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Symplectic quotients of unstable Morse strata for normsquares of moment maps

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Let K be a compact Lie group and fix an invariant inner product on its Lie algebra $\mathfrak{k}$. Given a Hamiltonian action of K on a compact symplectic manifold X with moment map $\mu : X \to \mathfrak{k}^*$, the normsquare $\|\mu\|^2$ of μ defines a Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of X by locally closed symplectic submanifolds of X such that the stratum to which any $x \in X$ belongs is determined by the limiting behaviour of its downwards trajectory under the gradient flow of $\|\mu\|^2$ with respect to a suitably compatible Riemannian metric on X. The open stratum S_0 retracts K-equivariantly via this gradient flow to the minimum $\mu^{-1}(0)$ of $\|\mu\|^2$ (if this is not empty). If $\beta \neq 0$ the usual 'symplectic quotient' $(S_\beta \cap \mu^{-1}(0))/K$ for the action of K on the stratum S_β is empty. Nonetheless, motivated by recent results in non-reductive geometric invariant theory, we find that the symplectic quotient construction can be modified to provide natural 'symplectic quotients' for the unstable strata with $\beta \neq 0$. There is an analogous infinite-dimensional picture for the Yang–Mills functional over a Riemann surface with strata determined by Harder–Narasimhan type.

This article is dedicated to Karen Uhlenbeck in celebration of her 75th birthday, with grateful thanks for her support, hospitality and many useful conversations when I started thinking about these topics several decades ago.

1. Introduction

Let (X, ω) be a compact symplectic manifold, and let $\mu : X \to \mathfrak{k}^*$ be a moment map for a Hamiltonian action of a compact Lie group K on (X, ω). Then the symplectic quotient (or Marsden–Weinstein reduction of X at 0 [36]) is given by $X/\!\!/K = \mu^{-1}(0)/K$ with its induced symplectic structure. Let us fix a K-invariant inner product on the Lie algebra $\mathfrak{k}$ of K; then the associated normsquare $\|\mu\|^2$ of μ can be considered as a Morse function on X. This is not in general a Morse function in the classical sense, nor even a Morse-Bott function, since the connected components of its set of critical points are not in general submanifolds of X. Nonetheless, given a suitably compatible K-invariant Riemannian metric on X, there is a Morse stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of X induced by $\|\mu\|^2$ such that each stratum S_β is a K-invariant locally closed symplectic submanifold of X [28]. Here the stratum to which $x \in X$ belongs is determined by the limit set of its path of steepest descent for $\|\mu\|^2$, and the index β is the intersection with a positive Weyl chamber $\mathfrak{t}_+$ for K of the co-adjoint orbit which is the image under μ of the corresponding critical set.

We can attempt to construct symplectic quotients for the restrictions of the Hamiltonian K-action to the strata S_β. However the usual construction, given by $(S_\beta \cap \mu^{-1}(0))/K$, is empty if $\beta \neq 0$. When $K = T$ is abelian we can deal with this problem by shifting the moment map by a suitable constant; a natural choice is to replace $\mu^{-1}(0)$ here with $\mu^{-1}((1+\epsilon)\beta)$ for $0 < \epsilon < < 1$. However when β is not central $\mu^{-1}((1+\epsilon)\beta)$ will not in general be K-invariant, so we must modify the construction. We will see in this paper that this can be done by recalling that there are natural identifications

$$S_\beta \cong K \times_{K_\beta} (Y_\beta \cap S_\beta)$$

where Y_β is the locally closed submanifold given for the Morse–Bott function $\mu_\beta(x) = \mu(x).\beta$ by

$$Y_\beta = \{y \in X : \text{the downward trajectory of } y \text{ for } \operatorname{grad}(\mu_\beta) \text{ has a limit point } x \text{ with } \mu_\beta(x) = \|\beta\|^2\},$$

and the stabiliser K_β of β under the adjoint action of K acts diagonally on the product of K with the open subset $Y_\beta \cap S_\beta$ of Y_β. For sufficiently small $\epsilon > 0$ we will see that $Y_\beta \cap \mu^{-1}((1+\epsilon)\beta) \subseteq S_\beta$ is compact and

$$S_\beta /\!\!/\!\!/ K := (Y_\beta \cap \mu^{-1}((1+\epsilon)\beta)/K_\beta$$

has an induced symplectic structure with which it can be regarded as a symplectic quotient for the K-action on the stratum S_β.

The motivation for this construction comes from the relationship between symplectic quotients and geometric invariant theory (GIT). Suppose that $X \subseteq \mathbb{P}^n$ is a nonsingular complex projective variety, that ω is the restriction to X of the Fubini-Study Kähler form on the complex projective space $\mathbb{P}^n$ and that K acts linearly on X via a unitary representation $\rho : K \to U(n+1)$. Then the open stratum S_0 coincides with the semistable locus X^{ss} in the sense of Mumford's GIT for the induced linear action on X of the complexification $G = K_{\mathbb{C}}$ of K, and the inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient map from X^{ss} to the GIT quotient $X/\!/G$ induces an identification of the symplectic quotient $X /\!/\!/ K = \mu^{-1}(0)/K$ with $X/\!/G$. (For this reason, even in the non-algebraic case, we will refer to the strata S_β for $\beta \neq 0$ as the unstable strata). The unstable strata S_β are G-invariant locally closed subvarieties of X and have descriptions of the form

$$S_\beta = KY_\beta^{ss} = GY_\beta^{ss} \cong G \times_{P_\beta} Y_\beta^{ss} \cong K \times_{K_\beta} Y_\beta^{ss}$$

where P_β is a parabolic subgroup of G, and $Y_\beta^{ss} = Y_\beta \cap S_\beta$ has an inductive description involving semistability for the action of a Levi subgroup L_β of P_β, after twisting the linearisation by a suitable rational character of P_β. Since such characters do not in general extend to G, in order to construct GIT quotients of the unstable strata S_β it is natural to consider quotients of the subvarieties Y_β by the action of the non-reductive groups P_β. Recent results have extended classical GIT to suitable non-reductive linear algebraic group actions on projective varieties [4, 5], and the modified symplectic quotient construction for unstable strata just described is suggested by these advances, together with links between non-reductive GIT and the symplectic implosion construction of Guillemin, Jeffrey and Sjamaar [20, 34]. In the algebraic setting the modified symplectic quotient construction coincides with a non-reductive GIT quotient construction for the P_β action on Y_β with an appropriately twisted linearisation.

In their fundamental paper [3] Atiyah and Bott observed that the Yang–Mills functional over a compact Riemann surface Σ plays the role of $\|\mu\|^2$ in an infinite-dimensional analogue of this picture (modulo a constant which

depends on the addition of a central constant to the moment map). Here the corresponding analogue of the GIT or symplectic quotient is a moduli space of semistable holomorphic bundles of fixed rank and degree over Σ, and the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is by the Harder–Narasimhan type of a holomorphic bundle. The primary motivation for considering the Yang–Mills functional in [3] (and $\|\mu\|^2$ in the finite-dimensional setting explored in [28]) as a Morse function was to study the cohomology (at least the Betti numbers) of the symplectic quotient. This was done by relating the equivariant cohomology of the compact symplectic manifold X, or its infinite-dimensional analogue in the Yang–Mills case, to the equivariant cohomology of the strata, and by describing the equivariant cohomology of the unstable strata inductively in terms of semistable strata for symplectic submanifolds of X acted on by compact subgroups of K (or subgroups of its infinite-dimensional analogue, the relevant gauge group). Later work [26, 27, 42] showed how related ideas could be used to study intersection pairings on $X/\!\!/K$ and the ring structure of its cohomology. In a future paper [11] we will show how to extend these results to symplectic quotients of unstable strata and other non-reductive GIT quotients.

The layout of this paper is as follows. In §2 we will review the Morse stratification for the normsquare of a moment map on a compact symplectic manifold with a compact Hamiltonian action. In §3 and §4 we will summarise the relevant results from non-reductive GIT and symplectic implosion. Finally §5 describes the construction of symplectic quotients of unstable strata for compact Hamiltonian actions on compact symplectic manifolds, with the main results summarised in Theorem 5.1, and §6 considers the infinite-dimensional Yang–Mills analogue.

2. Normsquares of moment maps and their Morse stratifications

Suppose that a compact Lie group K with Lie algebra $\mathfrak{k}$ acts smoothly on a symplectic manifold X and preserves the symplectic form ω. Any $a \in \mathfrak{k}$ determines a vector field $x \mapsto a_x$ on X defined by the infinitesimal action of a. A moment map for the action of K on X is a smooth K-equivariant map $\mu : X \to \mathfrak{k}^*$ which satisfies

$$d\mu(x)(\xi).a = \omega_x(\xi, a_x)$$

for all $x \in X$, $\xi \in T_xX$ and $a \in \mathfrak{k}$. Equivalently, if $\mu_a : X \to \mathbb{R}$ denotes the component of μ along $a \in \mathfrak{k}$ defined for all $x \in X$ by the pairing $\mu_a(x) =$

$\mu(x).a$ between $\mu(x) \in \mathfrak{k}^*$ and $a \in \mathfrak{k}$, then μ_a is a Hamiltonian function for the vector field on X induced by a.

If the stabiliser K_ζ of $\zeta \in \mathfrak{k}^*$ under the adjoint action of K acts with only finite stabilisers on $\mu^{-1}(\zeta)$, then $\mu^{-1}(\zeta)$ is a submanifold of X and the symplectic form ω induces a symplectic structure on the orbifold $\mu^{-1}(\zeta)/K_\zeta$ which is the Marsden–Weinstein reduction at ζ of the action of K on X. The symplectic quotient $X/\!\!/\!\!/K$ is the Marsden–Weinstein reduction $\mu^{-1}(0)/K$ at 0.

The reduction $\mu^{-1}(\zeta)/K_\zeta$ also inherits a symplectic structure when the action of K_ζ on $\mu^{-1}(\zeta)$ has positive-dimensional stabilisers, but in this case it is likely to have more serious singularities.

Remark 2.1. Let X be a nonsingular complex projective variety embedded in complex projective space $\mathbb{P}^n$, and let $G = K_{\mathbb{C}}$ be a reductive complex Lie group with maximal compact subgroup K acting on X via a representation $\rho : G \to GL(n+1;\mathbb{C})$. By choosing coordinates on $\mathbb{P}^n$ appropriately we can assume that ρ maps K into the unitary group $U(n+1)$. Then the Fubini-Study form ω on $\mathbb{P}^n$ restricts to a K-invariant Kähler form on X, and there is a moment map $\mu : X \to \mathfrak{k}^*$ defined (up to multiplication by a constant scalar factor depending on the convention chosen for the normalisation of the Fubini-Study form) by

$$\mu(x).a = \frac{\overline{\hat{x}}^t \rho_*(a)\hat{x}}{2\pi i \|\hat{x}\|^2} \tag{2.1}$$

for all $a \in \mathfrak{k}$, where $\hat{x} \in \mathbb{C}^{n+1} - \{0\}$ is a representative vector for $x \in \mathbb{P}^n$ and the representation $\rho : K \to U(n+1)$ induces $\rho_* : \mathfrak{k} \to \mathfrak{u}(n+1)$ and dually $\rho^* : \mathfrak{u}(n+1)^* \to \mathfrak{k}^*$.

In this situation the symplectic quotient $X/\!\!/\!\!/K = \mu^{-1}(0)/K$ coincides with the GIT quotient $X/\!/G$ in algebraic geometry described in §3 below [28]. Moreover $x \in X$ lies in the semistable locus X^{ss} if and only if the closure of its orbit Gx meets $\mu^{-1}(0)$, and x lies in the stable locus if and only if its orbit Gx meets the open subset $\mu^{-1}(0)_{\mathrm{reg}}$ of $\mu^{-1}(0)$ where $d\mu$ is surjective. The inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient map $X^{ss} \to X/\!/G$ is K-invariant and induces a bijection $\mu^{-1}(0)/K \to X/\!/G$ which can be used to identify the symplectic quotient $X/\!/\!/K = \mu^{-1}(0)/K$ with the GIT quotient $X/\!/G$.

When X is Kähler but not necessarily algebraic then $\mu^{-1}(0)/K$ inherits a Kähler structure (at least away from its singularities) by identifying $\mu^{-1}(0)_{\mathrm{reg}}/K$ with the quotient by G of the open subset $G\mu^{-1}(0)_{\mathrm{reg}}$ of X.

Let us fix a maximal torus T of K and an inner product on the Lie algebra $\mathfrak{k}$ which is invariant under the adjoint action of K, and which we will use to identify $\mathfrak{k}^*$ with $\mathfrak{k}$. We will assume that the inner product is chosen so that its restriction to the Lie algebra $\mathfrak{t}$ of T takes rational values on the lattice given by the derivatives at the identity of homomorphisms $S^1 \to T$. Then we can consider the associated normsquare $\|\mu\|^2$ of μ as a Morse function on X; it is not a Morse function in the classical sense, nor even a Morse–Bott function, but it is shown in [28] that $\|\mu\|^2$ is a 'minimally degenerate' Morse function. More precisely, the set of critical points for $f = \|\mu\|^2$ is a finite disjoint union of closed subsets $\{C_\beta : \beta \in \mathcal{B}\}$ along each of which f is *minimally degenerate* in the following sense.

Definition 2.1. A locally closed submanifold Σ_β containing C_β with orientable normal bundle in X is a *minimising submanifold* for $f = \|\mu\|^2$ if

1) the restriction of f to Σ_β achieves its minimum value exactly on C_β, and
2) the tangent space to Σ_β at any point $x \in C_\beta$ is maximal among subspaces of T_xX on which the Hessian $H_x(f)$ is non-negative.

If a minimising submanifold Σ_β exists, then f is called minimally degenerate along C_β.

In [28] it is shown that as a consequence $\|\mu\|^2$ induces a smooth stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of X such that, for a suitable choice of Riemannian metric (which can be taken to be the Kähler metric if (X, ω) is Kähler), $x \in X$ lies in the stratum S_β if its path of steepest descent for $\|\mu\|^2$ has a limit point in the critical subset C_β. The stratum S_β then coincides with Σ_β near C_β.

Remark 2.2. Here we choose a K-invariant Riemannian metric which is compatible with the symplectic structure in the sense that X has a K-invariant almost-complex structure such that if $\xi \in T_xX$ then $i\xi$ is the dual with respect to the metric of the linear form $\zeta \to \omega_x(\zeta, \xi)$ on T_xX. This implies that

$$\mathrm{grad}\mu_\beta(x) = i\beta_x$$

for all $x \in X$, where $\mu_\beta(x) = \mu(x).\beta$.

It is shown in [28] that in the situation of Remark 2.1 the open stratum S_0 of this stratification $\{S_\beta : \beta \in \mathcal{B}\}$ coincides with the semistable locus X^{ss},

and that each stratum S_β has the form

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{ss}$$

where Y_β^{ss} is a locally closed nonsingular subvariety of X and P_β is a parabolic subgroup of G ([28] Theorem 6.18). Moreover, there is a linear action of a Levi subgroup L_β of P_β on a nonsingular closed subvariety Z_β of X such that Y_β^{ss} retracts equivariantly onto the subset Z_β^{ss} of semistable points for this action. It is also shown in [28] that $\|\mu\|^2$ is equivariantly perfect, in the sense that its equivariant Morse inequalities are in fact equalities, and that this leads to an inductive procedure for calculating $\dim H_G^j(X^{ss};\mathbb{Q})$, which in good cases give the Betti numbers of the quotient variety $X//G$.

When X is merely a compact symplectic manifold acted on by a compact group K, the function $\|\mu\|^2$ still induces a smooth stratification of X and is K-equivariantly perfect, providing a formula for the Betti numbers of the symplectic quotient $\mu^{-1}(0)/K$ in the good case when 0 is a regular value of μ, which involves the K-equivariant cohomology of the critical subsets C_β. Indeed the same is true when $\|\mu\|^2$ is replaced with any convex function of μ (cf. [3] §§8,12).

The set $\mathcal{B}$ indexing the critical subsets C_β and the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ can be identified with a finite set of orbits of the adjoint representation of K on its Lie algebra $\mathfrak{k}$ (which is identified with its dual using the fixed invariant inner product). Each orbit in $\mathcal{B}$ is the image under the moment map $\mu : X \to \mathfrak{k}^\star \cong \mathfrak{k}$ of the critical subset which it indexes. If a choice is made of a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra of some maximal torus T of K, then each adjoint orbit intersects $\mathfrak{t}_+$ in a unique point, so $\mathcal{B}$ can also be identified with a finite set of points in $\mathfrak{t}_+$. In the situation of Remark 2.1 a point of $\mathfrak{t}_+$ lies in $\mathcal{B}$ if it is the closest point to the origin of the convex hull of a nonempty set of the weights of the unitary representation of K which defines its action on $X \subseteq \mathbb{P}^n$. The same is true more generally if we interpret weight here as the image under the T-moment map of a connected component of the fixed point set X^T.

When $\mathcal{B}$ is identified with a finite set of points in $\mathfrak{t}_+$, for $\beta \in \mathcal{B}$ the submanifold Z_β of X is the union of those components of the fixed point set of the subtorus T_β of K generated by β on which the moment map for T_β given by composing μ with the restriction map from $\mathfrak{k}^*$ to $\mathfrak{t}_\beta^*$ takes the value β. Then

$$C_\beta = K(Z_\beta \cap \mu^{-1}(\beta)) \cong K \times_{K_\beta} (Z_\beta \cap \mu^{-1}(\beta))$$

where the subgroup K_β is the stabiliser of β under the adjoint action of K on its Lie algebra, and in the Kähler case, the complexification L_β of K_β is a Levi subgroup of the parabolic subgroup P_β of $G = K_{\mathbb{C}}$.

Since the moment map is K-equivariant the image of Z_β under μ is contained in the Lie algebra of K_β, and thus $\mu|_{Z_\beta}$ can be regarded as a moment map for the action of K_β on Z_β. As moment maps are only determined up to the addition of a central constant, $\mu|_{Z_\beta} - \beta$ is also a moment map for the action of K_β on Z_β.

Remark 2.3. In the situation of Remark 2.1 this change of moment map corresponds to a modification of the linearisation of the action of K_β on Z_β, and we define Z_β^{ss} to be the set of semistable points of Z_β with respect to this modified linear action. Equivalently, Z_β^{ss} is the stratum labelled by 0 for the Morse stratification of the function $\|\mu - \beta\|^2$ on Z_β. Then

$$Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$$

where Y_β and $p_\beta : Y_\beta \to Z_\beta$ are given by $p_\beta(x) = \lim_{t\to\infty} \exp(-it\beta)x$ and

$$Y_\beta = \{y \in X \mid p_\beta(y) \in Z_\beta\}.$$

If B is the Borel subgroup of G associated to the choice of positive Weyl chamber $\mathfrak{t}_+$ and if P_β is the parabolic subgroup BK_β, then Y_β and Y_β^{ss} are P_β-invariant and we have $S_\beta = KY_\beta^{ss} \cong K \times_{K_\beta} Y_\beta^{ss} \cong G \times_{P_\beta} Y_\beta^{ss}$. Moreover when X is nonsingular Y_β is a nonsingular subvariety of X and $p_\beta : Y_\beta \to Z_\beta$ is a locally trivial fibration whose fibre is isomorphic to $\mathbb{C}^{m_\beta}$ for some $m_\beta \geq 0$.

An element g of G lies in the parabolic subgroup P_β if and only if $\exp(-it\beta)g\exp(it\beta)$ tends to a limit in G as $t \to \infty$, and this limit defines a surjection $q_\beta : P_\beta \to L_\beta$ such that

$$p_\beta(gy) = q_\beta(g)p_\beta(y)$$

for each $g \in P_\beta$ and $y \in Y_\beta$. Since $G = KB$ and $B \subseteq P_\beta$ we have $G\overline{Y_\beta} = K\overline{Y_\beta}$, which is compact, and hence

$$\overline{S_\beta} \subseteq G\overline{Y_\beta} \subseteq S_\beta \cup \bigcup_{\|\gamma\|>\|\beta\|} S_\gamma. \tag{2.2}$$

3. Non-reductive geometric invariant theory

3.1. GIT for reductive groups

In Mumford's classical Geometric Invariant Theory we choose a linearisation of an action of a reductive group G on a complex projective variety X; this is given by an ample line bundle L on X and a lift of the action to L. When L is very ample, so that X can be embedded in a projective space $\mathbb{P}^n$ such that L is the restriction of the hyperplane line bundle $\mathcal{O}(1)$, the action is given by a representation $\rho : G \to GL(n+1)$ and $\hat{\mathcal{O}}_L(X) = \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k})$ is $\Bbbk[x_0, \ldots, x_n]/\mathcal{I}_X$ where $\mathcal{I}_X$ is the ideal generated by the homogeneous polynomials which vanish on X.

$$\begin{array}{ccccc} (X,L) & \rightsquigarrow & \hat{\mathcal{O}}_L(X) & = & \bigoplus_{k=0}^{\infty} H^0(X, L^{\otimes k}) \\ \big| & & & & \\ \big| & & \bigcup\big| & & \\ \downarrow & & & & \\ X/\!/G & \hookleftarrow & \hat{\mathcal{O}}_L(X)^G & & \text{algebra of invariants.} \end{array}$$

Since G is reductive, the algebra of G-invariants $\hat{\mathcal{O}}_L(X)^G$ is finitely generated as a graded algebra and so defines a projective variety $X/\!/G = \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^G)$. The inclusion of $\hat{\mathcal{O}}_L(X)^G$ in $\hat{\mathcal{O}}_L(X)$ determines a rational map $X - - \to X/\!/G$ which fits into a diagram

$$\begin{array}{rccc} & X & - - \to & X/\!/G \quad \text{projective variety} \\ & \bigcup & & \| \\ \text{semistable} & X^{ss} & \overset{\text{onto}}{\longrightarrow} & X/\!/G \\ & \bigcup & & \bigcup \quad \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/G \end{array}$$

where X^s and X^{ss} are open subvarieties of X, the GIT quotient $X/\!/G$ is a categorical quotient for the action of G on X^{ss} via the G-invariant surjective morphism $\phi_G : X^{ss} \to X/\!/G$, and

$$\phi_G(x) = \phi_G(y) \Leftrightarrow \overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset.$$

Remark 3.1. A complex Lie group G is reductive if and only if it is the complexification $G = K_{\mathbb{C}}$ of a maximal compact subgroup K, and then $X/\!/G = \mu^{-1}(0)/K$ for a suitable moment map μ for the action of K (see Remark 2.1 above). Indeed, recall that in this situation the semistable locus X^{ss} coincides with the open stratum S_0, while $x \in X$ lies in $S_0 = X^{ss}$ if and

only if the closure of its orbit Gx meets $\mu^{-1}(0)$, and x lies in the stable locus if and only if its orbit Gx meets the open subset $\mu^{-1}(0)_{\mathrm{reg}}$ of $\mu^{-1}(0)$ where $d\mu$ is surjective. Then the inclusion $\mu^{-1}(0) \to X^{ss}$ composed with the quotient map $X^{ss} \to X//G$ induces an identification of the symplectic quotient $X/\!\!/\!\!/K = \mu^{-1}(0)/K$ with the GIT quotient $X//G$.

When X is Kähler but not necessarily algebraic then we can define an equivalence relation $\sim$ on the open stratum S_0 by $x \sim y$ if and only if $\overline{Gx} \cap \overline{Gy} \cap X^{ss} \neq \emptyset$; if x lies in the open subset $G\mu^{-1}(0)_{\mathrm{reg}}$ of S_0 then $x \sim y$ if and only if $y \in Gx$. Then the inclusion $\mu^{-1}(0) \to S_0$ induces an identification of the symplectic quotient $X/\!\!/\!\!/K = \mu^{-1}(0)/K$ with the topological quotient $S_0/\sim$. Thus $\mu^{-1}(0)/K$ inherits a stratified Kähler structure, with the complex structure induced from S_0 and the Kähler form given by the symplectic form on $\mu^{-1}(0)/K$ [25].

The subsets X^{ss} and X^s of X for a linear action of a reductive group G with respect to an ample linearisation $\mathcal{L}$ are characterised by the Hilbert–Mumford criteria [38, Chapter 2], [39]:

Proposition 3.2. *(i) A point $x \in X$ is semistable (respectively stable) for the action of G on X if and only if for every $g \in G$ the point gx is semistable (respectively stable) for the action of a fixed maximal torus T of G.*
(ii) A point $x \in X$ with homogeneous coordinates $[x_0 : \cdots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal torus T of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull

$$\mathrm{Conv}\{\alpha_i : x_i \neq 0\}$$

contains 0 *(respectively contains* 0 *in its interior).*

The projective GIT quotient $X//G$ contains as an open subset the geometric quotient X^s/G of the stable set X^s. When X is nonsingular then the singularities of X^s/G are very mild, since the stabilisers of stable points are finite subgroups of G. If $X^{ss} \neq X^s \neq \emptyset$ the singularities of $X//G$ are typically more severe, but $X//G$ has a 'partial desingularisation' $\tilde{X}//G$ which (if X is irreducible and $X^s \neq \emptyset$) is also a projective completion of X^s/G and is itself a geometric quotient

$$\tilde{X}//G = \tilde{X}^{ss}/G$$

by G of an open subset $\tilde{X}^{ss} = \tilde{X}^s$ of a G-equivariant blow-up $\tilde{X}$ of X [30]. $\tilde{X}^{ss}$ is obtained from X^{ss} by successively blowing up along the subvarieties

of semistable points stabilised by reductive subgroups of G of maximal dimension and then removing the unstable points in the resulting blow-up. Thus for irreducible X we have
i) when $X^{ss} = X^s \neq \emptyset$ the GIT quotient $X//G = X^s/G$ is a projective variety which is a geometric quotient of the open subvariety X^s of X;
ii) when $X^{ss} \neq X^s \neq \emptyset$ then the GIT quotient $X//G$ is a projective completion of the geometric quotient X^s/G, and X^s/G has another projective completion $\tilde{X}//G = \tilde{X}^s/G$ which is a 'partial desingularisation' of $X//G$ in the sense just described.

Remark 3.1. The GIT quotient $X//G$ has an ample line bundle which pulls back to a positive tensor power on X^{ss} of the line bundle L defining the linearisation $\mathcal{L}$.

Note that when we replace the linearisation $\mathcal{L}$ for the action of G on X by any positive tensor power of itself, the stable and semistable loci X^s and X^{ss} and the GIT quotient $X//G$ are unchanged. From a symplectic viewpoint the symplectic form and moment map (and the induced symplectic form on the symplectic quotient) are multiplied by a positive integer, but $\mu^{-1}(0)$ is unchanged. In particular this means that it makes sense to multiply a linearisation by a rational character χ/m of G, where $\chi : G \to \mathbb{C}^*$ is a character and m is a positive integer: from a GIT perspective we can interpret the result as multiplying the induced linearisation on $L^{\otimes m}$ by the character χ. From a symplectic viewpoint we are adding a central constant to the moment map.

Example 3.2. As we have seen, associated to the G-action on X with linearisation $\mathcal{L}$ and an invariant inner product on the Lie algebra of a maximal compact subgroup K of G, there is a stratification (the Morse stratification for $\|\mu\|^2$)

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

of X by locally closed subvarieties S_β, indexed by a partially ordered finite subset $\mathcal{B}$ of a positive Weyl chamber for the reductive group G, such that

(i) $S_0 = X^{ss}$,

and for each $\beta \in \mathcal{B}$

(ii) the closure of S_β is contained in $\bigcup_{\gamma \geqslant \beta} S_\gamma$, and

(iii) $S_\beta = KY_\beta^{ss} = GY_\beta^{ss} \cong G \times_{P_\beta} Y_\beta^{ss} \cong K \times_{K_\beta} Y_\beta^{ss}$

where P_β is a parabolic subgroup of G acting on a projective subvariety $\overline{Y}_\beta$ of X with an open subset Y_β^{ss} which is determined by the action of the Levi subgroup L_β of P_β with respect to a suitably twisted linearisation [21, 28].

Here the linearisation $\mathcal{L}$ is restricted to the action of the parabolic subgroup P_β over $\overline{Y}_\beta$, and then twisted by the rational character β of P_β.

To construct a quotient by G of (an open subset of) an unstable stratum S_β, we can study the linear action on $\overline{Y}_\beta$ of the parabolic subgroup P_β. In order to have a non-empty quotient we must modify the linearisation $\mathcal{L}$, and it is natural to do this by twisting it by a rational character; such a character may not extend to a character of G, which is why it makes sense to consider the action on $\overline{Y_\beta}$ of the non-reductive group P_β. Twisting by β (or subtracting β from the moment map for the maximal compact subgroup K_β of P_β) gives a categorical quotient $Z_\beta//L_\beta \cong (Z_\beta \cap \mu^{-1}(\beta))/K_\beta \cong C_\beta/K$ for the action of P_β on Y_β^{ss}, or equivalently for the action of G on S_β (cf. Remark 2.3), but in general this is far from being a geometric quotient. To have hope of a non-empty open subset of S_β with a geometric quotient (when $\beta \neq 0$) one can instead try twisting the action of P_β on $\overline{Y_\beta}$ by $(1+\epsilon)\beta$ where $0 < \epsilon << 1$, or by another perturbation of β whose restriction to T_β is of this form.

3.2. GIT for non-reductive groups

Motivated by Example 3.2, let us consider a complex projective variety X acted on linearly (with ample linearisation $\mathcal{L}$) by a linear algebraic group H which is not necessarily reductive. Then $H = U \rtimes R$ is the semi-direct product of its unipotent radical U by a reductive subgroup R; here R is a Levi subgroup of H and is unique up to conjugation by H.

An immediate difficulty arises when trying to extend classical GIT to non-reductive linear algebraic groups H; this is that in general we cannot define a projective variety $X//H = \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^H)$ because $\hat{\mathcal{O}}_L(X)^H$ is not necessarily finitely generated as a graded algebra. However in [4, 16] it is shown that given an H-action on X with linearisation $\mathcal{L}$ as above, X has open subvarieties X^s ('stable points') and X^{ss} ('semistable points') with a geometric quotient $X^s \to X^s/H$ and an 'enveloping quotient' $X^{ss} \to X//H$, with a diagram

$$\begin{array}{rccc} & X & & \\ & \bigcup & & \\ \text{semistable} & X^{ss} & \longrightarrow & X//H \\ & \bigcup & & \bigcup \quad \text{open} \\ \text{stable} & X^s & \longrightarrow & X^s/H \end{array}$$

where *if* $\hat{\mathcal{O}}_L(X)^H$ is finitely generated then $X//H = \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^H)$ as in the reductive case. However $X//H$ is not always a projective variety; moreover (even when $\hat{\mathcal{O}}_L(X)^H$ is finitely generated and so $X//H = \mathrm{Proj}(\hat{\mathcal{O}}_L(X)^H)$ is a projective variety) the H-invariant morphism $X^{ss} \to X//H$ is not necessarily a categorical quotient, and its image is not in general a subvariety of $X//H$, only a constructible subset. A final problem is that there are in general no obvious analogues in this situation of the Hilbert–Mumford criteria for (semi)stability.

However non-reductive GIT is better behaved when the unipotent radical U of $H = U \rtimes R$ is 'internally graded' in the sense that its Levi subgroup $R \cong H/U$ has a central one-parameter subgroup $\lambda : \mathbb{C}^* \to R$ whose adjoint action on the Lie algebra of U has only strictly positive weights. It is shown in [5, 6, 10] that, provided that we are willing to twist the linearisation for a linear action of H on a projective variety X by an appropriate (rational) character, many of the good properties of Mumford's GIT hold. Many non-reductive linear algebraic group actions arising in algebraic geometry are actions of linear algebraic groups with internally graded unipotent radicals: in particular, any parabolic subgroup of a reductive group has this form, as does the automorphism group of any complete simplicial toric variety [13], and the group of k-jets of germs of biholomorphisms of $(\mathbb{C}^p, 0)$ for any positive integers k and p [10].

Example 3.3. The automorphism group of the weighted projective plane $\mathbb{P}(1,1,2)$ with weights 1,1 and 2 is $\mathrm{Aut}(\mathbb{P}(1,1,2)) \cong R \ltimes U$ where $R \cong GL(2)$ acting on the two-dimensional weight space with weight 1 is reductive, and $U \cong (\mathbb{C}^+)^3$ is unipotent with elements given by $(x,y,z) \mapsto (x, y, z + \lambda x^2 + \mu xy + \nu y^2)$ for $(\lambda, \mu, \nu) \in \mathbb{C}^3$.

Definition 3.4. Let us call a unipotent linear algebraic group U *graded unipotent* if there is a homomorphism $\lambda : \mathbb{C}^* \to Aut(U)$ with the weights of the $\mathbb{C}^*$ action on $Lie(U)$ all strictly positive. For such a homomorphism λ let

$$\hat{U} = U \rtimes \mathbb{C}^* = \{(u,t) : u \in U, t \in \mathbb{C}^*\}$$

be the associated semi-direct product of U and $\mathbb{C}^*$ with multiplication $(u,t) \cdot (u',t') = (u(\lambda(t)(u')), tt')$. We will say that a linear algebraic group $H = U \rtimes R$ has *internally graded unipotent radical* U if the centre $Z(R)$ of R has a one-parameter subgroup $\lambda : \mathbb{C}^* \to Z(R)$ whose adjoint action grades U.

When L is very ample, and so induces an embedding of X in a projective space $\mathbb{P}^n$, we can choose coordinates on $\mathbb{P}^n$ such that the action of $\mathbb{C}^*$ on X

is diagonal, given by

$$t \mapsto \begin{pmatrix} t^{r_0} & 0 & \cdots & 0 \\ 0 & t^{r_1} & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & t^{r_n} \end{pmatrix}$$

where $r_0 \leq r_1 \leq \cdots \leq r_n$. The *lowest bounded chamber* for this linear $\mathbb{C}^*$-action is the closed interval $[r_0, r_j]$ where $r_0 = \cdots = r_{j-1} < r_j \leq \cdots \leq r_n$, with interior the open interval (r_0, r_j), unless the action of $\mathbb{C}^*$ on X is trivial; when the action is trivial so that $r_0 = r_1 = \cdots = r_n$ we will say that $[r_0, r_0]$ is the lowest bounded chamber and it is its own interior. Note that in the situation above, if $\mathbb{C}^*$ acts trivially then so does U.

Theorem 3.3 ([5, 6]). *Let H be a linear algebraic group with internally graded unipotent radical, acting linearly on a projective variety X with linearisation $\mathcal{L}$ on a very ample line bundle L. Let $\lambda : \mathbb{C}^* \to Z(R)$ define the internal grading of the unipotent radical U of $H = U \rtimes R$. Suppose also that semistability coincides with stability for the unipotent radical U, in the sense that*

$$x \in Z_{\min} \Rightarrow \mathrm{Stab}_U(x) = \{e\}$$

where $Z_{\min}$ is the union of those connected components of the fixed point set $X^{\lambda(\mathbb{C}^*)}$ where $\lambda(\mathbb{C}^*)$ acts on the fibres of L^* with minimum weight. Then the linearisation for the action of $\hat{U} = U \rtimes \lambda(\mathbb{C}^*)$ on X can be twisted by a rational character of $\hat{U}$ so that 0 lies in the interior of the lowest bounded chamber for the linear $\lambda(\mathbb{C}^*)$ action on X and

(i) the algebras $\hat{\mathcal{O}}_{\mathcal{L}^{\otimes c}}(X)^{\hat{U}} = \oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^{\hat{U}}$ of $\hat{U}$-invariants and $\hat{\mathcal{O}}_{\mathcal{L}^{\otimes c}}(X)^{H} = \oplus_{m=0}^{\infty} H^0(X, L^{\otimes cm})^{H}$ of H-invariants are finitely generated for any sufficiently divisible integer $c > 0$, so that the enveloping quotients $X//\hat{U} = Proj(\hat{\mathcal{O}}_{\mathcal{L}^{\otimes c}}(X)^{\hat{U}})$ and $X//H = (X//\hat{U})//(R/\lambda(\mathbb{C}^*))$ are projective varieties;

(ii) $X^{ss,\hat{U}} = X^{s,\hat{U}}$ and also $X^{ss,H}$ and $X^{s,\hat{U}}$ have Hilbert–Mumford descriptions, and $X//\hat{U} = X^{s,\hat{U}}/\hat{U}$ is a geometric quotient of $X^{s,\hat{U}}$ by $\hat{U}$.

Moreover, even when the condition that semistability should coincide with stability for the unipotent radical fails, there is

(iii) a projective variety, containing the geometric quotient $X^{s,\hat{U}}/\hat{U}$ as an open subset, which is a geometric quotient $\tilde{X}^{ss,\hat{U}}/\hat{U}$ by $\hat{U}$ of an open subset $\tilde{X}^{ss,\hat{U}}$ of a $\hat{U}$-equivariant blow-up $\tilde{X}$ of X, and

(iv) an induced linear action of $R/\lambda(\mathbb{C}^*)$ on $\tilde{X}^{ss,\hat{U}}/\hat{U}$ whose reductive GIT quotient is a projective variety which contains the geometric quotient $X^{s,H}/H$ as an open subset.

4. Symplectic implosion

The non-reductive GIT quotients described in §3 can be studied using symplectic techniques closely related to the 'symplectic implosion' construction of Guillemin, Jeffrey and Sjamaar [20, 34]. In this paper this link will be described for the special case of the unstable strata for the moment map normsquare; in [11] we will explore more general situations.

For the original construction [20] we suppose that U is a maximal unipotent subgroup of a complex reductive group G acting linearly (with respect to an ample line bundle L) on a complex projective variety X, and we assume that the linear action of U on X extends to a linear action of G. Then the algebra of invariants $\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^U$ is a finitely generated graded algebra and the enveloping quotient $X/\!/U$ is the associated projective variety $\mathrm{Proj}(\bigoplus_{k\geq 0} H^0(X, L^{\otimes k})^U)$ [19]. It is shown in [20] that if K is a maximal compact subgroup of G, and X is given a suitable K-invariant Kähler form, then $X/\!/U$ can be identified with the 'symplectic implosion' or 'imploded cross-section' X_{impl} of X by K. In this section we will recall this construction and its generalisation [34] to the situation when U is the unipotent radical of any parabolic subgroup P of G.

As before let (X, ω) be a symplectic manifold on which a compact connected Lie group K acts with a moment map $\mu : X \to \mathfrak{k}^*$ where $\mathfrak{k}$ is the Lie algebra of K, and fix an invariant inner product on $\mathfrak{k}$, using it to identify $\mathfrak{k}^*$ with $\mathfrak{k}$. Let T be a maximal torus of K with Lie algebra $\mathfrak{t} \subseteq \mathfrak{k}$ and Weyl group $W = N_K(T)/T$, and let $\mathfrak{t}^*_+ \cong \mathfrak{t}^*/W \cong \mathfrak{k}^*/\mathrm{Ad}^*(K)$ be a positive Weyl chamber in $\mathfrak{k}^*$. The *imploded cross-section* [20] of X is then

$$X_{\mathrm{impl}} = \mu^{-1}(\mathfrak{t}^*_+)/\approx$$

where $x \approx y$ if and only if $\mu(x) = \mu(y) = \zeta \in \mathfrak{t}^*_+$ and $x = ky$ for some element k of the commutator subgroup $[K_\zeta, K_\zeta]$ of the stabiliser K_ζ of ζ under the co-adjoint action of K. If Σ is the set of faces of $\mathfrak{t}^*_+$ then X_{impl} is the disjoint union

$$X_{\mathrm{impl}} \;=\; \coprod_{\sigma\in\Sigma} \frac{\mu^{-1}(\sigma)}{[K_\sigma, K_\sigma]} \;=\; \mu^{-1}((\mathfrak{t}^*_+)^\circ) \quad \sqcup \quad \coprod_{\substack{\sigma \in \Sigma \\ \sigma \neq (\mathfrak{t}^*_+)^\circ}} \frac{\mu^{-1}(\sigma)}{[K_\sigma, K_\sigma]}$$

where $K_\sigma = K_\zeta$ for any $\zeta \in \sigma$. We give X_{impl} the quotient topology induced from $\mu^{-1}(\mathfrak{t}_+^*)$, and it inherits a stratified symplectic structure, where the strata are the locally closed subsets $\mu^{-1}(\sigma)/[K_\sigma, K_\sigma]$. Each such stratum is the symplectic reduction by the action of $[K_\sigma, K_\sigma]$ of a locally closed symplectic submanifold

$$X_\sigma = K_\sigma \mu^{-1}(\bigcup_{\tau \in \Sigma, \bar{\tau} \supseteq \sigma} \tau)$$

of X; locally near every point $y \in X_{\mathrm{impl}}$ can be identified symplectically with the product of the stratum containing y and a normal cone [20]. The induced action of T on X_{impl} preserves this stratified symplectic structure and has a moment map

$$\mu_{\mathrm{impl}} : X_{\mathrm{impl}} \to \mathfrak{t}_+^* \subseteq \mathfrak{t}^*$$

induced by the restriction of μ to $\mu^{-1}(\mathfrak{t}_+^*)$. If $\zeta \in \mathfrak{t}_+^*$ the symplectic reduction of X_{impl} at ζ for the action of T is the symplectic reduction of X at ζ for the action of K:

$$\frac{\mu_{\mathrm{impl}}^{-1}(\zeta)}{T} = \frac{\mu^{-1}(\zeta)}{T.[K_\zeta, K_\zeta]} = \frac{\mu^{-1}(\zeta)}{K_\zeta}.$$

The universal imploded cross-section (or universal symplectic implosion) is the imploded cross-section

$$(T^*K)_{\mathrm{impl}} = K \times \mathfrak{t}_+^* / \approx$$

of the cotangent bundle $T^*K \cong K \times \mathfrak{k}^*$ with respect to the K-action given by the right action of K on itself, with an induced action of $K \times T$ from the left action of K on itself and the right action of T on K. Any other implosion X_{impl} can be constructed as the symplectic quotient of the product $X \times (T^*K)_{\mathrm{impl}}$ by the diagonal action of K [20].

The universal symplectic implosion $(T^*K)_{\mathrm{impl}}$ is always a complex affine variety and its symplectic structure is given by a Kähler form. As in [20] we can assume for simplicity that K is semisimple and simply connected; for general compact connected K one can reduce to this case by considering the product $\tilde{K}$ of the centre of K and the universal cover of its commutator subgroup $[K, K]$, and expressing K as $\tilde{K}/\Upsilon$, where Υ is a finite central subgroup of $\tilde{K}$. When B is a Borel subgroup of the complexification $G = K_c$ of K with $G = KB$ and $K \cap B = T$, and $U_{\max} \leq B$ is the unipotent radical of B (and hence a maximal unipotent subgroup of G), then $U_{\max}$ is a Grosshans subgroup of G [18]. This means that the quasi-affine variety $G/U_{\max}$ can be embedded as an open subset of an affine variety in such a

way that its complement has complex codimension at least two, and so the algebra of invariants $\mathcal{O}(G)^{U_{\max}}$ is finitely generated. By [20] Proposition 6.8 there is a natural $K \times T$-equivariant identification

$$(T^*K)_{\text{impl}} \cong \text{Spec}(\mathcal{O}(G)^{U_{\max}})$$

of the canonical affine completion $\text{Spec}(\mathcal{O}(G)^{U_{\max}})$ of $G/U_{\max}$ with $(T^*K)_{\text{impl}}$. It follows that if X is a complex projective variety on which G acts linearly with respect to an ample line bundle L, and ω is an associated K-invariant Kähler form on X, then the symplectic quotient X_{impl} of $X \times (T^*K)_{\text{impl}}$ by K can be identified with the non-reductive GIT quotient

$$X//U_{\max} = \text{Proj}(\hat{\mathcal{O}}_L(X)^{U_{\max}}) \cong (X \times \text{Spec}(\mathcal{O}(G)^{U_{\max}}))//G \cong X_{\text{impl}}.$$

Suppose now that U is the unipotent radical of a parabolic subgroup P of the complex reductive group G with Lie algebra $\mathfrak{p}$. By replacing P with a suitable conjugate in G, we can assume that P contains the Borel subgroup B of G and $U \leq U_{\max}$. Then $P = UL^{(P)} \cong U \rtimes L^{(P)}$, where the Levi subgroup $L^{(P)}$ of P contains the complex maximal torus T_c of G, and we can assume in addition that $L^{(P)}$ is the complexification of its intersection

$$K^{(P)} = L^{(P)} \cap K = P \cap K$$

with K. There is a subset S_P of the set S of simple roots such that P is the unique parabolic subgroup of G containing B with the property that if $\alpha \in S$ then the root space $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{p}$ if and only if $\alpha \in S_P$. The Lie algebra of $L^{(P)}$ is generated by the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for $\alpha \in S_P$ together with the Lie algebra $\mathfrak{t}_c = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ of the complexification T_c of T, and the Lie algebra of U is

$$\mathfrak{u} = \bigoplus_{\alpha \in R^+ : \mathfrak{g}_\alpha \not\subseteq \text{Lie}(L^{(P)})} \mathfrak{g}_\alpha$$

where R^+ is the set of positive roots for G. The Lie algebra of P is

$$\mathfrak{p} = \mathfrak{t}_c \oplus \bigoplus_{\alpha \in R(S_P)} \mathfrak{g}_\alpha$$

where $R(S_P)$ is the union of R^+ with the set of all roots which can be written as sums of negatives of the simple roots in S_P. We can decompose

$\mathfrak{k}^{(P)} = \mathrm{Lie}K^{(P)}$ and $\mathfrak{t}$ as

$$\mathfrak{k}^{(P)} = [\mathfrak{k}^{(P)}, \mathfrak{k}^{(P)}] \oplus \mathfrak{z}^{(P)} \quad \text{and} \quad \mathfrak{t} = \mathfrak{t}^{(P)} \oplus \mathfrak{z}^{(P)}$$

where $[\mathfrak{k}^{(P)}, \mathfrak{k}^{(P)}]$ is the Lie algebra of the semisimple part $Q^{(P)} = [K^{(P)}, K^{(P)}]$ of $K^{(P)}$, while $\mathfrak{t}^{(P)}$ is the Lie algebra of the maximal torus $T^{(P)} = T \cap [K^{(P)}, K^{(P)}]$ of $Q^{(P)}$, and $\mathfrak{z}^{(P)}$ is the Lie algebra of the centre $Z(K^{(P)})$ of $K^{(P)}$.

When $U = U_{\max}$ the Iwasawa decomposition $G = K \exp(i\mathfrak{t})\, U_{\max}$ allows us to identify $G/U_{\max}$ with $K\exp(i\mathfrak{t})$. More generally there is a decomposition

$$\begin{aligned} G &= K \times_{K^{(P)}} P = K \times_{K^{(P)}} L^{(P)}U \\ &= K \times_{K^{(P)}} K^{(P)} \exp(i\mathfrak{k}^{(P)})U = K\exp(i\mathfrak{k}^{(P)})U \end{aligned} \tag{4.1}$$

giving an identification of G/U with $K\exp(i\mathfrak{k}^{(P)})$.

U is a Grosshans subgroup of G [19], and so the algebra of invariants $\mathcal{O}(G)^U$ is finitely generated and G/U has a canonical affine completion

$$G/U \subseteq \overline{G/U}^{\mathrm{a}} = \mathrm{Spec}(\mathcal{O}(G)^U)$$

where the complement of the open subset G/U of the affine variety $\overline{G/U}^{\mathrm{a}}$ has complex codimension at least two. Therefore if G acts linearly on a complex projective variety X with linearisation $\mathcal{L}$, then the algebra of invariants

$$\hat{\mathcal{O}}_{\mathcal{L}}(X)^U \cong (\hat{\mathcal{O}}_{\mathcal{L}}(X) \otimes \mathcal{O}(G)^U)^G$$

is finitely generated, and the associated projective variety $X/\!/U = \mathrm{Proj}(\hat{\mathcal{O}}_{\mathcal{L}}(X)^U)$ is isomorphic to the GIT quotient $(\overline{G/U}^{\mathrm{a}} \times X)/\!/G$. It is shown in [34] that, just as in the case when $U = U_{\max}$, there is a K-invariant Kähler form on $\overline{G/U}^{\mathrm{a}}$ which gives us an identification of $X/\!/U$ with a symplectic quotient of $\overline{G/U}^{\mathrm{a}} \times X$ by K, and thus a symplectic description of $X/\!/U$ generalising the symplectic implosion construction of [20].

To describe this generalised universal symplectic implosion, let $\Lambda = \ker(\exp|_{\mathfrak{t}})$ be the exponential lattice in $\mathfrak{t}$, and let $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the weight lattice in $\mathfrak{t}^*$, so that $\Lambda^*_+ = \Lambda^* \cap \mathfrak{t}^*_+$ is the monoid of dominant weights. For $\lambda \in \Lambda^*_+$ let V_λ be the irreducible G-module with highest weight λ, and let $\Pi = \{\varpi_1, \ldots, \varpi_r\}$ be the set of fundamental weights, forming a $\mathbb{Z}$-basis of Λ^* and a minimal set of generators for Λ^*_+. Recall that there is an isomorphism

of $G \times G$-modules

$$\mathcal{O}(G) \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda \otimes V_{\iota\lambda} \tag{4.2}$$

which restricts to an isomorphism of $G \times T_c$-modules

$$\mathcal{O}(G)^{U_{\max}} \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^{(T)} \otimes V_\lambda^* \cong \bigoplus_{\lambda \in \Lambda^*_+} V_\lambda^*. \tag{4.3}$$

where $V_\lambda^{(T)}$ is the irreducible T_c-module with highest weight λ. The graded algebra $\mathcal{O}(G)^{U_{\max}}$ is generated by its finite-dimensional vector subspace $\bigoplus_{\varpi \in \Pi} V_\varpi^*$, which gives us a closed $G \times T_c$-equivariant embedding of $\overline{G/U}^{\mathrm{a}}_{\max} = \mathrm{Spec}(\mathcal{O}(G)^{U_{\max}})$ into the affine space $\bigoplus_{\varpi \in \Pi} V_\varpi$. It is shown in [20] that $(T^*K)_{\mathrm{impl}}$ can be identified with the image of this embedding, equipped with the restriction of a flat K-invariant Kähler structure on $\bigoplus_{\varpi \in \Pi} V_\varpi$.

To extend this construction to $\overline{G/U}^{\mathrm{a}}$ when U is the unipotent radical of a parabolic subgroup P as above, it is observed in [34] that $\mathcal{O}(G)^U$ is generated by the smallest (finite-dimensional) $K^{(P)}$-invariant subspace of $\mathcal{O}(G)$ which contains $\bigoplus_{\varpi \in \Pi} V_\varpi^* \cong \bigoplus_{\varpi \in \Pi} V_\varpi^{(T)} \otimes V_\varpi^*$. Here $K^{(P)}$ acts on $\mathcal{O}(G)$ via left multiplication on G. Let $E^{(P)}$ be the dual of this smallest such $K^{(P)}$-invariant subspace $(E^{(P)})^*$ of $\mathcal{O}(G)$; then $(E^{(P)})^*$ is fixed pointwise by U, and its inclusion in $\mathcal{O}(G)^U \subseteq \mathcal{O}(G)$ induces a closed $L^{(P)} \times G$-equivariant embedding of $\overline{G/U}^{\mathrm{a}} = \mathrm{Spec}(\mathcal{O}(G)^U)$ into the affine space $E^{(P)}$. Then $(E^{(P)})^*$ decomposes under the action of $K \times K^{(P)}$ as a direct sum of irreducible $K \times K^{(P)}$-modules

$$(E^{(P)})^* = \bigoplus_{\varpi \in \Pi} (V_\varpi^{(P)})^*$$

where $(V_\varpi^{(P)})^*$ is the smallest $K \times K^{(P)}$-invariant subspace of $\mathcal{O}(G)$ containing V_ϖ^*. Moreover $(V_\varpi^{(P)})^* \cong V_\varpi^{K^{(P)}} \otimes V_\varpi^*$ where $V_\varpi^{K^{(P)}}$ is the irreducible $K^{(P)}$-module with highest weight ϖ, so

$$E^{(P)} = \bigoplus_{\varpi \in \Pi} V_\varpi^{(P)} = \bigoplus_{\varpi \in \Pi} (V_\varpi^{K^{(P)}})^* \otimes V_\varpi.$$

If $v_\varpi^{(P)}$ is the vector in $V_\varpi^{(P)} \cong (V_\varpi^{K^{(P)}})^* \otimes V_\varpi$ representing the inclusion of $V_\varpi^{K^{(P)}}$ in V_ϖ then the embedding of $G/U \subseteq \overline{G/U}^{\mathrm{a}}$ in $E^{(P)}$ induced by the

inclusion of $(E^{(P)})^*$ in $\mathcal{O}(G)^U$ takes the identity coset U to $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$. Let

$$V_\varpi^{K^{(P)}} = \bigoplus_{\lambda\in\Lambda_\varpi^*} V_{\varpi,\lambda}^{K^{(P)}}$$

be the decomposition of $V_\varpi^{K^{(P)}}$ into weight spaces with weights $\lambda \in \mathfrak{t}^*$ under the action of the maximal torus T of $K^{(P)}$. Then $V_\varpi^{(P)}$ decomposes as a $K \times T$-module into a sum of irreducible $K \times T$-modules

$$V_\varpi^{(P)} \cong \bigoplus_\lambda V_\varpi \otimes (V_{\varpi,\lambda}^{K^{(P)}})^*$$

and $v_\varpi^{(P)} = \sum_\lambda v_{\varpi,\lambda}^{(P)}$ where $v_{\varpi,\lambda}^{(P)} \in V_\varpi \otimes (V_{\varpi,\lambda}^{K^{(P)}})^*$ represents the inclusion of $V_{\varpi,\lambda}^{K^{(P)}}$ in V_ϖ. In particular $v_{\varpi,\varpi}^{(P)}$ is a highest weight vector for the action of $K \times K^{(P)}$ on $V_\varpi^{(P)}$. The embedding of $G/U \subseteq \overline{G/U}^{\mathrm{a}}$ in $E^{(P)}$ induced by the inclusion of $(E^{(P)})^*$ in $\mathcal{O}(G)^U$ takes the identity coset to $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$. From the decomposition $G = K\exp(i\mathfrak{k}^{(P)})U$ (4.1) and the compactness of K it follows that the closure $\overline{G/U}^{\mathrm{a}}$ of the G-orbit of $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$ in $E^{(P)}$ is given by the K-sweep

$$\overline{G/U}^{\mathrm{a}} = K(\overline{\exp(i\mathfrak{k}^{(P)}) \sum_{\varpi\in\Pi} v_\varpi^{(P)}})$$

of the closure in $E^{(P)}$ of the $\exp(i\mathfrak{k}^{(P)})$-orbit of $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$. Similarly the closure of the $L^{(P)}$-orbit of $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$ is given by $K^{(P)}(\overline{\exp(i\mathfrak{k}^{(P)}) \sum_{\varpi\in\Pi} v_\varpi^{(P)}})$.

There is a unique $K \times K^{(P)}$-invariant Hermitian inner product on $E^{(P)} = \bigoplus_{\varpi\in\Pi} V_\varpi^{(P)}$ satisfying $\|v_{\varpi,\varpi}^{(P)}\| = 1$ for each $\varpi \in \Pi$, which is obtained from K-invariant Hermitian inner products on the irreducible K-modules V_ϖ and their restrictions to $K^{(P)}$-invariant Hermitian inner products on the irreducible $K^{(P)}$-modules $V_\varpi^{(P)}$. This gives $E^{(P)}$ a flat Kähler structure which is $K \times K^{(P)}$-invariant. If we identify $(V_\varpi^{K^{(P)}})^* \otimes V_\varpi^{K^{(P)}}$ with $\mathrm{End}(V_\varpi^{K^{(P)}})$ equipped with the Hermitian structure $\langle A, B\rangle = \mathrm{Trace}(AB^*)$ in the standard way, then $v_\varpi^{(P)}$ is identified with the identity map in $\mathrm{End}(V_\varpi^{K^{(P)}})$.

Definition 4.1. Let $\mathfrak{t}^*_{(P)+}$ be the cone in $\mathfrak{t}^*$ given by

$$\mathfrak{t}^*_{(P)+} = \bigcup_{w\in W^{(P)}} \mathrm{Ad}^*(w)\mathfrak{t}^*_+$$

where $W^{(P)}$ is the Weyl group of $Q^{(P)} = [K^{(P)}, K^{(P)}]$ (which is a subgroup of the Weyl group W of K).

It is shown in [34] that the restriction to the closure $\overline{\exp(i\mathfrak{t})\sum_{\varpi\in\Pi} v_\varpi^{(P)}}$ of the $\exp(i\mathfrak{t})$-orbit in $E^{(P)}$ of $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$ of the moment map $\mu_T^{E^{(P)}}$ for the action of T on $E^{(P)}$ is a homeomorphism onto the cone $\mathfrak{t}^*_{(P)+}$ in $\mathfrak{t}^*$. Its inverse provides a continuous injection

$$\mathcal{F}^{(P)} : \mathfrak{t}^*_{(P)+} \to \overline{G/U}^{\mathrm{a}} \subseteq E^{(P)} \tag{4.4}$$

such that $\mu_T^{E^{(P)}} \circ \mathcal{F}^{(P)}$ is the identity on $\mathfrak{t}^*_{(P)+}$. Moreover $\overline{\exp(i\mathfrak{t})\sum_{\varpi\in\Pi} v_\varpi^{(P)}}$ is the union of finitely many $\exp(i\mathfrak{t})$-orbits, each of the form

$$\mathcal{F}^{(P)}(\sigma) = \exp(i\mathfrak{t}) \sum_{\varpi\in\Pi,\lambda\in\Lambda^*_\varpi\cap\bar\sigma} v^{(P)}_{\varpi,\lambda}$$

where σ is an open face of $\mathfrak{t}^*_{(P)+}$. Furthermore the restriction of the $K^{(P)}$-moment map $\mu^{E^{(P)}} : E^{(P)} \to (\mathfrak{k}^{(P)})^*$ to the closure of the $\exp(i\mathfrak{k}^{(P)})$-orbit in $E^{(P)}$ of $\sum_{\varpi\in\Pi} v_\varpi^{(P)}$ is a homeomorphism from $\overline{\exp(i\mathfrak{k}^{(P)})\sum_{\varpi\in\Pi} v_\varpi^{(P)}}$ onto the closed subset

$$\mathfrak{k}_+^{(P)*} = \mathrm{Ad}^*(K^{(P)})\ \mathfrak{t}^*_{(P)+}$$

of $\mathfrak{k}^{(P)*}$, and $\overline{\exp(i\mathfrak{k}^{(P)})\sum_{\varpi\in\Pi} v_\varpi^{(P)}}$ is the union of finitely many $\exp(i\mathfrak{k}^{(P)})$-orbits which correspond under this homeomorphism to the open faces of $\mathfrak{k}_+^{(P)*}$.

The inverse of $\mu^{E^{(P)}} : \overline{\exp(i\mathfrak{k}^{(P)})\sum_{\varpi\in\Pi} v_\varpi^{(P)}} \to \mathfrak{k}_+^{(P)*}$ gives us a continuous $K^{(P)}$-equivariant map

$$\mathcal{F}^{(P)} : \mathfrak{k}_+^{(P)*} \to \overline{G/U}^{\mathrm{a}} \subseteq E^{(P)}$$

extending (4.4) such that $\mu_T^{E^{(P)}} \circ \mathcal{F}^{(P)}$ is the identity on $\mathfrak{k}_+^{(P)*}$. This in turn extends to a continuous $K \times K^{(P)}$-equivariant surjection

$$\mathcal{F}^{(P)} : K \times \mathfrak{k}_+^{(P)*} \to \overline{G/U}^{\mathrm{a}}.$$

Definition 4.2. If $\zeta \in \mathfrak{k}_+^{(P)*} = \mathrm{Ad}^*(K^{(P)})\mathfrak{t}^*_{(P)+} = \mathrm{Ad}^*(K^{(P)})\mathfrak{t}^*_+$ let $\zeta = \mathrm{Ad}^*(k)\xi$ with $k \in K^{(P)}$ and $\xi \in \mathfrak{t}^*_+$, and let σ_0 be the open face of $\mathfrak{t}^*_+$ containing ξ. Let $\sigma_0(P)$ be the open face of $\mathfrak{t}^*_+$ whose closure is

$$\overline{\sigma_0(P)} = \{\zeta \in \mathfrak{t}^* : \zeta \cdot \alpha = 0 \text{ for all } \alpha \in R_{\sigma_0} \setminus R^{(P)}\}$$

where R and $R^{(P)}$ are the sets of roots of K and $K^{(P)}$, and $R_{\sigma_0} = \{\alpha \in R : \zeta \cdot \alpha = 0 \text{ for all } \zeta \in \sigma_0\}$, so that $\sigma_0(P)$ is an open subset of the open face containing σ_0 of the cone $\mathfrak{t}^*_{(P)+}$. Finally let $K_\zeta(P) = kK_\xi k^{-1}$ where $K_\xi(P) = K_{\sigma_0(P)}$ is the stabiliser under the adjoint action of K of any element of $\sigma_0(P)$.

Remark 4.1. If ξ lies in the interior of $\mathfrak{t}^*_{(P)+}$ then $K_\zeta(P) = T$ and $[K_\zeta(P), K_\zeta(P)]$ is trivial.

This leads to the following definition given in [34] of the $K^{(P)}$-imploded cross-section (or generalised symplectic implosion) $X_{\mathrm{KimplK}^{(P)}}$.

Definition 4.3. Let (X, ω) be a symplectic manifold with a Hamiltonian action of K with moment map $\mu : X \to \mathfrak{k}^*$. Let

$$\mathfrak{k}_+^{(P)*} = \mathrm{Ad}^*(K^{(P)})\mathfrak{t}^*_{(P)+} = \mathrm{Ad}^*(K^{(P)})\mathfrak{t}^*_+ = \mathrm{Ad}^*(Q^{(P)})\mathfrak{t}^*_+ \subseteq \mathfrak{k}^{(P)*}$$

be the sweep of $\mathfrak{t}^*_+$ under the co-adjoint action of $K^{(P)}$ on $\mathfrak{k}^*$, and let $\Sigma^{(P)}$ be the set of open faces of $\mathfrak{k}_+^{(P)*}$. If $\zeta \in \mathfrak{k}^{(P)*}$ let $K_\zeta(P)$ be as in Definition 4.2. The $K^{(P)}$*-imploded cross-section* (or generalised symplectic implosion) of X is

$$X_{\mathrm{KimplK}^{(P)}} = \mu^{-1}(\mathfrak{k}_+^{(P)*}) / \approx_{K^{(P)}}$$

where $x \approx_{K^{(P)}} y$ if and only if $\mu(x) = \mu(y) = \zeta \in \mathfrak{k}_+^{(P)*}$ and $x = \kappa y$ for some $\kappa \in [K_\zeta(P), K_\zeta(P)]$.

The *universal* $K^{(P)}$*-imploded cross-section* (or universal generalised symplectic implosion for $K^{(P)} \subseteq K$) is the $K^{(P)}$-imploded cross-section

$$(T^*K)_{\mathrm{KimplK}^{(P)}} = K \times \mathfrak{k}_+^{(P)*} / \approx_{K^{(P)}}$$

for the cotangent bundle $T^*K \cong K \times \mathfrak{k}^*$ with respect to the K-action induced from the right action of K on itself.

The map $\mathcal{F}^{(P)} : K \times \mathfrak{k}_+^{(P)*} \to \overline{G/U}^{\mathrm{a}}$ induces a $K \times K^{(P)}$-equivariant homeomorphism

$$(T^*K)_{\mathrm{KimplK}^{(\mathrm{P})}} = K \times \mathfrak{k}_+^{(P)*} / \approx_{K^{(P)}} \to \overline{G/U}^{\mathrm{a}} \subseteq E^{(P)}. \tag{4.5}$$

Moreover under this identification of $K \times \mathfrak{k}_+^{(P)*} / \approx_{K^{(P)}}$ with $\overline{G/U}^{\mathrm{a}} \subseteq E^{(P)}$, the moment map for the action of $K \times K^{(P)}$ on $E^{(P)}$ is induced by the map $(K \times \mathfrak{k}_+^{(P)*}) / \approx_{K^{(P)}} \to \mathfrak{k}^* \times \mathfrak{k}^{(P)*}$ given by

$$(k, \zeta) \mapsto (Ad^*(k)(\zeta), \zeta)).$$

$\overline{G/U}^{\mathrm{a}}$ has an induced $K \times K^{(P)}$-invariant Kähler structure as a complex subvariety of $E^{(P)}$; it is stratified by its (finitely many) G-orbits, and the $K \times K^{(P)}$-invariant Kähler structure on $E^{(P)}$ restricts to a $K \times K^{(P)}$-invariant symplectic structure on each stratum. Under the homeomorphism $(T^*K)_{\mathrm{KimplK}^{(\mathrm{P})}} \to \overline{G/U}^{\mathrm{a}}$ of (4.5) these strata correspond to the locally closed subsets

$$\begin{aligned} \frac{K \times \mathrm{Ad}^*(K^{(P)})\sigma}{\approx_{K^{(P)}}} &\cong K^{(P)} \times_{K_\sigma \cap K^{(P)}} \left(\frac{K \times \sigma}{\approx_{K^{(P)}}} \right) \\ &\cong K^{(P)} \times_{K_\sigma \cap K^{(P)}} \left(\frac{K \times \sigma}{[K_{\sigma(P)}, K_{\sigma(P)}]} \right) \end{aligned}$$

of $(T^*K)_{\mathrm{KimplK}^{(\mathrm{P})}}$ where $\sigma \in \Sigma$ runs over the open faces of $\mathfrak{t}_+^*$.

By construction, when K acts on a symplectic manifold X with moment map $\mu : X \to \mathfrak{k}^*$, then the symplectic quotient of

$$\overline{G/U}^{\mathrm{a}} \times X = (T^*K)_{\mathrm{KimplK}^{(\mathrm{P})}} \times X$$

by the diagonal action of K can be identified via $\mathcal{F}^{(P)}$ with $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ (and in particular if X is a projective variety with a linear action of the complexification G of K, then $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ can be identified with the GIT quotient of $\overline{G/U}^{\mathrm{a}} \times X$ by the diagonal action of G). Thus $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ inherits a stratified $K \times K^{(P)}$-invariant symplectic structure

$$\begin{aligned} X_{\mathrm{KimplK}^{(\mathrm{P})}} &= \bigsqcup_{\sigma \in \Sigma} \frac{\mu^{-1}(\sigma)}{\approx_{K^{(P)}}} \\ &= \mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ) \sqcup \bigsqcup_{\substack{\sigma \in \Sigma \\ \sigma \neq (\mathfrak{t}_+^*)^\circ}} K^{(P)} \times_{K_\sigma \cap K^{(P)}} \left(\frac{\mu^{-1}(\sigma)}{[K_{\sigma(P)}, K_{\sigma(P)}]} \right) \end{aligned} \tag{4.6}$$

with strata indexed by the set Σ of open faces of $\mathfrak{t}_+^*$, which are locally closed symplectic submanifolds of $X_{\mathrm{KimplK}^{(\mathrm{P})}}$. The induced action of $K^{(P)}$ on $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ preserves this symplectic structure and has a moment map

$$\mu_{X_{\mathrm{KimplK}^{(\mathrm{P})}}} : X_{\mathrm{KimplK}^{(\mathrm{P})}} \to \mathfrak{k}_+^{(P)*} \subseteq \mathfrak{k}^{(P)*}$$

inherited from the restriction of μ to $\mu^{-1}(\mathfrak{k}_+^{(P)})$.

Remark 4.2. If $K^{(P)} = T$ and $\zeta \in \mathfrak{k}_+^{(P)*}$ then $K_\zeta(P) = K_\zeta$, and so X_{KimplT} is the standard imploded cross-section X_{impl} of [20]. On the other hand if $K^{(P)} = K$ then $K_\zeta(P)$ is conjugate to T and $[K_\zeta(P), K_\zeta(P)]$ is trivial for all $\zeta \in \mathfrak{k}_+^{(P)*}$, so $X_{\mathrm{KimplK}} = T^*K$.

Remark 4.3. When K acts holomorphically on a Kähler manifold X with moment map $\mu : X \to \mathfrak{k}^*$, then the action of K extends to a holomorphic action of its complexification $G = K_{\mathbb{C}}$. Since the generalised symplectic implosion $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ is the symplectic quotient of $\overline{G/U}^{\mathrm{a}} \times X = (T^*K)_{\mathrm{KimplK}^{(\mathrm{P})}} \times X$ by the diagonal action of K, it has an induced Kähler structure (cf. Remarks 2.1, 3.1). The open subset $\mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ)$ of $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ corresponds to the open subset $(G/U) \times X$ of $\overline{G/U}^{\mathrm{a}} \times X$, and provides a $K^{(P)}$-invariant slice for the action of U on the open subset $U\mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ)$ of X. Thus if Y is a P-invariant complex submanifold of X which meets $\mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ)$, then $Y \cap \mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ)$ is a $K^{(P)}$-invariant slice for the action of U on the open subset $U(Y \cap \mu^{-1}((\mathfrak{k}_+^{(P)*})^\circ))$ of Y, and its closure in $X_{\mathrm{KimplK}^{(\mathrm{P})}}$, which is the image of $Y \cap \mu^{-1}(\mathfrak{k}_+^{(P)*})$ in $X_{\mathrm{KimplK}^{(\mathrm{P})}}$, has an induced Kähler structure. However the singularities of this closure on the image of the boundary of $Y \cap \mu^{-1}(\mathfrak{k}_+^{(P)*})$ are likely to be more serious and harder to describe than in the case when Y is G-invariant (or equivalently K-invariant).

In the general case when K acts on a symplectic manifold (X, ω) with moment map $\mu : X \to \mathfrak{k}^*$, then we can choose a K-invariant almost complex structure which is compatible with ω as at Remark 2.2. If Y is a $K^{(P)}$-invariant almost complex submanifold of X which is invariant under the induced infinitesimal action of U, then just as in the case above the image of $Y \cap \mu^{-1}(\mathfrak{k}_+^{(P)*})$ in $X_{\mathrm{KimplK}^{(\mathrm{P})}}$ has an induced $K^{(P)}$-invariant symplectic structure, and almost complex structure, such that it can be regarded as an almost-Kähler quotient of Y by the infinitesimal action of U. There is an induced Hamiltonian action of $K^{(P)}$ (or any subgroup of $K^{(P)}$) with moment map μ_{Yimpl} induced by the restriction of μ to $Y \cap \mu^{-1}(\mathfrak{k}_+^{(P)})$, and we can shift this moment map by any constant in the Lie algebra $\mathfrak{z}^{(P)}$ of

the centre $Z(K^{(P)})$ of $K^{(P)}$. It follows from the definition of $\mathfrak{t}^*_{(P)+}$ that for a generic choice of η in $\mathfrak{z}^{(P)}$ we have $K_\eta = K^{(P)}$ and $\eta \in \mu^{-1}((\mathfrak{k}^{(P)*}_+)^\circ)$, so the symplectic quotient by $K^{(P)}$ is given by

$$\mu^{-1}_{\mathrm{Yimpl}}(\eta)/K^{(P)} = (Y \cap \mu^{-1}(\eta))/K^{(P)}.$$

When Y is K-invariant this simply recovers for us the symplectic reduction of Y at η by the action of K, but the viewpoint from symplectic implosion allows us to extend this construction to include submanifolds Y which are not K-invariant (cf. [22]).

5. Symplectic quotients of unstable strata

As before let X be a compact symplectic manifold with a Hamiltonian action of a compact group K with moment map $\mu : X \to \mathfrak{k}^*$, and choose a compatible K-invariant almost complex structure and Riemannian metric as at Remark 2.2. Fix an invariant inner product on $\mathfrak{k}$ with associated norm.

Let $\{S_\beta : \beta \in \mathcal{B}\}$ be the Morse stratification for the function $\|\mu\|^2$. Recall that the set $\mathcal{B}$ indexing the critical subsets C_β for $\|\mu\|^2$ and the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ can be identified with a finite subset of a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra of a maximal torus T of K, where a point of $\mathfrak{t}_+$ lies in $\mathcal{B}$ if it is the closest point to the origin of the convex hull of a nonempty set of the weights for the Hamiltonian action of K, and we interpret 'weight' as the image under the T-moment map of a connected component of the fixed point set X^T. Then for $\beta \in \mathcal{B}$ the submanifold Z_β of X is the union of those components of the fixed point set of the subtorus T_β of K generated by β on which the moment map for T_β given by composing μ with the restriction map from $\mathfrak{k}^*$ to $\mathfrak{t}^*_\beta$ takes the value β, and

$$C_\beta = K(Z_\beta \cap \mu^{-1}(\beta)) \cong K \times_{K_\beta} (Z_\beta \cap \mu^{-1}(\beta))$$

where the subgroup K_β is the stabiliser of β under the adjoint action of K on its Lie algebra.

Recall that $\mu|_{Z_\beta}$ can be regarded as a moment map for the action of K_β on Z_β, and so can $\mu|_{Z_\beta} - \beta$ since β is central in K_β. We can define Z^{ss}_β to be the stratum labelled by 0 for the Morse stratification of the normsquare $\|\mu - \beta\|^2$ of the moment map $\mu|_{Z_\beta} - \beta$ on Z_β. For $x \in X$ we let $p_\beta(x)$ be the

limit $\lim_{t\to\infty} \exp(-it\beta)x$ of the downward trajectory from x for the Morse–Bott function $\mu_\beta = \mu.\beta$ on X, and define

$$Y_\beta^{ss} = p_\beta^{-1}(Z_\beta^{ss})$$

where Y_β with $p_\beta : Y_\beta \to Z_\beta$ is given by

$$Y_\beta = \{y \in X \,|\, p_\beta(y) \in Z_\beta\}$$

(cf. Remark 2.3). Then Y_β and Y_β^{ss} are K_β-invariant almost-complex submanifolds of X and we have

$$S_\beta = KY_\beta^{ss} \cong K \times_{K_\beta} Y_\beta^{ss}.$$

Moreover $p_\beta : Y_\beta \to Z_\beta$ is a locally trivial fibration whose fibre is isomorphic to $\mathbb{C}^m$ for some $m \geq 0$ depending on β (and possibly also on the connected component of Z_β over which the fibre lies).

The locally closed almost-complex submanifold Y_β of X is invariant under the action of the maximal torus T of K, and hence so is its closure $\overline{Y_\beta}$. Therefore by a result of Atiyah [2] (see also [12, 28]) the image of $\overline{Y_\beta}$ under the T-moment map μ_T given by composing μ with restriction $\mathfrak{k}^* \to \mathfrak{t}^*$ is a convex polytope in $\mathfrak{t}^*$; indeed it is the convex hull of the (finitely many) images of the T-fixed points in $\overline{Y_\beta}$. Thus $\mu_T(\overline{Y_\beta})$ is contained in the half-space in $\mathfrak{t}^*$ consisting of those $\eta \in \mathfrak{t}^*$ satisfying $\eta.\beta \geqslant \|\beta\|^2$; since by assumption $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$ is non-empty, β is in fact the closest point to 0 in $\mathfrak{t}^* \cong \mathfrak{t}$ of this convex hull, and a point $y \in X$ lies in Y_β if and only if β is the closest point to 0 of the image under μ_T of its trajectory under the gradient flow of μ_β.

Recall that $g \in G = K_{\mathbb{C}}$ lies in the parabolic subgroup P_β if and only if $\exp(-it\beta)g\exp(it\beta)$ tends to a limit in G as $t \to \infty$, and this limit defines a surjective homomorphism $q_\beta : P_\beta \to L_\beta$ whose kernel is the unipotent radical U_β of P_β. The chosen almost-Kähler structure on X is K-invariant, and so by the definition of a moment map the gradient flow of μ_β is given by the vector field $x \mapsto i\beta_x$ where $x \mapsto \beta_x$ is the infinitesimal action of $\beta \in \mathfrak{t}$ on X. Thus Y_β is invariant under the infinitesimal action of P_β on X.

This means that we can apply the symplectic implosion construction associated to the unipotent radical U_β of P_β to $\overline{Y_\beta}$ as in Remark 4.3, and take a symplectic quotient of the result by the induced Hamiltonian action of the maximal compact subgroup $K^{(P_\beta)} = K_\beta$ of P_β. As discussed in Remark 4.3, we can shift the moment map for this induced Hamiltonian action by any constant in the Lie algebra $\mathfrak{z}^{(P_\beta)}$ of the centre $Z(K^{(P_\beta)})$ of $K^{(P_\beta)}$, and for a

generic choice of η in $\mathfrak{z}^{(P_\beta)}$ we have $K_\eta = K^{(P_\beta)}$ and the symplectic quotient by $K^{(P_\beta)} = K_\beta$ is given by $(\overline{Y_\beta} \cap \mu^{-1}(\eta))/K_\beta$.

By definition if $\eta = \beta$ or any nonzero scalar multiple of β then $K_\eta = K_\beta = K^{(P_\beta)}$ and this symplectic quotient is $(\overline{Y_\beta} \cap \mu^{-1}(\eta))/K_\beta$. It follows from the description of Y_β above that if η is a generic element of $\mathfrak{z}^{(P_\beta)}$ and $\eta.\beta < \|\beta\|^2$ (or if we have equality and $\eta \neq \beta$) then $(\overline{Y_\beta} \cap \mu^{-1}(\eta))/K_\beta$ is empty, while if $\eta = \beta$ then the symplectic quotient $(\overline{Y_\beta} \cap \mu^{-1}(\eta))/K_\beta = (Z_\beta \cap \mu^{-1}(\beta))/K_\beta = C_\beta/K$ collapses the stratum onto its critical subset. On the other hand if η is a sufficiently small perturbation of β in $\mathfrak{z}^{(P_\beta)}$ then $\overline{Y_\beta} \cap \mu^{-1}(\eta) \subseteq Y_\beta$ so

$$(\overline{Y_\beta} \cap \mu^{-1}(\eta))/K_\beta = (Y_\beta \cap \mu^{-1}(\eta))/K_\beta.$$

It is therefore natural to choose η to be $(1+\epsilon)\beta$ for some sufficiently small $\epsilon > 0$ and define

$$S_\beta /\!\!/\!\!/_\epsilon K := (Y_\beta \cap \mu^{-1}((1+\epsilon)\beta))/K_\beta. \tag{5.1}$$

This has a stratified symplectic structure and it follows from the theory of variation of symplectic quotients [17, 23, 35] (cf. [15, 41]) that $S_\beta /\!\!/\!\!/_\epsilon K$ is independent of ϵ up to diffeomorphism for $0 < \epsilon << 1$ and the induced symplectic structure varies in a predictable fashion with ϵ; we can also use this theory to study the variation if η is chosen to be a different perturbation of β.

We have thus proved our main result.

Theorem 5.1. *Let X be a compact symplectic manifold with a Hamiltonian action of a compact group K with moment map $\mu : X \to \mathfrak{k}^*$. Choose a compatible K-invariant almost complex structure and Riemannian metric on X, and an invariant inner product on $\mathfrak{k}$ with associated norm. Let $\{S_\beta : \beta \in \mathcal{B}\}$ be the Morse stratification for the function $\|\mu\|^2$. If $\beta \in \mathcal{B} \setminus \{0\}$ and $0 < \epsilon << 1$ then*

$$S_\beta /\!\!/\!\!/_\epsilon K = (Y_\beta \cap \mu^{-1}((1+\epsilon)\beta))/K_\beta$$

is a compact stratified symplectic space which can be interpreted as a symplectic quotient for the action of K on the stratum S_β.

When $X \subseteq \mathbb{P}_n$ is a complex projective variety equipped with the Fubini–Study Kähler form and a linear action of K defined by a unitary representation $K \to U(n+1)$, then when ϵ is rational $S_\beta /\!\!/\!\!/_\epsilon K$ can be identified with a compactification of a quotient of an open subset of S_β by $G = K_{\mathbb{C}}$, obtained

from non-reductive GIT applied to the action of the parabolic subgroup P_β on Y_β, with the linearisation twisted by the rational character $(1+\epsilon)\beta$ of P_β.

6. The Yang–Mills functional over a compact Riemann surface

Atiyah and Bott observed [3] that the Yang–Mills functional over a compact Riemann surface Σ plays the role of $\|\mu\|^2$ for an infinite-dimensional Hamiltonian action. Here the symplectic quotient can be identified with a moduli space of semistable holomorphic bundles of fixed rank and degree over Σ, and the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is by the Harder–Narasimhan type of a holomorphic bundle.

Let Σ be a compact Riemann surface of genus $g \geqslant 2$, and let $\mathcal{E}$ be a fixed C^∞ complex hermitian vector bundle of rank n and degree d over Σ. Let $\mathcal{C}$ be the space of all holomorphic structures on $\mathcal{E}$. Since Σ has complex dimension one there are no integrality conditions to be satisfied, so $\mathcal{C}$ can be identified with the space of unitary connections on $\mathcal{E}$, which is an infinite-dimensional complex affine space with a flat Kähler structure.

Let $\mathcal{G}_\mathbb{C}$ denote the group of all C^∞ complex automorphisms of $\mathcal{E}$. We can regard $\mathcal{G}_\mathbb{C}$ as the complexification of the gauge group $\mathcal{G}$ consisting of C^∞ unitary automorphisms of $\mathcal{E}$. The natural action of $\mathcal{G}_\mathbb{C}$ on $\mathcal{C}$ preserves its complex structure and the action of the gauge group $\mathcal{G}$ preserves the Kähler structure and is Hamiltonian with a moment map given by the curvature of a connection. The central subgroup $\mathbb{C}^*$ of $\mathcal{G}_\mathbb{C}$ given by scalar multiplication on $\mathcal{E}$ acts trivially on $\mathcal{C}$, so the moment map in the direction of the corresponding central S^1 in $\mathcal{G}$ is constant; it is essentially given by the ratio d/n. The Yang–Mills functional on $\mathcal{C}$ takes a connection to the normsquare of its curvature and hence plays the role of $\|\mu\|^2$ for the action of the gauge group on $\mathcal{C}$, except that it is more natural to choose the moment map μ so that $\mu^{-1}(0)$ is nonempty by adding a suitable central constant to the curvature. This means that the Yang–Mills functional differs from $\|\mu\|^2$ by a constant, so their Morse stratifications will coincide.

Atiyah and Bott [3] identified the symplectic quotient $\mu^{-1}(0)/\mathcal{G}$ of $\mathcal{C}$ by the gauge group with the moduli space $\mathcal{M}(n,d)$ of semistable holomorphic vector bundles of rank n and degree d on Σ (modulo S-equivalence). Recall that a holomorphic vector bundle E over Σ is semistable (respectively stable) if every holomorphic subbundle D of E satisfies $\mathrm{slope}(D) \leq \mathrm{slope}(E)$, (respectively $\mathrm{slope}D) < \mathrm{slope}(E)$), where $\mathrm{slope}(D) = \deg(D)/\mathrm{rank}(D)$ (and thus semistable bundles of coprime rank and degree are stable). Any semistable vector bundle E has a Jordan–Hölder filtration by sub-bundles of

the same slope as E whose successive subquotients are stable; its associated graded bundle is the direct sum of these successive subquotients (which is independent of the choice of Jordan–Hölder filtration), and two semistable bundles of rank n and degree d are S-equivalent if their associated graded bundles are isomorphic.

A holomorphic bundle E over Σ of rank n and degree d has a canonical Harder–Narasimhan filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{s-1} \subset E_s = E$$

such that $\mathrm{slope}(E_{j-1}) > \mathrm{slope}(E_j)$ and E_j/E_{j-1} is semistable for $1 \leq j \leq s$. The Harder–Narasimhan type of E is then given by the data provided by the ranks and degrees of the successive subquotients E_j/E_{j-1}; in [3] this is encoded in the vector

$$\lambda(E) = (d_1/n_1, \ldots, d_1/n_1, d_2/n_2, \ldots, d_s/n_s)$$

in which d_j/n_j occurs n_j times. It was shown in [40] that if a family of bundles of Harder–Narasimhan type $\lambda = (\lambda_1, \ldots, \lambda_n)$ degenerates to a bundle of type $\lambda' = (\lambda'_1, \ldots, \lambda'_n)$ then $\lambda' \geqslant \lambda$ in the sense that $\sum_{j \leq i} \lambda'_j \geqslant \sum_{j \leq i} \lambda_j$ for all i.

The main aim of [3] is to study the cohomology of the moduli space $\mathcal{M}(n, d)$ by showing that the Yang–Mills functional is equivariantly perfect as a Morse function. Because of the analytical difficulties created by working in infinite-dimensions and the singularities in the critical locus for the Yang–Mills functional, Morse theory is not applied directly to the Yang–Mills functional in [3] but instead the stratification is defined directly in terms of Harder–Narasimhan types; however the analytical difficulties were later overcome [14]. Let Λ denote the set of all Harder–Narasimhan types, and for any Harder–Narasimhan type λ, let $\mathcal{C}_\lambda$ denote the subset of $\mathcal{C}$ consisting of holomorphic structures on $\mathcal{E}$ with Harder–Narasimhan type λ. Atiyah and Bott showed that $\{\mathcal{C}_\lambda : \lambda \in \Lambda\}$ is a $\mathcal{G}$-equivariantly perfect stratification of $\mathcal{C}$. They conjectured that it coincides with the Morse stratification for the Yang–Mills functional, which was later proved by Daskalopoulos [14].

The moduli space $\mathcal{M}(n, d)$ can also be constructed as finite-dimensional symplectic or GIT quotients, and the inductive formulas of [3] for its Betti numbers can be rederived via these 'finite-dimensional approximations' to the Yang–Mills picture [1, 29]. In [24] it is shown that the moduli spaces $\mathcal{M}(n, d)$ (and more generally moduli spaces of sheaves over any fixed non-singular projective scheme) can be constructed as GIT quotients for actions of complex reductive groups on finite-dimensional complex varieties such

that, for any given Harder–Narasimhan type λ for bundles of rank n and degree d there is a choice of GIT construction of $\mathcal{M}(n,d)$ for which the bundles of Harder–Narasimhan type λ appear as a stratum in the associated stratification $\{S_\beta : \beta \in \mathcal{B}\}$. The results of non-reductive GIT described in §3 can then be used to construct moduli spaces of holomorphic bundles of fixed Harder–Narasimhan type [7, 9].

Alternatively we can attempt to use the infinite-dimensional Yang–Mills construction of the moduli space $\mathcal{M}(n,d)$ as a symplectic quotient of $\mathcal{C}$ by the gauge group and the methods of this paper to find an analogous symplectic construction of moduli spaces of holomorphic bundles of fixed Harder–Narasimhan type. Ignoring the analytical difficulties associated to working with infinite-dimensional spaces and groups, we might proceed as follows.

Let $\lambda(E) = (d_1/n_1, \dots, d_1/n_1, d_2/n_2, \dots, d_s/n_s)$ be a Harder–Narasimhan type and fix a C^∞ filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_{s-1} \subset \mathcal{E}_s = \mathcal{E} \tag{6.1}$$

of the C^∞ bundle $\mathcal{E}$ with $\deg(\mathcal{E}_j/\mathcal{E}_{j-1}) = d_j$ and $\operatorname{rank}(\mathcal{E}_j/\mathcal{E}_{j-1}) = n_j$ for $1 \le j \le s$. Define $\mathcal{Y}_\lambda$ to be the subset of $\mathcal{C}$ consisting of those holomorphic structures (or equivalently unitary connections) on $\mathcal{E}$ which are compatible with this filtration, in the sense that the subbundles $\mathcal{E}_j$ are all holomorphic subbundles, and define $\mathcal{Y}_\lambda^{ss}$ to consist of those holomorphic structures for which in addition the induced holomorphic structures on the subquotients $\mathcal{E}_j/\mathcal{E}_{j-1}$ are semistable, so that the holomorphic structure on $\mathcal{E}$ lies in $\mathcal{C}_\lambda$. Let $\mathcal{P}_\lambda$ be the subgroup of $\mathcal{G}_\mathbb{C}$ consisting of the complex C^∞-automorphisms of $\mathcal{E}$ which preserve the filtration (6.1) and let $\mathcal{U}_\lambda$ be the kernel of its induced action on the direct sum of the successive subquotients $\mathcal{E}_j/\mathcal{E}_{j-1}$. There is a C^∞ decomposition of $\mathcal{E}$ as the orthogonal direct sum of the successive subquotients $\mathcal{E}_j/\mathcal{E}_{j-1}$; let $\mathcal{L}_\lambda$ be the subgroup of $\mathcal{P}_\lambda$ preserving this direct sum decomposition and let $\mathcal{K}_\lambda$ be its intersection with the gauge group $\mathcal{G}$. Finally let $\mathcal{Z}_\lambda$ be the subset of $\mathcal{Y}_\lambda$ consisting of holomorphic structures for which this orthogonal direct sum decomposition of $\mathcal{E}$ is a holomorphic decomposition, and let $\mathcal{Z}_\lambda^{ss} = \mathcal{Z}_\lambda \cap \mathcal{Y}_\lambda^{ss}$.

Then $\mathcal{Y}_\lambda$, $\mathcal{Y}_\lambda^{ss}$, $\mathcal{Z}_\lambda$, $\mathcal{Z}_\lambda^{ss}$ and $\mathcal{P}_\lambda$, $\mathcal{U}_\lambda$, $\mathcal{L}_\lambda$, $\mathcal{K}_\lambda$ play the roles for the Hamiltonian action of the gauge group $\mathcal{G}$ on $\mathcal{C}$, and on its stratum $\mathcal{C}_\lambda$, which Y_β, Y_β^{ss}, Z_β, Z_β^{ss} and P_β, U_β, L_β and K_β play in the finite-dimensional setting for the Hamiltonian action of the compact group K on the compact symplectic (or Kähler) manifold X, and its stratum S_β. Note however that it is really $\lambda - (d/n, \dots, d/n)$, not λ itself, which plays the role of β, since the

central circle subgroup in the gauge group acts trivially, so

$$(1+\epsilon)\lambda - \varepsilon(d/n, \dots, d/n)$$

plays the role of $(1+\varepsilon)\beta$.

Thus by analogy with the finite-dimensional situation, and using the methods of [14], we expect the stratum $\mathcal{C}_\lambda$ to have a symplectic quotient

$$\mathcal{C}_\lambda /\!\!/_\varepsilon \mathcal{G} = (\mathcal{Y}_\lambda \cap \mathrm{curv}^{-1}((1+\epsilon)\lambda - \varepsilon(d/n, \dots, d/n)))/\mathcal{K}_\lambda$$

for $0 < \varepsilon \ll 1$, where curv assigns to a holomorphic structure, or equivalently a unitary connection, on $\mathcal{E}$ its curvature, appropriately normalised, and $\mathcal{Y}_\lambda$ and $\mathcal{K}_\lambda$ are defined as above. Away from its singularities we expect this symplectic quotient to be identifiable with a suitable moduli space of holomorphic bundles of Harder–Narasimhan type λ. Alternatively such moduli spaces of holomorphic bundles can be constructed by applying Theorem 5.1 to a suitable finite-dimensional approximation [29] to the Atiyah–Bott picture [3].

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