Communications in
Analysis and Geometry
Volume 28, Number 4, 781–835, 2020

Existence of harmonic maps into CAT(1) spaces

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We are honored to contribute this article to the volume commemorating Karen Uhlenbeck. Each of us has been inspired by Karen's distinguished career and lasting legacy. We are grateful for her interest in this problem, and for the helpful discussions we had with her during our visit to BIRS. In addition, we appreciate the great role model and mentor she has been to the next generation of mathematicians.

Let $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$ where Σ is a compact Riemann surface, X is a compact locally CAT(1) space, and $W^{1,2}(\Sigma, X)$ is defined as in Korevaar-Schoen. We use the technique of harmonic replacement to prove that either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or there exists a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$. To complete the argument, we prove compactness for energy minimizers and a removable singularity theorem for conformal harmonic maps.

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This work began as part of the workshop "Women in Geometry" (15w5135) at the Banff International Research Station in November of 2015. We are grateful to BIRS for the opportunity to attend and for the excellent working environment. CB, CM were supported in part by NSF grants DMS-1308420 and DMS-1406332 respectively, and LH was supported by NSF grants DMS-1308837 and DMS-1452477. AF was supported in part by an NSERC Discovery Grant. PS was supported in part by an NSERC PGS D scholarship and a UBC Four Year Doctoral Fellowship. YZ was supported in part by an AWM-NSF Travel Grant. This material is also based upon work supported by NSF DMS-1440140 while CB and AF were in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Spring 2016 semester.

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1. Introduction

In many existence theorems for harmonic maps, the key assumption is the non-positivity of the curvature of the target space. The prototype is the celebrated work of Eells and Sampson [ES] and Al'ber [A1], [A2] where the assumption of the non-positive sectional curvature of the target Riemannian manifold plays an essential role. The Eells-Sampson existence theorem has been extended to the equivariant case by Diederich-Ohsawa [DO], Donaldson [D], Corlette [C], Jost-Yau [JY] and Labourie [La]. Again, all these works assume non-positive sectional curvature on the target. For smooth Riemannian manifold domains and NPC targets (i.e. complete metric spaces with non-positive curvature in the sense of Alexandrov), existence theorems were obtained by Gromov-Schoen [GS] and Korevaar-Schoen [KS1], [KS2]. The generalization to the case when the domain is a metric measure space has been discussed by Jost ([J2] and the references therein) and separately by Sturm [St].

When the curvature of the target space is not assumed to be non-positive, the existence problem for harmonic maps becomes more complicated, and in many ways, more interesting. Although the general problem is not well understood, a breakthrough was achieved in the case of two-dimensional domains by Sacks and Uhlenbeck [SU1]. Indeed, they discovered a "bubbling phenomena" for harmonic maps; more specifically, they prove the following dichotomy: given a finite energy map from a Riemann surface into a compact Riemannian manifold, either there exists a harmonic map homotopic to the given map or there exists a branched minimal immersion of the 2-sphere. We also mention the related works of Lemaire [Le], Sacks-Uhlenbeck [SU2], and Schoen-Yau [SY].

The goal of this paper is to prove an analogous result when the target space is a compact CAT(1) space, i.e. a compact metric space of curvature bounded above by 1 in the sense of Alexandrov.

Theorem 1.1. *Let Σ be a compact Riemann surface, X a compact locally CAT(1) space and $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$. Then either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$.*

Sacks and Uhlenbeck used the perturbed energy method in the proof of Theorem 1.1 for Riemannian manifolds. In doing so, they rely heavily on a priori estimates procured from the Euler-Lagrange equation of the perturbed energy functional. One of the difficulties in working in the singular setting is that, because of the lack of local coordinates, one does not have a P.D.E. derived from a variational principle (e.g. harmonic map equation). In order to prove results in the singular setting, we cannot rely on P.D.E. methods. To this end, we use a 2-dimensional generalization of the Birkhoff curve shortening method [B1], [B2]. The local replacement process can be thought of as a discrete gradient flow. This idea was used by Schoen [Sc, Theorem 2.12] to give a short proof of the Eells-Sampson existence result, and by Jost [J1] to give an alternative proof of the Sacks-Uhlenbeck theorem in the smooth setting. More recently, in studying width and proving finite time extinction of the Ricci flow, Colding-Minicozzi [CM] further developed the local replacement argument and proved a new convexity result for harmonic maps and continuity of harmonic replacement; see also [Z1, Z2]. However, even these arguments rely on the harmonic map equation and hence do not translate to our case. The main accomplishment of our method is to eliminate the need for a P.D.E. by using the local convexity properties of the target CAT(1) space. (The necessary convexity properties of a CAT(1) space are given in Appendices A & B.)

For clarity, we provide a brief outline of the harmonic replacement construction. Given $\varphi : \Sigma \to X$, we set $\varphi = u_0^0$ and inductively construct a sequence of energy decreasing maps u_n^l where $n \in \mathbb{N} \cup \{0\}$, $l \in \{0, \dots, \Lambda\}$, and Λ depends on the geometry of Σ. The sequence is constructed inductively as follows. Given the map u_n^0, we determine the largest radius, r_n, in the domain on which we can apply the existence and regularity of Dirichlet solutions (see Lemma 2.2) for this map. Given a suitable cover of Σ by balls of this radius, we consider Λ subsets of this cover such that every subset consists of non-intersecting balls. The maps $u_n^l : \Sigma \to X$, $l \in \{1, \dots, \Lambda\}$ are determined by replacing u_n^{l-1} by its Dirichlet solution on balls in the l-th subset of the covering and leaving the remainder of the map unchanged. We then set $u_{n+1}^0 := u_n^\Lambda$ to continue by induction. There are now two possibilities, depending on $\liminf r_n = r$. If $r > 0$, we demonstrate that the sequence we constructed is equicontinuous and has a unique limit that is necessarily homotopic to φ. Compactness for minimizers (Lemma 2.4) then implies that the limit map is harmonic. If $r = 0$, then bubbling occurs. That is, after an appropriate rescaling of the original sequence, the new sequence is an equicontinuous family of harmonic maps from domains exhausting $\mathbb{C}$. As in the previous case, this sequence converges on compact sets to a limit harmonic map from $\mathbb{C}$ to X. We extend this map to $\mathbb{S}^2$ by a removable singularity theorem developed in section 3.

We now give an outline of the paper. In section 2, we introduce some notation and provide the results that are necessary in order to perform harmonic replacement and obtain a harmonic limit map. In particular, we state the existence and regularity results for Dirichlet solutions and prove compactness of energy minimizing maps into a CAT(1) space. In section 3, we prove our removable singularity theorem. Namely, in Theorem 3.6 we prove that any conformal harmonic map from a punctured surface into a CAT(1) space extends as a locally Lipschitz harmonic map on the surface. This theorem extends to CAT(1) spaces the removable singularity theorem of Sacks-Uhlenbeck [SU1] for a finite energy harmonic map into a Riemannian manifold, provided the map is conformal. The proof relies on two key ideas. First, for harmonic maps u_0 and u_1 into a CAT(1) space, while $d^2(u_0, u_1)$ is not subharmonic, a more complicated weak differential inequality holds if the maps are into a sufficiently small ball (Theorem B.4 in Appendix B, [Se1]). Using this inequality, we prove a local removable singularity theorem for harmonic maps into a small ball. The second key idea, Theorem 3.4, is a monotonicity of the area in extrinsic balls in the target space, for conformal harmonic maps from a surface to a CAT(1) space. This theorem extends the classical monotonicity of area for minimal surfaces in Riemannian manifolds to metric space targets. The proof relies on the fact that the distance function from a point in a CAT(1) space is almost convex on a small ball. In application, the monotonicity is used to show that a conformal harmonic map defined on $\Sigma\backslash\{p\}$ is continuous across p. Then the local removable singularity theorem can be applied at some small scale. Section 4 contains the harmonic replacement construction outlined above and the proof of the main theorem, Theorem 1.1. Finally, in Appendix A we give complete proofs of several difficult estimates for quadrilaterals in a CAT(1) space. The estimates are stated in the unpublished thesis [Se1] without proof. We apply these estimates in Appendix B to give complete proofs of some energy convexity, existence, uniqueness, and subharmonicity results (also stated in [Se1]) that are used throughout this paper.

2. Preliminary results

Throughout the paper we let (Ω, g) denote a Lipschitz Riemannian domain and (X, d) a locally CAT(1) space. We refer the reader to Section 2.2 of [BFHMSZ] for some background on CAT(1) spaces. A metric space (X, d) is said to be *locally* CAT(1) if every point of X has a geodesically convex CAT(1) neighborhood. Note that for a compact locally CAT(1) space, there exists a radius $r(X) > 0$ such that for all $y \in X$, $\overline{B_{r(X)}(y)}$ is a compact CAT(1) space.

We define the Sobolev space $W^{1,2}(\Omega, X) \subset L^2(\Omega, X)$ of finite energy maps. In particular, if $u \in W^{1,2}(\Omega, X)$, one can define its energy density $|\nabla u|^2 \in L^1(\Omega)$ and the total energy

$$^{d}E^u[\Omega] = \int_\Omega |\nabla u|^2 d\mu_g.$$

We often suppress the superscript d when the context is clear. We refer the reader to [KS1] for further details and background. We denote a geodesic ball in Ω of radius r centered at $p \in \Omega$ by $B_r(p)$ and a geodesic ball in X of radius ρ centered at $P \in X$ by $\mathcal{B}_\rho(P)$. Furthermore, given $h \in W^{1,2}(\Omega, X)$, we define

$$W^{1,2}_h(\Omega, X) = \{f \in W^{1,2}(\Omega, X) : Tr(h) = Tr(f)\},$$

where $Tr(u) \in L^2(\partial\Omega, X)$ denotes the trace map of $u \in W^{1,2}(\Omega, X)$ (see [KS1] Section 1.12).

Definition 2.1. We say that a map $u : \Omega \to X$ is *harmonic* if it is locally energy minimizing with locally finite energy; precisely, for every $p \in \Omega$, there exist $r > 0$, $\rho > 0$ and $P \in X$ such that $u(B_r(p)) \subset \mathcal{B}_\rho(P)$, where $\mathcal{B}_\rho(P)$ is geodesically convex, and $h = u\big|_{B_r(p)}$ has finite energy and minimizes energy among all maps in $W^{1,2}_h(B_r(p), \overline{\mathcal{B}_\rho(P)})$.

The following results will be used in the proof of the main theorem, Theorem 1.1.

Lemma 2.2 (Existence, Uniqueness and Regularity of the Dirichlet solution). *For any finite energy map $h : \Omega \to \overline{\mathcal{B}_\rho(P)} \subset X$, where $2\rho \in (0, \min\{r(X), \frac{\pi}{4}\})$, the Dirichlet solution exists. That is, there exists a unique element ${}^{Dir}h \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(P)})$ that minimizes energy among all maps in $W^{1,2}_h(\Omega, X)$. Moreover, if ${}^{Dir}h(\partial\Omega) \subset \overline{\mathcal{B}_\sigma(P)}$ for some $\sigma \in (0, \rho)$, then*

$\overline{{}^{Dir}h(\Omega)} \subset \overline{\mathcal{B}_\sigma(P)}$. *Finally, the solution* ${}^{Dir}h$ *is locally Lipschitz continuous with Lipschitz constant depending only on the total energy of the map and the metric on the domain.*

For further details see Lemma B.2 in Appendix B, [Se1], and [BFHMSZ].

Remark 2.3. In [BFHMSZ] we demonstrated that ${}^{Dir}h$ minimized energy among all maps in $W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(P)})$. In fact, ${}^{Dir}h$ minimizes energy among all maps in $W^{1,2}_h(\Omega, X)$. To see this, first note that for any $Q \in \mathcal{B}_{2\rho}(P)$, there exists a unique geodesic connecting P to Q. Denote this geodesic by γ_{PQ}. Let $C\gamma_{PQ}$ denote the point on γ_{PQ} at distance C from P. Now consider the continuous map $\pi : X \to \mathcal{B}_\rho(P)$ defined such that

$$\pi(Q) := \begin{cases} Q & \text{for } Q \in \mathcal{B}_\rho(P) \\ (2\rho - d(P,Q))\gamma_{PQ} & \text{for } Q \in \mathcal{B}_{2\rho}(P) \backslash \mathcal{B}_\rho(P) \\ P & \text{for } Q \in X \backslash \mathcal{B}_{2\rho}(P) \end{cases}$$

Since π is everywhere distance non-increasing, for any $u \in W^{1,2}_h(\Omega, X)$, $\pi \circ u \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(P)})$ and

$${}^dE^{{}^{Dir}h}[\Omega] \leq {}^dE^{\pi\circ u}[\Omega] \leq {}^dE^{u}[\Omega].$$

Thus the stronger result holds.

Lemma 2.4 (Compactness for minimizers into CAT(1) space). *Let* (X, d) *be a CAT(1) space and* $B_r \subset \Omega$ *a geodesic (and topological) ball of radius* $r > 0$ *where* (Ω, g) *is a Riemannian manifold. Let* $u_i : B_r \to X$ *be a sequence of energy minimizers with* $E^{u_i}[B_r] \leq \Lambda$ *for some* $\Lambda > 0$.

Suppose that u_i *converges uniformly to* u *on* B_r *and that there exists* $P \in X$ *such that* $u(B_r) \subset \mathcal{B}_{\rho/2}(P)$ *where* ρ *is as in Lemma 2.2. Then* u *is energy minimizing on* $B_{r/2}$.

Proof. We will follow the ideas of the proof of Theorem 3.11 [KS2]. Rather than prove the bridge principle for CAT(1) spaces, we will modify the argument and appeal directly to the bridge principle for NPC spaces (see Lemma 3.12 [KS2]).

Since $u_i \to u$ uniformly and $u(B_r) \subset \mathcal{B}_{\rho/2}(P)$, there exists I large such that for all $i \geq I$, $u_i(B_r) \subset \mathcal{B}_\rho(P)$. By Lemma 2.2, there exists $c > 0$ depending only on Λ and g such that for all $i \geq I$, $u_i|_{B_{3r/4}}$ is Lipschitz with Lipschitz constant c. It follows that for $t > 0$ small, there exists $C > 0$ depending on

c and the dimension of Ω such that

$$E^{u_i}[B_{r/2} \backslash B_{r/2-t}] \leq Ct. \tag{2.1}$$

For $\varepsilon > 0$, increase I if necessary so that for all $i \geq I$ and all $x \in B_{3r/4}$,

$$d^2(u_i(x), u(x)) < \varepsilon. \tag{2.2}$$

For notational ease, let $U_t := B_{r/2-t}$. Let $w_t : U_t \to X$ denote the energy minimizer $w_t := {}^{Dir}u|_{U_t} \in W^{1,2}_u(U_t, X)$, with existence guaranteed by Lemma 2.2. Following the argument in the proof of Theorem 3.11 [KS2], (2.1) and the lower semi-continuity of the energy imply that $\lim_{t\to 0} E^{w_t}[U_t] = E^{w_0}[B_{r/2}]$. Observe that by the lower semi-continuity of energy, Thm. 1.6.1 [KS1],

$${}^d E^u[B_{r/2}] \leq \liminf_{i\to\infty} {}^d E^{u_i}[B_{r/2}].$$

Thus, it will be enough to show that

$$\limsup_{i\to\infty} {}^d E^{u_i}[B_{r/2}] \leq {}^d E^{w_0}[B_{r/2}].$$

Let $v_t : B_{r/2} \to X$ be the map such that $v_t|_{U_t} = w_t$ and $v_t|_{B_{r/2}\backslash U_t} = u$. Given $\delta > 0$, choose $t > 0$ sufficiently small so that

$${}^d E^{v_t}[B_{r/2}] < {}^d E^{w_0}[B_{r/2}] + \delta. \tag{2.3}$$

Since v_t is not a competitor for u_i (i.e. $v_t|_{\partial B_{r/2}}$ is not necessarily equal to $u_i|_{\partial B_{r/2}}$), for each i we want to bridge from v_t to u_i for values near $\partial B_{r/2}$. Since we want to exploit a bridging lemma into NPC spaces, rather than bridge between v_t and u_i, we will bridge between their lifted maps in the cone $\mathcal{C}(X)$.

Let $\mathcal{C}(X) := (X \times [0,\infty)/X \times \{0\}, D)$ where

$$D^2([P,x],[Q,y]) = x^2 + y^2 - 2xy \cos \min(d(P,Q), \pi).$$

Then $\mathcal{C}(X)$ is an NPC space and we can identify X with $X \times \{1\} \subset \mathcal{C}(X)$. For any map $f : B_r \to X$, we let $\overline{f} : B_r \to X \times \{1\}$ such that $\overline{f}(x) = [f(x), 1]$. Note that for $f \in W^{1,2}(B_r, \mathcal{B}_\rho(Q))$, since

$$\lim_{P\to Q} \frac{D^2([P,1],[Q,1])}{d^2(P,Q)} = \lim_{P\to Q} \frac{2(1-\cos(d(P,Q)))}{d^2(P,Q)} = 1,$$

it follows that ${}^D E^{\overline{f}}[\Omega] = {}^d E^f[\Omega]$ for $\Omega \subset B_r$.

For each $i \geq I$, and a fixed $s, \rho > 0$ to be chosen later, define the map

$$v_i : \partial U_s \times [0, \rho] \to \mathcal{C}(X)$$

such that

$$v_i(x, z) := \left(1 - \frac{z}{\rho}\right) \overline{v}_t(x) + \frac{z}{\rho} \overline{u}_i(x).$$

The map v_i is a bridge between $\overline{v}_t|_{\partial U_s}$ and $\overline{u}_i|_{\partial U_s}$ in the NPC space $\mathcal{C}(X)$. That is, we are interpolating along geodesics connecting $\overline{v}_t(x), \overline{u}_i(x)$ in the NPC space $\mathcal{C}(X)$ and not along geodesics in X. By [KS2] (Lemma 3.12) and the equivalence of the energies for a map f and its lift $\overline{f}$,

$$\begin{aligned} {}^D E^{v_i}[\partial U_s \times [0, \rho]] &\leq \frac{\rho}{2} \left({}^D E^{\overline{v}_t}[\partial U_s] + {}^D E^{\overline{u}_i}[\partial U_s] \right) \\ &\quad + \frac{1}{\rho} \int_{\partial U_s} D^2([v_t, 1], [u_i, 1]) d\sigma \\ &= \frac{\rho}{2} \left({}^d E^{v_t}[\partial U_s] + {}^d E^{u_i}[\partial U_s] \right) \\ &\quad + \frac{1}{\rho} \int_{\partial U_s} D^2([v_t, 1], [u_i, 1]) d\sigma. \end{aligned}$$

By (2.1), and since $v_t = u$ on $B_{r/2} \backslash U_t$, for $s \in [2t/3, 3t/4]$ the average values of the tangential energies of v_t and u_i on ∂U_s are bounded above by $Ct/(3t/4 - 2t/3) = 12C$. Moreover, since $u_i(B_{r/2}), v_t(B_{r/2}) \subset \mathcal{B}_\rho(P)$, (2.2) implies that for all $x \in B_{r/2} \backslash U_t$,

$$D^2(\overline{u}_i(x), \overline{v}_t(x)) = 2(1 - \cos d(u_i(x), v_t(x))) \leq d^2(u_i(x), v_t(x)) < \varepsilon. \tag{2.4}$$

Thus, there exists $C' > 0$ depending only on g such that for every $s \in [2t/3, 3t/4]$,

$$\int_{\partial U_s} D^2([v_t, 1], [u_i, 1]) d\sigma < C' \varepsilon.$$

Note that for each $\varepsilon > 0$, the bound above depends on I but not on t. Now, we first choose an $s \in (2t/3, 3t/4)$ such that ${}^d E^{v_t}[\partial U_s] + {}^d E^{u_i}[\partial U_s] \leq 24C$. Next, pick $0 < \mu \ll 1$ such that $[s, s + \mu t] \subset [2t/3, 3t/4]$ and $12C\mu t < \delta/2$.

For this t, μ, decrease ε if necessary (by increasing I) such that

$$\begin{aligned} {}^{D}E^{v_i}[\partial U_s \times [0,\mu t]] &= \frac{\mu t}{2}\left({}^{d}E^{v_t}[\partial U_s] + {}^{d}E^{u_i}[\partial U_s]\right) \\ &\quad + \frac{1}{\mu t}\int_{\partial U_s} D^2([v_t,1],[u_i,1])d\sigma \\ &< 24C\mu t/2 + C'\varepsilon/(\mu t) \\ &< \delta. \end{aligned}$$

Now, define $\tilde{v}_i : B_{r/2} \to \mathcal{C}(X)$ such that on U_s, $\tilde{v}_i$ is the conformally dilated map of $\overline{v}_t$ so that $\tilde{v}_i|_{\partial U_{s+\mu t}} = \overline{v}_t|_{\partial U_s}$. On $U_s \backslash U_{s+\mu t}$, let $\tilde{v}_i$ be the bridging map v_i, reparametrized in the second factor from $[0,\mu t]$ to $[s, s+\mu t]$. Finally, on $B_{r/2}\backslash U_s$, let $\tilde{v}_i = \overline{u}_i$. Then, for all $i \geq I$,

$$ {}^{D}E^{\tilde{v}_i}[B_{r/2}] \leq {}^{d}E^{v_t}[B_{r/2}] + \delta + {}^{d}E^{u_i}[B_{r/2}\backslash U_s]. \tag{2.5}$$

While the map $\tilde{v}_i$ agrees with $\overline{u}_i$ on $\partial B_{r/2}$, it is not a competitor for u_i into X since $\tilde{v}_i$ maps into $\mathcal{C}(X)$. However, by defining $\underline{v}_i : B_{r/2} \to X$ such that $\tilde{v}_i(x) = [\underline{v}_i(x), h(x)]$, $\underline{v}_i$ is a competitor. Note that for all $x \in \partial U_s$, (2.4) implies that $h(x) \geq 1 - \sqrt{\varepsilon}$. Therefore, on the bridging strip we may estimate the change in energy under the projection map by first observing the pointwise bound

$$\begin{aligned} D^2(\tilde{v}_i(x), \tilde{v}_i(y)) &= D^2([\underline{v}_i(x), h(x)], [\underline{v}_i(y), h(y)]) \\ &= h(x)^2 + h(y)^2 - 2h(x)h(y)\cos(d(\underline{v}_i(x), \underline{v}_i(y))) \\ &= (h(x) - h(y))^2 + 2h(x)h(y)(1 - \cos(d(\underline{v}_i(x), \underline{v}_i(y)))) \\ &\geq 2(1-\sqrt{\varepsilon})^2(1 - \cos(d(\underline{v}_i(x), \underline{v}_i(y)))) \\ &= (1-\sqrt{\varepsilon})^2 D^2([\underline{v}_i(x), 1], [\underline{v}_i(y), 1]). \end{aligned}$$

Therefore,

$$ {}^{d}E^{\underline{v}_i}[B_{r/2}] = {}^{D}E^{[\underline{v}_i,1]}[B_{r/2}] \leq \left(1-\sqrt{\varepsilon}\right)^{-2} {}^{D}E^{\tilde{v}_i}[B_{r/2}]. \tag{2.6}$$

Since $\underline{v}_i$ is a competitor for u_i on $B_{r/2}$, (2.6), (2.5), (2.3), and (2.1) imply that

$$\begin{aligned} {}^{d}E^{u_i}[B_{r/2}] &\leq \left(1-\sqrt{\varepsilon}\right)^{-2} {}^{D}E^{\tilde{v}_i}[B_{r/2}] \\ &\leq \left(1-\sqrt{\varepsilon}\right)^{-2} \left({}^{d}E^{w_0}[B_{r/2}] + 2\delta + Ct\right) \end{aligned}$$

Since for any $\varepsilon, \delta > 0$, by choosing $t > 0$ sufficiently small and $I \in \mathbb{N}$ large enough, the previous estimate holds for all $i \geq I$, the inequality

$$\limsup_{i \to \infty} {}^d E^{u_i}[B_{r/2}] \leq {}^d E^{w_0}[B_{r/2}]$$

then implies the result. □

3. Monotonicity and removable singularity theorem

We first show the removable singularity theorem for harmonic maps into small balls. Note that the first theorem of this section is true for domains of dimension $n \geq 2$, but all other results require the domain dimension $n = 2$.

Theorem 3.1. *Let $u : B_r(p) \setminus \{p\} \to \mathcal{B}_\rho(P) \subset X$ be a finite energy harmonic map, where ρ is as in Lemma 2.2 and $\dim(B_r(p)) = n$. Then u can be extended on $B_r(p)$ as the unique energy minimizer among all maps in $W^{1,2}_u(B_r(p), \mathcal{B}_\rho(P))$.*

Proof. Let $v \in W^{1,2}_u(B_r(p), \mathcal{B}_\rho(P))$ minimize the energy. It suffices to show that $u = v$ on $B_r(p) \setminus \{p\}$. Since u is harmonic, there exists a locally finite countable open cover $\{U_i\}$ of $B_r(p) \setminus \{p\}$, and $\rho_i > 0, P_i \in \mathcal{B}_\rho(P)$ such that $u|_{U_i}$ minimizes energy among all maps in $W^{1,2}_u(U_i, \mathcal{B}_{\rho_i}(P_i))$. Let

$$F = \sqrt{\frac{1 - \cos d}{\cos R^u \cos R^v}}$$

where $d(x) = d(u(x), v(x))$ and $R^u = d(u, P), R^v = d(v, P)$. By Theorem B.4,

$$\operatorname{div}(\cos R^u \cos R^v \nabla F) \geq 0$$

holds weakly on each U_i. Therefore, for a partition of unity $\{\varphi_i\}$ subordinate to the cover $\{U_i\}$ and for any test function $\eta \in C_c^\infty(B_r(p) \setminus \{p\})$,

$$\begin{aligned} (3.1) \qquad & -\int_{B_r(p)\setminus\{p\}} \nabla\eta \cdot (\cos R^u \cos R^v \nabla F)\, d\mu_g \\ &= -\sum_i \int_{U_i} \nabla(\varphi_i \eta) \cdot (\cos R^u \cos R^v \nabla F)\, d\mu_g \\ &\geq 0, \end{aligned}$$

where we use $\sum_i \varphi_i = 1$ and $\sum_i \nabla\varphi_i = 0$.

Using polar coordinates in $B_r(p)$ centered at p, for $0 < \epsilon \ll 1$, we define

$$\phi_\epsilon = \begin{cases} 0 & r \le \epsilon^2 \\ \frac{\log r - \log \epsilon^2}{-\log \epsilon} & \epsilon^2 \le r \le \epsilon \\ 1 & \epsilon \le r. \end{cases}$$

Letting ω_{n-1} denote the volume of the unit $(n-1)$-dimensional sphere, note that

$$\int_{B_r(p)} |\nabla \phi_\epsilon|^2 \, d\mu_g = \frac{\omega_{n-1}}{(\log \epsilon)^2} \int_{\epsilon^2}^{\epsilon} r^{n-3} \, dr + o(\epsilon) \to 0 \qquad \text{as } \epsilon \to 0.$$

Therefore, for $\eta \in C_c^\infty(B_r(p))$,

$$\begin{aligned} &-\int_{B_r(p)} \phi_\epsilon \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\ &= -\int_{B_r(p)} \nabla(\eta \phi_\epsilon) \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\ &\quad + \int_{B_r(p)} \eta \nabla \phi_\epsilon \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\ &\ge \int_{B_r(p) \setminus \{p\}} \eta \nabla \phi_\epsilon \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \qquad \text{(by (3.1))} \\ &\ge -\left(\int_{B_r(p) \setminus \{p\}} |\nabla \phi_\epsilon|^2 \, d\mu_g \right)^{\frac{1}{2}} \left(\int_{B_r(p) \setminus \{p\}} \eta^2 |\cos R^u \cos R^v \nabla F|^2 \, d\mu_g \right)^{\frac{1}{2}} \\ &\quad \text{(by Hölder's inequality).} \end{aligned}$$

The last line converges to zero as $\epsilon \to 0$ because d, R^u, R^v are bounded by the compactness of $\overline{\mathcal{B}_\rho(P)}$ and $\int_{B_r(p) \setminus \{p\}} |\nabla F|^2 \, d\mu_g$ is bounded by energy convexity. We conclude that

$$\begin{aligned} &-\int_{B_r(p)} \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \\ &= -\lim_{\epsilon \to 0} \int_{B_r(p)} \phi_\epsilon \nabla \eta \cdot (\cos R^u \cos R^v \nabla F) \, d\mu_g \ge 0, \end{aligned}$$

and hence $\operatorname{div}(\cos R^u \cos R^v \nabla F) \ge 0$ holds weakly on $B_r(p)$.

Since $d(u(x), v(x)) = 0$ on $\partial B_r(p)$, by the maximum principle $d(u(x), v(x)) \equiv 0$ in $B_r(p)$. This implies that $u \equiv v$ is the unique energy minimizer. $\square$

Remark 3.2. Note that Theorem 3.1 implies that if $u : \Omega \to \mathcal{B}_\rho(P)$ is harmonic, then u is energy minimizing.

From this point on we assume our domain is of dimension 2. Recall the construction in [KS1] and [BFHMSZ] of a continuous, symmetric, bilinear, non-negative tensorial operator

$$\pi^u : \Gamma(T\Omega) \times \Gamma(T\Omega) \to L^1(\Omega) \tag{3.2}$$

associated with a $W^{1,2}$-map $u : \Omega \to X$ where $\Gamma(T\Omega)$ is the space of Lipschitz vector fields on Ω defined by

$$\pi^u(Z, W) := \frac{1}{4}|u_*(Z+W)|^2 - \frac{1}{4}|u_*(Z-W)|^2$$

where $|u_*(Z)|^2$ is the directional energy density function (cf. [KS1, Section 1.8]). This generalizes the notion of the pullback metric for maps into a Riemannian manifold, and hence we shall refer to $\pi = \pi^u$ also as the pullback metric for u.

Definition 3.3. If Σ is a Riemann surface, then $u \in W^{1,2}(\Sigma, X)$ is *(weakly) conformal* if

$$\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) = \pi\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) \quad \text{and} \quad \pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 0,$$

where $z = x_1 + ix_2$ is a local complex coordinate on Σ.

For a conformal harmonic map $u : \Sigma \to X$ with conformal factor $\lambda = \frac{1}{2}|\nabla u|^2$, and any open sets $S \subset \Sigma$ and $\mathcal{O} \subset X$, define

$$A(u(S) \cap \mathcal{O}) := \int_{u^{-1}(\mathcal{O}) \cap S} \lambda \, d\mu_g,$$

where $d\mu_g$ is the area element of (Σ, g).

Theorem 3.4 (Monotonicity). *There exist constants c, C such that if $u : \Sigma \to X$ is a non-constant conformal harmonic map from a Riemann surface Σ into a compact locally CAT(1) space (X, d), then for any $p \in \Sigma$ and $0 <$*

$\sigma < \sigma_0 = \min\{\rho, d(u(p), u(\partial\Sigma))\}$, *the following function is increasing:*

$$\sigma \mapsto \frac{e^{c\sigma^2} A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p)))}{\sigma^2},$$

and

$$A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p))) \geq C\sigma^2.$$

Proof. Since Σ is locally conformally Euclidean and the energy is conformally invariant, without loss of generality, we may assume that the domain is Euclidean. Fix $p \in \Sigma$ and let $R(x) = d(u(x), u(p))$. Since u is continuous and locally energy minimizing, by [Se1, Proposition 1.17], [BFHMSZ, Lemma 4.3] we have that the following differential inequality holds weakly on $u^{-1}(\mathcal{B}_\rho(u(p)))$:

$$\frac{1}{2}\Delta R^2 \geq (1 - O(R^2))|\nabla u|^2. \tag{3.3}$$

Let $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ be any smooth nonincreasing function such that $\zeta(t) = 0$ for $t \geq 1$, and let $\zeta_\sigma(t) = \zeta(\frac{t}{\sigma})$. By (3.3), for $\sigma < \sigma_0$ we have

$$\begin{aligned} -\int_\Sigma \nabla R^2 \cdot \nabla(\zeta_\sigma(R))\, dx_1 dx_2 &\geq 2\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))|\nabla u|^2\, dx_1 dx_2 \\ &= 4\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))\,\lambda\, dx_1 dx_2. \end{aligned}$$

Therefore,

$$\begin{aligned} 2\int_\Sigma \zeta_\sigma(R)\,(1 - O(R^2))\,\lambda\, dx_1 dx_2 &\leq -\int_\Sigma R\,\nabla R \cdot \nabla(\zeta_\sigma(R))\, dx_1 dx_2 \\ &= -\int_\Sigma \frac{R}{\sigma}\zeta'\left(\frac{R}{\sigma}\right)|\nabla R|^2\, dx_1 dx_2 \\ &\leq -\int_\Sigma \frac{R}{\sigma}\zeta'\left(\frac{R}{\sigma}\right)\frac{1}{2}|\nabla u|^2\, dx_1 dx_2 \\ &= -\int_\Sigma \frac{R}{\sigma}\zeta'\left(\frac{R}{\sigma}\right)\lambda\, dx_1 dx_2 \\ &= \int_\Sigma \sigma\frac{d}{d\sigma}(\zeta_\sigma(R))\,\lambda\, dx_1 dx_2 \\ &= \sigma\frac{d}{d\sigma}\int_\Sigma \zeta_\sigma(R)\,\lambda\, dx_1 dx_2, \end{aligned}$$

where in the second inequality we have used that $\zeta' \leq 0$ and $|\nabla R|^2 \leq \frac{1}{2}|\nabla u|^2$, since u is conformal. Set $f(\sigma) = \int_\Sigma \zeta_\sigma(R)\,\lambda\,dx_1dx_2$. We have shown that

$$2(1 - O(\sigma^2))f(\sigma) \leq \sigma f'(\sigma).$$

Integrating this, we conclude that there exist $c > 0$ such that the function

$$\sigma \mapsto \frac{e^{c\sigma^2} f(\sigma)}{\sigma^2} \tag{3.4}$$

is increasing for all $0 < \sigma < \sigma_0$. Approximating the characteristic function of $[-1, 1]$, and letting ζ be the restriction to $\mathbb{R}^+$, it then follows that

$$\frac{e^{c\sigma^2} A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p)))}{\sigma^2}$$

is increasing in σ for $0 < \sigma < \sigma_0$.

Since $\lambda = \frac{1}{2}|\nabla u|^2 \in L^1(\Sigma, \mathbb{R})$,

$$\lim_{r\to 0} \frac{\int_{B_r(x)} \lambda\,dx_1dx_2}{\pi r^2} = \lambda(x), \qquad \text{a.e. } x \in \Sigma \tag{3.5}$$

by the Lebesgue-Besicovitch Differentiation Theorem. Since u is conformal, for every $\omega \in \mathbb{S}^1$,

$$\lambda(x) = \lim_{t\to 0} \frac{d^2(u(x + t\omega), u(x))}{t^2}, \qquad \text{a.e. } x \in \Sigma \tag{3.6}$$

([KS1, Theorem 1.9.6 and Theorem 2.3.2]). Since u is locally Lipschitz [BFHMSZ, Theorem 1.2], by an argument as in the proof of Rademacher's Theorem ([EG, p. 83-84]),

$$\lambda(x) = \lim_{y\to x} \frac{d^2(u(y), u(x))}{|y - x|^2} \tag{3.7}$$

for almost every $x \in \Sigma$. To see this, choose $\{\omega_k\}_{k=1}^\infty$ to be a countable, dense subset of $\mathbb{S}^1$. Set

$$S_k = \left\{ x \in \Sigma\ : \lim_{t\to 0} \frac{d(u(x + t\omega_k), u(x))}{t} \text{ exists, and is equal to } \sqrt{\lambda(x)} \right\}$$

for $k = 1, 2, \ldots$ and let

$$S = \cap_{k=1}^\infty S_k.$$

Observe that $\mathcal{H}^2(\Sigma \setminus S) = 0$. Fix $x \in S$, and let $\varepsilon > 0$. Choose N sufficiently large such that if $\omega \in \mathbb{S}^1$ then

$$|\omega - \omega_k| < \frac{\varepsilon}{2\mathrm{Lip}(u)}$$

for some $k \in \{1, \dots, N\}$. Since

$$\lim_{t \to 0} \frac{d(u(x + t\omega_k), u(x))}{t} = \sqrt{\lambda(x)}$$

for $k = 1, \dots, N$, there exists $\delta > 0$ such that if $|t| < \delta$ then

$$\left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| < \frac{\varepsilon}{2}$$

for $k = 1, \dots, N$. Consequently, for each $\omega \in \mathbb{S}^1$ there exists $k \in \{1, \dots, N\}$ such that

$$\begin{aligned} &\left| \frac{d(u(x + t\omega), u(x))}{t} - \sqrt{\lambda(x)} \right| \\ &\leq \left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| \\ &\quad + \left| \frac{d(u(x + t\omega), u(x))}{t} - \frac{d(u(x + t\omega_k), u(x))}{t} \right| \\ &\leq \left| \frac{d(u(x + t\omega_k), u(x))}{t} - \sqrt{\lambda(x)} \right| + \left| \frac{d(u(x + t\omega), u(x + t\omega_k))}{t} \right| \\ &< \frac{\varepsilon}{2} + \mathrm{Lip}(u)|\omega - \omega_k| \\ &< \varepsilon. \end{aligned}$$

Therefore the limit in (3.7) exists, and (3.7) holds, for almost every $x \in \Sigma$.

The zero set of λ is of Hausdorff dimension zero by [M]. At points where $\lambda(x) \neq 0$ and (3.7) holds, we have that for any $\varepsilon > 0$

$$u(B_{\frac{\sigma}{(1+\varepsilon)\sqrt{\lambda}}}(x)) \subset u(\Sigma) \cap \mathcal{B}_\sigma(u(x))$$

if σ is sufficiently small. Therefore by (3.5),

$$\Theta(x) := \lim_{\sigma \to 0} \frac{A(u(\Sigma) \cap \mathcal{B}_\sigma(u(x)))}{\pi\sigma^2} \geq 1, \quad \text{a.e. } x \in \Sigma. \tag{3.8}$$

By the monotonicity of (3.4), $\Theta(x)$ exists for every $x \in \Sigma$, and $\Theta(x)$ is upper semicontinuous since it is a limit of continuous functions (the density at a given radius is a continuous function of x). Therefore, $\Theta(x) \geq 1$ for every $x \in \Sigma$. Together with the monotonicity of (3.4), it follows that

$$A(u(\Sigma) \cap \mathcal{B}_\sigma(u(p))) \geq C\sigma^2$$

for $0 < \sigma < \sigma_0$. □

Remark 3.5. Note that if $u : M \to \mathcal{B}_\rho(P)$ is a harmonic map from a compact Riemannian manifold M, then u must be constant. This follows from the maximum principle, since equation (3.3) implies that $R^2(x) = d^2(u(x), P)$ is subharmonic.

For a *conformal* harmonic map from a surface into a Riemannian manifold, continuity follows easily using monotonicity ([Sc, Theorem 10.4], [G], [J1, Theorem 9.3.2]). By Theorem 3.4, using this idea we can prove the following removable singularity result for conformal harmonic maps into a CAT(1) space.

Theorem 3.6 (Removable singularity). *If $u : \Sigma \setminus \{p\} \to X$ is a conformal harmonic map of finite energy from a Riemann surface Σ into a compact locally CAT(1) space (X, d), then u extends to a locally Lipschitz harmonic map $u : \Sigma \to X$.*

Proof. Let B_r denote $B_r(p)$, the geodesic ball of radius r centered at the point p in Σ, and let $C_r = \partial B_r$ denote the circle of radius r centered at p. By the Courant-Lebesgue Lemma, there exists a sequence $r_i \searrow 0$ so that

$$L_i = L(u(C_{r_i})) := \int_{C_{r_i}} \sqrt{\lambda}\, ds_g \to 0$$

as $i \to \infty$, where ds_g denotes the induced measure on $C_{r_i} = \partial B_{r_i}$ from the metric g on Σ. Since $E(u) < \infty$, $\lambda = \frac{1}{2}|\nabla u|^2$ is an L^1 function and, by the Dominated Convergence Theorem,

$$A_i = A(u(B_{r_i} \setminus \{p\})) := \int_{B_{r_i} \setminus \{p\}} \lambda\, d\mu_g \to 0$$

as $i \to \infty$.

First we claim that there exists $P \in X$ such that $u(C_{r_i}) \to P$ with respect to the Hausdorff distance as $i \to \infty$. Let $d_{i,j} = d(u(C_{r_i}), u(C_{r_j}))$. Suppose $i < j$ so $r_i > r_j$, and choose $Q \in u(B_{r_i} \setminus \bar{B}_{r_j})$ such that $d(Q, u(C_{r_i}) \cup$

$u(C_{r_j})) \geq d_{i,j}/2$. For $\sigma = \min\{\frac{d_{i,j}}{3}, \frac{\rho}{2}\}$, by monotonicity (Theorem 3.4),

$$A(u(B_{r_i} \setminus \bar{B}_{r_j}) \cap \mathcal{B}_\sigma(Q)) \geq C\sigma^2.$$

Since $A(u(B_{r_i} \setminus \bar{B}_{r_j}) \cap \mathcal{B}_\sigma(Q)) \leq A(u(B_{r_i} \setminus \{p\})) = A_i$, it follows that $\sigma \leq c\sqrt{A_i} \to 0$ as $i \to \infty$, and we must have $d_{i,j} \to 0$. Therefore any sequence of points $P_i \in u(C_{r_i})$ is a Cauchy sequence since

$$d(P_i, P_j) \leq d_{i,j} + L_i + L_j \to 0$$

as $i, j \to \infty$. Hence, there exists $P \in X$ independent of the sequence, such that $P_i \to P$.

Finally, we claim that $\lim_{x \to p} u(x) = P$. It follows from this that we may extend u continuously to Σ by defining $u(p) = P$. To prove the claim, consider a sequence $x_i \in \Sigma \setminus \{p\}$ such that $x_i \to p$. We want to show that $u(x_i) \to P$. Suppose $x_i \in B_{r_{j(i)}} \setminus \bar{B}_{r_{j(i)+1}}$ for some $j(i)$, and let $d_i = d(u(x_i), u(C_{r_{j(i)}}) \cup u(C_{r_{j(i)+1}}))$. For $\sigma = \min\{\frac{d_i}{3}, \frac{\rho}{2}\}$, by monotonicity (Theorem 3.4),

$$A(u(B_{r_{j(i)}} \setminus \bar{B}_{r_{j(i)+1}}) \cap \mathcal{B}_\sigma(u(x_i))) \geq C\sigma^2.$$

Therefore, $\sigma < c\sqrt{A_{j(i)}} \to 0$ as $i \to \infty$, and we must have $d(u(x_i), u(C_{r_{j(i)}}) \cup u(C_{r_{j(i)+1}})) \to 0$. It follows that $u(x_i) \to P$ and u extends continuously to Σ.

We may now apply Theorem 3.1 to show that u is energy minimizing at p. Since u is continuous, there exists $\delta > 0$ such that $u(B_\delta) \subset \mathcal{B}_\rho(Q) \subset X$. By Theorem 3.1, u is the unique energy minimizer in $W^{1,2}_u(B_\delta, \mathcal{B}_\rho(Q))$. Hence u is locally energy minimizing on Σ and by [BFHMSZ, Theorem 1.2], u is locally Lipschitz on Σ. □

The following is derived using only domain variations as in [Sc, Lemma 1.1] (using [KS1, Theorem 2.3.2] to justify the computations involving change of variables) and is independent of the curvature of the target space (see for example, [GS, (2.3) page 193]).

Lemma 3.7. *Let $u : \Sigma \to X$ be a harmonic map from a Riemann surface into a locally CAT(1) space. The Hopf differential*

$$\Phi(z) = \left[\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) - \pi\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) - 2i\pi\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\right] dz^2,$$

where $z = x_1 + ix_2$ is a local complex coordinate on Σ and π is the pull-back inner product, is holomorphic.

Corollary 3.8. *Let $u : \mathbb{C} \to X$ be a harmonic map of finite energy and (X, d) be a compact locally CAT(1) space. Then u extends to a locally Lipschitz harmonic map $u : \mathbb{S}^2 \to X$.*

Proof. Let $p : \mathbb{S}^2 \setminus \{n\} \to \mathbb{R}^2$ be stereographic projection from the north pole $n \in \mathbb{S}^2$. Set $\hat{u} = u \circ p : \mathbb{S}^2 \setminus \{n\} \to X$. We will show that n is a removable singularity.

Let $\varphi = \pi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}) - \pi(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}) - 2i\pi(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. By Lemma 3.7, the Hopf differential $\Phi(z) = \varphi(z)dz^2$ is holomorphic on $\mathbb{C}$. By assumption,

$$E(u) = \int_{\mathbb{R}^2} \left(\left\| u_* \left(\frac{\partial}{\partial x_1} \right) \right\|^2 + \left\| u_* \left(\frac{\partial}{\partial x_2} \right) \right\|^2 \right) \, dx_1 dx_2 < \infty$$

and therefore

$$\int_{\mathbb{R}^2} |\varphi| \, dx_1 dx_2 \leq 2E(u) < \infty.$$

Thus $|\varphi| \in L^1(\mathbb{C}, \mathbb{R})$ and is subharmonic, and hence $\varphi \equiv 0$ and u is conformal. Then by Theorem 3.6, u extends to a locally Lipschitz harmonic map $u : \mathbb{S}^2 \to X$. □

4. Harmonic replacement construction

In this section we prove the main theorem:

Theorem 4.1. *Let Σ be a compact Riemann surface, X a compact locally CAT(1) space and $\varphi \in C^0 \cap W^{1,2}(\Sigma, X)$. Then either there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a nontrivial conformal harmonic map $v : \mathbb{S}^2 \to X$.*

Lemma 4.2 (Jost's covering lemma, [J1] Lemma 9.2.6). *For a compact Riemannian manifold Σ, there exists $\Lambda = \Lambda(\Sigma) \in \mathbb{N}$ with the following property: for any covering*

$$\Sigma \subset \bigcup_{i=1}^{m} B_r(x_i)$$

by open balls such that

$$B_{\frac{r}{8}}(x_i) \cap B_{\frac{r}{8}}(x_j) = \emptyset, \quad \forall i \neq j,$$

there exists a partition $I^1, \dots I^\Lambda$ of the integers $\{1, \dots, m\}$ such that for any $l \in \{1, \dots, \Lambda\}$ and two distinct elements i_1, i_2 of I^l,

$$B_{4r}(x_{i_1}) \cap B_{4r}(x_{i_2}) = \emptyset.$$

Definition 4.3. For each $k = 0, 1, 2, \dots$, we fix a covering

$$\mathcal{O}_k = \{B_{2^{-k}}(x_{k,i})\}_{i=1}^{m_k}$$

of Σ by balls of radius 2^{-k}. Furthermore, let $I_k^1, \dots, I_k^\Lambda$ be the disjoint subsets of $\{1, \dots, m_k\}$ as in Lemma 4.2; in other words, for every $l \in \{1, \dots, \Lambda\}$,

$$B_{2^{-k+2}}(x_{k,i_1}) \cap B_{2^{-k+2}}(x_{k,i_2}) = \emptyset, \quad \forall i_1, i_2 \in I_k^l, \quad i_1 \neq i_2. \tag{4.1}$$

and

$$B_{2^{-k-3}}(x_{k,i_1}) \cap B_{2^{-k-3}}(x_{k,i_2}) = \emptyset, \quad \forall i_1, i_2 \in \{1, \dots, m_k\}, \quad i_1 \neq i_2. \tag{4.2}$$

Let Σ be a compact Riemann surface. By uniformization, we can endow Σ with a Riemannian metric of constant Gaussian curvature $+1$, 0 or -1. Let $\Lambda = \Lambda(\Sigma)$ be as in Lemma 4.2 and $\rho = \rho(X) > 0$ be as in Lemma 2.2. We inductively define a sequence of numbers

$$\{r_n\} \subset 2^{-\mathbb{N}} := \{1, 2^{-1}, 2^{-2}, \dots\}$$

and a sequence of finite energy maps

$$\{u_n^l : \Sigma \to X\}$$

for $l = 0, \dots, \Lambda$, $n = 1, \dots, \infty$ as follows:

Initial Step 0: Fix $\kappa_0 \in \mathbb{N}$ such that $B_{2^{-\kappa_0}}(x)$ is homeomorphic to a disk for all $x \in \Sigma$. Let $u_0^0 := \varphi \in C^0 \cap W^{1,2}(\Sigma, X)$, and let

$$r_0' = \sup\{r > 0 : \forall\, x \in \Sigma, \exists\, P \in X \text{ such that } u_0^0(B_{2r}(x)) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)\}$$

and $k_0' > 0$ be such that

$$2^{-k_0'} \leq r_0' < 2^{-k_0'+1}.$$

Define

$$r_0 = 2^{-k_0} = \min\{2^{-k_0'}, 2^{-\kappa_0}\},$$

and let

$$\mathcal{O}_{k_0} = \{B_{r_0}(x_{k_0,i})\}_{i=1}^{m_{k_0}} \text{ and } I_{k_0}^1, \ldots, I_{k_0}^\Lambda$$

be as in Definition 4.3. For $l \in \{1, \ldots, \Lambda\}$, assume

$$\forall x \in \Sigma,\ u_0^{l-1}(B_{2r_0}(x)) \subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \subset \mathcal{B}_\rho(P) \text{ for some } P \in X, \tag{4.3}$$

and define $u_0^l : \Sigma \to X$ from u_0^{l-1} by setting

$$u_0^l = \begin{cases} u_0^{l-1} & \text{in } \Sigma \backslash \bigcup_{i \in I_{k_0}^l} B_{2r_0}(x_{k_0,i}) \\ {}^{Dir}u_0^{l-1} & \text{in } B_{2r_0}(x_{k_0,i}),\ i \in I_{k_0}^l \end{cases} \tag{4.4}$$

where ${}^{Dir}u_0^{l-1}$ is the unique Dirichlet solution in $W^{1,2}_{u_0^{l-1}}(B_{2r_0}(x_{k_0,i}), \mathcal{B}_\rho(P))$ of Lemma 2.2. Since $B_{2r_0}(x_{k_0,i_1}) \cap B_{2r_0}(x_{k_0,i_2}) = \emptyset$, $\forall\, i_1, i_2 \in I_{k_0}^l$ with $i_1 \neq i_2$ (cf. (4.1)), there is no issue of interaction between the Dirichlet solutions for the different balls in the set $\{B_{2r_0}(x_{k_0,i})\}_{i \in I_{k_0}^l}$. Thus the map is well-defined.

Now note that since $r_0 \leq r_0'$,

$$\forall x \in \Sigma,\ u_0^0(B_{2r_0}(x)) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P) \subset \mathcal{B}_\rho(P) \text{ for some } P \in X.$$

Thus, the map u_0^1 can be defined by (4.4). For $l \in \{1, \ldots, \Lambda\}$, we claim that if we assume statement (4.3) and define u_0^l by (4.4), then

$$\forall x \in \Sigma,\ u_0^l(B_{2r_0}(x)) \subset \mathcal{B}_{3^{-\Lambda+l}\rho}(P) \subset \mathcal{B}_\rho(P) \text{ for some } P \in X. \tag{4.5}$$

Indeed, given $x \in \Sigma$, we have the following two possibilities:

Case 1. $B_{2r_0}(x) \cap B_{2r_0}(x_{k_0,i}) = \emptyset$ for all $i \in I_{k_0}^l$.

In this case, $u_0^l = u_0^{l-1}$ on $B_{2r_0}(x)$ and (4.3) implies

$$u_0^l(B_{2r_0}(x)) = u_0^{l-1}(B_{2r_0}(x)) \subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \subset \mathcal{B}_{3^{-\Lambda+l}\rho}(P) \text{ for some } P \in X.$$

Case 2. $B_{2r_0}(x) \cap B_{2r_0}(x_{k_0,i}) \neq \emptyset$ for some $i \in I_{k_0}^l$.

Since $B_{4r_0}(x_{k_0,i}) \cap B_{4r_0}(x_{k_0,j}) = \emptyset$ for any $j \in I_{k_0}^l$ by the definition of $I_{k_0}^l$, we conclude that i is the only element in $I_{k_0}^l$ satisfying the above property.

In other words,

$$B_{2r_0}(x) \cap B_{2r_0}(x_{k_0,j}) = \emptyset, \ \forall j \in I^l_{k_0} \backslash \{i\}.$$

This in turn implies that $u_0^l = u_0^{l-1}$ in $B_{2r_0}(x) \backslash B_{2r_0}(x_{_{k_0},i})$, and thus

$$\begin{aligned} u_0^l(B_{2r_0}(x) \backslash B_{2r_0}(x_{_{k_0},i})) &= u_0^{l-1}(B_{2r_0}(x) \backslash B_{2r_0}(x_{_{k_0},i})) \\ &\subset u_0^{l-1}(B_{2r_0}(x)) \\ &\subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P') \text{ for some } P' \in X \text{ by (4.3).} \end{aligned}$$

Furthermore, by (4.3) and Lemma 2.2, we have

$$u_0^l(B_{2r_0}(x_{_{k_0},i})) \subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \text{ for some } P \in X.$$

Therefore, we conclude

$$\begin{aligned} u_0^l(B_{2r_0}(x)) &\subset u_0^l(B_{2r_0}(x) \backslash B_{2r_0}(x_{_{k_0},i})) \cup u_0^l(B_{2r_0}(x_{_{k_0},i})) \\ &\subset \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P') \cup \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \\ &\subset \mathcal{B}_{3^{-\Lambda+l}\rho}(P) \end{aligned}$$

where the last line uses the fact that $\mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P') \cap \mathcal{B}_{3^{-\Lambda+(l-1)}\rho}(P) \neq \emptyset$. This conclude the proof of (4.5) which allows us to iteratively define $u_0^1, \dots, u_0^\Lambda$.

Inductive Step n: Having defined

$$r_0, \dots, r_{n-1} \in 2^{-\mathbb{N}},$$

and

$$u_\nu^0,\ u_\nu^1, \dots, u_\nu^\Lambda : \Sigma \to X, \quad \nu = 0, 1, \dots, n-1,$$

we set $u_n^0 = u_{n-1}^\Lambda$ and define

$$r_n \in 2^{-\mathbb{N}} \text{ and } \ u_n^1, \dots, u_n^\Lambda$$

as follows. Let

$$r_n' = \sup\{r > 0 : \forall\, x \in \Sigma, \exists\, P \in X \text{ such that } u_n^0(B_{2r}(x)) \subset \mathcal{B}_{3^{-\Lambda}\rho}(P)\}$$

and $k_n' \in \mathbb{N}$ be such that

$$2^{-k_n'} \leq r_n' < 2^{-k_n'+1}.$$

Define

$$r_n = 2^{-k_n} = \min\{2^{-k'_n}, 2^{-\kappa_0}\}.$$

Let

$$\mathcal{O}_{k_n} = \{B_{r_n}(x_{k_n,i})\}_{i=1}^{m_{k_n}} \text{ and } I^1_{k_n}, \dots, I^\Lambda_{k_n}$$

be as in Definition 4.3. Having defined $u_n^0, \dots, u_n^{l-1}$, we now define $u_n^l : \Sigma \to X$ by setting

$$u_n^l = \begin{cases} u_n^{l-1} & \text{in } \Sigma \backslash \bigcup_{i \in I^l_{k_n}} B_{2r_n}(x_{k_n,i}) \\ {}^{Dir}u_n^{l-1} & \text{in } B_{2r_n}(x_{k_n,i}),\ i \in I^l_{k_n} \end{cases}$$

where ${}^{Dir}u_n^{l-1}$ is the unique Dirichlet solution in $W^{1,2}_{u_n^{l-1}}(B_{2r_n}(x_{k_n,i}), \mathcal{B}_\rho(P))$ for some P of Lemma 2.2.

This completes the inductive construction of the sequence $\{u_n^l\}$. Note that

$$E(u_n^\Lambda) \le \cdots \le E(u_n^0) = E(u_{n-1}^\Lambda), \quad \forall\, n = 1, 2, \dots.$$

Thus, there exists E_0 such that

$$\lim_{n\to\infty} E(u_n^l) = E_0, \quad \forall\, l = 0, \dots, \Lambda. \tag{4.6}$$

We consider the following two cases separately:

CASE 1: $\liminf_{n\to\infty} r_n > 0$.

CASE 2: $\liminf_{n\to\infty} r_n = 0$.

For **CASE 1**, we prove that there exists a harmonic map $u : \Sigma \to X$ homotopic to $\varphi = u_0^0$. We will need the following two claims.

Claim 4.4. For any $l \in \{0, \dots \Lambda - 1\}$,

$$\lim_{n\to\infty} ||d(u_n^l, u_n^\Lambda)||_{L^2(\Sigma)} = 0.$$

Proof. Fix $l \in \{0, \dots, \Lambda - 1\}$. For $n \in \mathbb{N}$, $\lambda \in \{l+1, \dots, \Lambda\}$ and $i \in I^\lambda_{k_n}$, we apply Theorem B.1 with $u_0 = u_n^{\lambda-1}\big|_{B_{2r_n}(x_{k_n,i})}$, $u_1 = u_n^\lambda\big|_{B_{2r_n}(x_{k_n,i})}$ and $\Omega = B_{2r_n}(x_{k_n,i})$. Let $w : \Sigma \to X$ be the map defined as $w = u_n^\lambda = u_n^{\lambda-1}$ outside $\bigcup_{i \in I^\lambda_{k_n}} B_{2r_n}(x_{k_n,i})$ and the map corresponding to w in Theorem B.1 in each

$B_{2r_n}(x_{k_n,i})$. Then

$$(\cos^8 \rho) \int_{B_{2r_n}(x_{k_n,i})} \left| \nabla \frac{\tan \frac{1}{2} d(u_n^{\lambda-1}, u_n^{\lambda})}{\cos R} \right|^2 d\mu$$
$$\leq \frac{1}{2} \left(\int_{B_{2r_n}(x_{k_n,i})} |\nabla u_n^{\lambda-1}|^2 d\mu + \int_{B_{2r_n}(x_{k_n,i})} |\nabla u_n^{\lambda}|^2 d\mu \right)$$
$$- \int_{B_{2r_n}(x_{k_n,i})} |\nabla w|^2 d\mu.$$

Summing over i, using that $w = u_n^{\lambda} = u_n^{\lambda-1}$ outside $\bigcup_{i \in I_{k_n}^{\lambda}} B_{2r_n}(x_{k_n,i})$, and applying the Poincaré inequality, we obtain

$$\int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \leq C \left(\frac{1}{2} E(u_n^{\lambda-1}) + \frac{1}{2} E(u_n^{\lambda}) - E(w) \right),$$

where here and henceforth C is a constant independent of n. Since u_n^{λ} is harmonic in $\bigcup_{i \in I_{k_n}^{\lambda}} B_{2r_n}(x_{k_n,i})$, we have $E(u_n^{\lambda}) \leq E(w)$. Hence

$$\int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \leq C \left(\frac{1}{2} E(u_n^{\lambda-1}) - \frac{1}{2} E(u_n^{\lambda}) \right).$$

Thus,

$$\begin{aligned} \int_{\Sigma} d^2(u_n^l, u_n^{\Lambda}) d\mu &\leq \int_{\Sigma} \left(\sum_{\lambda=l+1}^{\Lambda} d(u_n^{\lambda-1}, u_n^{\lambda}) \right)^2 d\mu \\ &\leq (\Lambda - l)^2 \sum_{\lambda=l+1}^{\Lambda} \int_{\Sigma} d^2(u_n^{\lambda-1}, u_n^{\lambda}) d\mu \\ &\leq C \sum_{\lambda=l+1}^{\Lambda} \left(E(u_n^{\lambda-1}) - E(u_n^{\lambda}) \right) \\ &= C \left(E(u_n^l) - E(u_n^{\Lambda}) \right). \end{aligned}$$

This proves the claim since $\lim_{n\to\infty} \left(E(u_n^l) - E(u_n^{\Lambda}) \right) = 0$ by (4.6). □

Claim 4.5. *Let $\epsilon > 0$ such that $3^{-\Lambda}\epsilon < \rho$, $l \in \{1, \dots, \Lambda\}$ and $n \in \mathbb{N}$ be given. If $\delta \in (0, r_n)$ is such that*

$$\sqrt{\frac{8\pi E(u_0^0)}{\log \delta^{-2}}} \leq 3^{-\Lambda}\epsilon, \tag{4.7}$$

then

$$\forall\, x \in \bigcup_{\lambda=1}^{l} \bigcup_{i \in I_{k_n}^{\lambda}} B_{r_n}(x_{k_n,i}), \quad \exists\, P \in X \text{ such that } u_n^l(B_{\delta^\Lambda}(x)) \subset \mathcal{B}_{3\epsilon}(P).$$

In particular, for $l = \Lambda$, $\forall\, x \in \Sigma$, $\exists\, P \in X$ such that $u_n^\Lambda(B_{\delta^\Lambda}(x)) \subset \mathcal{B}_{3\epsilon}(P)$.

Proof. Fix ϵ, l, n and let δ be as in (4.7). For $x \in \bigcup_{\lambda=1}^{l} \bigcup_{i \in I_{k_n}^{\lambda}} B_{r_n}(x_{k_n,i})$, there exists $\lambda \in \{1, \dots, l\}$ such that $x \in B_{r_n}(x_{k_n,i})$ for some $i \in I_{k_n}^{\lambda}$ and hence

$$B_{r_n}(x) \subset B_{2r_n}(x_{k_n,i}).$$

Since u_n^λ is harmonic in $B_{2r_n}(x_{k_n,i})$, it is harmonic in $B_{r_n}(x)$. By the Courant-Lebesgue Lemma, there exists

$$R_1 \in (\delta^2, \delta)$$

such that

$$u_n^\lambda(\partial B_{R_1}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1) \text{ for some } P_1 \in X.$$

Since u_n^λ is a Dirichlet solution and $3^{-\Lambda}\epsilon < \rho$, Lemma 2.2 implies that

$$u_n^\lambda(B_{\delta^2}(x)) \subset u_n^\lambda(B_{R_1}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1). \tag{4.8}$$

Next, by the Courant-Lebesgue Lemma, there exists

$$R_2 \in (\delta^3, \delta^2)$$

such that

$$u_n^{\lambda+1}(\partial B_{R_2}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_2') \text{ for some } P_2' \in X. \tag{4.9}$$

There are two cases to consider:

Case a. $B_{R_2}(x) \cap B_{2r_n}(x_{k_n,i}) = \emptyset$, $\forall i \in I_{k_n}^{\lambda+1}$.

In this case, $u_n^{\lambda+1} = u_n^\lambda$ in $B_{R_2}(x)$. Thus,

$$u_n^{\lambda+1}(B_{R_2}(x)) = u_n^\lambda(B_{R_2}(x)) \subset u_n^\lambda(B_{\delta^2}(x)) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1)$$

by (4.8). In this case we let $P_2 = P_1$.

Case b. $B_{R_2}(x) \cap B_{2r_n}(x_{k_n,i}) \neq \emptyset$ for some $i \in I_{k_n}^{\lambda+1}$.

Since $B_{4r_0}(x_{k_n,i}) \cap B_{4r_0}(x_{k_n,j}) = \emptyset$ for any $j \in I_{k_n}^l \backslash \{i\}$ by the definition of $I_{k_n}^l$, we conclude that i is the only element in $I_{k_n}^l$ satisfying the above property. In other words,

$$B_{R_2}(x) \cap B_{2r_n}(x_{k_n,j}) = \emptyset, \ \forall j \in I_{k_n}^l \backslash \{i\}.$$

This in turn implies that $u_n^{\lambda+1} = u_n^\lambda$ in $B_{R_2}(x) \backslash B_{2r_n}(x_{k_n,i})$, and thus

$$\begin{aligned} u_n^{\lambda+1}(B_{R_2}(x) \backslash B_{2r_n}(x_{k_n,i})) &= u_n^\lambda(B_{R_2}(x) \backslash B_{2r_n}(x_{k_n,i})) \\ &\subset u_n^\lambda(B_{R_2}(x)) \\ &\subset u_n^\lambda(B_{\delta^2}(x)) \qquad (\text{since } R_2 \in (\delta^3, \delta^2)) \\ &\subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1) \text{ for some } P_1 \in X \text{ by (4.8).} \end{aligned}$$

Furthermore, the boundary of the region $B_{R_2}(x) \cap B_{2r_n}(x_{k_n,i})$ is the union of two smooth curves $\gamma = \overline{B_{R_2}(x)} \cap \partial B_{2r_n}(x_{k_n,i})$ and $\beta = \partial B_{R_2}(x) \cap \overline{B_{2r_n}(x_{k_n,i})}$. By construction, $u_n^{\lambda+1} = u_n^\lambda$ on γ, thus

$$u_n^{\lambda+1}(\gamma) \subset \mathcal{B}_{3^{-\Lambda}\epsilon}(P_1).$$

Moreover, (4.9) implies

$$u_n^{\lambda+1}(\beta) \subset \mathcal{B}_{3^{-\Lambda}\varepsilon}(P_2').$$

Note that

$$\mathcal{B}_{3^{-\Lambda}\epsilon}(P_1) \cap \mathcal{B}_{3^{-\Lambda}\epsilon}(P_2') \neq \emptyset,$$

and hence

$$u_n^{\lambda+1}(\gamma \cup \beta) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2').$$

Since $u_n^{\lambda+1}$ is harmonic in $B_{R_2}(x) \cap B_{2r_n}(x_{k_n,i})$,

$$u_n^{\lambda+1}(B_{R_2}(x) \cap B_{2r_n}(x_{k_n,i})) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2').$$

Let $P_2 = P_2'$.

In summary, we have shown that in either *Case a* or *Case b*,

$$u_n^{\lambda+1}(B_{\delta^3}(x)) \subset u_n^{\lambda+1}(B_{R_2}(x)) \subset \mathcal{B}_{3^{-\Lambda+1}\epsilon}(P_2).$$

After iterating this argument for $u_n^{\lambda+2}, \ldots, u_n^l$, we conclude that there exists $P_{l-\lambda+1} \in X$ such that

$$u_n^l(B_{\delta^\Lambda}(x)) \subset u_n^l(B_{\delta^{l-\lambda+2}}(x)) \subset \mathcal{B}_{3^{-\Lambda+l-\lambda}\epsilon}(P_{l-\lambda+1}) \subset \mathcal{B}_{3\epsilon}(P_{l-\lambda+1}).$$

Letting $P = P_{l-\lambda+1}$, we obtain the assertion of Claim 4.5. □

Since $\liminf_{n\to\infty} r_n > 0$, there exist $k \in \mathbb{N}$ and an increasing sequence $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ such that $r_{n_j} = 2^{-k}$ (or equivalently $k_{n_j} = k$). In particular, the covering used for STEP n_j in the inductive construction of $u_{n_j}^0, \ldots, u_{n_j}^\Lambda$ is the same for all $j = 1, 2, \ldots$. Thus, we can use the following notation for simplicity:

$$\mathcal{O} = \mathcal{O}_{k_j}, I^l = I^l_{k_j},\ B_i = B_{r_{n_j}}(x_{k_{n_j},i}) \text{ and } \ tB_i = B_{tr_{n_j}}(x_{k_{n_j},i}) \text{ for } t \in \mathbb{R}^+.$$

With this notation, Claim 4.5 implies that for a fixed $l \in \{1, \ldots, \Lambda\}$,

$$\{u_{n_j}^l\} \text{ is an equicontinuous family of maps on } B^l := \bigcup_{\lambda=1}^{l} \bigcup_{i \in I^\lambda} B_i. \tag{4.10}$$

In particular,$\{u_{n_j}^\Lambda\}$ is an equicontinuous family of maps in Σ. By taking a further subsequence if necessary, we can assume that

$$\exists\, u \in C^0(\Sigma, X) \text{ such that } u_{n_j}^\Lambda \rightrightarrows u. \tag{4.11}$$

We claim that for every $l \in \{1, \ldots, \Lambda\}$,

$$u_{n_j}^l \rightrightarrows u \text{ on } B^l \text{ where } u \text{ is as in (4.11)}. \tag{4.12}$$

Indeed, if (4.12) is not true, consider a subsequence of $\{u_{n_j}^l\}$ that does not converge to u. By (4.10), we can assume (by taking a further subsequence if necessary) that

$$\exists\, v : B^l \to X \text{ such that } u^l_{n_j} \rightrightarrows v \neq u|_{B^l}.$$

Combining this with (4.11) and Claim 4.4, we conclude that

$$||d(v,u)||_{L^2(B^l)} = \lim_{j\to\infty} ||d(u^l_{n_j}, u^\Lambda_{n_j})||_{L^2(B^l)} \leq \lim_{j\to\infty} ||d(u^l_{n_j}, u^\Lambda_{n_j})||_{L^2(\Sigma)} = 0$$

which in turn implies that $u = v$. This contradiction proves (4.12).

Finally, we are ready to prove the harmonicity of u. For an arbitrary point $x \in \Sigma$, there exists $r > 0$, $l \in \{1, \dots, \Lambda\}$, and $i \in I^l$ such that $B_{2r}(x) \subset B_i$. Since $u^l_{n_j}$ is energy minimizing in $B_{2r}(x)$ and $u^l_{n_j} \rightrightarrows u$ in B_i by (4.12), Lemma 2.4 implies that u is energy minimizing in $B_r(x)$.

The map u is homotopic to φ since it is a uniform limit of $u^\Lambda_{n_j}$ each of which is homotopic to φ. This completes the proof for **CASE 1** as u is the desired harmonic map homotopic to φ.

For **CASE 2**, we prove that there exists a non-constant harmonic map $u : \mathbb{S}^2 \to X$.

Recall that we have endowed Σ with a metric g of constant Gaussian curvature that is identically $+1$, 0 or -1. Fix

$$y_* \in \Sigma$$

and a local conformal chart

$$\pi : U \subset \mathbb{C} \to \pi(U) = B_1(y_*) \subset \Sigma$$

such that

$$\pi(0) = y_*$$

and the metric $g = (g_{ij})$ of Σ expressed with respect to this local coordinates satisfies

$$g_{ij}(0) = \delta_{ij}. \tag{4.13}$$

For each n, the definition of r_n implies that we can find $y_n, y'_n \in \Sigma$ with

$$2r_n \leq d_g(y_n, y'_n) \leq 4r_n$$

where d_g is the distance function on Σ induced by the metric g, and

$$d(u_n^0(y_n), u_n^0(y_n')) \geq 3^{-\Lambda}\rho.$$

Since Σ is a compact Riemannian surface of constant Gaussian curvature, there exists an isometry $\iota_n : \Sigma \to \Sigma$ such that $\iota_n(y_*) = y_n$. Define the conformal coordinate chart

$$\pi_n : U \subset \mathbb{C} \to \pi_n(U) = B_1(y_n) \subset \Sigma, \quad \pi_n(z) := \iota_n \circ \pi(z).$$

Thus,

$$\pi_n(0) = y_n.$$

Define the dilatation map

$$\Psi_n : \mathbb{C} \to \mathbb{C}, \quad \Psi_n(z) = r_n z$$

and set $\Omega_n := \Psi_n^{-1} \circ \pi_n^{-1}(B_1(y_n)) \subset \mathbb{C}$ and

$$\tilde{u}_n^l : \Omega_n \to X, \quad \tilde{u}_n^l := u_n^l \circ \pi_n \circ \Psi_n.$$

Since $\liminf_{n\to\infty} r_n = 0$, there exists a subsequence

$$\{r_{n_j}\} \text{ such that } \lim_{j\to\infty} r_{n_j} = 0. \tag{4.14}$$

Thus, $\Omega_{n_j} \nearrow \mathbb{C}$. Furthermore, (4.13) implies that

$$\lim_{j\to\infty} \frac{d_g(y_{n_j}', y_{n_j})}{|\pi_{n_j}^{-1}(y_{n_j}')|} = 1.$$

Hence, for $z_n = \Psi_n^{-1} \circ \pi_n^{-1}(y_n')$,

$$2 \leq \lim_{j\to\infty} |z_{n_j}| \leq 4 \tag{4.15}$$

and

$$d(\tilde{u}_{n_j}^0(z_{n_j}), \tilde{u}_{n_j}^0(0)) = d(u_{n_j}^0(y_{n_j}'), u_{n_j}^0(y_{n_j})) \geq 3^{-\Lambda}\rho. \tag{4.16}$$

Additionally, by the conformal invariance of energy, we have that

$$E(\tilde{u}_n^l) = E(u_n^l\big|_{B_1(y_n)}) \leq E(u_0^0). \tag{4.17}$$

For $R > 0$, let

$$D_R := \{z \in \mathbb{C} : |z| < R\}.$$

In **CASE 1**, we could choose a subsequence such that $k_{n_j} = k$ and thus the cover was fixed. In **CASE 2**, $r_{n_j} = 2^{-k_{n_j}} \to 0$ by (4.14). Therefore, as a first step we determine a fixed cover which will allow us to apply arguments similar to those of **CASE 1**.

Lemma 4.6. *Let $\mathcal{O}_{k_n}$ be as in Definition 4.3. Given $R > 0$, there exists $N \in \mathbb{N}$ and M independent of N such that for every $n \geq N$,*

$$|\{i : B_{2^{-k_n}}(x_{k_n,i}) \cap (\pi_n \circ \Psi_n(D_R)) \neq \emptyset\}| \leq M.$$

Proof. By (4.13),

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(\pi_n \circ \Psi_n(D_{2R}))}{4\pi R^2 2^{-2k_n}} = 1$$

and

$$\lim_{n\to\infty} \frac{\mathrm{Vol}(B_{2^{-k_n-3}}(x_{n,i}))}{\pi 2^{-2k_n-6}} = 1$$

where Vol is the volume in Σ. Let $\mathcal{J} \subset \{1, \dots, m_{k_n}\}$ be such that

$$\mathcal{J} = \{i : B_{2^{-k_n}}(x_{k_n,i}) \cap (\pi_n \circ \Psi_n(D_R)) \neq \emptyset\}.$$

By (4.2), we have that for sufficiently large k_n,

$$\begin{aligned} |\mathcal{J}|\pi 2^{-2k_n-6} &\leq 2\sum_{i\in\mathcal{J}} \mathrm{Vol}(B_{2^{-k_n-3}}(x_{k_n,i})) \\ &\leq 2\mathrm{Vol}(\pi_n \circ \Psi_n(D_{2R})) \\ &\leq 16\pi R^2 2^{-2k_n}. \end{aligned}$$

Hence $|\mathcal{J}| \leq R^2 2^{10}$ and $\{B_{2^{-k_n}}(x_{k_n,i})\}_{i\in\mathcal{J}}$ covers $\pi_n \circ \Psi_n(D_R)$. □

For each $B_{2^{-k_n}}(x_{k_n,i}) \in \mathcal{O}_{k_n}$, for notational simplicity let

$$\tilde{B}_{n,i} := \Psi_n^{-1} \circ \pi_n^{-1}(B_{2^{-k_n}}(x_{k_n,i}))$$

and

$$t\tilde{B}_{n,i} := \Psi_n^{-1} \circ \pi_n^{-1}(B_{t2^{-k_n}}(x_{k_n,i})) \text{ for } t \in \mathbb{R}^+.$$

After renumbering, Lemma 4.6 implies that there exists $M = M(R)$ such that

$$D_R \subset \bigcup_{i=1}^{M} \tilde{B}_{n,i}.$$

If we write

$$I^l_{k_n}(R) = \{i \in I^l_{k_n} : i \leq M\} \quad \forall\, l = 1, \ldots, \Lambda,$$

then

$$D_R \subset \bigcup_{l=1}^{\Lambda} \bigcup_{i \in I^l_{k_n}(R)} \tilde{B}_{n,i}.$$

Choose a subsequence of (4.14), which we will denote again by $\{n_j\}$, such that

$$\Psi^{-1}_{n_j} \circ \pi^{-1}_{n_j}(x_{k_{n_j},i}) \to \tilde{x}_i \quad \forall\, i \in \{1, \ldots, M\}$$

and such that for each $l = 1, \ldots, \Lambda$, the sets

$$\tilde{I}^l := I^l_{k_{n_j}}(R) = \{i \in I^l_{k_{n_j}} : i \leq M\}$$

are equal for all k_{n_j}. Again, note that unlike **CASE 1**, where $B_{r_{n_j}}(x_{k_{n_j},i})$ is the same ball B_i for all j, the sets $\tilde{B}_{n_1,i}, \tilde{B}_{n_2,i}, \ldots$ are not necessarily the same.

Since the component functions of the pullback metric $(\pi_{n_j} \circ \Psi_{n_j})^* g$ converge uniformly to those of the standard Euclidean metric g_0 on $\mathbb{C}$ by (4.13) and $\tilde{B}_{n_j,i}$ with respect to $(\pi_{n_j} \circ \Psi_{n_j})^* g$ is a ball of radius 1, $\tilde{B}_{n_j,i}$ with respect to g_0 is close to being a ball of radius 1 in the following sense: for all $\epsilon > 0$, there exists J large enough such that for all $j \geq J$, $B_{1-\epsilon}(\tilde{x}_i) \subset \tilde{B}_{n_j,i}$ for $i = 1, \ldots, M$. Moreover, for $\epsilon > 0$ sufficiently small we have that

$$D_R \subset \bigcup_{i=1}^{M} B_{1-\epsilon}(\tilde{x}_i). \tag{4.18}$$

Choose J as above. Set

$$\tilde{B}_i := \bigcap_{j \geq J} \tilde{B}_{n_j,i} \supset B_{1-\epsilon}(\tilde{x}_i) \quad \text{and} \quad t\tilde{B}_i := \bigcap_{j \geq J} t\tilde{B}_{n_j,i} \text{ for } t \in \mathbb{R}^+.$$

Then

$$D_R \subset \bigcup_{i=1}^{M} \tilde{B}_i = \bigcup_{\lambda=1}^{\Lambda} \bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i. \tag{4.19}$$

Claim 4.7. *For* $l \in \{1, \ldots, \Lambda\}$,

$$\{\tilde{u}^l_{n_j}\} \textit{ is equicontinuous on } \bigcup_{\lambda=1}^{l} \bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i. \tag{4.20}$$

Proof. We demonstrate the equicontinuity by modifying the proof of Claim 4.5 to this new cover.

Let $\tilde{\epsilon} > 0$ such that $3^{-\Lambda}\tilde{\epsilon} < \rho$, $l \in \{1, \ldots, \Lambda\}$, and $\delta \in (0, 1-\epsilon)$ such that

$$\sqrt{\frac{8\pi E(u_0^0)}{\log \delta^{-2}}} \leq 3^{-\Lambda}\tilde{\epsilon},$$

where ϵ is given by (4.18). For $x \in \bigcup_{\lambda=1}^{l} \bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i$, there exists $\lambda \in \{1, \ldots, l\}$ and $i \in \tilde{I}^\lambda$ such that $x \in \tilde{B}_i$. By definition,

$$B_{1-\epsilon}(x) \subset 2\tilde{B}_i \subset 2\tilde{B}_{n_j,i} \quad \text{for all} \quad n_j.$$

Therefore $\tilde{u}_{n_j}^\lambda$ is harmonic on $B_{1-\epsilon}(x)$ for all n_j.

From this point forward, the proof proceeds as in the proof of Claim 4.5, noting in particular that while the $R_k(x)$ in the proof of Claim 4.5 now depend upon n_j, each of them is still bounded below uniformly by δ^{k+1} and δ is independent of n_j. Equicontinuity then follows immediately. □

By Claim 4.7, $\{\tilde{u}_{n_j}^\Lambda\}$ is equicontinuous on $\bigcup_{\lambda=1}^{\Lambda} \bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i$ and thus, perhaps taking a further subsequence,

$$\exists \tilde{u}_R \in C^0(D_R, X) \text{ such that } \tilde{u}_{n_j}^\Lambda \rightrightarrows \tilde{u}_R \text{ in } D_R. \tag{4.21}$$

Claim 4.8. *There exists a further subsequence such that for each $l \in \{1, \ldots, \Lambda\}$,*

$$\tilde{u}_{n_j}^l \rightrightarrows \tilde{u}_R \text{ on } D_R \cap \left(\bigcup_{\alpha=1}^{l} \bigcup_{i \in \tilde{I}^\alpha} \tilde{B}_i \right) := D_R^l.$$

Proof. Fix $l \in \{0, \ldots, \Lambda - 1\}$. By the equicontinuity of $\tilde{u}_{n_j}^l$ on D_R^l there exists a subsequence and a $v_R : D_R^l \to X$ such that $\tilde{u}_{n_j}^l \rightrightarrows v_R$. Fix $\lambda \in \{l+1, \ldots, \Lambda\}$ and apply Theorem B.1 with $\Omega = \tilde{B}_i$, $i \in \tilde{I}^\lambda$, and $u_0 = \tilde{u}_{n_j}^{\lambda-1}|_{\tilde{B}_i}$, $u_1 = \tilde{u}_{n_j}^\lambda|_{\tilde{B}_i}$. Let $\tilde{w} : \bigcup_{\alpha=1}^{\Lambda} \bigcup_{i \in \tilde{I}^\alpha} \tilde{B}_i \to X$ be the map corresponding to w in Theorem B.1 on each $\tilde{B}_i$, $i \in \tilde{I}^\lambda$, and equal to $\tilde{u}_{n_j}^\lambda$ elsewhere. Following Claim 4.4, as $B_{1-\epsilon}(\tilde{x}_i) \subset \tilde{B}_i = \bigcap_{j \geq J} \tilde{B}_{n_j,i}$, there exists $C > 0$ independent of j and i such that

$$\int_{\bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i} d^2(\tilde{u}_{n_j}^{\lambda-1}, \tilde{u}_{n_j}^\lambda) d\mu$$
$$\leq C \left(\frac{1}{2} E(\tilde{u}_{n_j}^{\lambda-1}|_{\bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i}) + \frac{1}{2} E(\tilde{u}_{n_j}^\lambda|_{\bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i}) - E(\tilde{w}|_{\bigcup_{i \in \tilde{I}^\lambda} \tilde{B}_i}) \right)$$

where $d\mu$ denotes the Euclidean volume form.

By construction, $\tilde{u}_{n_j}^{\lambda}$ is harmonic on $\bigcup_{i\in\tilde{I}^\lambda}\tilde{B}_i$ and $\tilde{u}_{n_j}^{\lambda-1}=\tilde{u}_{n_j}^{\lambda}=\tilde{w}$ outside $\bigcup_{i\in\tilde{I}^\lambda}\tilde{B}_i$. It follows that

$$\int_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i} d^2(\tilde{u}_{n_j}^{\lambda-1},\tilde{u}_{n_j}^{\lambda})d\mu \\ \leq C\left(\frac{1}{2}E(\tilde{u}_{n_j}^{\lambda-1}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})-\frac{1}{2}E(\tilde{u}_{n_j}^{\lambda}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})\right).$$

Therefore, following the proof of Claim 4.4,

$$\int_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i} d^2(\tilde{u}_{n_j}^{l},\tilde{u}_{n_j}^{\Lambda})d\mu \\ \leq C(E(\tilde{u}_{n_j}^{l}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})-E(\tilde{u}_{n_j}^{\Lambda}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})).$$

By conformal invariance of energy and (4.6)

$$E(\tilde{u}_{n_j}^{l}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})-E(\tilde{u}_{n_j}^{\Lambda}|_{\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i})\leq E(u_{n_j}^{l})-E(u_{n_j}^{\Lambda})\to 0.$$

It follows that

$$\begin{aligned}\|d(v_R,\tilde{u}_R)\|_{L^2(D_R^l)} &= \lim_{j\to\infty}\|d(\tilde{u}_{n_j}^{l},\tilde{u}_{n_j}^{\Lambda})\|_{L^2(D_R^l)} \\ &\leq \lim_{j\to\infty}\|d(\tilde{u}_{n_j}^{l},\tilde{u}_{n_j}^{\Lambda})\|_{L^2(\bigcup_{\alpha=1}^{\Lambda}\bigcup_{i\in\tilde{I}^\alpha}\tilde{B}_i)}=0.\end{aligned}$$

Thus, $v_R=\tilde{u}_R$. □

We now demonstrate that $\tilde{u}_R$ is harmonic on D_R. Let $x\in D_R$. There exist $r>0$, $l\in\{1,\dots,\Lambda\}$, and $i\in\tilde{I}^l$ such that $B_{2r}(x)\in\tilde{B}_i$ by (4.19). Since harmonicity is invariant under conformal transformations of the domain, $\tilde{u}_{n_j}^{l}$ is a energy minimizing on $2\tilde{B}_{n_j,i}$. Since $\tilde{B}_i\subset\tilde{B}_{n_j,i}\subset 2\tilde{B}_{n_j,i}$ and $\tilde{u}_{n_j}^{l}\rightrightarrows\tilde{u}_R$ on $\tilde{B}_i$ by Claim 4.8, Lemma 2.4 implies that $\tilde{u}_R$ is energy minimizing on $B_r(x)$. Since x is an arbitrary point in D_R, we have shown that $\tilde{u}_R$ is harmonic on D_R.

Finally, by the conformal invariance of energy, $E(\tilde{u}_{n_j}^{l})=E(u_{n_j}^{l}\big|_{B_1(y_{n_j})})\leq E(u_0^0)$. By the lower semicontinuity of energy and (4.17), we have

$$E(\tilde{u}_R)\leq E(u_0^0). \tag{4.22}$$

By considering a compact exhaustion $\{D_{2^m}\}_{m=1}^{\infty}$ of $\mathbb{C}$ and a diagonalization procedure, we prove the existence of a harmonic map $\tilde{u}:\mathbb{C}\to X$. By

(4.22),

$$E(\tilde{u}) \leq E(u_0^0).$$

It follows from (4.15) and (4.16) that $\tilde{u}$ is nonconstant. Thus, **CASE 2** is complete by applying the removable singularity result Corollary 3.8.

Appendix A. Quadrilateral estimates

In this section, we include several estimates for quadrilaterals in a CAT(1) space. The estimates are stated in the unpublished thesis [Se1] without proof. As the calculations were not obvious, we include our proofs for the convenience of the reader. References to the location of each estimate in [Se1] are also included.

The first lemma is a result of Reshetnyak which will be essential in later estimates.

Lemma A.1 ([R, Lemma 2]). *Let $\square PQRS$ be a quadrilateral in X. Then the sum of the length of diagonals in $\square PQRS$ can be estimated as follows:*

$$\begin{aligned} \cos d_{PR} + \cos d_{QS} \geq & -\frac{1}{2}(d_{PQ}^2 + d_{RS}^2) \\ & + \frac{1}{4}(1 + \cos d_{PS})(d_{QR} - d_{PS})^2 \\ & + \cos d_{QR} + \cos d_{PS} \\ & + \mathrm{Cub}\,(d_{PQ}, d_{RS}, d_{QR} - d_{SP})\,. \end{aligned} \tag{A.1}$$

Proof. It suffices to prove the inequality holds for a quadrilateral $\square PQRS$ in $\mathbb{S}^2$. By viewing $\mathbb{S}^2$ as a unit sphere in $\mathbb{R}^3$, the points P, Q, R, S determine a quadrilateral in $\mathbb{R}^3$. Applying the identity for the quadrilateral in $\mathbb{R}^3$ (cf. [KS1, Corollary 2.1.3]),

$$\overline{PR}^2 + \overline{QS}^2 \leq \overline{PQ}^2 + \overline{QR}^2 + \overline{RS}^2 + \overline{SP}^2 - (\overline{SP} - \overline{QR})^2$$

where $\overline{AB}$ denotes the Euclidean distance between A and B in $\mathbb{R}^3$. To prove this, consider the vectors $A = Q - P, B = R - Q, C = S - R, D = P - S$.

Then

$$\begin{aligned}\overline{PR}^2 + \overline{QS}^2 &= \frac{1}{2}\left(|A+B|^2 + |C+D|^2 + |B+C|^2 + |D+A|^2\right) \\ &= |A|^2 + |B|^2 + |C|^2 + |D|^2 + (A\cdot B + C\cdot B + D\cdot A + D\cdot C) \\ &= |A|^2 + |B|^2 + |C|^2 + |D|^2 - |B+D|^2 \\ &\quad \text{since } A+B+C+D=0 \\ &\leq |A|^2 + |B|^2 + |C|^2 + |D|^2 - ||B|-|D||^2 .\end{aligned}$$

Note that $\overline{AB}^2 = 2 - 2\cos d_{AB}$, we obtain

$$\begin{aligned}\cos d_{PR} + \cos d_{QS} = -2 &+ \cos d_{PQ} + \cos d_{RS} + \cos d_{QR} + \cos d_{PS} \\ &+ \frac{1}{2}\left(\sqrt{2-2\cos d_{QR}} - \sqrt{2-2\cos d_{SP}}\right)^2 .\end{aligned}$$

The lemma follows from the following Taylor expansion:

$$\begin{aligned}&-2 + \cos d_{PQ} + \cos d_{RS} \\ &= -\frac{1}{2}d_{PQ}^2 - \frac{1}{2}d_{RS}^2 + O(d_{RS}^4 + d_{PQ}^4), \\ &\left(\sqrt{2-2\cos d_{QR}} - \sqrt{2-2\cos d_{SP}}\right)^2 \\ &= \left(\frac{\sin d_{SP}}{\sqrt{2-2\cos d_{SP}}}(d_{QR}-d_{SP}) + O\left((d_{QR}-d_{SP})^2\right)\right)^2 \\ &= \frac{1+\cos d_{PS}}{2}(d_{QR}-d_{SP})^2 + O\left((d_{QR}-d_{SP})^3\right).\end{aligned}$$

□

Lemma A.2 ([Se1, Estimate I, Page 11]). *Let $\square PQRS$ be a quadrilateral in the CAT(1) space X. Let $P_{\frac{1}{2}}$ be the mid-point between P and S, and let $Q_{\frac{1}{2}}$ be the mid-point between Q and R. Then*

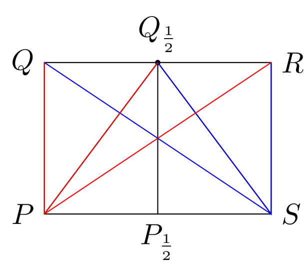
$$\begin{aligned}\cos^2\left(\frac{d_{PS}}{2}\right) d^2(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \leq \frac{1}{2}(d_{PQ}^2 + d_{RS}^2) &- \frac{1}{4}(d_{QR}-d_{PS})^2 \\ &+ \mathrm{Cub}\left(d_{PQ}, d_{RS}, d(P_{\frac{1}{2}}, Q_{\frac{1}{2}}), d_{QR}-d_{SP}\right).\end{aligned}$$

Proof. As a direct consequence of law of cosine (see also the figure below), we have the following inequalities

$$\begin{aligned}\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) &\geq \alpha \left(\cos d(Q_{\frac{1}{2}}, S) + \cos d(Q_{\frac{1}{2}}, P)\right)\\ \cos d(Q_{\frac{1}{2}}, S) &\geq \beta\left(\cos d_{RS} + \cos d_{QS}\right)\\ \cos d(Q_{\frac{1}{2}}, P) &\geq \beta\left(\cos d_{RP} + \cos d_{QP}\right)\end{aligned}$$

where

$$\alpha = \frac{1}{2\cos\left(\frac{d_{PS}}{2}\right)} \quad \text{and} \quad \beta = \frac{1}{2\cos\left(\frac{d_{QR}}{2}\right)}.$$



Combining the above inequalities yields

$$\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}}) \geq \alpha\beta\left(\cos d_{RS} + \cos d_{QS} + \cos d_{RP} + \cos d_{QP}\right).$$

We apply (A.1) for the sum of diagonals $\cos d_{QS} + \cos d_{RP}$ and Taylor expansion for $\cos d_{RS}$ and $\cos d_{QP}$. It yields

$$\begin{aligned}&\cos d(Q_{\frac{1}{2}}, P_{\frac{1}{2}})\\ &\geq \alpha\beta\left(2 - (d_{PQ}^2 + d_{RS}^2) + \frac{1}{4}(1+\cos d_{PS})(d_{QR}-d_{PS})^2 + \cos d_{QR} + \cos d_{PS}\right)\\ &\quad + \operatorname{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right)\\ &= \alpha\beta\left(2 + \cos d_{QR} + \cos d_{PS} + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2\right)\\ &\quad - \alpha\beta(d_{PQ}^2 + d_{RS}^2) + \operatorname{Cub}\left(d_{PQ}, d_{RS}, d_{QR} - d_{SP}\right).\end{aligned}$$

Note that

$$
\begin{aligned}
& 2+\cos d_{QR}+\cos d_{PS}+\frac{1}{4}(1+\cos d_{PS})(d_{QR}-d_{PS})^2 \\
&= 2(\cos^2\frac{d_{QR}}{2}+\cos^2\frac{d_{PS}}{2})+\frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR}-d_{PS})^2 \\
&= 2\left(\cos\frac{d_{QR}}{2}-\cos\frac{d_{PS}}{2}\right)^2+4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2}+\frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR}-d_{PS})^2 \\
&= \frac{1}{2}\sin^2\frac{d_{PS}}{2}(d_{QR}-d_{PS})^2+4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2}+\frac{1}{2}\cos^2\frac{d_{PS}}{2}(d_{QR}-d_{PS})^2 \\
&\quad + O(|d_{QR}-d_{PS}|^3) \\
&= \frac{1}{2}(d_{QR}-d_{PS})^2+4\cos\frac{d_{QR}}{2}\cos\frac{d_{PS}}{2}+O(|d_{QR}-d_{PS}|^3).
\end{aligned}
$$

Since $\alpha\beta=\alpha^2+O(|d_{QR}-d_{PS}|)$, we have

$$
\begin{aligned}
\cos d(Q_{\frac{1}{2}},P_{\frac{1}{2}}) \geq 1-\alpha^2(d_{PQ}^2+d_{RS}^2)+\frac{1}{2}\alpha^2(d_{QR}-d_{PS})^2 \\
+\operatorname{Cub}\left(d_{PQ},d_{RS},d_{QR}-d_{SP}\right).
\end{aligned}
$$

The lemma follows as

$$
\cos d(Q_{\frac{1}{2}},P_{\frac{1}{2}}) = 1-\frac{d^2(Q_{\frac{1}{2}},P_{\frac{1}{2}})}{2}+O(d^4(Q_{\frac{1}{2}},P_{\frac{1}{2}})). \qquad \square
$$

Definition A.3. Given a metric space (X,d) and a geodesic γ_{PQ} with $d_{PQ}<\pi$, for $\tau\in[0,1]$ let $(1-\tau)P+\tau Q$ denote the point on γ_{PQ} at distance τd_{PQ} from P. That is

$$
d((1-\tau)P+\tau Q,P)=\tau d_{PQ}.
$$

Lemma A.4 (cf. [Se1, Estimate II, Page 13]). *Let ΔPQS be a triangle in the CAT(1) space X. For a pair of numbers $0\leq\eta,\eta'\leq 1$ define*

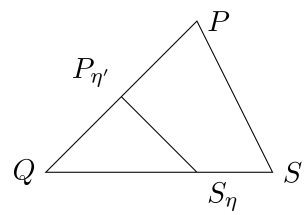
$$
\begin{aligned}
P_{\eta'} &= (1-\eta')P+\eta' Q \\
S_\eta &= (1-\eta)S+\eta Q.
\end{aligned}
$$

Then

$$
\begin{aligned}
d^2(P_{\eta'},S_\eta) \leq \frac{\sin^2((1-\eta)d_{QS})}{\sin^2 d_{QS}}&(d_{PS}^2-(d_{QS}-d_{QP})^2) \\
&+\left((1-\eta)(d_{QS}-d_{QP})+(\eta'-\eta)d_{QS}\right)^2 \\
&+\operatorname{Cub}\left(d_{PS},d_{QS}-d_{QP},\eta-\eta'\right).
\end{aligned}
$$

Proof. Again we prove the inequality for a quadrilateral on $\mathbb{S}^2$. Denote $x = d_{QS}$ and $y = d_{QP}$. Denote

$$\alpha_\eta = \frac{\sin(\eta d_{QS})}{\sin d_{QS}} = \frac{\sin(\eta x)}{\sin x}, \qquad \beta_{\eta'} = \frac{\sin(\eta' d_{QP})}{\sin d_{QP}} = \frac{\sin(\eta' y)}{\sin y}.$$



By the law of cosines on the sphere (see the figure above),

$$\begin{aligned}\cos d_{PS} &= \cos x \cos y + \sin x \sin y \cos\theta \\ &= \cos(x-y) + \sin x \sin y(\cos\theta - 1), \\ \cos d(P_{\eta'}, S_\eta) &\geq \cos((1-\eta)x)\cos((1-\eta')y) \\ &\quad + \sin((1-\eta)x)\sin((1-\eta')y)\cos\theta \\ &= \cos((1-\eta)x - (1-\eta')y) \\ &\quad + \sin((1-\eta)x)\sin((1-\eta')y)(\cos\theta - 1),\end{aligned}$$

where θ denotes the angle $\angle PQS$ on $\mathbb{S}^2$. Substituting the term $(\cos\theta - 1)$ of the second inequality with the one in the first identity, we obtain

$$\begin{aligned}\cos d(P_{\eta'}, S_\eta) &\geq \cos((1-\eta)x - (1-\eta')y) \\ &\quad + \alpha_{1-\eta}\beta_{1-\eta'}(\cos d_{PS} - \cos(x-y)) \\ &= \cos\big((1-\eta)(x-y) + (\eta'-\eta)x + (\eta'-\eta)(y-x)\big) \\ &\quad + \alpha_{1-\eta}^2(\cos d_{PS} - \cos(x-y)) \\ &\quad + \alpha_{1-\eta}(\beta_{1-\eta'} - \alpha_{1-\eta})(\cos d_{PS} - \cos(x-y)).\end{aligned}$$

Using the Taylor expansion $\cos a = 1 - \frac{a^2}{2} + O(a^4)$ and $(\beta_{1-\eta'} - \alpha_{1-\eta}) = O(|\eta'-\eta| + |x-y|)$, we derive

$$\begin{aligned}\cos d(P_{\eta'}, S_\eta) &\geq 1 - \frac{((1-\eta)(x-y) + (\eta'-\eta)x)^2}{2} \\ &\quad + \alpha_{1-\eta}^2\left(-\frac{d_{PS}^2}{2} + \frac{(x-y)^2}{2}\right) + \mathrm{Cub}\left(|\eta'-\eta|, |x-y|, d_{PS}\right).\end{aligned}$$

It implies that

$$d^2(P_{\eta'}, S_\eta) \leq \alpha^2_{1-\eta}(d^2_{PS} - (x-y)^2) + ((1-\eta)(x-y) + (\eta'-\eta)x)^2 \\ + \mathrm{Cub}\left(|\eta'-\eta|, |x-y|, d_{PS}\right).$$

□

Corollary A.5. *Let $u : \Omega \to \mathcal{B}_\rho(Q)$ be a finite energy map and $\eta \in C^\infty_C(\Omega, [0,1])$. Define $\hat{u} : \Omega \to \mathcal{B}_\rho(Q)$ as*

$$\hat{u}(x) = (1-\eta(x))u(x) + \eta(x)Q.$$

Then $\hat{u}$ has finite energy, and for any smooth vector field $W \in \Gamma(\Omega)$ we have

$$|\hat{u}_*(W)|^2 \leq \left(\frac{\sin(1-\eta)R^u}{\sin R^u}\right)^2 (|u_*(W)|^2 - |\nabla_W R^u|^2) + |\nabla_W((1-\eta)R^u)|^2,$$

where $R^u(x) = d(u(x), Q)$.

Note that every error term that appeared in Lemma A.4 will converge to the product of an L^1 function and a term that goes to zero. So all error terms vanish when taking limits.

Lemma A.6 (cf. [Se1, Estimate III, page 19]). *Let $\square PQRS$ be a quadrilateral in a CAT(1) space X. For $\eta', \eta \in [0,1]$ define*

$$Q_{\eta'} = (1-\eta')Q + \eta' R, \quad P_\eta = (1-\eta)P + \eta S.$$

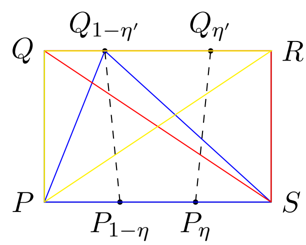
Then

$$\begin{aligned} & d^2(Q_{\eta'}, P_\eta) + d^2(Q_{1-\eta'}, P_{1-\eta}) \\ \leq & \left(1 + 2\eta d_{PS}\tan\left(\frac{1}{2}d_{PS}\right)\right)(d^2_{PQ} + d^2_{RS}) \\ & - 2\eta\left(1 + \frac{1}{2}d_{PS}\tan\left(\frac{1}{2}d_{PS}\right)\right)(d_{QR} - d_{PS})^2 \\ & + 2(2\eta-1)(\eta'-\eta)d_{PS}(d_{QR} - d_{PS}) \\ & + \eta^2 \mathrm{Quad}(d_{PQ}, d_{RS}, d_{QR} - d_{PS}) \\ & + \mathrm{Cub}\left(d_{QR} - d_{PS}, d_{PQ}, d_{RS}, \eta - \eta'\right) \end{aligned}$$

Proof. For notation simplicity, we denote

$$x = d_{PS}, \qquad y = d_{QR}, \qquad \alpha_\eta = \frac{\sin(\eta x)}{\sin x}, \qquad \beta_{\eta'} = \frac{\sin(\eta' y)}{\sin y}.$$

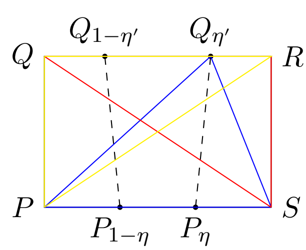
Apply [Se1, Definition 1.6] to each of the blue, red, and yellow triangles below.



We derive

$$\begin{aligned}\cos d(Q_{1-\eta'}, P_{1-\eta}) &\geq \alpha_\eta \cos d(Q_{1-\eta'}, S) + \alpha_{1-\eta} \cos d(Q_{1-\eta'}, P) \\ &\geq \alpha_\eta(\beta_{\eta'} \cos d_{SR} + \beta_{1-\eta'} \cos d_{SQ}) \\ &\quad + \alpha_{1-\eta}(\beta_{\eta'} \cos d_{PR} + \beta_{1-\eta'} \cos d_{PQ}).\end{aligned}$$

Compute similarly for $d(Q_{\eta'}, P_\eta)$ for the highlighted triangles below:



We derive

$$\begin{aligned}\cos d(Q_{\eta'}, P_\eta) &\geq \alpha_\eta \cos d(Q_{\eta'}, P) + \alpha_{1-\eta} \cos d(Q_{\eta'}, S) \\ &\geq \alpha_\eta(\beta_{\eta'} \cos d_{PQ} + \beta_{1-\eta'} \cos d_{PR}) \\ &\quad + \alpha_{1-\eta}(\beta_{\eta'} \cos d_{SQ} + \beta_{1-\eta'} \cos d_{SR}).\end{aligned}$$

Adding the above two inequalities, we obtain

$$\begin{aligned}\text{(A.2)}\qquad &\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_{\eta}) \\ &\geq (\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'})(\cos d_{PQ} + \cos d_{SR}) \\ &\quad + (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos d_{PR} + \cos d_{SQ}).\end{aligned}$$

Applying (A.1) to the term $\cos d_{PR} + \cos d_{SQ}$ and using Taylor expansion, the inequality (A.2) becomes

$$\begin{aligned}&\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_{\eta}) \\ &\geq (\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'})\left(2 - \frac{d_{PQ}^2}{2} - \frac{d_{SR}^2}{2}\right) \\ &\quad + (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})\bigg(-\frac{1}{2}(d_{PQ}^2 + d_{SR}^2) \\ &\qquad + \frac{1}{4}(1+\cos d_{PS})(d_{QR} - d_{PS})^2 + \cos d_{QR} + \cos d_{PS}\bigg) \\ &\quad + \mathrm{Cub}\,(d_{PQ}, d_{RS}, d_{QR} - d_{SP})\,.\end{aligned}$$

Hence,

$$\begin{aligned}&\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_{\eta}) \\ \text{(A.3)}\quad &\geq -\frac{1}{2}(\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(d_{PQ}^2 + d_{SR}^2) \\ \text{(A.4)}\quad &\quad + 2(\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) + (\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos d_{QR} + \cos d_{PS}) \\ \text{(A.5)}\quad &\quad + \frac{1}{4}(\alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(1+\cos d_{PS})(d_{QR} - d_{PS})^2 \\ &\quad + \mathrm{Cub}\,(d_{PQ}, d_{RS}, d_{QR} - d_{SP})\,.\end{aligned}$$

We need the following elementary trigonometric identities to compute (A.3), (A.4), (A.5):

$$\begin{aligned}\alpha_\eta \beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} &= \frac{\sin(\eta-\frac{1}{2})x \sin(\eta'-\frac{1}{2})y}{2\sin\frac{1}{2}x \sin\frac{1}{2}y} + \frac{\cos(\eta-\frac{1}{2})x \cos(\eta'-\frac{1}{2})y}{2\cos\frac{1}{2}x \cos\frac{1}{2}y} \\ \alpha_\eta \beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'} &= -\frac{\sin(\eta-\frac{1}{2})x \sin(\eta'-\frac{1}{2})y}{2\sin\frac{1}{2}x \sin\frac{1}{2}y} + \frac{\cos(\eta-\frac{1}{2})x \cos(\eta'-\frac{1}{2})y}{2\cos\frac{1}{2}x \cos\frac{1}{2}y} \\ \left(\frac{\cos(\eta-\frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2 &= 1 + 2\eta x \tan\frac{1}{2}x + O(\eta^2).\end{aligned}$$

Noting that

$$\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'} = \frac{\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y}{\cos\frac{1}{2}x\cos\frac{1}{2}y}$$
$$= \left(\frac{\cos(\eta-\frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2 + O(|\eta-\eta'|+|x-y|)$$
$$= 1 + 2\eta x\tan\left(\frac{1}{2}x\right) + O(\eta^2+|\eta-\eta'|+|x-y|),$$

we obtain for (A.3)

$$\begin{aligned}
&-\frac{1}{2}(\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'} + \alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(d_{PQ}^2+d_{SR}^2)\\
&= -\frac{1}{2}\left(1+2\eta x\tan\left(\frac{1}{2}x\right)\right)(d_{PQ}^2+d_{SR}^2)\\
&\quad + O\left((\eta^2+|\eta-\eta'|+|x-y|)(d_{PQ}^2+d_{SR}^2)\right).
\end{aligned}$$

Lemma A.7. *We can compute* (A.4) *as follows:*

$$\begin{aligned}
&2(\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) + (\alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos x+\cos y)\\
&= 2 - \left(\left(\eta-\frac{1}{2}\right)(y-x) + (\eta'-\eta)x\right)^2 + \frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2\left(\frac{1}{2}x\right)(x-y)^2\\
&\quad + \frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}\sin^2\left(\frac{1}{2}x\right)(x-y)^2 + O(|x-y|^2(|x-y|+|\eta'-\eta|)).
\end{aligned}$$

Proof.

$$\begin{aligned}
&2(\alpha_\eta\beta_{\eta'} + \alpha_{1-\eta}\beta_{1-\eta'}) + (\alpha_\eta\beta_{1-\eta'} + \alpha_{1-\eta}\beta_{\eta'})(\cos x+\cos y)\\
&= \frac{\sin(\eta-\frac{1}{2})x\sin(\eta'-\frac{1}{2})y}{2\sin\frac{1}{2}x\sin\frac{1}{2}y}(2-\cos x-\cos y)\\
&\quad + \frac{\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y}{2\cos\frac{1}{2}x\cos\frac{1}{2}y}(2+\cos x+\cos y).
\end{aligned}$$

Note that

$$
\begin{aligned}
2-\cos x-\cos y &= 2\left(\sin\frac{1}{2}x\right)^2+2\left(\sin\frac{1}{2}y\right)^2\\
&= 2\left(2\sin\frac{1}{2}x\sin\frac{1}{2}y+\left(\sin\frac{1}{2}x-\sin\frac{1}{2}y\right)^2\right)\\
&= 4\sin\frac{1}{2}x\sin\frac{1}{2}y+\frac{1}{2}\left(\cos\frac{1}{2}x\right)^2(x-y)^2+O(|x-y|^3)\\
2+\cos x+\cos y &= 2\left(\cos\frac{1}{2}x\right)^2+2\left(\cos\frac{1}{2}y\right)^2\\
&= 2\left(2\cos\frac{1}{2}x\cos\frac{1}{2}y+\left(\cos\frac{1}{2}x-\cos\frac{1}{2}y\right)^2\right)\\
&= 4\cos\frac{1}{2}x\cos\frac{1}{2}y+\frac{1}{2}\left(\sin\frac{1}{2}x\right)^2(x-y)^2+O(|x-y|^3),
\end{aligned}
$$

where we apply Taylor expansion in the last equality. Hence we have

$$
\begin{aligned}
&2(\alpha_\eta\beta_{\eta'}+\alpha_{1-\eta}\beta_{1-\eta'})+(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(\cos x+\cos y)\\
&=2\left(\sin\left(\eta-\frac{1}{2}\right)x\sin\left(\eta'-\frac{1}{2}\right)y+\cos\left(\eta-\frac{1}{2}\right)x\cos\left(\eta'-\frac{1}{2}\right)y\right)\\
&\quad+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\left(\cos\frac{1}{2}x\right)^2(x-y)^2\\
&\quad+\frac{\cos^2\left(\eta-\frac{1}{2}\right)x}{4\cos^2\frac{1}{2}x}\left(\sin\frac{1}{2}x\right)^2(x-y)^2+O(|x-y|^2(|x-y|+|\eta'-\eta|)).
\end{aligned}
$$

Here we use the estimates

$$
\frac{\sin(\eta-\frac{1}{2})x\sin(\eta'-\frac{1}{2})y}{2\sin\frac{1}{2}x\sin\frac{1}{2}y}-\frac{\sin^2(\eta-\frac{1}{2})x}{2\sin^2\frac{1}{2}x}=O(|\eta-\eta'|+|x-y|)
$$

and

$$
\frac{\cos(\eta-\frac{1}{2})x\cos(\eta'-\frac{1}{2})y}{2\cos\frac{1}{2}x\cos\frac{1}{2}y}-\frac{\cos^2(\eta-\frac{1}{2})x}{2\cos^2\frac{1}{2}x}=O(|\eta-\eta'|+|x-y|).
$$

Observe that

$$\left(\sin\left(\eta-\frac{1}{2}\right)x\sin\left(\eta'-\frac{1}{2}\right)y+\cos\left(\eta-\frac{1}{2}\right)x\cos\left(\eta'-\frac{1}{2}\right)y\right)$$
$$=\cos\left(\left(\eta-\frac{1}{2}\right)(y-x)+(\eta'-\eta)x+(\eta'-\eta)(y-x)\right)$$

and use $\cos a = 1-\frac{a^2}{2}+O(a^4)$. □

Lemma A.8. *Adding the terms in the previous computational lemma that contain* $(x-y)^2$ *to* (A.5), *we have the following estimate:*

$$\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2-\left(\eta-\frac{1}{2}\right)^2(x-y)^2$$
$$+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2\left(\frac{1}{2}x\right)(x-y)^2+\frac{\cos^2\left(\eta-\frac{1}{2}\right)x}{4\cos^2\frac{1}{2}x}\sin^2\left(\frac{1}{2}x\right)(x-y)^2$$
$$=\eta\left(1+\frac{1}{2}x\tan\frac{1}{2}x\right)(x-y)^2+O(|x-y|^2(\eta^2+|x-y|+|\eta-\eta'|)).$$

Proof. Noting that $1+\cos x = 2\cos^2(\frac{1}{2}x)$, we have that

$$\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2$$
$$=\frac{1}{4}\left(-\left(\frac{\sin(\eta-\frac{1}{2})x}{\sin\frac{1}{2}x}\right)^2+\left(\frac{\cos(\eta-\frac{1}{2})x}{\cos\frac{1}{2}x}\right)^2\right)\cos^2\left(\frac{1}{2}x\right)(x-y)^2$$
$$+O(|x-y|^2(|\eta-\eta'|+|x-y|)).$$

Therefore,

$$\frac{1}{4}(\alpha_\eta\beta_{1-\eta'}+\alpha_{1-\eta}\beta_{\eta'})(1+\cos x)(x-y)^2-\left(\eta-\frac{1}{2}\right)^2(x-y)^2$$
$$+\frac{\sin^2(\eta-\frac{1}{2})x}{4\sin^2\frac{1}{2}x}\cos^2\left(\frac{1}{2}x\right)(x-y)^2+\frac{\cos^2(\eta-\frac{1}{2})x}{4\cos^2\frac{1}{2}x}\sin^2\left(\frac{1}{2}x\right)(x-y)^2$$
$$=\left(\frac{\cos^2\left(\eta-\frac{1}{2}\right)x}{4\cos^2\frac{1}{2}x}-\left(\eta-\frac{1}{2}\right)^2\right)(x-y)^2+O(|x-y|^2(|\eta-\eta'|+|x-y|))$$
$$=\left(\frac{1}{4}+\frac{1}{2}\eta x\tan\frac{1}{2}x-\left(-\eta+\frac{1}{4}\right)\right)(x-y)^2$$
$$+O(|x-y|^2(\eta^2+|\eta-\eta'|+|x-y|)).$$

□

Combing the above computations, we have that

$$\begin{aligned}
&\cos d(Q_{1-\eta'}, P_{1-\eta}) + \cos d(Q_{\eta'}, P_\eta)\\
&\geq 2 - \frac{1}{2}\left(1 + 2\eta d_{PS} \tan\left(\frac{1}{2} d_{PS}\right)\right)(d_{PQ}^2 + d_{SR}^2)\\
&+ \eta\left(1 + \frac{1}{2} d_{PS} \tan \frac{1}{2} d_{PS}\right)(d_{QR} - d_{PS})^2\\
&- (2\eta - 1)(\eta' - \eta) d_{PS}(d_{QR} - d_{PS})\\
&+ \eta^2 \mathrm{Quad}(d_{PQ}, d_{RS}, d_{QR} - d_{PS})\\
&+ \mathrm{Cub}\left(d_{QR} - d_{PS}, d_{PQ}, d_{RS}, \eta' - \eta\right).
\end{aligned}$$

Taylor expansion gives the result. □

Corollary A.9. *Given a pair of finite energy maps* $u_0, u_1 \in W^{1,2}(\Omega, X)$ *with images* $u_i(\Omega) \subset \mathcal{B}_\rho(Q)$ *and a function* $\eta \in C_c^1(\Omega)$, $0 \leq \eta \leq \frac{1}{2}$, *define the maps*

$$\begin{aligned}
u_\eta(x) &= (1 - \eta(x)) u_0(x) + \eta(x) u_1(x)\\
u_{1-\eta}(x) &= \eta(x) u_0(x) + (1 - \eta(x)) u_1(x)\\
d(x) &= d(u_0(x), u_1(x)).
\end{aligned}$$

Then $u_\eta, u_{1-\eta} \in W^{1,2}(\Omega, X)$ *and*

$$\begin{aligned}
|\nabla u_\eta|^2 + |\nabla u_{1-\eta}|^2 \leq &\left(1 + 2\eta d \tan \frac{d}{2}\right)(|\nabla u_0|^2 + |\nabla u_1|^2)\\
&- 2\eta\left(1 + \frac{1}{2} d \tan \frac{d}{2}\right)|\nabla d|^2 - 2d \nabla\eta \cdot \nabla d\\
&+ \mathrm{Quad}(\eta, |\nabla \eta|).
\end{aligned}$$

Appendix B. Energy convexity, existence, uniqueness, and subharmonicity

As with the previous section, the results in this section are stated in [Se1]. Excepting the first theorem, they are stated without proof. As, again, the calculations are non-trivial and tedious, we verify them for the reader.

Theorem B.1 ([Se1, Proposition 1.15]). *Let* $u_0, u_1 : \Omega \to \overline{\mathcal{B}_\rho(O)}$ *be finite energy maps with* $\rho \in (0, \frac{\pi}{2})$*. Denote by*

$$d(x) = d(u_0(x), u_1(x))$$
$$R(x) = d(u_{\frac{1}{2}}(x), O).$$

Then there exists a continuous function $\eta(x) : \Omega \to [0, 1]$ *such that the function* $w : \Omega \to \overline{\mathcal{B}_\rho(O)}$ *defined by*

$$w(x) = (1 - \eta(x))u_{\frac{1}{2}}(x) + \eta(x)O$$

is in $W^{1,2}(\Omega, \overline{B_\rho(O)})$ *and satisfies*

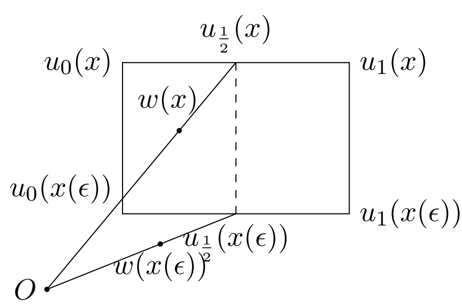
$$(\cos^8 \rho) \int_\Omega \left| \nabla \frac{\tan \frac{1}{2} d}{\cos R} \right|^2 d\mu_g \leq \frac{1}{2} \left(\int_\Omega |\nabla u_0|^2 d\mu_g + \int_\Omega |\nabla u_1|^2 d\mu_g \right) - \int_\Omega |\nabla w|^2 d\mu_g.$$

Proof. Once the estimates in Lemma A.2 and Lemma A.4 are established, we proceed as in [Se1]. Choose η to satisfy

$$\frac{\sin((1 - \eta(x))R(x))}{\sin R(x)} = \cos \frac{d(x)}{2}.$$

Note that $0 \leq \eta \leq 1$ and η is as smooth as $d(x), R(x)$. It is straightforward to verify that $w \in L^2_h(\Omega, \overline{B_\rho(O)})$.

For $W \in \Gamma(\Omega)$, consider the flow $\epsilon \mapsto x(\epsilon)$ induced by W.



Applying Lemma A.2 to the quadrilateral determined by $P = u_0(x(\epsilon)), Q = u_0(x), R = u_1(x), S = u_1(x(\epsilon))$, divided by ϵ^2, and integrate the resulting inequality against $f \in C_c^\infty(\Omega)$ and taking $\epsilon \to 0$, we obtain

$$\left(\cos\frac{d(x)}{2}\right)^2 |(u_{\frac{1}{2}})_*(W)|^2 \le \frac{1}{2}\left(|(u_0)_*(W)|^2 + |(u_1)_*(W)|^2\right) - \frac{1}{4}|\nabla_W d|^2.$$

Note that the cubic terms vanish in the limit as every cubic term will be the product of an L^1 function and $d(x) - d(x(\epsilon))$ or $d(u_i(x), u_i(x(\epsilon)))$, $i = 0, \frac{1}{2}, 1$.

Applying Lemma A.4 to the triangle determined by $Q = O$, $P = u_{\frac{1}{2}}(x)$, $S = u_{\frac{1}{2}}(x(\epsilon))$ yields

$$\begin{aligned}|(w)_*(W)|^2 &\le \left(\frac{\sin(1-\eta)R}{\sin R}\right)^2 (|(u_{\frac{1}{2}}) * (W)|^2 - |\nabla_W R|^2) + |\nabla_W((1-\eta)R)|^2 \\ &= \left(\cos\frac{d(x)}{2}\right)^2 (|(u_{\frac{1}{2}})_*(W)|^2 - |\nabla_W R|^2) + |\nabla_W((1-\eta)R)|^2.\end{aligned}$$

The above two inequalities imply

$$\begin{aligned}|w_*(W)|^2 \le{}& \frac{1}{2}\left(|(u_0)_*(W)|^2 + |(u_1)_*(W)|^2\right) \\ &- \frac{1}{4}|\nabla_W d|^2 - \left(\cos\frac{d(x)}{2}\right)^2 |\nabla_W R|^2 + |\nabla_W\left((1-\eta)R\right)|^2.\end{aligned}$$

By direct computation,

$$\begin{aligned}&-\frac{1}{4}|\nabla_W d|^2 - \left(\cos\frac{d(x)}{2}\right)^2 |\nabla_W R|^2 + |\nabla_W\left((1-\eta)R\right)|^2 \\ &= -\frac{\cos^4 R(x)\cos^4\frac{d(x)}{2}}{1-\sin^2 R(x)\cos^2\frac{d(x)}{2}}\left|\nabla\frac{\tan\frac{d(x)}{2}}{\cos R(x)}\right|^2.\end{aligned}$$

The lemma follows from estimating

$$\frac{\cos^4 R(x)\cos^4\frac{d(x)}{2}}{1-\sin^2 R(x)\cos^2\frac{d(x)}{2}} \ge \cos^4 R(x)\cos^4\frac{d(x)}{2} \ge \cos^8\rho,$$

dividing the resulting inequality by ϵ^2, integrating over $\mathbb{S}^{n-1}$, letting $\epsilon \to 0$, and then integrating over Ω. □

Theorem B.2 (Existence Theorem). *For any $\rho \in (0, \frac{\pi}{4})$ and for any finite energy map $h : \Omega \to \overline{\mathcal{B}_\rho(O)} \subset X$, there exists a unique element ${}^{Dir}h \in W_h^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$ which minimizes energy amongst all maps in $W_h^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$.*

Moreover, for any $\sigma \in (0, \rho)$, if ${}^{Dir}h(\partial\Omega) \subset \overline{\mathcal{B}_\sigma(O)}$ then ${}^{Dir}h(\Omega) \subset \overline{\mathcal{B}_\sigma(O)}$.

Proof. Denote by

$$E_0 = \inf\{E(u) : u \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})\}.$$

Let $u_i \in W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(P)})$ such that $E(u_i) \to E_0$. By Theorem B.1, we have that

$$(\cos^8 \rho) \int_\Omega \left| \nabla \frac{\tan \frac{1}{2} d(u_k(x), u_\ell(x))}{\cos R} \right| d\mu_g \leq \frac{1}{2}\left(E(u_k) + E(u_\ell)\right) - E(w_{k\ell}),$$

where $w_{k\ell}$ is the interpolation map defined by Theorem B.1. The above right hand side goes to 0 as $k, \ell \to \infty$. By the Poincaré inequality,

$$\int_\Omega d(u_k, u_\ell)\, d\mu_g \to 0.$$

Thus the sequence $\{u_k\}$ is Cauchy and $u_k \to u$ for some $u \in W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$ because $W^{1,2}(\Omega, \overline{\mathcal{B}_\rho(O)})$ is a complete metric space. By trace theory, $u \in W^{1,2}_h(\Omega, \overline{\mathcal{B}_\rho(O)})$. By lower semi-continuity of the energy, $E(u) = E_0$. The energy minimizer is unique by energy convexity.

Finally, since $\rho < \frac{\pi}{4}$, for any $\sigma \in (0, \rho]$, the ball $\mathcal{B}_\sigma(O)$ is geodesically convex. Therefore, the projection map $\pi_\sigma : \overline{\mathcal{B}_\rho(O)} \to \overline{\mathcal{B}_\sigma(O)}$ is well-defined and distance decreasing. Thus, since ${}^{Dir}h(\Omega) \subset \overline{\mathcal{B}_\rho(O)}$, we can prove the final statement by contradiction using the projection map to decrease energy. □

Lemma B.3 (cf. [Se1, (2.5)]). *Let $u_0, u_1 : \Omega \to \mathcal{B}_\rho(Q) \subset X$ be finite energy maps (possibly with different boundary values). For any given $\eta \in C^\infty_c(\Omega)$ with $0 \leq \eta < 1/2$, there exists finite energy maps $u_\eta, \hat{u}_\eta \in W^{1,2}_{u_0}(\Omega, \mathcal{B}_\rho(Q))$ and $u_{1-\eta}, \hat{u}_{1-\eta} \in W^{1,2}_{u_1}(\Omega, \mathcal{B}_\rho(Q))$ such that*

$$\begin{aligned}&|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2 \\ &\leq -2 \cos R^{u_\eta} \cos R^{u_{1-\eta}} \nabla\left(\frac{d}{\sin d}\eta F_\eta\right) \cdot \nabla F_\eta + \mathrm{Quad}(\eta, \nabla\eta),\end{aligned}$$

where

$$\begin{aligned} d(x) &= d(u_0(x), u_1(x)) \\ R^{u_\eta}(x) &= d(u_\eta(x), Q) \\ R^{u_{1-\eta}}(x) &= d(u_{1-\eta}(x), Q) \end{aligned}$$

and

$$F_\eta = \sqrt{\frac{1-\cos d}{\cos R^{u_\eta}\cos R^{u_{1-\eta}}}}.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$ satisfy $0 \le \eta < 1/2$. For $0 \le \phi, \psi \le 1$ that will be determined below, we define the comparison maps

$$\hat{u}_\eta = (1-\phi(x))u_\eta(x) + \phi(x)Q$$
$$\hat{u}_{1-\eta} = (1-\psi(x))u_{1-\eta}(x) + \psi(x)Q,$$

where

$$u_\eta(x) = (1-\eta(x))u_0(x) + \eta(x)u_1(x)$$

and

$$u_{1-\eta}(x) = \eta(x)u_0(x) + (1-\eta(x))u_1(x).$$

By Corollary A.5,

$$\begin{aligned}
&|\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 \\
&\le \left(\frac{\sin(1-\phi)R^{u_\eta}}{\sin R^{u_\eta}}\right)^2 (|\pi(u_\eta)|^2 - |\nabla R^{u_\eta}|^2) + |\nabla((1-\phi)R^{u_\eta})|^2 \\
&\quad + \left(\frac{\sin(1-\psi)R^{u_{1-\eta}}}{\sin R^{u_{1-\eta}}}\right)^2 (|\pi(u_{1-\eta})|^2 - |\nabla R^{u_{1-\eta}}|^2) + |\nabla((1-\psi)R^{u_{1-\eta}})|^2.
\end{aligned}$$

Define ϕ and ψ so that

$$\frac{\sin^2((1-\phi)R^{u_\eta})}{\sin^2 R^{u_\eta}} = 1 - 2\eta d \tan\frac{d}{2} + O(\eta^2)$$
$$\frac{\sin^2((1-\psi)R^{u_{1-\eta}})}{\sin^2 R^{u_{1-\eta}}} = 1 - 2\eta d \tan\frac{d}{2} + O(\eta^2).$$

Since $\frac{\sin(1-a)\theta}{\sin\theta} = 1 - a\theta\cot\theta + O(a^2)$, we solve

$$\phi = \eta\frac{\tan R^{u_\eta}}{R^{u_\eta}} d\tan\frac{d}{2} \quad \text{and} \quad \psi = \eta\frac{\tan R^{u_{1-\eta}}}{R^{u_{1-\eta}}} d\tan\frac{d}{2}.$$

Note that in particular $u_\eta, \hat{u}_\eta \in W^{1,2}_{u_0}(\Omega, \mathcal{B}_\rho(Q))$ and $u_{1-\eta}, \hat{u}_{1-\eta} \in W^{1,2}_{u_1}(\Omega, \mathcal{B}_\rho(Q))$.

Together with the estimate for $|\pi(u_\eta)|^2+|\pi(u_{1-\eta})|^2$ in Corollary A.9 (which also explains the choice of ϕ and ψ in order to eliminate the coefficient), we have

$$\begin{aligned}
&|\pi(\hat u_\eta)|^2+|\pi(\hat u_{1-\eta})|^2-|\pi(u_0)|^2-|\pi(u_1)|^2\\
&\le -2\eta\left(1+\frac{1}{2}d\tan\frac{d}{2}\right)|\nabla d|^2-2d\nabla\eta\cdot\nabla d\\
&\quad-\left(1-2\eta d\tan\frac{d}{2}\right)(|\nabla R^{u_\eta}|^2+|\nabla R^{u_{1-\eta}}|^2)\\
&\quad+\left|\nabla\left(1-\eta\frac{\tan R^{u_\eta}}{R^{u_\eta}}d\tan\frac{d}{2}\right)R^{u_\eta}\right|^2\\
&\quad+\left|\nabla\left(1-\eta\frac{\tan R^{u_{1-\eta}}}{R^{u_{1-\eta}}}d\tan\frac{d}{2}\right)R^{u_{1-\eta}}\right|^2+\mathrm{Quad}(\eta,|\nabla\eta|).
\end{aligned}$$

Simplifying the expression and using $1-\sec^2\theta=-\tan^2\theta$, we obtain

$$\begin{aligned}
\text{(B.1)}\qquad &\frac{1}{2}\left(|\pi(\hat u_\eta)|^2+|\pi(\hat u_{1-\eta})|^2-|\pi(u_0)|^2-|\pi(u_1)|^2\right)\\
&\le\eta\Bigg(-\left(1+\frac{1}{2}d\tan\frac{d}{2}\right)|\nabla d|^2\\
&\quad-d\tan\frac{d}{2}(\tan^2R^{u_\eta}|\nabla R^{u_\eta}|^2+\tan^2R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2)\\
&\quad-\nabla\left(d\tan\frac{d}{2}\right)\cdot(\tan R^{u_\eta}\nabla R^{u_\eta}+\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})\Bigg)\\
&\quad+\nabla\eta\cdot\left(-d\nabla d-\tan R^{u_\eta}d\tan\frac{d}{2}\nabla R^{u_\eta}-\tan R^{u_{1-\eta}}d\tan\frac{d}{2}\nabla R^{u_{1-\eta}}\right)\\
&\quad+\mathrm{Quad}(\eta,\nabla\eta).
\end{aligned}$$

We hope to find a,b,F_η which are functions of d,R^{u_η} and $R^{u_{1-\eta}}$ such that the right hand side above is $\le a\nabla(b\eta F_\eta)\cdot\nabla F_\eta$.

Since $a\nabla(b\eta F_\eta)\cdot\nabla F_\eta=\eta(ab|\nabla F_\eta|^2+\frac{a}{2}\nabla b\cdot\nabla F_\eta^2)+\frac{ab}{2}\nabla\eta\cdot\nabla F_\eta^2$, by comparing the terms involving $\nabla\eta$ in (B.1), we solve

$$\begin{aligned}
&\frac{ab}{2}\nabla\eta\cdot\nabla F_\eta^2\\
&=\nabla\eta\cdot\left(-d\nabla d-\tan R^{u_\eta}d\tan\frac{d}{2}\nabla R^{u_\eta}-\tan R^{u_{1-\eta}}d\tan\frac{d}{2}\nabla R^{u_{1-\eta}}\right)\\
&=-d\tan\frac{d}{2}\nabla\eta\cdot\left(\nabla\log\sin^2\frac{d}{2}-\nabla\log\cos R^{u_\eta}-\nabla\log\cos R^{u_{1-\eta}}\right)\\
&=-\frac{d}{\sin d}\cos R^{u_\eta}\cos R^{u_{1-\eta}}\nabla\eta\cdot\nabla\frac{1-\cos d}{\cos R^{u_\eta}\cos R^{u_{1-\eta}}},
\end{aligned}$$

where we use $2\sin^2\frac{d}{2}=(1-\cos d)$ and $\tan\frac{d}{2}=\frac{1-\cos d}{\sin d}$. It suggests us to choose

$$\frac{ab}{2}=-\frac{d}{\sin d}\cos R^{u_\eta}\cos R^{u_{1-\eta}}\quad\text{and}\quad F_\eta=\sqrt{\frac{1-\cos d}{\cos R^{u_\eta}\cos R^{u_{1-\eta}}}}.$$

We then compute the term $\eta(ab|\nabla F_\eta|^2+\frac{a}{2}\nabla b\cdot\nabla F_\eta^2)$ for the above choices of a, b, and F_η. For the term $ab|\nabla F_\eta|^2$, we compute

$$\begin{aligned}
ab|\nabla F_\eta|^2&=-\frac{d}{2\sin d(1-\cos d)}\\
&\quad\times|\sin d\nabla d+(1-\cos d)(\tan R^{u_\eta}\nabla R^{u_\eta}+\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})|^2\\
&\geq-\left(\frac{d\sin d}{2(1-\cos d)}|\nabla d|^2+d\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta}+\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})\right.\\
&\qquad\left.+\frac{d(1-\cos d)}{\sin d}(\tan^2 R^{u_\eta}|\nabla R^{u_\eta}|^2+\tan^2 R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2)\right),
\end{aligned}$$

where we expand the quadratic term and use the AM-GM inequality to handle the cross term $(\tan R^{u_\eta}\nabla R^{u_\eta})\cdot(\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}})$. For the term $\frac{a}{2}\nabla b\cdot\nabla F_\eta^2$, we assume $b=b(d)$ and compute:

$$\begin{aligned}
&\frac{a}{2}\nabla b\cdot\nabla F_\eta^2=\frac{ab}{2}\nabla\log b\cdot\nabla F_\eta^2\\
&=-d\frac{b'}{b}|\nabla d|^2-\frac{d(1-\cos d)}{\sin d}\frac{b'}{b}\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta}+\tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}}).
\end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned}
&ab|\nabla F_\eta|^2 + \frac{a}{2}\nabla b \cdot \nabla F_\eta^2 \\
&\geq -\left[\left(\frac{d\sin d}{2(1-\cos d)} + d\frac{b'}{b}\right)|\nabla d|^2\right. \\
&\quad + \left(d + \frac{d(1-\cos d)}{\sin d}\frac{b'}{b}\right)\nabla d\cdot(\tan R^{u_\eta}\nabla R^{u_\eta} + \tan R^{u_{1-\eta}}\nabla R^{u_{1-\eta}}) \\
&\quad \left. + \frac{d(1-\cos d)}{\sin d}(\tan^2 R^{u_\eta}|\nabla R^{u_\eta}|^2 + \tan^2 R^{u_{1-\eta}}|\nabla R^{u_{1-\eta}}|^2)\right].
\end{aligned}$$

Comparing to (B.1), we solve

$$\begin{aligned}
\frac{d\sin d}{2(1-\cos d)}\nabla d + d\nabla\log b &= \left(1 + \frac{1}{2}d\tan\frac{d}{2}\right)\nabla d \\
d\nabla d + \frac{d(1-\cos d)}{\sin d}\nabla\log b &= \nabla\left(d\tan\frac{d}{2}\right).
\end{aligned}$$

which implies that $b = \frac{d}{\sin d}$, and hence $a = -2\cos R^{u_\eta}\cos R^{u_{1-\eta}}$. □

Theorem B.4 (cf. [Se1, Corollary 2.3]). *Let $u_0, u_1 : \Omega \to \mathcal{B}_\rho(P) \subset X$ be a pair of energy minimizing maps (possibly with different boundary values). Let $d(x) = d(u_0(x), u_1(x))$ and $R^{u_i} = d(u_i, P)$. Then the function*

$$F = \sqrt{\frac{1-\cos d}{\cos R^{u_0}\cos R^{u_1}}}$$

satisfies the differential inequality weakly

$$\operatorname{div}(\cos R^{u_0}\cos R^{u_1}\nabla F) \geq 0.$$

Proof. Let $\eta \in C_c^\infty(\Omega)$ with $\eta \geq 0$. For $t > 0$ sufficiently small, we have $0 \leq t\eta < 1/2$. Let $\hat{u}_{t\eta}$ and $\hat{u}_{1-t\eta}$ be the corresponding maps defined as in Lemma B.3. Since u_0 and u_1 minimize the energy among maps of the same boundary values, we have

$$\begin{aligned}
0 &\leq \int_\Omega |\pi(\hat{u}_\eta)|^2 + |\pi(\hat{u}_{1-\eta})|^2 - |\pi(u_0)|^2 - |\pi(u_1)|^2\, d\mu_g \\
&\leq \int_\Omega -2\cos R^{u_{t\eta}}\cos R^{u_{1-t\eta}}\nabla\left(\frac{d}{\sin d}t\eta F_{t\eta}\right)\cdot\nabla F_{t\eta}\, d\mu_g + t^2\mathrm{Quad}(\eta,\nabla\eta).
\end{aligned}$$

Dividing the inequality by t and let $t \to 0$, since $R^{u_{t\eta}} \to R^{u_0}$ and $R^{u_{1-t\eta}} \to R^{u_1}$ and $F_{t\eta} \to F$, we derive

$$\begin{aligned} 0 &\leq \int_\Omega -2\cos R^{u_0} \cos R^{u_1} \nabla \left(\frac{d}{\sin d} \eta F \right) \cdot \nabla F \, d\mu_g \\ &= 2 \int_\Omega \left(\frac{d}{\sin d} \eta F \right) \operatorname{div} \left(\cos R^{u_0} \cos R^{u_1} \nabla F \right) \, d\mu_g. \end{aligned}$$

□

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Received January 14, 2018
Accepted January 15, 2018