathbb{N}$ with $n = 0$, or $n \neq 0$ and $\gcd(|n|, m) = 1$, and $(\mathfrak{I}) \in \{(\mathrm{II}), (\mathrm{III})\}$, we define

$$(5.1)\quad \mathcal{S}^{(\mathfrak{I})}_{n,m} := \left\{ (u, v) \mid (u, v) \text{ satisfies (3.18), (3.19) and } m\Lambda^{(\mathfrak{I})}_{p;\lambda,d} = 2n\pi \right\}.$$

By Theorem 4, each curve in $\mathcal{S}^{(\mathfrak{I})}_{n,m}$ is closed, p-elastic, it moves round $\mathbf{S}^2(G)$ n-times in the positive direction and it closes up after the time of m-period of ω. We give some pictures of normal, closed p-elastic curves in Figures 7–9. In Figures 7 and 8, each blue curve and each red curve represent half of a minimal period curve.

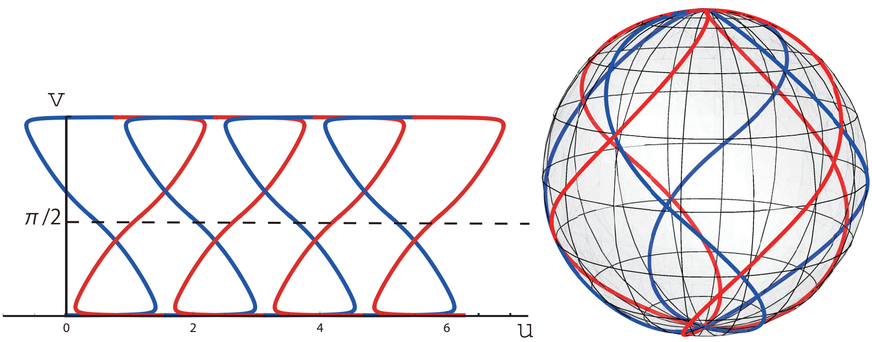

Figure 7: An MP–crossing, normal, wavelike, closed 3-elastic curve belonging to $\mathcal{S}^{(\mathrm{III})}_{1,4}$; $\lambda = 3, d \sim 6.762, G = 1$.

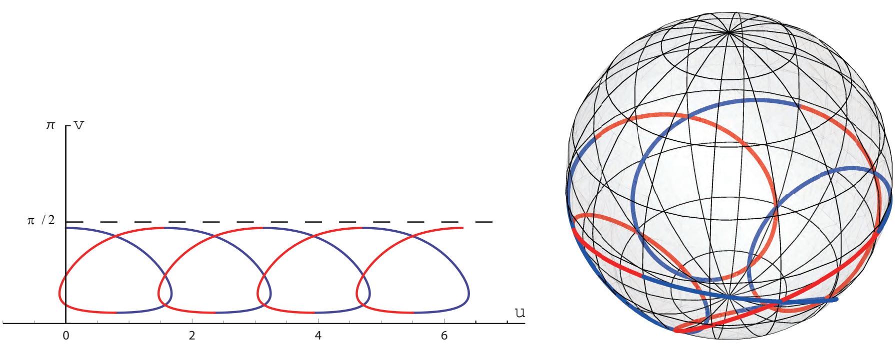

Figure 8: An MP–crossing, normal, orbitlike, closed 3-elastic curve belonging to $\mathcal{S}^{(\mathrm{II})}_{1,4}$; $\lambda = 10$, $d \sim -2.653, G = 1$.

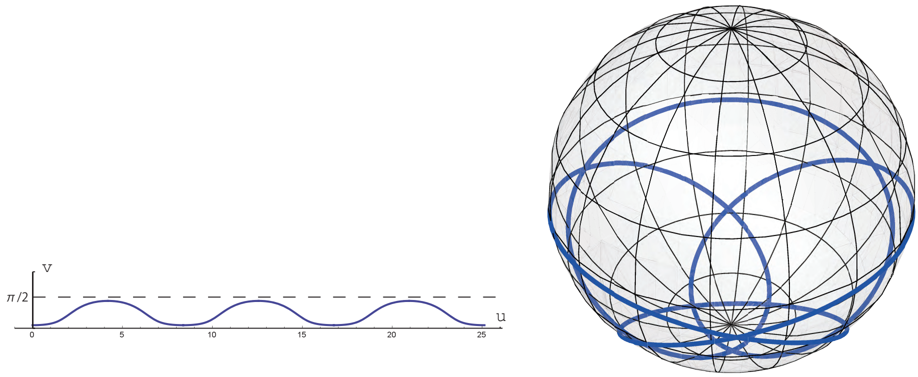

Figure 9: A non MP–crossing, normal, orbitlike, closed 3-elastic curve belonging to $\mathcal{S}^{(\mathrm{III})}_{4,3}$; $\lambda = 5$, $d \sim -0.401, G = 1$.

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