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A gap theorem of four-dimensional gradient shrinking solitons

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In this paper, we will prove a gap theorem on four-dimensional gradient shrinking soliton. More precisely, we will show that any complete four-dimensional gradient shrinking soliton with nonnegative and bounded Ricci curvature, satisfying a pinched Weyl curvature, either is flat, or $\lambda_1 + \lambda_2 \geq c_0 R > 0$ at all points, where $c_0 \approx 0.29167$ and $\{\lambda_i\}$ are the two least eigenvalues of Ricci curvature. Furthermore, we can improve our estimate to $\lambda_1 + \lambda_2 \geq \frac{1}{3} R > 0$ under a stronger pinched condition. We point out that the lower bound $\frac{1}{3}R$ is sharp.

1. Introduction

A Riemannian manifold (M, g), couple with a smooth function f, is called gradient Ricci soliton, if there is a constant ρ, such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}.$$

The soliton is called shrinking, steady, or expanding, if $\rho > 0$, $\rho = 0$, or $\rho < 0$, respectively. Gradient shrinking solitons (GSS for short) play an important role in the Ricci flow, as they correspond to self-similar solutions, and often arise naturally as limits of dilations of Type I singularities of Ricci flow. They are also generalizations of Einstein metrics. Thus it is a central issue to understand and classify GSS.

The GSS are complete classified in dimension 2 (see [10]) and 3 (see [3, 11, 17, 18]), and in dimension $n \geq 4$ with vanishing Weyl tensor (see [17, 19, 22]). In recent years, there are some other attention to the classification of complete GSS (see [1, 8, 12, 15, 21]).

For a better understanding and ultimately for the classifications of GSS in higher dimension, one tries to obtain some curvature estimates and other geometric structures on GSS. In particular, on a complete non-compact GSS, Chen [7] showed that it will have nonnegative scalar curvature. In addition, Cao-Zhu[5] showed that it has infinite volume (or see [2] Theorem 3.1). While

Cao-Zhou[4] obtained a rather precise estimate on asymptotic behavior of the potential function f, and showed that it must have at most Euclidean volume growth.

If the GSS further satisfies some curvature assumptions, then we can get some more precise characteristics. For example, Carrillo-Ni [6] showed that any GSS with nonnegative Ricci curvature must have zero asymptotic volume ratio, and Munteanu-Wang [14] proved that GSS with nonnegative sectional curvature and positive Ricci curvature must be compact. In [13], Munteanu-Wang obtained some curvature estimates on four-dimensional GSS with bounded scalar curvature. In this paper, we obtain a gap theorem on four-dimensional GSS with pinched curvature.

Let (M^n, g) be a complete Riemannian manifold, we denote by Ric and R the Ricci tensor and scalar curvature respectively. It is well known that the Riemannian curvature tensor Rm can be decomposed into the orthogonal components :

$$Rm = W \oplus \frac{2}{n-2}\overset{\circ}{Ric} \wedge g \oplus \frac{R}{n(n-1)} g \wedge g,$$

where W is the Weyl tensor, and $\overset{\circ}{Ric} = Ric - \frac{R}{n} g$ is the traceless Ricci curvature. Now we can state our main theorem.

Theorem 1.1. *Let (M^4, g) be a complete four-dimensional GSS with bounded and nonnegative Ricci curvature $0 \le Ric \le C$, satisfying*

$$(*) \qquad |W| \le \gamma \left| |\overset{\circ}{Ric}| - \frac{1}{2\sqrt{3}} R \right|$$

for some constant $\gamma < 1 + \sqrt{3}$. Then either the soliton is flat, or

$$\lambda_1 + \lambda_2 \ge c_0 R > 0$$

at all points, where $c_0 = \frac{(1+2\sqrt{3}) - \sqrt{5+4\sqrt{3}}}{2\sqrt{3}} \approx 0.29167$, λ_1 and λ_2 are the least two eigenvalues of the Ricci curvature.

Remark 1.2. In view of the round cylinder $\mathbb{S}^2 \times \mathbb{R}^2$ with constant scalar curvature, the pinched constant $\gamma < 1 + \sqrt{3}$ in $(*)$ is necessary. Indeed, $\mathbb{S}^2 \times \mathbb{R}^2$ is a non-flat GSS with Ricci curvature $0 \le Ric \le \frac{1}{2} R$. Furthermore,

$|\overset{\circ}{Ric}| = \frac{1}{2}R$, and the Weyl tensor satisfies

$$|W| = \frac{1}{\sqrt{3}}R = (1+\sqrt{3})\left||\overset{\circ}{Ric}| - \frac{1}{2\sqrt{3}}R\right|.$$

But the least two eigenvalues of the Ricci curvature $\lambda_1 + \lambda_2 \equiv 0$ at all points.

Follow by a similar argument, we can show a better result under a stronger pinched condition as follow.

Theorem 1.3. *Let (M^4, g) be a complete four-dimensional GSS with bounded and nonnegative Ricci curvature $0 \le Ric \le C$, satisfying*

$$(**) \qquad |W| \le \gamma\left||\overset{\circ}{Ric}| - \frac{1}{2\sqrt{3}}R\right|$$

for some constant $\gamma \le \frac{1+\sqrt{3}}{\sqrt{3}}$. Then either the soliton is flat, or

$$\lambda_1 + \lambda_2 \ge \frac{1}{3}R > 0$$

at all points, where λ_1 and λ_2 are the least two eigenvalues of the Ricci curvature.

Remark 1.4. Our conclusion $\lambda_1 + \lambda_2 \ge \frac{1}{3}R > 0$ is sharp due to the example of round cylinder $\mathbb{S}^3 \times \mathbb{R}$. Since $\mathbb{S}^3 \times \mathbb{R}$ is also a non-flat GSS with Ricci curvature $0 \le Ric \le \frac{1}{3}R$, and $|\overset{\circ}{Ric}| = \frac{1}{2\sqrt{3}}R$, $|W| = 0$. These facts imply that the pinched condition $(*)$ holds. But the least two eigenvalues of the Ricci curvature $\lambda_1 + \lambda_2 \equiv \frac{1}{3}R$ at all points.

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2. Preliminaries

Le (M^4, g_{ij}) be a complete Riemannian manifold with bounded curvature. We deform the metric with the Ricci flow equation

$$\begin{cases} \frac{\partial g_{ij}(x,t)}{\partial t} = -2R_{ij}(x,t), & x \in M^4,\ t > 0, \\ g_{ij}(x,0) = g_{ij}(x), & x \in M^4. \end{cases}$$

Since the curvature is bounded, it is well known [20] that there exist a complete solution $g(t)$ of the Ricci flow on a time interval $[0, T)$ with bounded curvature for each t. On the other hand, the Ricci curvature tensor R_{ij} and the scalar curvature R evolve by the (PDE) system (cf. Hamilton [9]):

$$\text{(PDE)} \qquad \begin{cases} \frac{\partial}{\partial t} R_{ij} = \triangle R_{ij} + 2\sum\limits_{k,l} R_{ikjl} R_{kl}, \\ \frac{\partial}{\partial t} R = \triangle R + 2|Ric|^2. \end{cases}$$

Next we want to give a basic estimate of eigenvalues of Ricci tensor. Recall that a tensor evolves by a nonlinear heat equation may be controlled by a corresponding (ODE) system (cf. Hamilton [9]), while the (ODE) system corresponding to the above (PDE) is the following

$$\text{(ODE)} \qquad \begin{cases} \frac{d}{dt} R_{ij} = 2\sum\limits_{k,l} R_{ikjl} R_{kl}, \\ \frac{d}{dt} R = 2|Ric|^2. \end{cases}$$

By a direct computation, we have the following lemma.

Lemma 2.1. *Let $b = (\lambda_3 + \lambda_4) - (\lambda_1 + \lambda_2)$, where $\{\lambda_i\}$ are eigenvalues of the Ricci tensor with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$. Then under the (ODE) system, we have*

$$\frac{1}{2}\frac{d}{dt} b \leq 2b\left(\frac{R}{3} + W_{1212}\right) + (\lambda_1^2 + \lambda_2^2) - (\lambda_3^2 + \lambda_4^2).$$

Proof. Indeed, since $R_{ijij} = W_{ijij} + \frac{\lambda_i + \lambda_j}{2} - \frac{R}{6}$, and $W_{ijij} = W_{klkl}$, $\sum\limits_j W_{ijij} = 0$ for any orthonormal four-frame $\{e_i, e_j, e_k, e_l\}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\lambda_1 + \lambda_2) &\geq \sum_{k=2,3,4} \lambda_k \left(W_{1k1k} + \frac{\lambda_1 + \lambda_k}{2} - \frac{R}{6}\right) \\ &\quad + \sum_{l=1,3,4} \lambda_l \left(W_{2l2l} + \frac{\lambda_2 + \lambda_l}{2} - \frac{R}{6}\right) \\ &= (\lambda_1 + \lambda_2)\left(W_{1212} + \frac{\lambda_1 + \lambda_2}{2} - \frac{R}{6}\right) \\ &\quad + \lambda_3\left(-W_{1212} + \frac{\lambda_1 + \lambda_2}{2} + \lambda_3 - \frac{R}{3}\right) \\ &\quad + \lambda_4\left(-W_{1212} + \frac{\lambda_1 + \lambda_2}{2} + \lambda_4 - \frac{R}{3}\right) \\ &= \left(W_{1212} + \frac{R}{3}\right)(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) + \lambda_3^2 + \lambda_4^2. \end{aligned}$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\lambda_3+\lambda_4\Big) \le \left(W_{3434}+\frac{R}{3}\right)(\lambda_3+\lambda_4-\lambda_1-\lambda_2)+\lambda_1^2+\lambda_2^2.$$

The desired result follow from the difference of the above two inequalities. □

3. A key pinched estimate

In this section, we will give a pinched estimate, which implies that the curvature b described in Lemma 2.1 can become better under the Ricci flow.

Lemma 3.1. *Suppose we have a solution of Ricci flow $g(t)_{t\in[0,T]}$ on a four-manifold with uniformly bounded and nonnegative Ricci curvature, and satisfying the pinched condition $(*)$ at all $t\in[0,T]$.*

Assume at $t=0$, $R\ge r_0$ and $b\le \eta_0 R\le R$ for some positive constant $r_0>0$ and $\eta_0>\tilde{c}$, where $\tilde{c}=\frac{\sqrt{5+4\sqrt{3}}-(1+\sqrt{3})}{\sqrt{3}}\approx 0.41666$. Then there exist a positive constant $\delta=\delta(r_0,\eta_0,\gamma)\in(0,1]$, such that

$$b\le(\eta_0-\delta t)R$$

holds at all points and all $t\in[0,T']$, where $T'=\min\{T,\frac{\eta_0-\tilde{c}}{2}\}$.

Proof. Note that both the Ricci curvature tensor and the Weyl tensor are uniformly bounded, hence $g(t)$ has uniformly bounded curvature.

Consider the set $\Omega(t)_{t\in[0,T']}$ of matrices defined by the inequalities

$$\Omega(t):\begin{cases} R\ge r_0,\\ b\le(\eta_0-\delta t)R.\end{cases}$$

The constant $\delta\in(0,1]$ will be chosed later.

It is easy to see that $\Omega(t)$ is closed, convex and $O(n)$-invariant. By the assumptions at $t=0$ and the Hamilton's maximum principle for tensor, we only need to show the set $\Omega(t)$ is preserved by the (ODE) system. Indeed, we only need to look at points on the boundary of the set.

From the (ODE) system, we have

$$\frac{d}{dt}R = 2|Ric|^2 \geq 0,$$

which implies that $R \geq r_0$ for all $t \geq 0$. Thus the first inequality is preserved. To prove the second inequality, we only need to show that

$$\frac{1}{2}b' \leq (\eta_0 - \delta t)\frac{1}{2}R' - \frac{\delta}{2}R = \eta \cdot \frac{1}{2}R' - \frac{\delta}{2}R,$$

where $b = (\eta_0 - \delta t)R = \eta R$.

By Lemma 2.1 and the (ODE) system, it is suffice to show that

$$2b\left(\frac{R}{3} + W_{1212}\right) + (\lambda_1^2 + \lambda_2^2) - (\lambda_3^2 + \lambda_4^2) \leq \eta \sum_i \lambda_i^2 - \frac{\delta}{2}R.$$

It is equivalent to show that

$$\begin{aligned} I = {} & (1+\eta)(\lambda_3^2 + \lambda_4^2) - (1-\eta)(\lambda_1^2 + \lambda_2^2) \\ & - 2\eta R\left(\frac{R}{3} + W_{1212}\right) \\ \geq {} & \frac{\delta}{2}R. \end{aligned} \tag{3.1}$$

Now $b = \eta R$, thus $\lambda_3 + \lambda_4 = \frac{1+\eta}{2}R$ and $\lambda_1 + \lambda_2 = \frac{1-\eta}{2}R$. Denote by $x = \frac{\lambda_2 - \lambda_1}{2}$ and $y = \frac{\lambda_4 - \lambda_3}{2}$, which satisfies

$$0 \leq x \leq \frac{1-\eta}{4}R, \quad y \geq 0, \quad x + y \leq \frac{\eta}{2}R.$$

And then

$$\begin{aligned} \lambda_1 &= \frac{1-\eta}{4}R - x, & \lambda_2 &= \frac{1-\eta}{4}R + x, \\ \lambda_3 &= \frac{1+\eta}{4}R - y, & \lambda_4 &= \frac{1+\eta}{4}R + y. \end{aligned}$$

Meanwhile, by a direct computation, we have

$$W_{1212}^2 \leq \frac{2}{3}\sum W_{1k1k}^2 \leq \frac{2}{3} \cdot \frac{1}{8}|W|^2 \leq \frac{1}{12}\gamma^2\left(|\overset{\circ}{Ric}| - \frac{1}{2\sqrt{3}}R\right)^2.$$

In the following, we divide the argument into two cases.

Case 1: $|\mathring{Ric}| \geq \frac{R}{2\sqrt{3}}$. In this case,

$$|\mathring{Ric}|^2 = \sum_i \left(\frac{R}{4} - \lambda_i\right)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 - \frac{1}{4}R^2$$
$$= \left(\frac{1-\eta}{2}\right)^2 R^2 - 2\lambda_1\lambda_2 + 2\left(\frac{1+\eta}{4}\right)^2 R^2 + 2y^2 - \frac{1}{4}R^2$$
$$\leq \frac{1}{8}\cdot(3\eta^2 - 2\eta + 1)R^2 + 2y^2.$$

Denote by $t = \sqrt{\frac{1}{8}\cdot(3\eta^2 - 2\eta + 1)R^2 + 2y^2}$. Thus

$$|W_{1212}| \leq \frac{\gamma}{2\sqrt{3}}\cdot\left(t - \frac{R}{2\sqrt{3}}\right).$$

So I defined in (3.1) can be calculated as follow :

$$I = (1+\eta)\left[2\left(\frac{1+\eta}{4}R\right)^2 + 2y^2\right] - (1-\eta)\left[\left(\frac{1-\eta}{2}R\right)^2 - 2\lambda_1\lambda_2\right]$$
$$- \frac{2}{3}\eta R^2 - 2\eta R W_{1212}$$
$$\geq \frac{1}{24}(-3 + 11\eta - 9\eta^2 + 9\eta^3)R^2 + 2(1+\eta)y^2$$
$$- 2\eta R\cdot\frac{\gamma}{2\sqrt{3}}\cdot\left(t - \frac{R}{2\sqrt{3}}\right).$$

To get a lower bound of I, we rewrite the RHS as follow

$$RHS = \frac{1}{24}(-3 + 11\eta - 9\eta^2 + 9\eta^3)R^2$$
$$+ (1+\eta)\left[t^2 - \frac{1}{8}\cdot(3\eta^2 - 2\eta + 1)R^2\right] - \frac{\gamma\eta R}{\sqrt{3}}\cdot\left(t - \frac{R}{2\sqrt{3}}\right)$$
$$= (1+\eta)t^2 - \frac{\gamma\eta R}{\sqrt{3}}\cdot t$$
$$+ \frac{1}{24}(-3 + 11\eta - 9\eta^2 + 9\eta^3)R^2$$
$$- \frac{1}{8}\cdot(1+\eta)\cdot(3\eta^2 - 2\eta + 1)R^2 + \frac{\gamma\eta}{6}R^2.$$

Obviously, the RHS is a quadratic function of t, and we will see that that it is a increasing function of t. Indeed, we only need to show that $2(1+\eta)t - \frac{\gamma\eta R}{\sqrt{3}} > 0$.

It is easy to see that $12(\sqrt{3}-1) > (1+\sqrt{3})^2 > \gamma^2$, and then we have

$$\frac{(1+\eta)\sqrt{3\eta^2-2\eta+1}}{\eta} = \left(\frac{1}{\sqrt{\eta}}+\sqrt{\eta}\right)\sqrt{3\eta-2+\frac{1}{\eta}} \geq 2\sqrt{2\sqrt{3}-2} > \gamma.$$

Thus $2(1+\eta)t \geq \frac{R}{\sqrt{2}}\cdot(1+\eta)\sqrt{3\eta^2-2\eta+1} > \frac{R}{\sqrt{2}}\cdot\eta\gamma > \frac{\gamma\eta R}{\sqrt{3}}$.

Follow by the above monotonic property, the RHS achieves its minimal value if t takes its minimal value $\frac{R}{2\sqrt{2}}\sqrt{3\eta^2-2\eta+1}$, i.e. $y=0$. Hence we have

$$\begin{aligned}
I &\geq \frac{1}{24}(-3+11\eta-9\eta^2+9\eta^3)R^2 - \frac{\gamma\eta R^2}{\sqrt{3}}\cdot\left(\frac{1}{2\sqrt{2}}\sqrt{3\eta^2-2\eta+1}-\frac{1}{2\sqrt{3}}\right)\\
&= \frac{R^2}{24}\left[9\left(\eta-\frac{1}{3}\right)\left(\left(\eta-\frac{1}{3}\right)^2+\frac{8}{9}\right)-4\gamma\eta\cdot\left(\sqrt{1+\frac{9}{2}\left(\eta-\frac{1}{3}\right)^2}-1\right)\right]\\
&= \frac{3(\eta-\frac{1}{3})R^2}{8}\left[\left(\eta-\frac{1}{3}\right)^2+\frac{8}{9}-2\gamma\cdot\frac{\eta\left(\eta-\frac{1}{3}\right)}{\sqrt{1+\frac{9}{2}\left(\eta-\frac{1}{3}\right)^2}+1}\right]\\
&= \frac{3(\eta-\frac{1}{3})R^2}{8}\left[II+2(1+\sqrt{3}-\gamma)\cdot\frac{\eta(\eta-\frac{1}{3})}{\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}+1}\right],
\end{aligned}$$

where

$$\begin{aligned}
II &= (\eta-\frac{1}{3})^2+\frac{8}{9}-2(1+\sqrt{3})\cdot\frac{\eta(\eta-\frac{1}{3})}{\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}+1}\\
&= \eta^2-\frac{2}{3}\eta+1-2\eta(\eta-\frac{1}{3})\\
&\quad -2\eta(\eta-\frac{1}{3})\cdot\left[\frac{1+\sqrt{3}}{\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}+1}-1\right]\\
&= (1-\eta)(1+\eta)-2\eta(\eta-\frac{1}{3})\cdot\frac{\sqrt{3}-\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}}{\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}+1}\\
&= (1-\eta)(1+\eta)\\
&\quad -2\eta\left(\eta-\frac{1}{3}\right)\cdot\frac{1}{\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}+1}\cdot\frac{\frac{3}{2}(1-\eta)(1+3\eta)}{\sqrt{3}+\sqrt{1+\frac{9}{2}(\eta-\frac{1}{3})^2}}.
\end{aligned}$$

Note that $\eta = \eta_0 - \delta t \in [\frac{\eta_0+\tilde{c}}{2}, 1] \subset (\frac{1}{3}, 1]$, thus

$$\begin{aligned} II &\geq (1+\eta)(1-\eta) - 2\eta \cdot \frac{2}{3} \cdot \frac{1}{1+1} \cdot \frac{\frac{3}{2}(1-\eta)\cdot 4}{\sqrt{3}+1} \\ &= (1-\eta)\left(1+\eta - \frac{4}{1+\sqrt{3}}\eta\right) \geq 0, \end{aligned}$$

and then

$$\begin{aligned} I &\geq \frac{3(\eta-\frac{1}{3})R^2}{8} \cdot 2(1+\sqrt{3}-\gamma) \cdot \frac{\eta(\eta-\frac{1}{3})}{\sqrt{3}+1} \\ &\geq C_1(\eta_0,\gamma)R^2 \geq C_2(c_0,\eta_0,\gamma)R \end{aligned}$$

for some positive constant $C_2(r_0, \eta_0, \gamma) > 0$.

Case 2: $|\overset{\circ}{Ric}| < \frac{R}{2\sqrt{3}}$. In this case,

$$\begin{aligned} |\overset{\circ}{Ric}|^2 &= 2\left(\frac{1-\eta}{4}\right)^2 R^2 + 2\left(\frac{1-\eta}{4}\right)^2 R^2 - \frac{1}{4}R^2 + 2y^2 + 2x^2 \\ &\geq \frac{1}{4}\eta^2 R^2 + 2x^2. \end{aligned}$$

Denote by $\tau = \sqrt{\frac{1}{4}\eta^2 R^2 + 2x^2}$, then

$$|W_{1212}| \leq \frac{\gamma}{2\sqrt{3}} \cdot \left(\frac{R}{2\sqrt{3}} - \tau\right).$$

By a direct computation, we have

$$\begin{aligned} I &\geq \frac{(1+\eta)^3}{8}R^2 - \frac{(1-\eta)^3}{8}R^2 - \frac{2}{3}\eta R^2 + 2(1+\eta)y^2 - 2(1-\eta)x^2 \\ &\quad - 2\eta R \cdot \frac{\gamma}{2\sqrt{3}} \cdot \left(\frac{R}{2\sqrt{3}} - \tau\right) \\ &\geq \frac{\eta}{12}(3\eta^2+1)R^2 - 2(1-\eta)x^2 + \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(\tau - \frac{R}{2\sqrt{3}}\right). \end{aligned}$$

Similarly, to get a lower bound of I, we rewrite the RHS as follow

$$RHS = \frac{\eta}{12}(3\eta^2+1)R^2 - (1-\eta)(\tau^2 - \frac{1}{4}\eta^2 R^2) + \frac{\gamma\eta R}{\sqrt{3}} \cdot \left(\tau - \frac{R}{2\sqrt{3}}\right).$$

Since the RHS is a quadratic function of τ, it is easy to see that the RHS will get the minimum value on the boundary, i.e. $x = 0$ or $x = \frac{1-\eta}{4}R$.

If $x = \frac{1-\eta}{4}R$. Then

$$I \geq \frac{\eta R^2}{12}(3\eta^2+1) - \frac{1}{8}(1-\eta)^3 R^2 + \frac{\gamma\eta R^2}{\sqrt{3}} \cdot \left(\frac{1}{2\sqrt{2}}\sqrt{3\eta^2-2\eta+1} - \frac{1}{2\sqrt{3}}\right).$$

Since $\eta \geq \frac{\eta_0+\tilde{c}}{2} > \frac{1}{3}$, we have $\frac{1}{2\sqrt{2}}\sqrt{3\eta^2-2\eta+1} > \frac{1}{2\sqrt{2}}\sqrt{\frac{2}{3}} = \frac{1}{2\sqrt{3}}$. Thus

$$\begin{aligned} I &> \frac{\eta R^2}{12}(3\eta^2+1) - \frac{1}{8}(1-\eta)^3 R^2 \\ &= \frac{R^2}{24}\left[2\eta \cdot (3\eta^2+1) - 3(1-\eta)^3\right] \\ &= \frac{R^2}{24}(9\eta^3 - 9\eta^2 + 11\eta - 3) \\ &= \frac{R^2}{24}\left(\eta - \frac{1}{3}\right)\left[(3\eta-1)^2+8\right] = C_3(\eta_0, c_0)R. \end{aligned}$$

If $x = 0$. Then

$$\begin{aligned} I &\geq \frac{\eta}{12}(3\eta^2+1)R^2 + \frac{\gamma\eta R^2}{\sqrt{3}} \cdot \left(\frac{\eta}{2} - \frac{1}{2\sqrt{3}}\right) \\ &= \frac{\eta R^2}{12}\left[3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1)\right]. \end{aligned} \tag{3.2}$$

Note that $\frac{R}{2\sqrt{3}} > |\mathring{Ric}| \geq \frac{\eta}{2}R$, which implies that $\eta < \frac{1}{\sqrt{3}}$. So we have

$$\begin{aligned} &3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1) \\ &> 3\eta^2 + 1 + 2(1+\sqrt{3}) \cdot (\sqrt{3}\eta - 1) \\ &= 3\eta^2 + (6+2\sqrt{3})\eta - (1+2\sqrt{3}) \\ &= 3\left(\eta - \frac{\sqrt{5+4\sqrt{3}} - (1+\sqrt{3})}{\sqrt{3}}\right) \cdot \left(\eta + \frac{\sqrt{5+4\sqrt{3}} + (1+\sqrt{3})}{\sqrt{3}}\right). \end{aligned}$$

Thus

$$I \geq C_4(\eta_0, c_0)R.$$

Combine the above argument, we have

$$I \geq C_5(c_0, \eta_0)R.$$

So by choosing $\delta = \delta(c_0, \eta_0, \gamma) = \min\{1, 2C_2, 2C_5\}$, the inequality (3.1) holds. The proof of Lemma 3.1 is complete. $\square$

4. A gap theorem of four-dimensional GSS

Suppose (M^4, g) is a complete GSS. Then there are a smooth function f and a positive constant ρ, such that

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}.$$

It is well known that there exist a self-similar solution of Ricci flow as follow

$$g(t) = \tau(t)\varphi_t^*(g),\ t \in \left(-\infty, \frac{1}{2\rho}\right),$$

where $\tau(t) = 1 - 2\rho t$, and φ_t is a family of diffeomorphisms.

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is well known that any GSS with nonnegative Ricci curvature either is flat, or has positive scalar curvature $R \geq r_0 > 0$ for some positive constant $r_0 = r_0(g)$. In the following, we always assume the soliton has positive scalar curvature $R \geq r_0 > 0$ (cf. [16]).

We will argue by contradiction. Denote by

$$\eta_0 = \sup_{x \in (M^4, g)} \frac{b(x)}{R(x)} \leq 1.$$

If $\eta_0 \leq \tilde{c}$, then we have $\lambda_1 + \lambda_2 \geq \frac{1-\tilde{c}}{2} R = c_0 R$, and we have done. If not, then $\eta_0 > \tilde{c}$. By the assumptions, we see that the self-similar solution $g(t)_{t \in [0, \frac{1}{10\rho}]}$ has nonnegative and uniformly bounded Ricci curvature with $g(0) = g$.

Then by Lemma 3.1, there exist a positive constant $\delta = \delta(r_0, \eta_0, \gamma) \in (0, 1]$, such that

$$b \leq (\eta_0 - \delta t) R$$

is preserved under the Ricci flow at all points and all small $t \in [0, T']$, where $T' = \min\{\frac{1}{10\rho}, \frac{\eta_0 - \tilde{c}}{2}\}$.

Hence we have

$$b \leq (\eta_0 - \delta T') R$$

at all points. But this is impossible. Since at $t = 0$, there exist some point $p \in M$, such that $b(p) \geq (\eta_0 - \frac{\delta}{2} T') R(p)$. Note that $g(t)$ only changes by scaling and a diffeomorphism on M^4. So at time $t = T'$, there is some point $q \in M$, such that

$$
\begin{aligned}
b(q, T') &= \frac{1}{1 - 2\rho T'} b(p) \\
&\geq \frac{1}{1 - 2\rho T'} \left(\eta_0 - \frac{\delta}{2} T' \right) R(p) = \left(\eta_0 - \frac{\delta}{2} T' \right) R(q, T'),
\end{aligned}
$$

which is contradictive with $b(q, T') \leq (\eta_0 - \delta T') R(q, T')$.

And we complete the proof of Theorem 1.1. □

Next, we follow a similar argument to prove Theorem 1.3.

Proof of Theorem 1.3. Obviously, we only need to show that

$$\eta_0 = \sup_{x \in (M^4, g)} \frac{b(x)}{R(x)} \leq \frac{1}{3}.$$

If not, $\eta_0 > \frac{1}{3}$. Follow from a similar argument as Claim 3.1, we can prove the following assertion.

Claim 4.1. *Suppose we have a solution of Ricci flow $g(t)_{t \in [0,T]}$ on a four-manifold with uniformly bounded and nonnegative Ricci curvature, and satisfying the pinched condition $(**)$ at all $t \in [0, T]$.*

Assume at the initial time $t = 0$, $R \geq r_0$ and $b \leq \eta_0 R$ for some positive constant $r_0 > 0$ and $\eta_0 > \frac{1}{3}$. Then there exist a positive constant $\delta = \delta(r_0, \eta_0, \gamma) \in (0, 1]$, such that

$$b \leq (\eta_0 - \delta t) R$$

holds at all points and all $t \in [0, T']$, where $T' = \min\{T, \frac{\eta_0 - \frac{1}{3}}{2}\}$.

For the proof of Claim 4.1, we check the argument of Lemma 3.1. Then we only need to get a positive lower bound of (3.2). Indeed,

$$
\begin{aligned}
3\eta^2 + 1 + 2\gamma(\sqrt{3}\eta - 1) &\geq 3\eta^2 + 1 + 2 \cdot \frac{1 + \sqrt{3}}{\sqrt{3}} \cdot (\sqrt{3}\eta - 1) \\
&= 3(\eta - \frac{1}{3}) \cdot \left(\eta + \frac{2 + \sqrt{3}}{\sqrt{3}} \right) \geq C(\eta_0).
\end{aligned}
$$

Thus Claim 4.1 holds. But this assertion will develop a contradiction like the proof of Theorem 1.1. And then we obtain Theorem 1.3. □

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