Gronwall's conjecture for 3-webs with infinitesimal symmetries

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We study non-flat planar 3-webs with infinitesimal symmetries. Using multi-dimensional Schwarzian derivative we give a criterion for linearization of such webs and present a projective classification thereof. Using this classification we show that the Gronwall conjecture is true for 3-webs admitting infinitesimal symmetries.

1. Introduction

A planar 3-web $\mathcal{W}_3$ is formed by three foliations in the plane. A 3-web $\mathcal{L}_3$ is linear if the leaves of all its foliations are rectilinear. A 3-web $\mathcal{W}_3$ is locally linearizable if there is a local diffeomorphism mapping $\mathcal{W}_3$ into some linear web $\mathcal{L}_3$. If this diffeomorphism can be chosen so that the lines of each foliation of $\mathcal{L}_3$ are parallel, then the web is called flat. Any projective transformation maps a linear 3-web into a linear 3-web therefore a linearization, if it exists, is not unique.

Graf and Sauer (see [9]) provided the following very elegant complete description of linear flat 3-webs. Locally, to each of three foliations there corresponds a curve arc in the dual plane, thus we have 3 arcs. A linear web $\mathcal{L}_3$ is flat if and only if these 3 arcs belong to some (possibly singular) cubic. Since any two flat 3-webs are locally diffeomorphic by definition, but not any two cubics are projectively equivalent, there are diffeomorphic but projectively non-equivalent linear 3-webs. For example, the 3-web formed by tangent lines to the curve dual to a smooth cubic is not projectively equivalent to the web formed by three families of parallel lines, whose corresponding cubic is decomposed in three lines.

More than hundred years ago Gronwall conjectured in [11] that any two local linearizations φ, ψ of a non-flat 3-web are projectively equivalent, i.e. there exists a projective transformation $g \in PGL(3, \mathbb{C})$ such that $\psi = g \circ \varphi$. In spite of many efforts (see, for example, [2–6, 8, 10, 12, 19, 20, 22, 23]) the conjecture is still open. Note that for a d-web with $d \geq 4$, there exists at most one projective equivalence class of linearizations, (see [2]). The reason

is that a 4-web determines a unique projective connection such that the web leaves are geodesic and the linearizability of this web amounts to flatness of the connection (see [2, 13]) for the classical approach and [1, 12, 17] for a modern exposition).

Bol proved in [4] that there are at most 17 essentially different (i.e. projectively non-equivalent) linearizations of a non-flat 3-web. Borůvka in [6] improved this bound to 16. In the short note [22], Vaona gave a sketchy proof that it is 11. Grifone, Muzsnay and Saab studied the linearizability of 3-webs in [10] and proved that the bound is 15.

There are also results on the Gronwall conjecture with some restrictions on the web and on the map. For instance, Bol [5] demonstrated the following Theorems:

1) a local diffeomorphism, mapping a pencil of lines into a pencil of lines and preserving linearity of some two other families, is projective,

2) a local diffeomorphism, mapping a linear 3-web, whose two families of lines are tangents of some conic, to some 3-web of the same type, is projective,

3) a local diffeomorphism, mapping a linear 3-web, whose two families of lines are tangents of some conic and the 3d family is arbitrary, to some linear 3-web so that the image of the 3d family is a pencil of lines, is projective.

Wang in [23] proved that a 3-web admitting two projectively non-equivalent linearizations is flat, provided that the Chern curvature of the web vanishes to order three at some point.

In a short note [19], Smirnov rediscovered the Bol result 1) mentioned above, and suggested a line of attack on the general case, claiming that it is reducible to a web with a pencil of lines.[1] Later Smirnov published a more detailed paper [20] proving the Bol result 1), but never returned to his project outlined in [19].

Actually, we will need only the fact that there are finitely many distinct linearizations.

This paper is motivated by the following result of Cartan [7]: the symmetry algebra of a 3-web is either three-dimensional, and then the web is flat; or one-dimensional; or trivial. Thus the uniqueness of linearization does not hold true for the 3-webs admitting symmetry algebra of largest possible dimensions and it is quite natural to look into the class of 3-webs with one-dimensional symmetry algebra. (A similar strategy was successfully applied in [14] to studying planar webs of maximal rank.)

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[1]The reviewers of both Mathematical Reviews and Zentralblatt MATH erroneously accepted this very non-precise program as a proof of the conjecture.

To distinguish between essentially different linearizations one needs projective differential invariants of maps, namely multi-dimensional Schwarzian derivatives (see [18]). Unlike the one-dimensional case, a complete invariant has more components than the map, thus leading to some differential relations between the Schwarzian components. These relations, which one can consider as differential syzygies, were used by Hénaut (see [12]) to characterize the linearizability of planar webs in terms of solutions to these syzygies. In fact, these equations, written for a special normalization of the vector fields tangent to the web leaves, were also the main tool of Vaona in [22]. The principal difficulty in analyzing the linearizability of 3-webs lies in the fact that a 3-web does not determine uniquely a projective connection, like in the case of 4-webs. Therefore the "differential syzygies", manifesting the flatness of a "rectifying" projective connection, give a weakly under-determined system of nonlinear PDEs. Compatibility conditions for this system are so hopelessly involved that are intractable even with the help of modern symbolic computation software.

If a linearizable non-flat 3-web has an infinitesimal symmetry, then the Schwarzian derivative of the linearization is also invariant with respect to this symmetry. Thus we reduce the difficult problem of studying a weakly over-determined system of non-linear PDEs to a simpler problem of compatibility of non-linear ODEs. This compatibility condition amounts to vanishing of a resultant of two polynomials, one being of degree 5 and the other of degree 6.

We also give a geometric characterization of one-dimensional symmetry of a non-flat 3-web.

Since there is a finite number of essentially different linearizations of a non-flat 3-web, any infinitesimal symmetry of such web is necessarily projective. This observation reduces the classification problem to the Jordan classification of 3×3 matrices.

Using the obtained classification, we prove the Gronwall conjecture for linear 3-webs with infinitesimal symmetries. The proof is based on the fact that the conjecture is true for planar d-webs with $d > 3$. The key observations are the following:

- any of the classified 3-webs $\mathcal{W}$ admits an analytic extension to a global (singular) linear d-web $\widetilde{\mathcal{W}}$ on the projective plane with $d \geq 4$, the case $d = \infty$ (of countably many 1-parameter families of lines) being possible,

- if a diffeomorphism preserves linearity of such 3-web $\mathcal{W}$ then it preserves linearity of some 4-subweb of the d-web $\widetilde{\mathcal{W}}$.

This paper treats local properties of 3-webs, all the objects (webs, maps, infinitesimal symmetries) are defined in some open set Ω.

2. Non-projective morphisms of linear webs

Definition 1. A morphism of a linear d-web is a local diffeomorphism that maps this d-web to a linear d-web.

Let us recall some basic facts about projective differential invariants (for the detail see, for example, [18]). Two local diffeomorphisms $\varphi, \widetilde{\varphi} : \Omega \subset \mathbb{C}^2 \to \mathbb{C}^2$ are projectively equivalent, i.e. there exists a projective transformation $g \in PGL(3, \mathbb{C})$ such that $\widetilde{\varphi} = g \circ \varphi$ if and only if their Schwarzian derivatives

$$S_{ij}^k = \widetilde{S_{ij}^k}$$

coincide, where the components of the Schwarzian of the map $(u^1, u^2) \mapsto (\varphi^1, \varphi^2)$ are defined as follows:

$$
(1) \qquad S_{ij}^k(u) = \sum_{l=1}^{2} \frac{\partial^2 \varphi^l}{\partial u^i \partial u^j} \frac{\partial u^k}{\partial \varphi^l} - \frac{1}{3} \delta_i^k \frac{\partial}{\partial u^j} \ln \left(\det \frac{\partial \varphi}{\partial u} \right)
$$
$$
- \frac{1}{3} \delta_j^k \frac{\partial}{\partial u^i} \ln \left(\det \frac{\partial \varphi}{\partial u} \right).
$$

They satisfy the following linear relations

$$S_{ij}^k(u) = S_{ji}^k(u), \quad \sum_{l=1}^{2} S_{il}^l(u) = 0.$$

To avoid working with a lot of indices let us choose the following notation for linearly independent components of Schwarzian derivative:

$$
(2) \qquad K = S_{11}^1, \quad L = S_{22}^1, \quad M = S_{11}^2, \quad N = S_{22}^2,
$$

and $x = u^1$, $y = u^2$ for the local coordinates.

Four functions K, L, M, N are the above components of the Schwarzian derivative of some map $\varphi : \Omega \subset \mathbb{C}^2 \to \mathbb{C}^2$ if and only if they satisfy the following non-linear PDEs:

(3)
$$2K_{xy} + M_{yy} + N_{xx} - 6KK_y + 2ML_x$$
$$+ LM_x + 3NM_y - 3KN_x + 3MN_y = 0,$$

$$K_{yy} + L_{xx} + 2N_{xy} + 3LK_x - 3NK_y$$
$$+ 3KL_x + ML_y + 2LM_y - 6NN_x = 0.$$

As "differential syzygies" of projective invariants, these equations were written explicitly by Tresse in [21]. In a slightly different but related context they appeared a bit earlier in the Liouville studies of projective connections [13].

Let $\mathcal{W}_3$ be a 3-web in some domain $\Omega \subset \mathbb{C}^2$ and $V_i = \partial_x + \lambda_i \partial_y$, $i = 1, 2, 3$ vector fields tangent to the web leaves. The web is linear if and only if

(4)
$$V_i(\lambda_i) = 0, \quad i = 1, 2, 3.$$

Hénaut in [12] proved that $\mathcal{W}_3$ is linearizable by a diffeomorphism $\varphi = (\varphi^1, \varphi^2)$ (i.e. φ maps $\mathcal{W}_3$ into some linear 3-web $\mathcal{L}_3$) if and only if the components of the Schwarzian derivative of φ verify

(5)
$$L\lambda_i^3 - 3N\lambda_i^2 + 3K\lambda_i - M = V_i(\lambda_i), \quad i = 1, 2, 3.$$

Since φ is projective if and only if $K = L = M = N = 0$, we have the following proposition.

Proposition 1. *A linear 3-web admits a non-projective morphism if and only if there exists a non-vanishing solution to system (3) such that for the vector fields $V_i = \partial_x + \lambda_i \partial_y$, $i = 1, 2, 3$, tangent to the web leaves, holds true $L\lambda_i^3 - 3N\lambda_i^2 + 3K\lambda_i - M = 0$.*

Remark. One can choose the local coordinates to normalize the vector fields to ∂_x, ∂_y, $\partial_x + \lambda \partial_y$. Then equations (5) amount to $L = M = 0$ and $3\lambda(K - \lambda N) = \lambda_x + \lambda \lambda_y$ (for the detail see [12]).

3. Linearization of 3-webs with infinitesimal symmetries

Definition 2. An infinitesimal symmetry of a d-web is a vector field whose local flow preserves the web.

All the infinitesimal symmetries of a web form a Lie algebra with respect to the Lie bracket. As an example of his general theory of continuous transformations, Cartan considered infinitesimal symmetries of 3-webs, formed by coordinate lines and integral curves of one ODE in the plane, and gave criteria for existing of nontrivial symmetry algebra. (See the original paper [7] of Cartan. For a modern treatment the reader can look up in [8].) Following Cartan, we choose local coordinates x, y so that the symmetry takes the form

$$(6) \qquad\qquad Y = \partial_x + \partial_y,$$

and the web leaves are tangent to the following vector fields

$$(7) \qquad\qquad v_1 = \partial_x, \quad v_2 = \partial_y, \quad v_3 = \partial_x + S(t)\partial_y,$$

where $t = x - y$ is an invariant of the group action. Then equations (5) give $M = L = 0$ and

$$(8) \qquad\qquad 3S(K - SN) = (1 - S)S',$$

where $S' = \frac{dS}{dt}$ (see the Remark in the end of Section 2). Since there is only a finite number of projectively non-equivalent morphisms of a non-flat linear 3-web, one concludes immediately that K, N are invariant with respect to the symmetry, i.e. they are functions of t. Now equations (3) take the form

$$(9) \qquad \begin{aligned} -2K'' + N'' + 6KK' - 3KN' &= 0, \\ K'' - 2N'' + 3NK' - 6NN' &= 0. \end{aligned}$$

This system has the following two integrals:

$$(10) \qquad \begin{aligned} I_1 &= N' - K' + K^2 + N^2 - NK, \\ I_2 &= 3(N'K - K'N) - 2(K^3 + N^3) + 3(K^2N + N^2K). \end{aligned}$$

Using the main result of [12], we conclude that the web is linearizable if and only if there is at least one solution to equations (9) subject to relation (8). Due to the nonlinearity, the direct approach to finding the compatibility condition for the innocent looking overdetermined system (8, 9) leads to very involved expressions. We simplify them by symmetrizing the system using the natural action of the symmetric group S_3, which permutes the vector fields (7). Transposing the vector fields v_1 and v_2 induces the transposition

of x and y and the following action on our normalization of the web:

$$x \mapsto y, \quad y \mapsto x, \quad t \mapsto -t,$$
$$S \mapsto \tfrac{1}{S}, \quad S' \mapsto \tfrac{S'}{S^2},$$
$$K \mapsto N, \quad N \mapsto K.$$

Similarly, transposing y, which is an integral of v_1, with z, the integral of v_3, defined by $dz = \frac{S dx}{S-1} - \frac{dy}{S-1}$ to satisfy $Y(z) = 1$, one gets, using the formula for the Schwarzian derivative of a composition (see [18]), the following substitutions:

$$x \mapsto x, \quad y \mapsto z, \quad dt \mapsto \tfrac{dt}{1-S},$$
$$S \mapsto \tfrac{S}{S-1}, \quad S' \mapsto \tfrac{S'}{S-1},$$
$$K \mapsto K - 2NS + \tfrac{2}{3}S', \quad N \mapsto -N(S-1) + \tfrac{1}{3}S'.$$

The above two transpositions generate the whole group action of S_3 that permutes the fist integrals x, y and z of the foliations.

For a non-flat web holds true $\frac{d^2}{dt^2}(\log S) \neq 0$, therefore one can take the following symmetrization of S as an independent variable:

$$(11) \qquad\qquad X = \frac{1}{3}\frac{(S^2 - S + 1)^3}{(S-1)^2 S^2},$$

which is nothing else as the symmetrization of the cross-ratio of the directions of Y, v_1, v_2, v_3. Symmetrizing K and substituting S' from equation (8) we define

$$(12) \qquad\qquad U := -\frac{K}{S-1} + \frac{SN}{S-1},$$

similarly, applying the same procedure to $\frac{S^2 K}{S+1}$ we define another invariant

$$(13) \qquad\qquad V := \frac{1}{3}\frac{(S^2 - S + 1)^2 (N - K)}{(S-1)(S-2)(2S-1)(S+1)}.$$

The first integrals (10) now take the forms

$$I_1 = 9U(4X - 9)\frac{dV}{dX} + \frac{9(4X - 9)}{X}V^2 - \frac{6U(X - 9)}{X}V + U^2,$$

$$I_2 = 27U^2(4X - 9)\frac{dV}{dX} + \frac{27(4X - 9)^2}{X^2}V^3 - \frac{27U(4X - 9)}{X}V^2$$
$$- \frac{9U}{X}\left[2XU - 18U + (12X^2 - 27X)\frac{dU}{dX}\right]V + 2U^3.$$

Note that the the system (8, 9) admits the symmetry algebra $\{\partial_t, -t\partial_t + K\partial_K + N\partial_N\}$. We have already used the first symmetry, choosing X as a new variable. Now let us use the second one and set $V = ZU$, $U' = FU$. Now $'$ means derivative by X. Substituting these expressions into the first integrals, taking total derivatives of them by X and equating the results to zero, we get the following two equations:

$$9X^2(4X-9)Z'' + 18X(4X-9)ZZ'$$
$$+ 3X[9X(4X-9)F + 10X + 18]Z'$$
$$+ [81 + 18X(4X-9)F]Z^2$$
$$+ [18X^2(4X-9)F^2 + 12X(9+2X)F$$
(14)
$$- 54 + 9X^2(4X-9)F']Z + 2X^2F = 0,$$

$$9X^3(4X-9)Z'' + 27X(4X-9)^2Z^2Z' - 18X^2(4X-9)ZZ'$$
$$+ 3X^2[9X(4X-9)F + 10X + 18]Z'$$
$$+ 27(4X-9)[6 + X(4X-9)F]Z^3 - 27X[3 + X(4X-9)F]Z^2$$
$$- 18X[3 + X(X-9)F]Z + 2X^3F = 0.$$

For a non-flat web one has $F \neq 0$ since due to (8) holds true $U = \frac{S'}{3S}$. Then a given non-flat 3-web has so many distinct linearizations as many solutions $Z(X)$ has the system (14). Equations (14) give Z' and Z'' as functions of Z, X, F, F'. In particular,

(15) $\quad Z' = \dfrac{-9(4X-9)[6 + X(4X-9)F]Z^2 + 3X[18 + 5X(4X-9)F]Z}{3X(4X-9)[3(4X-9)Z - 4X]}$
$$+ \dfrac{X^2[6X(4X-9)F^2 + (14X-18)F + 3X(4X-9)F']}{3X(4X-9)[3(4X-9)Z - 4X]}.$$

Expressions for Z' and Z'' have the factor $3(4X-9)Z - 4X$ in the denominators, but the equation $(12X - 27)Z = 4X$ is not compatible with (14).

The compatibility condition $\frac{dZ'}{dX} = Z''$ has the form $E(Z) = 0$, where E is a polynomal in Z of degree 5

$$E(Z) = E_5Z^5 + E_4Z^4 + E_3Z^3 + E_2Z^2 + E_1Z + E_0,$$

whose coefficients E_i are given in Appendix.

Equating the total derivative of $E(Z)$ to zero, one obtains a polynomial equation of degree 6 in Z

$$H(Z) = H_6Z^6 + H_5Z^5 + H_4Z^4 + H_3Z^3 + H_2Z^2 + H_1Z + H_0 = 0,$$

where the coefficients H_i are given in Appendix. Let us define

$$R(X, F, F', F'', F''')$$

as the resultant of polynomials $E(Z)$ and $H(Z)$, the functions ρ, ω by

$$\rho(X, F, F') := X(4X - 9)^2 F' + 2X(4X - 9)^2 F^2$$
$$+ 6(X - 1)(4X - 9)F - 8,$$
$$\omega(X, F, F', F'', F''') := \frac{R(X, F, F', F'', F''')}{(4X - 9)^{20} X^{26} \rho^8(X, F, F')}.$$

Theorem 1. *A non-flat 3-web admitting infinitesimal symmetry is linearizable if and only if the invariant $F(X)$ satisfies $\omega(X, F, F', F'', F''') = 0$.*

Proof. Our 3-web is linearizable if and only if the equations $E(Z) = 0$, $H(Z) = 0$ have a common solution $Z(X)$, i.e. if and only if the resultant of $E(Z)$ and $H(Z)$ vanishes. The resultant factors out as $(4X - 9)^{20} X^{26} \rho^8 \omega$. If $\rho(X, F, F') = 0$ then one finds F' from this equation and computes F'' and F'''. With this expressions for the derivatives of F, the polynomials $E(Z)$ and $H(Z)$ have the common factor $\sigma(X, Z) = [3(4X - 9)Z - 4X]^3$. As was mentioned above, the equation $\sigma(X, Z) = 0$ is not compatible with system (14) for Z. Now the resultant of $\frac{E(Z)}{\sigma(X,Z)}$ and $\frac{H(Z)}{\sigma(X,Z)}$ is the polynomial in F:

$$\tau(F) := 36X^6(4X - 9)^6[(84X - 189)F - 20][7X^2(4X - 9)^3 F^3$$
$$+ 21X(51X - 4)(4X - 9)^2 F^2 - 12(596X - 21)(4X - 9)F + 11856].$$

A direct computation shows that $\tau(F) = 0$ is not compatible with F' defined by $\rho(X, F, F') = 0$. $\qquad\square$

Remark 1. The computation in the proof of the above theorem was made with the help of symbolic computation software, namely Maple. The polynomial ω is quintic in the highest derivative F''', the coefficient of $(F''')^5$ being equal to $X^{11}(4X - 9)^8$. The expressions for the other coefficients are hopelessly involved to be presented here in their generality.

Remark 2. One is tempted to apply the Euclid algorithm to $H(Z)$ and $E(Z)$ to prove that there is at most one linearization. It works only for the first step, i.e. one can check by Maple that there is at most 4 linearizations, but the next step is out of Maple's reach.

Remark 3. Normalization (7) is defined by the web up to permutations of the first integrals x, y, z of foliations and up to scaling and translating of the parameter t. These transformations generate a group G acting on parameterized curves $(t, S(t))$, differential expressions X, F being invariants of this action. (Note that $U = \frac{S'}{3S}$ and therefore F do not depend on the linearizing diffeomorphism.) Then the parameterized curve $(X, F(X))$ is an analog of the signature curve considered by Olver (see [16]) for the action of the Euclidean group in the plane.

Finally, let us give a geometric characterization of one-dimensional symmetry of a non-flat 3-web (for a analytical criterion of existence in terms of differential invariants see, for instance, [8]).

Theorem 2. *A non-flat 3-web formed by integral curves of three vector fields v_1, v_2, v_3 has an infinitesimal symmetry Y if and only if the three 3-webs generated similarly by the 3-tuples of vector fields $\{Y, v_2, v_3\}$, $\{v_1, Y, v_3\}$, and $\{v_1, v_2, Y\}$ are flat.*

Proof. If a non-flat 3-web has a symmetry Y then, using the Cartan normalization (6, 7), one easily sees that the 3-web generated by $\{v_1, v_2, Y\}$ is flat. This proves the flatness of the three 3-webs of $\{Y, v_2, v_3\}$, $\{v_1, Y, v_3\}$, and $\{v_1, v_2, Y\}$.

To prove the converse claim note that the direction of the vector field Y cannot be tangent to the web leaves; otherwise one normalizes the vector fields to take the form $Y = v_1 = \partial_x$, $v_2 = \mu(y)\partial_x + \partial_y$, $v_3 = \nu(y)\partial_x + \partial_y$ and our 3-web is obviously flat. Let us again choose the coordinates so that the vector field Y takes the form (6) and v_1, v_2 are as in (7). This is possible due to the flatness of the web $\{v_1, v_2, Y\}$. Now the 3-web $\{v_1, v_2, v_3 = \partial_x + \lambda\partial_y\}$ can be defined by three 1-forms as follows

$$\omega_1 = dy, \qquad \omega_2 = dx, \qquad \omega_3 = dy - \lambda dx.$$

Further, the flatness of the web $\{v_1, Y, v_3\}$ implies

$$\lambda(\lambda - 1)(\lambda_{xx} + \lambda_{xy}) = (2\lambda - 1)\lambda_x(\lambda_x + \lambda_y).$$

Similarly, the flatness of $\{Y, v_2, v_3\}$ gives

$$(\lambda - 1)(\lambda_{xy} + \lambda_{yy}) = \lambda_y(\lambda_x + \lambda_y).$$

Computing the compatibility conditions for the above two equations for λ, we arrive at

$$(\lambda_x + \lambda_y)[\lambda_x^2(1 - 2\lambda) - \lambda\lambda_x\lambda_y + \lambda_{xx}\lambda(\lambda - 1)] = 0.$$

If the factor in the square brackets vanishes then one finds all the second order derivatives of λ, in particular $\lambda_{xy} = \frac{\lambda_x\lambda_y}{\lambda}$, which implies that the web of $\{v_1, v_2, v_3\}$ is flat. Thus this factor is not zero and therefore $\lambda_x + \lambda_y = 0$. That means that the foliation by the integral curves of the equation $\omega_3 = 0$ is also invariant by Y. □

4. Linear 3-webs with one-dimensional infinitesimal symmetry

In this section, we give a classification of non-flat 3-webs with infinitesimal symmetries up to projective transformations. A symmetry algebra is called projective if it generates a subgroup of the projective group.

Lemma 1. *If a linear non-flat 3-web admits a one-dimensional infinitesimal symmetry algebra then this symmetry algebra is projective.*

Proof. Let g^t be the local flow of the symmetry. For each t the map g^t is a morphism of our web. Since there is only a finite number of projectively non-equivalent morphisms for a non-flat linear 3-web (see [4]), one has $g^t \in PGL(3, \mathbb{C})$. □

Let $\mathcal{L}_3$ be a linear 3-web with a one-dimensional infinitesimal symmetry. Its ith foliation is a one-parameter family of lines in the form

(16) $$y = p_i(t)x + q_i(t),$$

or the pencil $x = cst$. Thus we obtain parameterized curves $(p_i(t), q_i(t))$ or a segment of the line at infinity l_∞ in the dual plane. In what follows we call them dual focal curves and denote them by Φ_i. Recall that a 3-web $\mathcal{L}_3$ is flat if and only if these three dual focal curves are arcs of a same cubic curve (Graf and Sauer Theorem [9]). Our projective symmetry acts also in the dual space. Obviously, the dual focal curves are (locally) invariant.

In the dual plane, the action of the projective group is generated by the following operators (see, for instance, [15]):

(17)
$$T_1 = \partial_p, \quad T_2 = \partial_q, \quad A_1 = p\partial_q, \quad A_2 = q\partial_p, \quad D_1 = p\partial_p,$$
$$D_2 = q\partial_q, \quad \Pi_1 = p^2\partial_p + pq\partial_q, \quad \Pi_2 = pq\partial_p + q^2\partial_q,$$

where p, q are affine coordinates of the line $y = px + q$.

Now let us classify one-dimensional subalgebras of the Lie algebra $pgl(3, \mathbb{C})$. This classification is provided by the Jordan normal forms of 3×3 matrices with zero traces. One can also normalize one of the non-zero eigenvalues to 1. Thus one can take the following matrices as the orbit representatives:

$$\Xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Xi_{2,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Xi_{2,3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$\Xi_{3,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \Xi_{3,3} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -(a+b) \end{pmatrix},$$

where $a \neq 0$, $b \neq 0$, $a + b \neq 0$. The first subindex is the number of Jordan's blocks and the second (if any) is the matrix rank. Let $l = (P, Q, R)$ be homogeneous coordinates of a line $RY = PX + QZ$. If the matrix representation of the infinitesimal action is Ml^T, then in the affine coordinates the operator is

(18)
$$\{(M_{11} - M_{33})p + M_{12}q + M_{13} - M_{31}p^2 - M_{32}pq\}\partial_p$$
$$+ \{M_{21}p + (M_{22} - M_{33})q + M_{23} - M_{31}pq - M_{32}q^2\}\partial_q.$$

For example, the operator corresponding to the matrix Ξ_1 is $q\partial_p + \partial_q$. We take as a representative of the same orbit the operator $\xi_1 = \partial_p + p\partial_q$. The invariant curves for this operator are the ones parameterized by $(p, p^2/2 + \lambda)$ with some constant λ and the line at infinity l_∞. To the line l_∞ there corresponds the foliation by parallel lines $x = cst$. We can move these curves around by the stabilizer of the algebra $\{\xi_1\}$. This stabilizer is spanned by the following operators: $\{T_2, T_1 + A_1, D_1 + 2D_2\}$. As a line intersects an invariant curve at most at 2 points, the dual focal curves Φ_i either are pieces of pairwise distinct invariant curves or two Φ_i's are pieces of the same invariant curve. In the first case, using the stabilizer we can bring them to the

following parameterized forms:

$$(p_1, p_1^2/2), \quad (p_2, p_2^2/2 + 1), \quad (p_3, p_3^2/2 + \lambda), \quad \lambda \neq 0, 1,$$

and in the second case one sets $\lambda = 0$. The corresponding linear 3-webs are given by the following families of lines:

$$y = p_1 x + \frac{p_1^2}{2}, \quad y = p_2 x + \frac{p_2^2}{2} + 1, \quad y = p_3 x + \frac{p_3^2}{2} + \lambda.$$

Finally, if one of the dual focal curves is included in l_∞, we can parameterized the other two either as follows:

$$(p_1, p_1^2/2), \quad (p_2, p_2^2/2 + 1),$$

whose 3-web is

$$y = p_1 x + \frac{p_1^2}{2}, \quad y = p_2 x + \frac{p_2^2}{2} + 1, \quad x + p_3 = 0,$$

or

$$(p_1, p_1^2/2), \quad (p_2, p_2^2/2),$$

with the hexagonal web

$$(19) \qquad y = p_1 x + \frac{p_1^2}{2}, \quad y = p_2 x + \frac{p_2^2}{2}, \quad x + p_3 = 0.$$

To write the corresponding symmetry operator $\check{\xi}$ in "geometric" coordinates we note that the passage to the dual plane is a contact transform given by the following formulae:

$$p = \frac{dy}{dx}, \quad q = y - \frac{dy}{dx}x, \quad \frac{dq}{dp} = -x,$$
$$x = -\frac{dq}{dp}, \quad y = q - \frac{dq}{dp}p, \quad \frac{dy}{dx} = p.$$

Therefore one has $\check{\xi}_1 = -\partial_x + x\partial_y$

Proceeding with this scheme, one obtains the classification below, where the parameters λ, μ in the parametrization of dual focal curves are supposed to be pairwise distinct and different from the corresponding constants in the normalized curves.

Theorem 3. *Any non-flat linear 3-web with infinitesimal symmetry is projectively equivalent to one in Table 1. Moreover, these normal forms are projectively nonequivalent.*

Type	Operator	Dual Focal Curves
Ξ_1	$\partial_p + p\partial_q,$ $-\partial_x + x\partial_y$	1) $(p_1, p_1^2/2)$ $(p_2, p_2^2/2+1)$ $(p_3, p_3^2/2+\lambda)$ 2) $(p_1, p_1^2/2)$ $(p_2, p_2^2/2)$ $(p_3, p_3^2/2+1)$ 3) $(p_1, p_1^2/2)$ $(p_2, p_2^2/2+1)$ l_∞
$\Xi_{3,2}$	$p\partial_p - q\partial_q,$ $-2x\partial_x - y\partial_y$	1) $(p_1, 1/p_1)$ $(p_2, \lambda/p_2)$ $(p_3, \mu/p_3)$ 3) $(p_1, 1/p_1)$ $(p_2, \lambda/p_2)$ $(p_3, 0)$ 5) $(p_1, 1/p_1)$ $(p_2, \lambda/p_2)$ l_∞ 2) $(p_1, 1/p_1)$ $(p_2, 1/p_2)$ $(p_3, \lambda/p_3)$ 4) $(p_1, 1/p_1)$ $(p_2, 0)$ $(0, 1/p_3)$ 6) $(p_1, 1/p_1)$ $(p_2, 0)$ l_∞
$\Xi_{2,3}$	$p\partial_p + (p+q)\partial_q,$ $-\partial_x + y\partial_y$	1) $(p_1, p_1 \ln p_1)$ $(p_2, p_2[\ln p_2 + \lambda])$ $(p_3, p_3[\ln p_3 + \mu])$ 4) $(p_1, p_1 \ln p_1)$ $(p_2, p_2[\ln p_2 + \lambda])$ $(0, p_3)$ 7) $(p_1, p_1 \ln p_1)$ $(p_2, p_2 \ln p_2)$ l_∞ 2) $(p_1, p_1 \ln p_1)$ $(p_2, p_2 \ln p_2)$ $(p_3, p_3[\ln p_3 + \lambda])$ 5) $(p_1, p_1 \ln p_1)$ $(p_2, p_2 \ln p_2)$ $(0, p_3)$ 8) $(p_1, p_1 \ln p_1)$ $(0, p_2)$ l_∞ 3) $(p_1, p_1 \ln p_1)$ $(p_2, p_2 \ln p_2)$ $(p_3, p_3 \ln p_3)$ 6) $(p_1, p_1 \ln p_1)$ $(p_2, p_2[\ln p_2 + \lambda])$ l_∞
$\Xi_{3,3}$	$p\partial_p + \beta q\partial_q,$ $(\beta - 1)x\partial_x + y\partial_y$ $\beta \neq 0, 1, -1, 2, \frac{1}{2}$	1) (p_1, p_1^β) $(p_2, \lambda p_2^\beta)$ $(p_3, \mu p_3^\beta)$ 4) (p_1, p_1^β) $(p_2, \lambda p_2^\beta)$ $(p_3, 0)$ 7) (p_1, p_1^β) $(p_2, \lambda p_2^\beta)$ l_∞ 2) (p_1, p_1^β) (p_2, p_2^β) $(p_3, \lambda p_3^\beta)$ 5) (p_1, p_1^β) (p_2, p_2^β) $(p_3, 0)$ 8) (p_1, p_1^β) (p_2, p_2^β) l_∞ 3) (p_1, p_1^β) (p_2, p_2^β) (p_3, p_3^β) $\beta \neq 3, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$ 6) (p_1, p_1^β) $(p_2, 0)$ $(0, p_3)$ 9) (p_1, p_1^β) $(p_2, 0)$ l_∞

Table 1: Classification of non-flat symmetric 3-webs.

Proof. To the matrix $\Xi_{2,1}$ there corresponds the operator ∂_q whose stabilizer is spanned by $\{T_1, T_2, A_1, D_1, D_2\}$; the invariant curves are rectilinear (lines $p = cst$ and l_∞) and the web is flat.

The invariant curves of the operator $p\partial_p - q\partial_q$, corresponding to $\Xi_{3,2}$, are hyperbolas $pq = cst$ and the lines $p = 0$, $q = 0$, l_∞. The stabilizer is spanned by $\{D_1, D_2\}$. Here the web is flat if and only if either its two dual focal curves are arcs of the same hyperbola and the third is a segment of one of the lines, or all its dual focal curves are rectilinear. The forms in the table are clearly not projectively equivalent since the lines $p = 0$, $q = 0$ are tangent to the hyperbolas, whereas the line at infinity l_∞ cuts them at 2 points.

For the symmetry type $\Xi_{2,3}$ we choose the operator $p\partial_p + (p + q)\partial_q$ as a representative (one easily gets this operator applying formula (18), transposing p, q, and then scaling q). The invariant curves are $q/p - \ln p = cst$ and the lines $p = 0$, l_∞. Here $\ln$ is the multivalued analytical function, the above formula being the parametrization of the line family by points of the Riemann surface of $\ln$. The stabilizer is spanned by $\{A_1, D_1 + D_2\}$. A projective transform, mapping the line at infinity l_∞ in the line $p = 0$ does not preserve the symmetry hence the forms in the table are not projectively equivalent.

To the symmetry type $\Xi_{3,3}$ we get the operator $\alpha p\partial_p + \beta q\partial_q$, where $\alpha = 2a + b$ and $\beta = a + 2b$. Therefore $[\alpha : \beta] \neq [1 : 2], [2 : 1], [1 : -1]$. As the cases $\alpha = 0$ (or $\beta = 0$) and $\alpha = \beta$ give flat 3-webs, one can choose $\alpha = 1$ and $\beta \neq 0, 1, -1, 2, \frac{1}{2}$. The invariant curves are $q = \lambda p^\beta$ (here again p^β is multivalued for non-integer β) or lines $p = 0$, $q = 0$, l_∞. The stabilizer is spanned by $\{D_1, D_2\}$. The invariant curve is a cubic if and only if $\beta \in \{3, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}\}$, therefore these values are excluded for the case 3, where the dual focal curves are included in the same invariant curve. (Recall the Graf and Sauer Theorem.) Again, a projective transform, mapping the line at infinity l_∞ in the line $p = 0$ (or $q = 0$), does not preserve the symmetry hence the forms in the table are not projectively equivalent.   □

Remark. As a by-product of the above proof, we obtain normal forms up to projective equivalence for the triple of dual focal curves of flat linear 3-webs with projective infinitesimal symmetries, as well as the corresponding algebras of infinitesimal simmetries of this type. Namely, one can choose them as follows:

1) the cuspidal cubic $p = q^3$, one-dimensional algebra $\{3p\partial_p + q\partial_q\}$,

2) the conic $pq = 1$ and its secant l_∞, one-dimensional algebra $\{p\partial_p - q\partial_q\}$,

3) the conic $2q = p^2$ and its tangent line l_∞, two-dimensional algebra $\{\partial_p + p\partial_q, p\partial_p + 2q\partial_q\}$,

4) three non-concurrent lines $p = 0$, $q = 0$, l_∞, two-dimensional algebra $\{p\partial_p, q\partial_q\}$,

5) three concurrent lines $p = 0$, $p = 1$, $p = -1$, three-dimensional algebra $\{\partial_q, p\partial_q, q\partial_q\}$.

5. Gronwall's conjecture for 3-webs with infinitesimal symmetries

The following theorem implies that the Gronwall conjecture is true for 3-webs with infinitesimal symmetries.

Theorem 4. *The normal forms in Table 1 are pairwise not diffeomorphic. Moreover, any diffeomorphism, preserving a normal form, is projective.*

Proof. Let us fix some normal form. The three dual focal curves Φ_i are arcs of three curves invariant under the symmetry. Some of this invariant curves are permitted to coincide, but for each normal form a generic line in the dual space intersects this collection of invariant curves in more then 3 points. This is obvious for the symmetry types Ξ_1, $\Xi_{3,2}$ and $\Xi_{3,3}$ with a rational value of the parameter β, where the invariant curves are algebraic. For the the symmetry type $\Xi_{2,3}$, a generic line $q = kp + l$, $k, l = cst$ intersects the invariant curve $q/p - \ln p = cst$ in infinitely many points: substituting $p = e^z$, one gets this conclusion from the theorem on values of a holomorphic function in a neighborhood of essential singularity, the singularity being $z = \infty$. For the type $\Xi_{3,3}$ with an irrational β, one applies the same trick with the substitution $p = e^z$.

Therefore any of the classified 3-webs $\mathcal{W}$ admits an analytic extension to a global (singular) linear d-web $\widetilde{\mathcal{W}}$ on the projective plane with $d \geq 4$, the case $d = \infty$ (of countably many 1-parameter families of lines) being possible. The key observation: if a local diffeomorphism preserves linearity of our 3-web then it also preserves linearity of some 4-subweb of that d-web. Hence the diffeomorphism is a projective transform, since the Gronwall conjecture is true for 4-webs.

Below we present the scheme for proving the observation and work out all the details for the web in the normal form $\Xi_1, 3$, which will be denoted by $\mathcal{W}(\Xi_1, 3)$.

The leaves of each foliation of the web are locally parameterized. At a non-singular point, where the leaves are transverse, one can choose two of these parameters as local coordinates. Then the parameter of the third family is a function of the chosen ones, determined implicitly by some equation

relating all three parameters. This relation is called the web equation of the web. (Note that its form depends on the choice of parameters.) The web equation, relating the chosen parameters t_i along the dual focal curves (16), reads as

$$(20) \qquad\qquad \det \begin{pmatrix} 1 & p_1 & q_1 \\ 1 & p_2 & q_2 \\ 1 & p_3 & q_3 \end{pmatrix} = 0.$$

The geometric meaning of this equation is that the three lines (from different foliations) corresponding to the parameters t_i, satisfying the equation, are concurrent. If there are k values, say, of t_3 satisfying this equation for fixed t_1, t_2 and giving different points (p_3, q_3) in the dual plane by virtue of (16), then there are k lines of the third family passing through the intersection point of the two lines, one from the first family with t_1 and the other of the second one with t_2, and our local web can be extended to d-web with $d \geq k + 2$.

As the local parameters for $\mathcal{W}(\Xi_1, 3)$, let as choose p_1 and p_3, then the web equation reads as

$$p_2^2 - 2p_3p_2 + (2 + 2p_1p_3 - p_1^2) = 0.$$

The values of p_3 and p_1 fix the point (x, y), one of the solutions p_2 to this quadratic web equation gives the third line of the web. (Note that this equation defines, in fact, a 5-web $\widetilde{\mathcal{W}}(\Xi_1, 3)$: there are 2 solutions for p_2 and there are 2 lines of the first family passing through (x, y).)

Web equation locally defines some surface M_1 in 3-dimensional space with the parameters t_1, t_2, t_3 as local coordinates. Moving the lines of the web, the infinitesimal symmetry generates also an action on this surface M_1. Moreover, this action is the restriction of the local flow of some vector field in 3-dimensional space of parameters. For the web $\mathcal{W}(\Xi_1, 3)$ this vector field is $\partial_{p_1} + \partial_{p_2} + \partial_{p_3}$, for the other normal forms the operators are presented in the column "Symmetry" of Table 2.

This vector field has two first integrals. Being invariant under the symmetry, the web equation can be written in terms of these two invariants. This defines a curve in a two-dimensional space. Thus to each of the normal forms there corresponds a Riemann surface S_1 (one-dimensional analytic manifold). For the web $\mathcal{W}(\Xi_1, 3)$, choosing the invariants $w = p_2 - p_1, \;\; z = p_3 - p_1$ one gets the Riemann surface S_1:

$$w^2 - 2zw + 2 = 0$$

as the symmetry reduction of the web equation. The invariants for the other normal forms one finds in the column "Invariants" of Table 2. The equations, defining the corresponding Riemann surfaces are given in the column "Riemann surfaces" of Table 2.

Note that the parameters z, w on these Riemann surfaces are chosen so that:

1) each pair of lines l_1, l_2 of our linear 3-web fixes a value of z and, by duality, two points on the invariant curves,

2) S_1 becomes the Riemann surface of a multi-valued analytic function $\widetilde{w}(z)$, defined by some equation $f(z, w) = 0$ (see the column "Riemann surfaces" of Table 2 for the explicit formulas for f),

3) one of the values of $\widetilde{w}(z)$ gives the third line l_3, passing through the intersection point of l_1 and l_2,

4) this third line l_3 defines the third point on one of the invariant curves.

Suppose a local diffeomorphism maps the chosen linear 3-web from the list to some (possibly the same) linear 3-web from the list. This fixes a pair of symmetry types (Ξ_*, Ξ_*), where the first element is the symmetry type of the first web and the second is the symmetry type of the second web.

This diffeomorphism induces a map between the corresponding surfaces M_1, M_2, defined by the web equations of the webs. On the surfaces M_1 and M_2 our webs are represented as the 3-webs cut by the planes $t_i = cst$, $T_j = cst$, where T_j are parameters along the dual focal curves of the second web. Mapping the 3-web on M_1 to the 3-web on M_2, the diffeomorphism takes the form $T_{\pi(i)} = g_i(t_i)$, where π is some permutation of 3 indices. Moreover, the diffeomorphism relates the corresponding symmetry operators. This condition gives ODEs of the first order for g_i. Resolving these ODEs and taking into account that the symmetry operator on the first surface M_1 is mapped to some multiple of the symmetry operator on the second surface M_2, one finds g_i up to 4 constants.

Consider a diffeomorphism of the web $\mathcal{W}(\Xi_1, 3)$, say, to the web $\mathcal{W}(\Xi_1, 1)$ whose web equation and symmetry are

$$(P_2 - P_1)P_3^2 + (P_1^2 - P_2^2 - 2)P_3 + [P_1 P_2^2 - P_1^2 P_2 + 2\lambda(P_2 - P_1) + 2P_1] = 0$$

and $\partial_{P_1} + \partial_{P_2} + \partial_{P_3}$ respectively. Permuting the indices, if necessary, we conclude that the diffeomorphism takes the form $P_1 = g_1(p_1)$, $P_2 = g_3(p_3)$. Since the symmetry is preserved, we have

$$g_1(p_1) = kp_1 + c_1, \quad g_3(p_3) = kp_3 + c_3,$$

where k, c_1, c_3 are some constants. Moreover, under this diffeomorphism holds true $P_3 = kp_2 + c_2$ for some constant c_2. For the other pairs of symmetry types, the formulas for g_i are presented below.

Further, our diffeomorphism between the surfaces M_1 and M_2 maps the orbits of the symmetry on M_1 to the orbits of the symmetry on M_2. Therefore the diffeomorphism is lowerable to a local biholomorphism of the corresponding Riemann surfaces S_1 defined by $f(z, w) = 0$ and S_2 defined by $F(Z, W) = 0$. In particular, this biholomorphism has the form $Z = a(z)$, $W = b(w)$. Using the explicit formulas for the functions g_i, one easily checks that $a(z), b(w)$ either are analytic for any $z, w \in \mathbb{C}$ or have a branch point at zero. Thus $F(a(z), b(w(z))) \equiv 0$ on some neighborhood of $(z_0, w_0) \in S_1$.

The corresponding symmetry reduction of the web equation for $\mathcal{W}(\Xi_1, 1)$ is the Riemann surface S_2:

$$ZW^2 - (Z^2 + 2)W + 2\lambda Z = 0,$$

where $Z = P_2 - P_1$, $W = P_3 - P_1$. Our local diffeomorphism induces the following local biholomorphism

$$(21) \qquad\qquad Z = kz + (c_3 - c_1), \quad W = kw + (c_2 - c_1).$$

Using the formulas from the column "Riemann surfaces" of Table 2, one easily checks that the locally defined analytic function $w(z)$ is extended to some multi-valued function $\tilde{w}(z)$.

Lemma 2. *For each of the following pairs of symmetry types (Ξ_1, Ξ_1), $(\Xi_{3,2}, \Xi_1)$, $(\Xi_{3,2}, \Xi_{3,2})$, $(\Xi_{2,3}, \Xi_1)$, $(\Xi_{2,3}, \Xi_{3,2})$, $(\Xi_{2,3}, \Xi_{2,3})$, $(\Xi_{3,3}, \Xi_1)$, $(\Xi_{3,3}, \Xi_{3,2})$, $(\Xi_{3,3}, \Xi_{2,3})$,$(\Xi_{3,3}, \Xi_{3,3})$, one can choose a closed path $\gamma \subset \mathbb{C}$, $z_0 \in \gamma$ so that:*

1) *the analytic continuation of $a(z)$ along γ does not change the branch,*

2) *the analytic continuation of $w(z)$ along γ changes the branch of $\tilde{w}(z)$,*

3) *the value w_1 of $\tilde{w}(z)$ on this new branch defines a 4th line passing through the intersection point of the concurrent lines l_1, l_2, l_3.*

The proof the Lemma is presented after the proof of the Theorem.

Since $F(a(z), b(w(z))) \equiv 0$ on some neighborhood of z_0 and $F(a(z), b(w(z)))$ is analytic, we conclude that $F(a(z), b(\tilde{w}(z))) \equiv 0$ along the path. Thus the same map $Z = a(z)$, $W = b(w)$ maps some neighborhood Σ_1 of $(z_0, w_1) \in S_1$, where w_1 is the value of $\tilde{w}(z)$ on the new branch,

to same neighborhood Σ_2 of the Riemann surface S_2. The neighborhood Σ_1 defines locally some "additional" family of lines, one of which passes through the point determined by z_0, and similarly, Σ_2 defines locally some family of lines, one of which passes through the point determined by Z_0. This means that the diffeomorphism maps the "additional" family of lines of the first d-web $\widetilde{\mathcal{W}}$ to some "additional" family of lines of the second one.

For the web $\mathcal{W}(\Xi_1, 3)$, let us choose a closed path γ with $z_0 \in \gamma$ so that it goes around one of the branch points $z_b = \pm\sqrt{2}$ of S_1. (Here z_0 corresponds to the base point of our web.) When we come back to z_0, we change the branch of $\tilde{w}(z)$. Along the path holds true $F(kz + (c_3 - c_1), kw + (c_2 - c_1)) \equiv 0$, where $F(Z, W) = ZW^2 - (Z^2 + 2)W + 2\lambda Z = 0$ defines the Riemann surface S_2. Thus to the value $w_1 = \overline{p_2} - p_1$ on the new branch corresponds a line different from l_1, l_2, l_3, since $w_1 = \overline{p_2} - p_1 \neq w = p_2 - p_1$ implies $\overline{p_2} \neq p_2$. Therefore some 4th family of lines of the web $\widetilde{\mathcal{W}}(\Xi_1, 3)$ is mapped by the diffeomorphism to some family of lines of the web $\widetilde{\mathcal{W}}(\Xi_1, 1)$, which is in fact 6-web. (Note that the dual focal curves of the web $\widetilde{\mathcal{W}}(\Xi_1, 1)$ are three different parabolas and a generic line intersects them in 6 points.) Since any morphism of linear 4-web is projective, our diffeomorphism (if it exists) is also projective. But the forms $\mathcal{W}(\Xi_1, 1)$ and $\mathcal{W}(\Xi_1, 3)$ are not projectively equivalent.

Now let us prove that any diffeomorphism preserving $\mathcal{W}(\Xi_1, 3)$ is projective. Again this diffeomorphism is of the form

$$P_{\pi(i)} = kp_i + c_i, \quad i = 1, 2, 3.$$

It generates an automorphism (21) of S_1, where the new parameters on S_1 are chosen as $W = P_{\pi(2)} - P_{\pi(1)}, \quad Z = P_{\pi(3)} - P_{\pi(1)}$. Repeating the trick with analytic continuation along the closed path we conclude that the diffeomorphism is projective. $\square$

Proof of Lemma 2. The case of the pair (Ξ_1, Ξ_1) was considered in the proof of the Theorem: the function $a(z)$ is one-valued, the path γ goes around one of the branch points of $\widetilde{w}(z)$, which are not zero.

A diffeomorphism of a web with the symmetry type $\Xi_{3,2}$ to some web with the symmetry type $(\Xi_1, 1)$ would have the form

$$P_{\pi(i)} = k \ln(p_i) + c_i, \quad i = 1, 2, 3,$$

except for the forms 5 and 6, where $P_{\pi(3)} = -\frac{k}{2} \ln(p_3) + c_3$. For such diffeomorphisms we choose a path γ going around one of the branch points of $\widetilde{w}(z)$ but not around the origin.

Type	Symmetry	Invariants	Riemann Surface	Branch Point
$\Xi_1, 1$	$\sum_i \partial_{p_i}$	$z = p_2 - p_1$ $w = p_3 - p_1$	$zw^2 - (z^2+2)w + 2\lambda z = 0$	$z_b^4 + (4-8\lambda)z_b^2 + 4 = 0$
$\Xi_1, 2$	$\sum_i \partial_{p_i}$	$z = p_2 - p_1$ $w = p_3 - p_1$	$w^2 - zw + 2 = 0$	$z_b^2 - 8 = 0$
$\Xi_1, 3$	$\sum_i \partial_{p_i}$	$z = p_3 - p_1$ $w = p_2 - p_1$	$w^2 - 2zw + 2 = 0$	$z_b^2 - 2 = 0$
$\Xi_{3,2}, 1$	$\sum_i p_i \partial_{p_i}$	$z = p_2/p_1$ $w = p_3/p_1$	$(z-\lambda)w^2 + (\lambda - z^2)w +$ $-\mu z(z-1) = 0$	$z_b^4 + 4\mu z_b^3 +$ $-2(\lambda + 2\mu\lambda + 2\mu)z_b^2 +$ $4\mu\lambda z_b + \lambda^2 = 0$
$\Xi_{3,2}, 2$	$\sum_i p_i \partial_{p_i}$	$z = p_2/p_1$ $w = p_3/p_1$	$w^2 - (z+1)w - \lambda z = 0$	$z_b^2 + (4\lambda + 2)z_b + 1 = 0$
$\Xi_{3,2}, 3$	$\sum_i p_i \partial_{p_i}$	$z = p_3/p_1$ $w = p_2/p_1$	$w^2 - wz + \lambda(z-1) = 0$	$z_b^2 - 4\lambda z_b + 4\lambda = 0$
$\Xi_{3,2}, 4$	$\sum_i p_i \partial_{p_i}$	$z = p_3/p_2$ $w = p_1/p_2$	$w^2 - w + z = 0$	$1 - 4z_b = 0$
$\Xi_{3,2}, 5$	$\sum_i^2 p_i \partial_{p_i} +$ $-2p_3\partial_{p_3}$	$z = p_3 p_2^2$ $w = p_1/p_2$	$zw^2 + (\lambda - z)w - 1 = 0$	$z_b^2 + (4-2\lambda)z_b + \lambda^2 = 0$
$\Xi_{3,2}, 6$	$\sum_i^2 p_i \partial_{p_i} +$ $-2p_3\partial_{p_3}$	$z = p_3 p_2^2$ $w = p_1/p_2$	$zw^2 - zw - 1 = 0$	$z_b + 4 = 0$
$\Xi_{2,3}, 1$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$we^w(1-e^z) + ze^z(e^w-1) +$ $(\lambda - \mu)e^w + \mu e^z - \lambda e^{z+w} = 0$	
$\Xi_{2,3}, 2$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$we^w(1-e^z) + ze^z(e^w-1) +$ $\lambda(e^z - e^w) = 0$	
$\Xi_{2,3}, 3$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$we^w(1-e^z) + ze^z(e^w-1) = 0$	
$\Xi_{2,3}, 4$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$e^w - e^z + (w - z + \lambda)e^{z+w} = 0$	
$\Xi_{2,3}, 5$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$e^w - e^z + (w - z)e^{z+w} = 0$	
$\Xi_{2,3}, 6$	$\sum_i \partial_{t_i} + \partial_{p_3},$ $p_i = e^{t_i},$ $i = 1, 2$	$z = t_1 - p_3$ $w = t_2 - p_3$	$we^w - ze^z + \lambda e^w = 0$	
$\Xi_{2,3}, 7$	$\sum_i \partial_{t_i} + \partial_{p_3},$ $p_i = e^{t_i},$ $i = 1, 2$	$z = t_1 - p_3$ $w = t_2 - p_3$	$we^w - ze^z = 0$	
$\Xi_{2,3}, 8$	$\sum_i \partial_{t_i} + \partial_{p_3},$ $p_i = e^{t_i},$ $i = 1, 2$	$z = t_2 - p_3$ $w = t_1 - p_3$	$we^w - e^z = 0$	
$\Xi_{3,3}, 1$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_2 - t_1$ $w = t_3 - t_1$	$\mu e^{\beta w}(e^z - 1) - \lambda e^{\beta z}(e^w - 1) +$ $e^w - e^z = 0$	
$\Xi_{3,3}, 2$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_2 - t_1$ $w = t_3 - t_1$	$\lambda e^{\beta w}(e^z - 1) - e^{\beta z}(e^w - 1) +$ $e^w - e^z = 0$	
$\Xi_{3,3}, 3$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_2 - t_1$ $w = t_3 - t_1$	$e^{\beta w}(e^z - 1) - e^{\beta z}(e^w - 1) +$ $e^w - e^z = 0$	

Table 2: Reduction of web equations.

A diffeomorphism of a web with the symmetry type $\Xi_{3,2}$ to some web with the same symmetry would have the form

$$P_{\pi(i)} = c_i p_i^k, \quad i = 1, 2, 3.$$

Type	Symmetry	Invariants	Riemann Surface	Branch Point
$\Xi_{3,3}, 4$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$\lambda e^{\beta w}(e^z - 1) - e^{\beta z}(e^w - 1) = 0$	
$\Xi_{3,3}, 5$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$e^{\beta w}(e^z - 1) - e^{\beta z}(e^w - 1) = 0$	
$\Xi_{3,3}, 6$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$ $p_3 = e^{\beta t_3}$	$z = t_2 - t_3$ $w = t_1 - t_3$	$e^z(e^{\beta w} - 1) + e^w = 0$	
$\Xi_{3,3}, 7$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$ $p_3 = e^{(\beta-1)t_3}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$\lambda e^{\beta w} - e^w + e^z - e^{\beta z} = 0$	
$\Xi_{3,3}, 8$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$ $p_3 = e^{(\beta-1)t_3}$	$z = t_1 - t_3$ $w = t_2 - t_3$	$e^{\beta w} - e^w + e^z - e^{\beta z} = 0$	
$\Xi_{3,3}, 9$	$\sum_i \partial_{t_i},$ $p_i = e^{t_i}$ $p_3 = e^{(\beta-1)t_3}$	$z = t_1 - t_2$ $w = t_3 - t_2$	$e^{(\beta-1)w}(1 - e^z) + e^{\beta z} = 0$	

Table 2: Reduction of web equations.

(For the forms 5 and 6 one adjusts the exponent in the obvious way.) Again we choose a path γ going around a branch point of $\widetilde{w}(z)$ but not around the origin.

In the above considered cases of the pairs of symmetry types, the claim of the third item of Lemma 2 is obviously true, since the invariant w is linear in the inclination p of web lines.

A diffeomorphism of a web with the symmetry type $\Xi_{2,3}$ to some web with the symmetry type Ξ_1 would have the form

$$P_{\pi(i)} = kt_i + c_i, \quad i = 1, 2, 3.$$

(For the forms 6,7,8 one substitutes $t_3 = p_3$.) Here $a(z)$ is one-valued, we choose a path γ going around one of the branch points of $\widetilde{w}(z)$.

A diffeomorphism of a web with the symmetry type $\Xi_{2,3}$ to some web with the symmetry type $\Xi_{3,2}$ would be (up to substitution $t_3 = p_3$ for the forms 6,7,8)

$$P_{\pi(i)} = c_i e^{kt_i}, \quad i = 1, 2, 3.$$

(For the forms $\Xi_{3,2}, 5$ and $\Xi_{3,2}, 6$ one again adjusts the exponent.) Again $a(z)$ is one-valued, the path γ goes around one of the branch points of $\widetilde{w}(z)$.

A diffeomorphism of a web with the symmetry type $\Xi_{2,3}$ to some web with the same symmetry type would be (up to substitution $t_3 = p_3$ for the

forms 6,7,8)

$$T_{\pi(i)} = kt_i + c_i, \quad i = 1, 2, 3.$$

Again $a(z)$ is one-valued, the path γ goes around one of the branch points of $\widetilde{w}(z)$.

To prove the third item of Lemma 2 note that: 1) parameter t_i defining the line l_4 is related to the inclination p of web lines by $p_i = e^{t_i}$, 2) the value w_1 on the new branch cannot be related to the value w defining the line l_3 by the formula $w_1 = w + 2\pi n$, $n \in \mathbb{Z}$, since from the equations defining the Riemann surface S_1 one easily sees that $(z, w) \in S_1 \Rightarrow (z, w + 2\pi n) \notin S_1$.

Now we have all the formulas written and the reader can easily check the statement of the Lemma for the pairs $(\Xi_{3,3}, \Xi_*)$. For irrational β one proves the third item of Lemma 2 exactly as in the case of pairs $(\Xi_{2,3}, \Xi_*)$. For rational β one observes that $(\overline{Z}, \overline{W}) = (e^z, e^{\widetilde{w}(z)})$ satisfies some algebraic equation and defines an algebraic function $\overline{W} = \mathcal{A}(\overline{Z})$. Now one chooses the path γ so that $\mathcal{A}(e^z)$ changes the branch, which ensures that l_4 is different from l_1, l_2, l_3. We present equations for the branch point z_b only for the forms with the symmetry type Ξ_1 and for the cases when it is important that $z_b \neq 0$. $\qquad \square$

5.1. Acknowledgement

6. Appendix: coefficients of $E(Z)$ and $H(Z)$

$$E_5 = 81(4X - 9)^3,$$
$$E_4 = -324X(4X - 9)^2,$$
$$E_3 = -27X^2(4X - 9)[X(4X - 9)^2 F' + 2X(4X - 9)^2 F^2$$
$$+ (6X - 13)(4X - 9)F - 12],$$
$$E_2 = -3X^2(4X - 9)[3X^2(4X - 9)^2 F'' + 21X^2(4X - 9)^2 FF'$$
$$+ 6X(4X - 9)(2X - 9)F' + 18X^2(4X - 9)^2 F^3$$
$$+ 42X(4X - 9)(X - 3)F^2 + (-108X^2 + 144X + 54)F - 8],$$

$$E_1 = 3X^3[8X^2(4X-9)^2F'' + 53X^2(4X-9)^2FF'$$
$$+ 2X(40X-69)(4X-9)F' + 42X^2(4X-9)^2F^3$$
$$+ 2X(95X-153)(4X-9)F^2 + (108-180X)F - 16],$$

$$E_0 = X^4[-16X^2(4X-9)F'' + 3X^2(4X-9)^2(F')^2 + 12X^2(4X-9)^2F^2F'$$
$$- 4X(16X+9)(4X-9)FF' - 24X(8X-9)F' + 12X^2(4X-9)^2F^4$$
$$- 72X(4X-9)F^3 + (108+240X-300X^2)F^2 - (48X+16)F],$$

$$H_6 = -243(4X-9)^4[5X(4X-9)F - 12X + 30],$$

$$H_5 = 81X(4X-9)^3[73X(4X-9)F - 192X + 486],$$

$$H_4 = 27X^2(4X-9)^2[-3X^2(4X-9)^3F'' - 3X^2(4X-9)^3FF'$$
$$- 9X(10X-21)(4X-9)^2F' + 18X^2(4X-9)^3F^3$$
$$- 3X(30X-61)(4X-9)^2F^2 - 2(4X-9)(180X^2-578X+747)F$$
$$+ 1008X - 2592],$$

$$H_3 = 9X^2(4X-9)[-3X^3(4X-9)^4F''' + -15X^3(4X-9)^4FF''$$
$$- 6X^2(14X-33)(4X-9)^3F'' - 21X^3(4X-9)^4(F')^2$$
$$- 12X^3(4X-9)^4F^2F' - 3X^2(215X-384)(4X-9)^3FF'$$
$$+ 6X(30X^2+121X-360)(4X-9)^2F'$$
$$+ 36X^3(4X-9)^4F^4 - 102X^2(5X-6)(4X-9)^3F^3$$
$$- 9X(102X^2-461X+534)(4X-9)^2F^2$$
$$+ 2(4X-9)(1800X^3-6548X^2+5445X+810)F$$
$$+ 2160 + 2712X - 1344X^2],$$

$$H_2 = -3(4X-9)X^3[-36X^3(4X-9)^3F''' - 189X^3(4X-9)^3FF''$$
$$- 18X^2(72X-133)(4X-9)^2F'' - 216X^3(4X-9)^3(F')^2$$
$$- 117X^3(4X-9)^3F^2F' - 18X^2(424X-737)(4X-9)^2FF'$$
$$- 72X(4X-9)(90X^2-370X+357)F' + 414X^3(4X-9)^3F^4$$
$$- 18X^2(221X-359)(4X-9)^2F^3$$
$$- 18X(4X-9)(1002X^2-3484X+2985)F^2$$
$$+ (8640X^3-22040X^2-5328X+16524)F - 2448 + 960X],$$

$$H_1 = -X^4[144X^3(4X-9)^3F''' + 792X^3(4X-9)^3FF''$$
$$+ 48X^2(116X-201)(4X-9)^2F'' + 54X^3(4X-9)^4F(F')^2$$
$$+ 18X^2(34X+15)(4X-9)^3(F')^2 + 216X^3(4X-9)^4F^3F'$$
$$+ 9X^2(61X+48)(4X-9)^3F^2F'$$
$$+ 12X(2336X^2-3921X-270)(4X-9)^2FF'$$
$$+ 36X(4X-9)(1040X^2-3560X+2709)F' +$$

$$+ 216X^3(4X-9)^4F^5 - 18X^2(43X+12)(4X-9)^3F^4$$
$$+ 18X(619X^2 - 1027X - 252)(4X-9)^2F^3$$
$$+ 24(4X-9)(3054X^3 - 9961X^2 + 7146X + 405)F^2$$
$$+ (11520X^3 - 83264X^2 + 148536X - 27864)F + 6048 - 2304X],$$
$$H_0 = X^5[64X^3(4X-9)^2F''' + 368X^3(4X-9)^2FF''$$
$$+ 160X^2(16X-27)(4X-9)F'' + 63X^3(4X-9)^3F(F')^2$$
$$+ 14X^2(8X+27)(4X-9)^2(F')^2 + 252X^3(4X-9)^3F^3F'$$
$$+ 4X^2(64X+189)(4X-9)^2F^2F'$$
$$+ 8X(4X-9)(1336X^2 - 2025X - 567)FF'$$
$$+ 16X(300X-269)(4X-9)F' + 252X^3(4X-9)^3F^5$$
$$+ 192X^3(4X-9)^2F^4 + 4X(7X+9)(95X-189)(4X-9)F^3$$
$$+ (13608 - 86904X^2 + 41904X + 28800X^3)F^2$$
$$+ (3840X^2 - 7840X - 2016)F].$$

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