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Remarks on complete noncompact Einstein warped products

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The purpose of this article is to investigate the structure of complete non-compact quasi-Einstein manifolds. We show that complete noncompact quasi-Einstein manifolds with $\lambda = 0$ are connected at infinity. In addition, we provide some conditions under which quasi-Einstein manifolds with $\lambda < 0$ are f-non-parabolic. In particular, we obtain estimates on volume growth of geodesic balls for such manifolds.

1. Introduction

A fascinating problem in differential geometry is to study Einstein manifolds. In large part, this is because Einstein manifolds have connections with mathematics and physics. In the last decades, much efforts have been devoted to study the geometry and classifications of Einstein manifolds. There is a wealth of classical literature in this subject, we refer the readers to the book [4] for a comprehensive treatment of Einstein manifolds. The m-Bakry-Emery Ricci tensor, which appeared previously in [1] and [21] as a modification of the classical Bakry-Emery tensor

$$Ric_f = Ric + \nabla^2 f,$$

is a powerful tool to study Einstein warped product. More precisely, the m-Bakry-Emery Ricci tensor is given by

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df, \tag{1.1}$$

where f is a smooth function on M^n and $\nabla^2 f$ stands for the Hessian of f. It is also used to study the weighted measure $d\mu = e^{-f}dx$, where dx is the Riemann-Lebesgue measure determined by the metric.

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According to [3, 7, 12, 22, 24] and [26], a Riemannian manifold (M^n, g), $n \geq 2$, will be called *m-quasi-Einstein manifold*, or simply *quasi-Einstein manifold*, if there exist a smooth potential function f on M^n and a constant λ satisfying the following fundamental equation

$$Ric_f^m = Ric + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g. \tag{1.2}$$

It is easy to see that a ∞-quasi-Einstein manifold means a gradient Ricci soliton. Ricci solitons model the formation of singularities in the Ricci flow and correspond to self-similar solutions, i.e., solutions which evolve along symmetries of the flow, see [6] and references therein for more details. In this article, we focus in the case $m < \infty$. Moreover, when m is a positive integer it corresponds to a warped product Einstein metric (cf. [7] and [12]). Following the terminology adopted in [3, 7, 22, 24] and [26], an m-quasi-Einstein manifold will be called *trivial* if its potential function f is constant, otherwise it will be *nontrivial*. Notice that the triviality implies that M^n is an Einstein manifold.

The remarkable motivation to study quasi-Einstein metrics on a Riemannian manifold is its direct relation with Einstein warped product, which also have different properties compared with the gradient Ricci solitons. In this approach, it is interesting to recall that, on a quasi-Einstein manifold, there is a crucial constant μ so that

$$\Delta f - |\nabla f|^2 = m\lambda - m\mu e^{\frac{2}{m}f}. \tag{1.3}$$

We refer to [3, 4, 8, 12, 22] and [26] for more details. It is also important to mention that some examples of m-quasi-Einstein manifolds with $\lambda < 0$ and arbitrary μ as well as quasi-Einstein manifolds with $\lambda = 0$ and $\mu > 0$ were built in [2, 4] and [26]. In [8], Case has shown that m-quasi-Einstein manifolds with $\lambda = 0$ and $\mu \leq 0$ are trivial. While Qian [21] proved that m-quasi-Einstein manifolds with $\lambda > 0$ must be compact. Moreover, by Kim and Kim [12] the converse statement is also true. Thereby, it follows that a nontrivial quasi-Einstein manifold is compact if and only if $\lambda > 0$. An example of nontrivial m-quasi-Einstein manifold with $\lambda > 0$, $m > 1$ and $\mu > 0$ was obtained in [18], see also [23] for further related results.

For our purposes it is very important to recall some known terminology. First of all, we consider a compact subset D of a complete noncompact manifold M^n. So, we would like to recall that an *end* of M^n with respect to D is a connected unbounded component of $M^n \backslash D$. The number of ends with respect to D is the number of unbounded connected component of $M^n \backslash D$.

Moreover, if $\{\Omega_i\}$ is a compact exhaustion of M^n, then the number of ends with respect to Ω_i is a monotonically nondecreasing sequence. In particular, M^n is said to have *finitely many ends* if there exists $1 \leq k < \infty$, such that, for any $D \subset M$, the number of ends is at most k. We further recall that a *Green's function* $G(x, y)$ is a function defined on

$$(M \times M) \backslash \{(x, x)\}$$

such that $G(x, y) = G(y, x)$ and $\Delta_y G(x, y) = -\delta_x(y)$, for all $x \neq y$.

It is well-known that every complete manifold admits a Green's function. In addition, a complete manifold M^n is said to be *non-parabolic* if it admits a positive Green's function. Otherwise, it is said to be *parabolic*. A manifold is non-parabolic if and only if there exists a positive superharmonic function whose infimum is achieved at infinity (cf. [16]).

In order to proceed, let us also point out that a complete manifold M^n is non-parabolic if and only if M^n has a non-parabolic end. Notice that the definition of parabolicity is essentially analytic, but there are geometric descriptions of parabolicity; for more details see, for instance, [11]. We also remark that the same definitions can be extended for f-Laplacian

$$\Delta_f = \Delta - \nabla f,$$

which is a self-adjoint operator on the space of square integrable functions on M^n with respect to the measure $e^{-f}dx$. In particular, a function h is called f-harmonic if $\Delta_f h = 0$. For a comprehensive reference on such a subject, we indicate, for instance [15].

In [31], S.-T. Yau proved brightly that every smooth positive harmonic function defined on a complete manifold with nonnegative Ricci curvature must be constant. This stimulated many interesting works. In fact, it is definitely important issue to investigate the existence of harmonic functions on complete manifolds. It is known that the existence of certain classes of harmonic functions is related to the existence of ends of the manifold. For instance, when the manifold M^n is non-parabolic the number of ends is bounded from above by dimension of the space spanned by the set of all positive harmonic function on M^n(cf. [16]). For the purpose of application, it is important to recall that X.-D. Li [17] was able to show that every smooth positive f-harmonic function defined on a complete manifold with $Ric_f^m \geq 0$ must be constant. These results play crucial role in this work.

1.1. Structure of Complete quasi-Einstein Manifolds

Inspired by the historical development on the study of Einstein warped products, in this article, we shall investigate the geometry of complete, noncompact, quasi-Einstein manifolds, that is, complete noncompact manifolds satisfying (1.2) and (1.3). In this case, as it was previously mentioned, λ must be nonpositive.

A classical result obtained by Cheeger and Gromoll [9] asserts that if M^n has nonnegative Ricci curvature, then either $M^n = N \times \mathbb{R}$, for some compact manifold N with nonnegative Ricci curvature, or M has only one end. It has been shown that a complete manifold M satisfying $Ric_f^m \geq 0$ is either

1) a product $N \times \mathbb{R}$ with N compact, or
2) M^n is connected at infinity.

For more details see [10], see also [29, 30]. Here, we shall improve this conclusion by showing that a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda = 0$ and $m \in (1, \infty)$ must be connected at infinity.

After these preliminary remarks we may state our first result as follows.

Theorem 1. *Let $\big(M^n,\, g,\, f\big)$ be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda = 0$ and $m \in (1, \infty)$. Then M^n is connected at infinity.*

One question that naturally arises from the previous result is to determine whether a similar result occurs for $\lambda < 0$. An example obtained by Wang (cf. [26], Example 2.1) indicates that, in special cases, noncompact quasi-Einstein manifold with $\lambda < 0$ can be connected at infinity. In 2012, Wang [27] has shown a splitting theorem for complete smooth measure manifolds whose m-Bakry-Emery tensor is bounded from below by a negative multiple of the lower bound of the weighted spectrum. More precisely, he proved that given a complete noncompact manifold M^n satisfying

$$Ric_f^m \geq -\frac{m+n-1}{m+n-2}\lambda_1(\Delta_f),$$

where $\lambda_1(\Delta_f)$ stands for the positive lower bound of the spectrum of the weighted Laplacian on M^n, then either M^n has at most one end with infinite weighted volume, or $M = \mathbb{R} \times N$ with the product metric

$$g_M = dt^2 + \cosh^2 \sqrt{\frac{\lambda_1(M)}{m+n-2}} t g_N^2,$$

where N is an $(n-1)$-dimensional compact manifold.

Following up on our previous discussion we proceed to deal with the case $\lambda < 0$. In this case, we have established the following result.

Theorem 2. *Let (M^n, g, f) be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda < 0$, $m \in (1, \infty)$ and $\mu \geq 0$. Then M^n is f-non-parabolic.*

Before to announce our next results let us mention that it is possible for a non-parabolic manifold to have many parabolic ends. Moreover, the key fact here that should be emphasized is that when $\lambda < 0$ and $\mu \geq 0$ we have a linear pinching estimate for $\sup_{\partial B_p(r)} e^{-\frac{f}{m}}$, which allows us to deduce that

$$\limsup_{x \to \infty} e^{-\frac{f}{m}} = +\infty.$$

However, if $\lambda < 0$ and $\mu < 0$, it is not hard to verify that

$$f \leq \frac{m}{2} \ln \left(\frac{m\lambda}{\mu} \right)$$

on M^n (cf. Wang [26]). Therefore, in this case, we are not able to guarantee that $e^{-\frac{f}{m}}$ converge to infinity.

For what follows, we remember that the scalar curvature R of a quasi-Einstein manifold with $\lambda \leq 0$ must to satisfy $R \geq \lambda n$ (cf. [26]). Here, we shall use this data to obtain the following result.

Theorem 3. *Let (M^n, g, f) be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda < 0$, $m \in (1, \infty)$ and $\mu < 0$. Then M^n is f-non-parabolic or Einstein.*

Our next result concerns the number of ends of a noncompact m-quasi-Einstein manifold with $\lambda < 0$. More precisely, we have the following result.

Theorem 4. *Let $\big(M^n,\, g,\, f\big)$ be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda < 0$ and $m \in (1, \infty)$. Suppose that*

$$R \geq \lambda\Big(n - \frac{m}{m-1}\Big).$$

Then M^n has only one f-non-parabolic end.

1.2. Bounds on Volume Growth

A remarkable result by Calabi [5] and Yau [32] asserts that the geodesic balls of complete noncompact manifolds with nonnegative Ricci tensor have at least linear growth, that is,

$$Vol(B_p(r)) \geq cr \tag{1.4}$$

for any $r > r_0$, where r_0 is a positive constant, $B_p(r)$ is the geodesic ball of radius r centered at $p \in M^n$ and c is a positive constant that does not depend on r. Indeed, volume growth rate is an important piece of geometric information. In this spirit, Munteanu and Sesum [19] showed that gradient Ricci solitons with $\lambda = 0$ must to satisfy (1.4). Recently, Barros, Batista and Ribeiro [3] were able to prove the same result for m-quasi-Einstein manifolds with $\lambda = 0$. Moreover, they showed that a noncompact m-quasi-Einstein manifold with $\lambda < 0$, $m \in (1, \infty)$ and $\mu \leq 0$, such that its potential function is bounded from below, must also to satisfy (1.4).

As an application of Theorem 2, we obtain the following result concerning the growth of f-volume (or weighted volume) of geodesic balls for noncompact quasi-Einstein manifolds with $\lambda < 0$.

Theorem 5. *Let $\big(M^n,\, g,\, f\big)$ be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda < 0$, $m \in (1, \infty)$ and $\mu \geq 0$. Then there exist positive constants c and r_0 such that for any $r > r_0$*

$$Vol_f(B_p(r)) \geq cr^2. \tag{1.5}$$

Next, we shall show that when $\mu = 0$ the assumption on the potential function considered by Barros et al. [3] (cf. Theorem 3 in [3]) can be removed. More precisely, in this case, we have the following estimate for volume growth of geodesic balls on noncompact quasi-Einstein manifolds with $\lambda < 0$.

Theorem 6. *Let $\big(M^n, g, f\big)$ be a nontrivial complete noncompact m-quasi-Einstein manifold with $\lambda < 0$, $m \in (1, \infty)$ and $\mu = 0$. Then there exist positive constants c and r_0 such that for any $r > r_0$*

$$Vol(B_p(r)) \geq ce^{-\sqrt{\frac{-\lambda}{m-1}}r}. \tag{1.6}$$

2. Proof of the Main Results

2.1. Proof of Theorem 1

Proof. Firstly, since $m \in (1, \infty)$, we may set the function

$$u = e^{-\frac{f}{m}}$$

on M^n. Notice moreover that

$$\nabla u = -\frac{u}{m}\nabla f$$

and

$$\nabla^2 f - \frac{1}{m} df \otimes df = -\frac{m}{u}\nabla^2 u. \tag{2.1}$$

Therefore, for some $a > 0$ (to be chosen later) we have

$$\begin{aligned}\Delta_f u^{-a} &= \Delta u^{-a} - \langle \nabla u^{-a}, \nabla f\rangle \\ &= -au^{-a-1}\Delta u + a(a+1)|\nabla u|^2 u^{-a-2} + au^{-a-1}\langle \nabla u, \nabla f\rangle.\end{aligned}$$

Substituting the trace of fundamental equation (1.2) into (2.1) we arrive at

$$\Delta u = \frac{u}{m}R,$$

where R stands for the scalar curvature of M^n. This substituted in the above expression yields

$$\begin{aligned}\Delta_f u^{-a} = &-\frac{a}{m}u^{-a-1}uR + a(a+1)|\nabla u|^2 u^{-a-2} \\ &- mau^{-a-2}|\nabla u|^2,\end{aligned} \tag{2.2}$$

which can be written succinctly as

$$\Delta_f u^{-a} = -\frac{a}{m}u^{-a}R - a\Big(m - (a+1)\Big)|\nabla u|^2 u^{-a-2}. \tag{2.3}$$

Lower bound estimates for scalar curvatures of noncompact quasi-Einstein manifolds were obtained in [26]. Among those estimates, we already know that if $\lambda \leq 0$, then $R \geq \lambda n$. From this, every noncompact quasi-Einstein manifold with $\lambda = 0$ has nonnegative scalar curvature. In particular, choosing $a = \frac{m-1}{2}$, which is positive, we immediately deduce from (2.3) that

$$\Delta_f u^{-a} \leq 0.$$

Proceeding, we remember that Wang [28] was able to show that the potential function of a noncompact quasi-Einstein manifold with $\lambda = 0$ and $m > 1$ must to satisfy

$$\frac{m-1}{n+m-1}\ln r - C_1 \leq \sup_{x\in\partial B_p(r)} (-f)(x) \leq m\ln r + C_2, \tag{2.4}$$

for $r > 1$, where C_1 and C_2 are constants. Of which we easily see that

$$c_0 r^{\frac{m-1}{m(m+n-1)}} \leq \sup_{x\in\partial B_p(r)} u(x) \leq C_0 r, \tag{2.5}$$

where c_0 and C_0 are positive constants.

Therefore, u^{-a} is a positive function on M^n such that

$$\Delta_f u^{-a} \leq 0,$$

and from (2.5) we also have

$$\liminf_{x\to\infty} u^{-a}(x) = 0.$$

So, it suffices to apply Li-Tam theorem [16] to conclude that M^n is f-non-parabolic. This means that M^n has one f-non-parabolic end (cf. [15]). Next, since $Ric_f^m = 0$ we immediately have $Ric_f \geq 0$ and then we may invoke Lemma 4.1 in [20] to conclude that M^n has only one f-non-parabolic end.

On the other hand, a result by X.-D. Li (cf. [17], Theorem 1.3) guarantees that every complete manifold M^n satisfying

$$Ric_f^m \geq 0$$

must to satisfy the strong Liouville property, that is, every smooth positive f-harmonic function on M^n must be constant. At the same time, we recall that Li and Tam [16] showed that when the manifold M^n is non-parabolic the number of ends is bounded from above by dimension of the space spanned by the set of all positive harmonic function on M^n. Hence, it follows that M^n has only one end and then M^n is connected at infinity. So the proof is completed. □

2.2. Proof of Theorem 2

Proof. First of all, we take trace of fundamental equation (1.2) jointly with (2.1) to infer

$$\Delta u = \frac{u}{m}(R - \lambda n).$$

This immediately gives

$$\Delta_f u^{-a} = -\frac{a}{m}u^{-a}(R - \lambda n) - a\big[m - (a+1)\big]|\nabla u|^2 u^{-a-2}. \tag{2.6}$$

We again recall that, by Wang [26], if $\lambda \leq 0$, then $R \geq \lambda n$, which combined with (2.6) ensures

$$\Delta_f u^{-a} \leq 0,$$

provided that $a = \frac{m-1}{2}$. Therefore, it remains to prove that

$$\liminf_{x \to \infty} u^{-a}(x) = 0.$$

To that end, we recall that the potential function of a noncompact quasi-Einstein manifold with $\lambda < 0$, $m > 1$ and $\mu > 0$ satisfies

$$\begin{aligned} \frac{2m}{\sqrt{n+5m-1} + \sqrt{n+m-1}}\sqrt{-\lambda}r - c \leq& \sup_{x \in \partial B_p(r)} (-f)(x) \\ \leq& \frac{m}{\sqrt{m-1}}\sqrt{-\lambda}r + C \end{aligned} \tag{2.7}$$

for $r > 1$, where c and C are constants (cf. [28], Theorem 3.1). Moreover, if $\mu = 0$, we then have, for $r > 1$, the following estimate

$$\frac{m}{\sqrt{n+m-1}}\sqrt{-\lambda}r - c_0 \leq \sup_{x \in \partial B_p(r)} (-f)(x) \leq \frac{m}{\sqrt{m-1}}\sqrt{-\lambda}r + C_0, \tag{2.8}$$

where c_0 and C_0 are constants (cf. [28], Theorem 4.4).

Rearranging these inequalities we achieve

$$C_1 e^{C_2 r} \leq \sup_{x\in\partial B_p(r)} u(x) \leq C_3 e^{C_4 r}, \tag{2.9}$$

where $u = e^{-\frac{f}{m}}$ and $C_{i's}$ are positive constants. From this it follows that

$$\liminf_{x\to\infty} u^{-a}(x) = 0.$$

Now, it suffices to apply again Li-Tam theorem [16] to conclude that M^n is f-non-parabolic. The proof is completed. □

2.3. Proof of Theorem 3

Proof. Firstly, for $a = \frac{m-1}{2}$, one easily verifies that

$$\begin{aligned}
\Delta_f u^{-a} &= -\frac{a}{m}u^{-a}(R-\lambda n) - a[m-(a+1)]|\nabla u|^2 u^{-a-2}\\
&= -\frac{m-1}{2m}(R-\lambda n)u^{-a} - \frac{(m-1)^2}{4}|\nabla u|^2 u^{-a-2}\\
&\leq -\frac{m-1}{2m}(R-\lambda n)u^{-a}. \qquad (2.10)
\end{aligned}$$

We then invoke [14] to infer

$$\int_M \frac{m-1}{2m}(R-\lambda n)\phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f}, \tag{2.11}$$

for all compactly supported function $\phi \in C_0^\infty(M)$. But weighted Poincaré inequalities are known to be equivalent to the manifold being f-non-parabolic, provided that $R - \lambda n$ is not identically zero; for more details see [14].

On the other hand, it is now well-known that every quasi-Einstein manifold satisfies

$$\begin{aligned}
\frac{1}{2}\Delta R - \frac{m+2}{2m}\langle\nabla f, \nabla R\rangle &= -\frac{m-1}{m}\left|Ric - \frac{R}{n}g\right|^2\\
&\quad - \frac{n+m-1}{mn}(R-n\lambda)\left(R - \frac{n(n-1)}{n+m-1}\lambda\right), \qquad (2.12)
\end{aligned}$$

(cf. [26] and [7]). Therefore, if $R - \lambda n$ is identically zero, we may use (2.12) to conclude that M^n is Einstein. This finishes the proof of the theorem. □

2.4. Proof of Theorem 4

Proof. The first part of the proof looks like that one of the previous theorem. For $a = m - 1$, which is positive, we immediately have

$$\Delta_f u^{-a} = -\frac{m-1}{m}(R - \lambda n)u^{-a}.$$

On the other hand, it is not hard to check that our assumption on the scalar curvature yields $R > \lambda n$. Therefore, we may use [14] to deduce

$$\int_M \Big[\frac{m-1}{m}(R - \lambda n)\Big]\phi^2 e^{-f} \leq \int_M |\nabla\phi|^2 e^{-f}, \tag{2.13}$$

for all $\phi \in C_0^\infty(M)$. Moreover, using once more our assumption on the scalar curvature we achieve

$$\frac{m-1}{m}(R - \lambda n) \geq -\lambda,$$

and hence, we invoke (2.13) (see also, for instance, [14]) to get

$$\lambda_1(\Delta_f) \geq \frac{m-1}{m}(R - \lambda n) \geq -\lambda. \tag{2.14}$$

We already know from Theorem 2 that there is at least one f-non-parabolic on M^n. From now on we argue by contradiction, assuming that there are at least two f-non-parabolic ends on M^n. Then, there is a bounded nonconstant f-harmonic function h on M^n such that

$$\int_M |\nabla h|^2 e^{-f} < \infty.$$

Next, from the Böchner formula we have

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\nabla^2 h|^2 + Ric_f(\nabla h, \nabla h) + \langle\nabla h, \nabla\Delta_f h\rangle. \tag{2.15}$$

Moreover, the well known *Kato's inequality* states

$$|\nabla^2 h|^2 \geq |\nabla|\nabla h||^2.$$

This jointly with (2.15) and using that h is f-harmonic we infer

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq |\nabla|\nabla h||^2 + Ric_f(\nabla h, \nabla h).$$

Hence, it follows from the fundamental equation (1.2) that

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq |\nabla|\nabla h||^2 + \lambda|\nabla h|^2. \tag{2.16}$$

At the same time, one easily verifies that

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\nabla|\nabla h||^2 + |\nabla h|\big(\Delta_f|\nabla h|\big)$$

and hence, we may use this data into (2.16) to deduce

$$\Delta_f|\nabla h| \geq \lambda|\nabla h|.$$

Therefore, it suffices to repeat the same arguments developed in [13] to deduce

$$\lambda_1(\Delta_f) \leq -\lambda. \tag{2.17}$$

Consequently, it follows from (2.17) and (2.14) that

$$R = \lambda\left(n - \frac{m}{m-1}\right).$$

In order to obtain a contradiction it therefore suffices to use (2.12). This contradiction argument finishes the proof of Theorem 4. □

2.5. Proof of Theorem 5

Proof. To begin with, we invoke Theorem 2 to deduce that M^n is f-non-parabolic. So, we can apply Varapoulos [25] to infer

$$\int_1^\infty A_f^{-1}(r)dr < \infty, \tag{2.18}$$

where $A_f(r) = Area_f(\partial B_p(r))$. Therefore, for $\overline{r} > 1$, we use Cauchy-Schwarz inequality to arrive at

$$\begin{aligned} \overline{r} - 1 &= \int_1^{\overline{r}} A_f^{-\frac{1}{2}}(r)A_f^{\frac{1}{2}}(r)dr \\ &\leq \left(\int_1^{\overline{r}} A_f^{-1}(r)\right)^{\frac{1}{2}}\left(\int_1^{\overline{r}} A_f(r)\right)^{\frac{1}{2}} \\ &\leq C(Vol_f(B_p(\overline{r})))^{\frac{1}{2}}. \end{aligned} \tag{2.19}$$

Hence, for all $\overline{r} \geq 2$, we obtain

$$Vol_f(B_p(\overline{r}))^{\frac{1}{2}} \geq c\overline{r}, \tag{2.20}$$

where c is a constant. This gives the requested result. □

2.6. Proof of Theorem 6

Proof. We start combining the trace of the fundamental equation (1.2) with (1.3) to obtain

$$R + \frac{m-1}{m}|\nabla f|^2 + (m-n)\lambda = m\mu e^{\frac{2f}{m}}. \tag{2.21}$$

Remember that for $\lambda < 0$ and $\mu = 0$ we have

$$R \geq \lambda n,$$

which allows us to deduce

$$|\nabla f| \leq \frac{m}{\sqrt{m-1}}\sqrt{-\lambda}. \tag{2.22}$$

From here it follows that

$$-f(x) \leq \frac{m}{\sqrt{m-1}}\sqrt{-\lambda}r(x) + c_1. \tag{2.23}$$

On the other hand, by means of (1.3) it is easy to show that

$$\Delta e^{-f} = (-\Delta f + |\nabla f|^2)e^{-f} = -\lambda m e^{-f}. \tag{2.24}$$

Upon integrating (2.24) over $B_p(r)$ we use (2.22) to achieve

$$\begin{aligned} -\lambda m \int_{B_p(r)} e^{-f} d\sigma &= \int_{B_p(r)} \Delta e^{-f} d\sigma \\ &= \int_{\partial B_p(r)} \frac{\partial}{\partial \eta}(e^{-f}) ds \\ &\leq \frac{m\sqrt{-\lambda}}{\sqrt{m-1}} \int_{\partial B_p(r)} e^{-f} ds. \end{aligned} \tag{2.25}$$

Let us set

$$\psi(r) := Vol_f(B_p(r)) = \int_{B_p(r)} e^{-f} d\sigma. \tag{2.26}$$

Thus, we may invoke (2.25) to get

$$\psi'(r) \geq \sqrt{-\lambda(m-1)}\psi(r). \tag{2.27}$$

Then, by integrating this inequality from 1 to r we deduce

$$\psi(r) = \int_{B_p(r)} e^{-f} d\sigma \geq c e^{\sqrt{-\lambda(m-1)}r}.$$

This immediately yields

$$\begin{aligned} Ce^{m\sqrt{\frac{-\lambda}{m-1}}r} Vol(B_p(r)) &\geq \sup_{B_p(r)} e^{-f} \int_{B_p(r)} d\sigma \\ &\geq \int_{B_p(r)} e^{-f} d\sigma \\ &\geq c e^{\sqrt{-\lambda(m-1)}r}, \end{aligned} \tag{2.28}$$

where we have used (2.23). Finally, (2.28) can be written succinctly as

$$Vol(B_p(r)) \geq c e^{-\sqrt{\frac{-\lambda}{m-1}}r}, \tag{2.29}$$

which finishes the proof of the theorem. □

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