

Stability of Einstein metrics under Ricci flow

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We prove dynamical stability and instability theorems for compact Einstein metrics under the Ricci flow. We give a nearly complete characterization of dynamical stability and instability in terms of the conformal Yamabe invariant and the Laplace spectrum. In particular, we prove dynamical stability of some classes of Einstein manifolds for which it was previously not known. Additionally, we show that the complex projective space with the Fubini-Study metric is surprisingly dynamically unstable.

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1. Introduction

Let M^n , $n \geq 2$ be a manifold. A Ricci flow on M is a curve of metrics $g(t)$ on M satisfying the evolution equation

$$(1.1) \quad \dot{g}(t) = -2\text{Ric}_{g(t)}.$$

The Ricci flow was first introduced by Hamilton in [Ham82]. Since then it has become an important tool in Riemannian geometry. It was not only an essential tool in the proof of the famous Poincaré conjecture [Per02, Per03] but also for proving other recent results like the differentiable sphere theorem [BS09].

The Ricci flow is not a gradient flow in the strict sense, but Perelman made the remarkable discovery that it can be interpreted as the gradient of the λ -functional

$$(1.2) \quad \lambda(g) = \inf_{\substack{f \in C^\infty(M) \\ \int_M e^{-f} dV_g = 1}} \int_M (\text{scal}_g + |\nabla f|_g^2) e^{-f} dV_g$$

on the space of metrics modulo diffeomorphisms [Per02].

Ricci-flat metrics are the stationary points of the Ricci flow and Einstein metrics remain unchanged under the Ricci flow up to rescaling. It is now natural to ask how the Ricci flow behaves as a dynamical system close to Einstein metrics. A stability result for compact Einstein metrics assuming positivity of the Einstein operator was proven in [Ye93]. Stability results for compact Ricci-flat metrics assuming nonnegativity of the Lichnerowicz Laplacian and integrability of infinitesimal Einstein deformations were proven by Sesum and Haslhofer in [Ses06, Has12], generalizing an older result in [GIK02]. Recently, Haslhofer and Müller [HM14] were able to get rid of the integrability condition and proved the following: A compact Ricci-flat manifold is dynamically stable if it is a local maximizer of λ and dynamically unstable, if this is not the case. Because of monotonicity of λ along the Ricci flow, the converse implications hold in both cases.

The aim of the present paper is to generalize these results to the Einstein case and to give geometric stability and instability conditions in terms of the conformal Yamabe invariant and the Laplace spectrum. Throughout, any manifold will be compact. The Yamabe invariant of a conformal class is defined by

$$Y(M, [g]) = \inf_{\tilde{g} \in [g]} \text{vol}(M, \tilde{g})^{2/n-1} \int_M \text{scal}_{\tilde{g}} dV_{\tilde{g}},$$

where $[g]$ denotes the conformal class of the metric g . By the solution of the Yamabe problem, this infimum is always realized by a metric of constant scalar curvature [Sch84]. Let M be a manifold and \mathcal{M} be the set of smooth metrics on M . We call the map $\mathcal{M} \ni g \mapsto Y(M, [g])$ the Yamabe functional and the real number

$$Y(M) = \sup_{g \in \mathcal{M}} Y(M, [g])$$

the smooth Yamabe invariant of M . A metric g on M is called supreme if it realizes the conformal Yamabe invariant in its conformal class and the smooth Yamabe invariant of the manifold.

It is a hard problem to compute the smooth Yamabe invariant of a given compact manifold and only for a few examples (including the round sphere), it is explicitly known. An interesting question is whether a compact manifold admits a supreme metric and whether it is Einstein. For more details concerning these questions, see e.g. [LeB99].

Any Einstein metric g_E is a critical point of the Yamabe functional and it is a local maximum of the Yamabe functional if and only if g_E is a local maximum of the Einstein-Hilbert action restricted to the set of constant scalar curvature metrics of volume $\text{vol}(M, g_E)$. This follows from [BWZ04, Theorem C]. A sufficient condition for this is that the Einstein operator

$$\Delta_E = \nabla^* \nabla - 2\mathring{R}$$

is positive on all nonzero transverse traceless tensors, i.e. the symmetric $(0, 2)$ -tensors satisfying $\text{tr}h = 0$ and $\delta h = 0$ ([Böh05, p. 279] and [Bes08, p. 131]). Here, \mathring{R} denotes the natural action of the curvature tensor on symmetric $(0, 2)$ -tensors. Conversely, if g_E is a local maximum of the Yamabe functional, the Einstein operator is necessarily nonnegative on transverse traceless tensors. The Einstein operator and its spectrum were studied in [Koi78, Koi83, IN05, DWW05, DWW07] and also in a recent paper by the author [Krö15]. We find the following relation to the λ -functional which will be proven in Section 5.

Theorem 1.1. *A Ricci-flat metric g_{RF} is a local maximizer of λ if and only if it is a local maximizer of the Yamabe functional, i.e. there are no metrics of positive scalar curvature close to g_E .*

Remark 1.2. This condition is automatic if M is spin and if $\hat{A}(M) \neq 0$ [Hit74, p. 46] because the existence of positive scalar curvature metrics is excluded.

Since Einstein metrics are not stationary points of the Ricci flow in the strict sense form we consider the volume-normalized Ricci flow

$$(1.3) \quad \dot{g}(t) = -2\text{Ric}_{g(t)} + \frac{2}{n} \left(\int_M \text{scal}_{g(t)} dV_{g(t)} \right) \cdot g(t).$$

This allows us to define appropriate notions of dynamical stability and instability for Einstein metrics.

Definition 1.3. A compact Einstein manifold (M, g_E) is called dynamically stable if for any $k \geq 3$ and any C^k -neighbourhood \mathcal{U} of g_E in the space of metrics, there exists a C^{k+2} -neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that for any $g_0 \in \mathcal{V}$, the normalized Ricci flow starting at g_0 exists for all $t \geq 0$ and converges modulo diffeomorphism to an Einstein metric in \mathcal{U} as $t \rightarrow \infty$.

We call a compact Einstein manifold (M, g_E) dynamically unstable if there exists a nontrivial normalized Ricci flow defined on $(-\infty, 0]$ which converges modulo diffeomorphism to g_E as $t \rightarrow -\infty$.

It is well-known that the round sphere is dynamically stable [Ham82, Hui85]. From now on, we assume that $(M^n, g) \neq (S^n, g_{st})$ and that $n \geq 3$. The round sphere is an exceptional case because it is the only compact Einstein space which admits conformal Killing vector fields. Now we can state the main theorems of this paper.

Theorem 1.4 (Dynamical stability). *Let (M, g_E) be a compact Einstein manifold with Einstein constant μ . Suppose that (M, g_E) is a local maximizer of the Yamabe functional and that the smallest nonzero eigenvalue of the Laplacian satisfies $\lambda > 2\mu$. Then (M, g_E) is dynamically stable.*

Theorem 1.5 (Dynamical instability). *Let (M, g_E) be a compact Einstein manifold with Einstein constant μ . Suppose that (M, g_E) is a not local maximizer of the Yamabe functional or the smallest nonzero eigenvalue of the Laplacian satisfies $\lambda < 2\mu$. Then (M, g_E) is dynamically unstable.*

Apart from the case $\lambda = 2\mu$, this gives a complete description of the Ricci flow as a dynamical system close to a compact Einstein metric. The converse implications nearly hold: If an Einstein manifold is dynamically stable, then it is a local maximizer of the Yamabe functional and the smallest nonzero eigenvalue of the Laplacian satisfies $\lambda \geq 2\mu$. If it is dynamically unstable it is not a local maximizer of the Yamabe functional or the smallest nonzero Laplace eigenvalue satisfies $\lambda \leq 2\mu$. This follows from Theorem 5.1 resp.

Theorem 8.3 and the monotonicity of the functionals μ_+ and ν_- along the corresponding variants of the Ricci flow.

The Ricci-flat case is already covered by the results in [HM14] and Theorem 1.1, so it remains to consider the cases of positive and negative Einstein constant. Both cases will be proven separately. We use the functional μ_+ in the negative case and ν_- in the positive case. Both are analogues of the λ -functional. The negative case will be treated more extensively. In the positive case, the strategy is basically the same and so we will skip the details there.

Observe that the case of nonpositive Einstein constant is easier to handle with because the eigenvalue condition drops there. In fact, all known compact nonpositive Einstein manifolds satisfy the assumptions of Theorem 1.4. For certain classes of nonpositive Einstein manifolds we actually know that these assumptions hold: By [LeB99, Theorem 3.6], any 4-dimensional Kähler-Einstein manifold with nonpositive scalar curvature realizes the smooth Yamabe invariant of M . Thus, we have

Corollary 1.6. *Any compact four-dimensional Kähler-Einstein manifold with nonpositive scalar curvature is dynamically stable.*

In contrast, there are many positive Einstein metrics, which satisfy one of the conditions of Theorem 1.5. For example, any product of two positive Einstein metrics does not maximize the Yamabe functional because the Einstein operator admits negative eigenvalues. On the other hand, there are some symmetric spaces of compact type (e.g. $\mathbb{H}P^n$ for $n \geq 3$, see [CH15, Table 2]), which are local maxima of the Yamabe functional and satisfy the eigenvalue condition of Theorem 1.5.

However, some interesting examples are not covered by the above two theorems because they are local maxima of the Yamabe functional but the smallest nonzero eigenvalue of the Laplacian satisfies $\lambda = 2\mu$. These examples include the symmetric spaces G_2 , $\mathbb{C}P^n$, $SO(n+2)/(SO(n) \times SO(2))$, $n \geq 5$, $S(2n)/U(n)$, $n \geq 5$, $E_6/(SO(10) \cdot SO(2))$, $E_7/(E_6 \cdot SO(2))$ with their standard metric, see [CH15, Table 1 and Table 2]. For such manifolds, we prove dynamical instability under an additional condition.

Theorem 1.7 (Dynamical instability). *Let (M^n, g_E) , $n \geq 3$ be a compact Einstein manifold with Einstein constant μ . Suppose that there exists a function $v \in C^\infty(M)$ satisfying $\Delta v = 2\mu v$ and $\int_M v^3 dV \neq 0$. Then (M, g_E) is dynamically unstable.*

We construct an eigenfunction on $\mathbb{C}P^n$ with its standard metric satisfying this condition and thus, we have

Corollary 1.8. *The manifold $(\mathbb{C}P^n, g_{st})$, $n > 1$ is dynamically unstable.*

This result is quite unexpected since the complex projective space is linearly stable. In particular, it raises the question whether the round sphere is the only positive Einstein metric in four dimensions which is dynamically stable (c.f. [Cao10, p. 29]).

Acknowledgement. This article is based on a part of the authors' PhD-thesis. The author would like to thank Christian Bär, Christian Becker and Robert Haslhofer for helpful discussions. Moreover, the author thanks the Max-Planck Institute for Gravitational Physics for financial support.

2. Notation and conventions

We define the Laplace-Beltrami operator acting on functions by $\Delta = -\text{tr}\nabla^2$. For the Riemann curvature tensor, we use the sign convention such that $R_{X,Y}Z = \nabla_X^2 Y - \nabla_Y^2 X$. Given a fixed metric, we equip the bundle of (r, s) -tensor fields (and any subbundle) with the natural scalar product induced by the metric. By $S^p M$, we denote the bundle of symmetric $(0, p)$ -tensors. The divergence $\delta : \Gamma(S^p M) \rightarrow \Gamma(S^{p-1} M)$ and its formal adjoint $\delta^* : \Gamma(S^{p-1} M) \rightarrow \Gamma(S^p M)$ are given by

$$\begin{aligned} \delta T(X_1, \dots, X_{p-1}) &= - \sum_{i=1}^n \nabla_{e_i} T(e_i, X_1, \dots, X_{p-1}), \\ \delta^* T(X_1, \dots, X_p) &= \frac{1}{p} \sum_{i=0}^{p-1} \nabla_{X_{1+i}} T(X_{2+i}, \dots, X_{p+i}), \end{aligned}$$

where the sums $1 + i, \dots, p + i$ are taken modulo p . For $\omega \in \Omega^1(M)$, we have $\delta^* \omega = \mathcal{L}_{\omega^\sharp} g$ where ω^\sharp is the sharp of ω . Thus, $\delta^*(\Omega^1(M))$ is the tangent space of the manifold $g \cdot \text{Diff}(M) = \{\varphi^* g \mid \varphi \in \text{Diff}(M)\}$. The Einstein operator Δ_E and the Lichnerowicz Laplacian Δ_L , both acting on $\Gamma(S^2 M)$, are defined by

$$\begin{aligned} \Delta_E h &= \nabla^* \nabla h - 2\mathring{R}h, \\ \Delta_L h &= \nabla^* \nabla h + \text{Ric} \circ h + h \circ \text{Ric} - 2\mathring{R}h. \end{aligned}$$

Here, $\mathring{R}h(X, Y) = \sum_{i=1}^n h(R_{e_i, X} Y, e_i)$ and \circ denotes the composition of symmetric $(0, 2)$ -tensors, considered as endomorphisms on TM .

3. The expander entropy

When considering the Ricci flow close to negative Einstein manifolds we may restrict to the case where the Einstein constant is equal to -1 . Such metrics are stationary points of the flow

$$(3.1) \quad \dot{g}(t) = -2(\text{Ric}_{g(t)} + g(t)).$$

This flow is homothetically equivalent to the standard Ricci flow. In fact,

$$\tilde{g}(t) = e^{-2t}g \left(\frac{1}{2}(e^{2t} - 1) \right)$$

is a solution of (3.1) starting at g_0 if and only if $g(t)$ is a solution of (1.1) starting at g_0 . Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Define

$$\mathcal{W}_+(g, f) = \int_M \left[\frac{1}{2}(|\nabla f|^2 + \text{scal}) - f \right] e^{-f} dV.$$

This is a simpler variant of the expander entropy $\mathcal{W}_+(g, f, \sigma)$ introduced in [FIN05].

Lemma 3.1. *The first variation of \mathcal{W}_+ at a tuple (g, f) equals*

$$\begin{aligned} \mathcal{W}'_+(h, v) = \int_M \left[-\frac{1}{2} \left\langle \text{Ric} + \nabla^2 f - \left(-\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}\text{scal} - f \right) g, h \right\rangle \right. \\ \left. - \left(-\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}\text{scal} - f + 1 \right) v \right] e^{-f} dV. \end{aligned}$$

Proof. Let $g_t = g + th$ and $f_t = f + tv$. We have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \mathcal{W}_+(g_t, f_t) &= \int_M \left[\frac{1}{2}(|\nabla f_t|_{g_t}^2 + \text{scal}_{g_t}) - f_t \right]' e^{-f} dV \\ &\quad + \int_M \left[\frac{1}{2}(|\nabla f|^2 + \text{scal}) - f \right] \left(-v + \frac{1}{2}\text{tr}h \right) e^{-f} dV. \end{aligned}$$

By the variational formula of the scalar curvature (see Lemma A.1),

$$\begin{aligned} & \int_M \left[\frac{1}{2} (|\nabla f|_{g_t}^2 + \text{scal}_{g_t}) - f \right]' e^{-f} dV \\ &= \int_M \left(-\frac{1}{2} \langle h, \nabla f \otimes \nabla f \rangle + \langle \nabla f, \nabla v \rangle \right) e^{-f} dV \\ & \quad + \int_M \left[\frac{1}{2} (\Delta \text{tr} h + \delta(\delta h) - \langle \text{Ric}, h \rangle) - v \right] e^{-f} dV. \end{aligned}$$

By integration by parts,

$$\int_M \langle \nabla f, \nabla v \rangle e^{-f} dV = \int_M (\Delta f + |\nabla f|^2) v e^{-f} dV$$

and

$$\begin{aligned} & \int_M \frac{1}{2} (\Delta \text{tr} h + \delta(\delta h)) e^{-f} dV \\ &= \int_M \frac{1}{2} [\text{tr} h \Delta(e^{-f}) + \langle h, \nabla^2(e^{-f}) \rangle] dV \\ &= \int_M \frac{1}{2} [\text{tr} h (-\Delta f - |\nabla f|^2) + \langle h, -\nabla^2 f + \nabla f \otimes \nabla f \rangle] e^{-f} dV. \end{aligned}$$

Thus,

$$\begin{aligned} & \int_M \left[\frac{1}{2} (|\nabla f|_{g_t}^2 + \text{scal}_{g_t}) - f \right]' e^{-f} dV \\ &= \int_M \left[-\frac{1}{2} \langle h, \nabla^2 f + \text{Ric} + (\Delta f + |\nabla f|^2) g \rangle + (\Delta f + |\nabla f|^2 - 1) v \right] e^{-f} dV. \end{aligned}$$

The second term of above can be written as

$$\begin{aligned} & \int_M \left[\frac{1}{2} (|\nabla f|^2 + \text{scal}) - f \right] \left(-v + \frac{1}{2} \text{tr} h \right) e^{-f} dV \\ &= \int_M \left[\frac{1}{2} \left\langle \left[\frac{1}{2} (|\nabla f|^2 + \text{scal}) - f \right] g, h \right\rangle - \left[\frac{1}{2} (|\nabla f|^2 + \text{scal}) - f \right] v \right] e^{-f} dV. \end{aligned}$$

By adding up these two terms, we obtain the desired formula. \square

Now we consider the functional

$$(3.2) \quad \mu_+(g) = \inf \left\{ \mathcal{W}_+(g, f) \mid f \in C^\infty(M), \int_M e^{-f} dV = 1 \right\}.$$

It was shown in [FIN05, Thm 1.7] that given any smooth metric, the infimum is always uniquely realized by a smooth function. We call the minimizer f_g . The minimizer depends smoothly on the metric. From Lemma 3.1, we can show that f_g satisfies the Euler-Lagrange equation

$$(3.3) \quad -\Delta f_g - \frac{1}{2} |\nabla f_g|^2 + \frac{1}{2} \text{scal}_g - f_g = \mu_+(g).$$

Remark 3.2. Observe that $\mathcal{W}_+(\varphi^*g, \varphi^*f) = \mathcal{W}_+(g, f)$ for any diffeomorphism φ and thus, $\mu_+(g)$ is invariant under diffeomorphisms.

Lemma 3.3 (First variation of μ_+). *The first variation of $\mu_+(g)$ is given by*

$$(3.4) \quad \mu_+(g)'(h) = -\frac{1}{2} \int_M \langle \text{Ric} + g + \nabla^2 f_g, h \rangle e^{-f_g} dV,$$

where f_g realizes $\mu_+(g)$. As a consequence, μ_+ is nondecreasing under the Ricci flow (3.1).

Proof. The first variational formula follows from Lemma 3.1 and (3.3). By diffeomorphism invariance,

$$\mu_+(g)'(\nabla^2 f_g) = \frac{1}{2} \mu_+'(g)(\mathcal{L}_{\text{grad} f_g} g) = 0.$$

Thus, if $g(t)$ is a solution of (3.1),

$$\frac{d}{dt} \mu_+(g(t)) = \int_M |\text{Ric}_{g(t)} + g(t) + \nabla^2 f_{g(t)}|^2 e^{-f_{g(t)}} dV_{g(t)} \geq 0. \quad \square$$

Remark 3.4. We call metrics gradient Ricci solitons if $\text{Ric}_g + \nabla^2 f = cg$ for some $f \in C^\infty(M)$ and $c \in \mathbb{R}$. In the compact case, any such metric is already Einstein if $c \leq 0$ (see [Cao10, Proposition 1.1]). By the first variational formula of μ_+ , we conclude that Einstein metrics with constant -1 are precisely the critical points of μ_+ .

Lemma 3.5. *Let (M, g_E) be an Einstein manifold with constant -1 . Furthermore, let $h \in \delta_{g_E}^{-1}(0)$. Then*

- (i) $f_{g_E} \equiv \log \operatorname{vol}(M, g_E)$,
- (ii) $\frac{d}{dt} \Big|_{t=0} f_{g_E+th} = \frac{1}{2} \operatorname{tr}_{g_E} h$,
- (iii) $\frac{d}{dt} \Big|_{t=0} (\operatorname{Ric}_{g_E+th} + g_E + th + \nabla_{g_E+th}^2 f_{g_E+th}) = \frac{1}{2} \Delta_E h$,

where Δ_E is the Einstein operator.

Proof. By substituting $w = e^{-f/2}$, we see that $w_{g_E} = e^{-f_{g_E}/2}$ is the minimizer of the functional

$$\widetilde{\mathcal{W}}(w) = \int_M 2|\nabla w|^2 + \frac{1}{2} \operatorname{scal} w^2 + w^2 \log w^2 \, dV$$

under the constraint

$$\|w\|_{L^2} = 1.$$

By Jensen’s inequality, we have a lower bound

$$(3.5) \quad \widetilde{\mathcal{W}}(w) \geq \frac{1}{2} \inf_{p \in M} \operatorname{scal}(p) - \log(\operatorname{vol}(M, g_E)),$$

which is realized by the constant function $w_{g_E} \equiv \operatorname{vol}(M, g_E)^{-1/2}$ since the scalar curvature is constant on M . This proves (i). To prove (ii), we differentiate the Euler-Lagrange equation (3.3) in the direction of h . We obtain

$$0 = (-\Delta f)' - \frac{1}{2} (|\nabla f|^2)' + \frac{1}{2} \operatorname{scal}' - f' = -(\Delta + 1)f' + \frac{1}{2}(\Delta + 1)\operatorname{tr}h.$$

Here we used that f_{g_E} is constant and $\delta h = 0$. The second assertion follows. It remains to show (iii). By straightforward differentiation,

$$(\operatorname{Ric} + g + \nabla^2 f)' = \frac{1}{2} \Delta_L h - \frac{1}{2} \nabla^2 \operatorname{tr}h + h + \frac{1}{2} \nabla^2 \operatorname{tr}h = \frac{1}{2} \Delta_E h.$$

Here we used Lemma A.1, (i) and (ii). □

Proposition 3.6 (Second variation of μ_+). *The second variation of μ_+ at an Einstein metric satisfying $\operatorname{Ric}_{g_E} = -g_E$ is given by*

$$\mu_+(g_E)''(h) = \begin{cases} -\frac{1}{4} \overline{f}_M \langle \Delta_E h, h \rangle \, dV, & \text{if } h \in \delta^{-1}(0), \\ 0, & \text{if } h \in \delta^*(\Omega^1(M)), \end{cases}$$

where \overline{f} denotes the averaging integral, i.e. the integral divided by the volume.

Proof. Recall that the space of symmetric $(0, 2)$ -tensors splits as $\Gamma(S^2M) = \delta^*(\Omega^1(M)) \oplus \delta^{-1}(0)$. Since μ_+ is a Riemannian functional, the Hessian restricted to $\delta^*(\Omega^1(M))$ vanishes. Now let $h \in \delta^{-1}(0)$. By the first variational formula and Lemma 3.5 (i) and (iii),

$$\mu_+(g_E)''(h) = -\frac{1}{4} \int_M \langle \Delta_E h, h \rangle dV.$$

Since $\delta(\Delta_E h) = \delta((\Delta_L + 2)h) = (\Delta_H + 2)(\delta h)$ [Lic61, pp. 28-29], Δ_E preserves $\delta^{-1}(0)$. Here, Δ_H is the Hodge-Laplacian acting on one-forms. Thus, the splitting $\delta^*(\Omega^1(M)) \oplus \delta^{-1}(0)$ is orthogonal with respect to μ_+'' . \square

4. Some technical estimates

In this section, we will establish bounds on μ_+ , f_g and their variations in terms of certain norms of the variations. These estimates are needed in proving the main theorems of the next two sections.

Lemma 4.1. *Let (M, g_E) be an Einstein manifold such that $\text{Ric} = -g_E$. Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} in the space of metrics such that the minimizers f_g are uniformly bounded in $C^{2,\alpha}$, i.e. there exists a constant $C > 0$ such that $\|f_g\|_{C^{2,\alpha}} \leq C$ for all $g \in \mathcal{U}$. Moreover, for each $\epsilon > 0$, we can choose \mathcal{U} so small that $\|\nabla f_g\|_{C^0} \leq \epsilon$ for all $g \in \mathcal{U}$.*

Proof. As in the proof of Lemma 3.5 (i), we use the fact that

$$\begin{aligned} (4.1) \quad \mu_+(g) &= \inf_{w \in C^\infty(M)} \widetilde{\mathcal{W}}(g, w) \\ &= \inf \int_M 2|\nabla w|^2 + \frac{1}{2} \text{scal} w^2 + w^2 \log w^2 dV \end{aligned}$$

under the constraint

$$\|w\|_{L^2} = 1.$$

There exists a unique minimizer of this functional which we denote by w_g . We have $w_g = e^{-f_g/2}$ and w_g satisfies the Euler-Lagrange equation

$$(4.2) \quad 2\Delta w_g + \frac{1}{2} \text{scal}_g w_g - 2w_g \log w_g = \mu_+(g)w_g.$$

We will now show that there exists a uniform bound $\|w_g\|_{C^{2,\alpha}} \leq C$ for all metrics g in a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E . For this purpose, we first remark

that all the Sobolev constants that will appear below are uniformly bounded on \mathcal{U} . Now observe that by (4.1),

$$2 \|\nabla w_g\|_{L^2} \leq \mu_+(g) - C_1 \text{vol}(M, g) - \frac{1}{2} \inf_{p \in M} \text{scal}_g(p),$$

since the function $x \mapsto x \log x$ has a lower bound. By testing with suitable functions, one sees that $\mu_+(g)$ is bounded from above on \mathcal{U} . Therefore, the H^1 -norm of ω_g is bounded and by Sobolev embedding, the same holds for the $L^{2n/(n-2)}$ -norm. Let $p = 2n/(n - 2)$ and choose some q slightly smaller than p . By (4.2) and elliptic regularity,

$$\|w_g\|_{W^{2,q}} \leq C_2(\|w_g \log w_g\|_{L^q} + \|w_g\|_{L^q}).$$

Since $x \mapsto x \log x$ grows slower than $x \mapsto x^\beta$ for any $\beta > 1$ as $x \rightarrow \infty$, we have the estimate

$$\|w_g \log w_g\|_{L^q} \leq C_3(\text{vol}(M, g)) + \|w_g\|_{L^p}.$$

This yields an uniform bound $\|w_g\|_{W^{2,q}} \leq C(q)$.

By Sobolev embedding, we have uniform bounds on $\|w_g\|_{L^{p'}}$ for some $p' > p$ and by applying elliptic regularity on (4.2), we have bounds on $\|w_g\|_{W^{2,q'}}$ for every $q' < p'$. Iterating this procedure, we obtain uniform bounds $\|w_g\|_{W^{2,p}} \leq C(p)$ for each $p \in (1, \infty)$. By choosing p large enough, we can bound the $C^{0,\alpha}$ -norm of ω_g and by elliptic regularity,

$$\begin{aligned} \|w_g\|_{C^{2,\alpha}} &\leq C_4(\|w_g \log w_g\|_{C^{0,\alpha}} + \|w_g\|_{C^{0,\alpha}}) \\ &\leq C_5[(\|w_g\|_{C^{0,\alpha}})^\gamma + \|w_g\|_{C^{0,\alpha}}] \leq C_6 \end{aligned}$$

Next, we show that the $C^{2,\alpha}$ -norms of f_g are uniformly bounded. First, we claim that we may choose a smaller neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that for $g \in \mathcal{V}$, the functions w_g are bounded away from zero (recall that any $w_g = e^{-f_g/2}$ is positive). Suppose this is not the case. Then there exists a sequence $g_i \rightarrow g_E$ in $C^{2,\alpha}$ such that $\min_p w_{g_i}(p) \rightarrow 0$ for $i \rightarrow \infty$. Since $\|w_{g_i}\|_{C^{2,\alpha}} \leq C$ for all i , there exists a subsequence, again denoted by w_{g_i} such that $w_{g_i} \rightarrow w_\infty$ in $C^{2,\alpha'}$ for some $\alpha' < \alpha$. Obviously, the right hand side of (4.1) converges. Since μ_+ is bounded from below by (3.5) and from above, a suitable choice of the subsequence ensures that also the left hand side of (4.1) converges. Therefore, w_∞ equals the minimizer of $\widetilde{\mathcal{W}}(g_E, w)$, so $w_\infty = w_{g_E} = \text{vol}(M, g_E)^{-1/2}$. In particular, $\min_p w_{g_i}(p) \rightarrow \text{vol}(M, g_E)^{-1/2} \neq$

0 which contradicts the assumption. Now we have

$$\|f_g\|_{C^{2,\alpha}} = \|-2 \log(w_g)\|_{C^{2,\alpha}} \leq C(\log \inf w_g, 1/(\inf w_g)) \|w_g\|_{C^{2,\alpha}} \leq C_7.$$

It remains to prove that for each $\epsilon > 0$, we may choose \mathcal{U} so small that $\|\nabla f_g\|_{C^0} < \epsilon$. We again use a subsequence argument. Suppose this is not possible. Then there exists a sequence of metrics $g_i \rightarrow g$ in $C^{2,\alpha}$ and some $\epsilon_0 > 0$ such that for the corresponding f_{g_i} , the estimate $\|\nabla f_{g_i}\|_{C^0} \geq \epsilon_0$ holds for all i . Because of the bound $\|f_g\|_{C^{2,\alpha}} \leq C$, we may choose a subsequence, again denoted by f_i converging to some f_∞ in $C^{2,\alpha'}$ for $\alpha' < \alpha$. By the same arguments as above, $f_\infty = f_{g_E} \equiv -\log(\text{vol}(M))$. In particular, $\|\nabla f_{g_i}\|_{C^0} \rightarrow 0$, a contradiction. \square

Lemma 4.2. *Let (M, g_E) be an Einstein manifold such that $\text{Ric}_{g_E} = -g_E$. Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E in the space of metrics and a constant $C > 0$ such that for all $g \in \mathcal{U}$, we have*

$$\left\| \frac{d}{dt} \Big|_{t=0} f_{g+th} \right\|_{C^{2,\alpha}} \leq C \|h\|_{C^{2,\alpha}}, \quad \left\| \frac{d}{dt} \Big|_{t=0} f_{g+th} \right\|_{H^i} \leq C \|h\|_{H^i}, \quad i = 1, 2.$$

Proof. Recall that f_g satisfies the Euler-Lagrange equation

$$-\Delta f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}\text{scal} - f = \mu_+(g).$$

Differentiating this equation in the direction of h yields

$$-\dot{\Delta}f - \Delta \dot{f} + \frac{1}{2}h(\text{grad}f, \text{grad}f) - \langle \nabla f, \nabla \dot{f} \rangle + \frac{1}{2}\dot{\text{scal}} - \dot{f} = \dot{\mu}_+(g).$$

By Lemma A.2 and Lemma A.1 the variational formulas for the Laplacian and the scalar curvature are

$$\begin{aligned} \dot{\Delta}f &= \langle h, \nabla^2 f \rangle - \langle \delta h + \frac{1}{2}\nabla \text{tr}h, \nabla f \rangle, \\ \dot{\text{scal}} &= \Delta(\text{tr}h) + \delta(\delta h) - \langle \text{Ric}, h \rangle. \end{aligned}$$

Because $\Delta + 1$ is invertible, we can apply elliptic regularity and we obtain

$$\begin{aligned} \|\dot{f}\|_{C^{2,\alpha}} &\leq C_1 \|(\Delta + 1)\dot{f}\|_{C^{0,\alpha}} \\ &\leq C_1 \|\nabla f\|_{C^0} \|\nabla \dot{f}\|_{C^{0,\alpha}} \\ &\quad + C_1 \left\| -\dot{\Delta}f + \frac{1}{2}h(\nabla f, \nabla f) + \frac{1}{2}\dot{\text{scal}} - \dot{\mu}_+(g) \right\|_{C^{0,\alpha}}. \end{aligned}$$

By Lemma 4.1, we may choose \mathcal{U} so small that $\|\nabla f\|_{C^0} < \epsilon$ for some small $\epsilon < \min\{C_1^{-1}, 1\}$. Then we have

$$\begin{aligned} (1 - \epsilon) \|f\|_{C^{2,\alpha}} &\leq C_1 \left\| -\dot{\Delta} f + \frac{1}{2} h(\nabla f, \nabla f) + \frac{1}{2} \text{scal} - \dot{\mu}_+(g) \right\|_{C^{0,\alpha}} \\ &\leq (C_2 \|f_g\|_{C^{2,\alpha}} + C_3) \|h\|_{C^{2,\alpha}}. \end{aligned}$$

The last inequality follows from the variational formulas of the Laplacian, the scalar curvature and μ_+ . By the uniform bound on $\|f_g\|_{C^{2,\alpha}}$, the first estimate of the lemma follows. The estimate of the H^i -norm is shown similarly. \square

Proposition 4.3 (Estimate of the second variation of μ_+). *Let (M, g_E) be an Einstein manifold with constant -1 . Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E and a constant $C > 0$ such that*

$$\left| \frac{d^2}{dsdt} \Big|_{s,t=0} \mu_+(g + th + sk) \right| \leq C \|h\|_{H^1} \|k\|_{H^1}$$

for all $g \in \mathcal{U}$.

Proof. By the formula of the first variation,

$$\begin{aligned} &\frac{d^2}{dsdt} \Big|_{s,t=0} \mu_+(g + th + sk) \\ &= -\frac{d}{ds} \Big|_{s=0} \frac{1}{2} \int_M \langle \text{Ric}_{g_s} + g_s - \nabla^2 f_{g_s}, h \rangle_{g_s} e^{-f_{g_s}} dV_{g_s} \\ &= (1) + (2) + (3), \end{aligned}$$

and we estimate these three terms separately. The first term comes from differentiating the scalar product:

$$|(1)| = \left| \int_M \langle \text{Ric}_g + g - \nabla^2 f_g, k \circ h \rangle_g e^{-f_g} dV_g \right| \leq C_1 \|h\|_{H^1} \|k\|_{H^1}.$$

This estimate holds since the functions f_g are uniformly bounded in a small $C^{2,\alpha}$ -neighbourhood of g_E . The second term comes from differentiating the gradient:

$$\begin{aligned}
 |(2)| &= \left| \frac{1}{2} \int_M \left\langle \frac{d}{ds} \Big|_{s=0} (\text{Ric}_{g_s} + g_s - \nabla^2 f_{g_s}), h \right\rangle_g e^{-f_g} dV_g \right| \\
 &= \left| \frac{1}{2} \int_M \left\langle \frac{1}{2} \Delta_L k - \delta^*(\delta k) - \frac{1}{2} \nabla^2 \text{tr} k + k - (\nabla^2)' f_g - \nabla^2 f'_g, e^{-f_g} h \right\rangle_g dV_g \right| \\
 &\leq C_2 \|k\|_{H^1} \|h\|_{H^1}.
 \end{aligned}$$

The inequality follows from integration by parts, Lemma A.2, Lemma 4.2 and from the uniform bound on the f_g . The third term appears when we differentiate the weighted volume measure:

$$\begin{aligned}
 |(3)| &= \left| \frac{1}{2} \int_M \langle \text{Ric}_g + g - \nabla^2 f_g, h \rangle_g \left(-f'_g + \frac{1}{2} \text{tr} k \right) e^{-f_g} dV_g \right| \\
 &\leq C_3 \|h\|_{H^1} \|k\|_{H^1}.
 \end{aligned}$$

Here we again used Lemma 4.2 in the last step. □

Lemma 4.4. *Let (M, g_E) be an Einstein manifold with constant -1 . Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E and a constant $C > 0$ such that*

$$\left\| \frac{d^2}{dt ds} \Big|_{t,s=0} f_{g+sk+th} \right\|_{H^i} \leq C \|h\|_{C^{2,\alpha}} \|k\|_{H^i}, \quad i = 1, 2.$$

Proof. In the proof, we denote t -derivatives by dot and s -derivatives by prime. Differentiating (3.3) twice yields

$$\begin{aligned}
 -\Delta \dot{f}' - \dot{\Delta} f' - \Delta' \dot{f} - \dot{\Delta}' f + h(\text{grad} f, \text{grad} f') + k(\text{grad} f, \text{grad} \dot{f}) \\
 - \langle \nabla f, \nabla \dot{f}' \rangle - \langle \nabla \dot{f}, \nabla f' \rangle + \frac{1}{2} \text{scal}' - \dot{f}' = \dot{\mu}'_+.
 \end{aligned}$$

By elliptic regularity, we have

$$(4.3) \quad \left\| \dot{f}' \right\|_{H^i} \leq C_1 \left\| (\Delta + 1) \dot{f}' \right\|_{H^{i-2}} \leq C_1 \|\nabla f\|_{C^0} \left\| \nabla \dot{f}' \right\|_{L^2} + C_1 \|(A)\|_{H^{i-2}},$$

where

$$\begin{aligned}
 (A) &= -\dot{\Delta} f' - \Delta' \dot{f} - \dot{\Delta}' f + h(\text{grad} f, \text{grad} f') + k(\text{grad} f, \text{grad} \dot{f}) \\
 &\quad - \langle \nabla \dot{f}, \nabla f' \rangle + \frac{1}{2} \text{scal}' - \dot{\mu}'_+.
 \end{aligned}$$

By the first variation of the Laplacian and the scalar curvature and the estimates we already developed for \dot{f} and f' in Lemma 4.2, we have

$$\begin{aligned} & \left\| -\dot{\Delta}f' - \Delta' \dot{f} + h(\text{grad}f, \text{grad}f') + k(\text{grad}f, \text{grad}\dot{f}) - \langle \nabla \dot{f}, \nabla f' \rangle \right\|_{L^2} \\ & \leq C_3 \|h\|_{C^{2,\alpha}} \|k\|_{H^1}. \end{aligned}$$

Now we consider the occurent second variational formulas of the Laplacian and the scalar curvature. By Lemma A.3, they can be schematically written as

$$\begin{aligned} \dot{\Delta}'f &= \nabla k * h * \nabla f + k * \nabla h * \nabla f, \\ \text{scal}' &= \nabla^2 k * h + k * \nabla^2 h + \nabla k * \nabla h + R * k * h. \end{aligned}$$

Here, $*$ is Hamilton's notation for a combination of tensor products with contractions. Now, Lemma 4.3, integration by parts and the Hölder inequality yield

$$\left\| -\dot{\Delta}'f + \frac{1}{2}\text{scal}' - \dot{\mu}'_+ \right\|_{H^{i-2}} \leq C_2 \|h\|_{C^{2,\alpha}} \|k\|_{H^i}.$$

We obtain

$$\|(A)\|_{H^{i-2}} \leq C_4 \|h\|_{C^{2,\alpha}} \|k\|_{H^i}.$$

Since $\|\nabla f\|_{C^0}$ can be assumed to be arbitrarily small, we bring this term to the left hand side of (4.3) and obtain the result. \square

Proposition 4.5 (Estimates of the third variation of μ_+). *Let (M, g_E) be an Einstein manifold with constant -1 . Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E and a constant $C > 0$ such that*

$$\left| \frac{d^3}{dt^3} \Big|_{t=0} \mu_+(g + th) \right| \leq C \|h\|_{H^1}^2 \|h\|_{C^{2,\alpha}}$$

for all $g \in \mathcal{U}$.

Proof. We have, by the first variational formula,

$$\begin{aligned}
 \frac{d^3}{dt^3} \Big|_{t=0} \mu_+(g + th) &= -\frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} \int_M \langle \text{Ric} + g + \nabla^2 f_g, h \rangle e^{-f} dV \\
 &= -\frac{1}{2} \int_M \langle (\text{Ric} + g + \nabla^2 f_g)'', h \rangle e^{-f} dV \\
 &\quad - 3 \int_M \langle \text{Ric} + g + \nabla^2 f_g, h \circ h \circ h \rangle e^{-f} dV \\
 &\quad - \frac{1}{2} \int_M \langle \text{Ric} + g + \nabla^2 f_g, h \rangle (e^{-f} dV)'' \\
 &\quad + 2 \int_M \langle (\text{Ric} + g + \nabla^2 f_g)', h \circ h \rangle e^{-f} dV \\
 &\quad - \int_M \langle (\text{Ric} + g + \nabla^2 f_g)', h \rangle (e^{-f} dV)' \\
 &\quad + 2 \int_M \langle \text{Ric} + g + \nabla^2 f_g, h \circ h \rangle (e^{-f} dV)'.
 \end{aligned}$$

Let us deal with the first term which contains the second derivative of the gradient of μ_+ . We have the schematic expressios

$$\begin{aligned}
 (\text{Ric} + g)'' &= \nabla^2 h * h + \nabla h * \nabla h + R * h * h, \\
 (\nabla^2 f_g)'' &= (\nabla^2)'' f_g + 2(\nabla^2)' f_g' + \nabla^2 f_g'', \\
 &= \nabla f * \nabla h * h + \nabla f' * \nabla h + \nabla^2 f_g'',
 \end{aligned}$$

see Lemma A.2 and Lemma A.3. From these expressions we obtain, by applying Lemma 4.2, Lemma 4.4 and the Hölder inequality,

$$\left| \int_M \langle (\text{Ric} + g + \nabla^2 f_g)'', h \rangle e^{-f} dV \right| \leq C \|h\|_{H^1}^2 \|h\|_{C^{2,\alpha}}.$$

The estimates of the other terms are straightforward from the variational formulas in the appendix, Lemma 4.2 and Lemma 4.4. □

5. Local maximum of λ and the expander entropy

Here we give characterizations of local maximality of λ and μ_+ . We prove Theorem 1.1 using the theory developed for the Yamabe problem. For μ_+ , we use Koiso’s local decomposition theorem of the space of metrics [Koi79] and the observation that the μ_+ -functional can be explicitly evaluated on metrics of constant scalar curvature.

Proof of Theorem 1.1. Suppose that g_{RF} is not a local maximum of the Yamabe invariant. Then there exists a metric g close to g_{RF} such that $Y(M, [g]) > 0$ and by the solution of the Yamabe problem [Sch84], it admits a positive scalar curvature metric \tilde{g} realizing $Y(M, [g])$ which is also close to g_{RF} by [Koi79, Theorem 2.5]. Then by definition, $\lambda(\tilde{g}) > 0$, i.e. g_{RF} is not a local maximizer of λ .

Conversely, suppose that g_{RF} is a local maximum of the Yamabe functional, i.e. the Yamabe invariant of any conformal class close to $[g_{RF}]$ is nonpositive. By the solution of the Yamabe problem, the sign of the smallest eigenvalue of the Yamabe operator $\Delta_Y = 4\frac{n-1}{n-2}\Delta + \text{scal}$ determines the sign of the Yamabe invariant. Thus, the smallest eigenvalue of Δ_Y is nonpositive on any metric close to g_{RF} . Since λ is the smallest eigenvalue of the operator $4\Delta + \text{scal}$ we necessarily have $\lambda \leq 0$ for these metrics. \square

Theorem 5.1. *Let (M, g_E) be a compact Einstein manifold with constant -1 . Then g_E is a maximum of the μ_+ -functional in a $C^{2,\alpha}$ -neighbourhood if and only if g is a local maximum of the Yamabe functional in a $C^{2,\alpha}$ -neighbourhood. In this case, any metric sufficiently close to g_E with $\mu_+(g) = \mu_+(g_E)$ is Einstein with constant -1 .*

Proof. Let $c = \text{vol}(M, g_E)$ and write

$$\begin{aligned} \mathcal{C} &= \{g \in \mathcal{M} \mid \text{scal}_g \text{ is constant}\}, \\ \mathcal{C}_c &= \{g \in \mathcal{M} \mid \text{scal}_g \text{ is constant and } \text{vol}(M, g) = c\}. \end{aligned}$$

Since $\frac{\text{scal}_{g_E}}{n-1} \notin \text{spec}_+(\Delta_{g_E})$, [Koi79, Theorem 2.5] asserts that the map

$$\begin{aligned} \Phi: C^\infty(M) \times \mathcal{C}_c &\rightarrow \mathcal{M}, \\ (v, g) &\mapsto v \cdot g, \end{aligned}$$

is a local ILH-diffeomorphism around $(1, g_E)$. Recall also that by [BWZ04, Theorem C], any metric $g \in \mathcal{C}$ sufficiently close to g_E is a Yamabe metric.

By the proof of Lemma 3.5 (i), the minimizer $f_{\bar{g}}$ realizing $\mu_+(\bar{g})$ is constant if $\bar{g} \in \mathcal{C}$ and by the constraint in the definition, it equals $\log(\text{vol}(M, \bar{g}))$. Thus, $\mu_+(\bar{g}) = \frac{1}{2}\text{scal}_{\bar{g}} - \log(\text{vol}(M, \bar{g}))$. If g_E is not a local maximum of the Yamabe functional, there exist metrics $g_i \in \mathcal{C}_c$, $g_i \rightarrow g_E$ in $C^{2,\alpha}$ which have the same volume but larger scalar curvature than g_E . Thus, also $\mu_+(g_i) > \mu_+(g_E)$ which causes the contradiction.

If g_E is a local maximum of the Yamabe functional, it is a local maximum of μ_+ restricted to \mathcal{C}_c . Any other metric $\bar{g} \in \mathcal{C}_c$ satisfying $\mu_+(\bar{g}) = \mu_+(g_E)$ is

also a local maximum of the Yamabe functional. In particular, \bar{g} is a critical point of the total scalar curvature restricted to \mathcal{C}_c and the scalar curvature is equal to $-n$. By Proposition [Bes08, Proposition 4.47], \bar{g} is an Einstein manifold with constant -1 . For $\alpha \cdot \bar{g}$, where $\alpha > 0$ and $\bar{g} \in \mathcal{C}_c$ sufficiently close to g_E , we have

$$\begin{aligned} \mu_+(\alpha \cdot \bar{g}) &= \frac{1}{2\alpha} \text{scal}_{\bar{g}} - \frac{n}{2} \log(\alpha) - \log(\text{vol}(M, \bar{g})) \\ &\leq -\frac{n}{2} - \log(\text{vol}(M, g_E)) = \mu_+(g_E), \end{aligned}$$

which shows that g_E is also a local maximum of μ_+ restricted to \mathcal{C} and equality occurs if and only if $\alpha = 1$ and $\mu_+(\bar{g}) = \mu_+(g_E)$.

It remains to investigate the variation of μ_+ in the direction of volume-preserving conformal deformations. Let $h = v \cdot \bar{g}$, where $\bar{g} \in \mathcal{C}$ and $v \in C^\infty(M)$ with $\int_M v \, dV_{\bar{g}} = 0$. Then

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \mu_+(\bar{g} + th) &= -\frac{1}{2} \int_M \langle \text{Ric}_{\bar{g}} + \bar{g}, h \rangle e^{-f_{\bar{g}}} \, dV \\ &= -\frac{1}{2} \int_M (\text{scal}_{\bar{g}} + n)v \, dV = 0, \end{aligned}$$

since $f_{\bar{g}}$ is constant. The second variation equals

$$\begin{aligned} \left. \frac{d^2}{dt^2} \right|_{t=0} \mu_+(\bar{g} + th) &= -\frac{1}{2} \int_M \left\langle \left. \frac{d}{dt} \right|_{t=0} (\text{Ric}_{\bar{g}+th} + \bar{g} + th + \nabla^2 f_{\bar{g}+th}), h \right\rangle_{\bar{g}} \, dV_{\bar{g}} \\ &\quad + \int_M \langle \text{Ric}_{\bar{g}} + \bar{g}, h \circ h \rangle_{\bar{g}} \, dV_{\bar{g}} \\ &\quad - \frac{1}{2} \int_M \langle \text{Ric}_{\bar{g}} + \bar{g}, h \rangle \left(-f' + \frac{1}{2} \text{tr}h \right) \, dV_{\bar{g}}. \end{aligned}$$

By the first variation of the Ricci tensor,

$$-\frac{1}{2} \int_M \langle \text{Ric}' + h, h \rangle \, dV_{\bar{g}} = -\frac{n-1}{2} \int_M |\nabla v|^2 \, dV_{\bar{g}} - \frac{n}{2} \int_M v^2 \, dV_{\bar{g}}.$$

By differentiating Euler-Lagrange equation (3.3), we have

$$(5.1) \quad (\Delta + 1)f' = \frac{1}{2}((n-1)\Delta v - \text{scal}_{\bar{g}}v).$$

Thus,

$$-\frac{1}{2} \int_M \left\langle \frac{d}{dt} \Big|_{t=0} \nabla^2 f_{\bar{g}+th}, h \right\rangle e^{-f_{\bar{g}}} dV = \frac{1}{4} \int_M [(n-1)\Delta v - \text{scal}_{\bar{g}}v]v dV - \frac{1}{2} \int_M f' \cdot v dV.$$

Adding up, we obtain

$$\begin{aligned} & -\frac{1}{2} \int_M \left\langle \frac{d}{dt} \Big|_{t=0} (\text{Ric}_{\bar{g}+th} + \bar{g} + th + \nabla^2 f_{\bar{g}+th}), h \right\rangle e^{-f_{\bar{g}}} dV_{\bar{g}} \\ &= -\frac{1}{4} \int_M |\nabla v|^2 dV_{\bar{g}} - \frac{1}{2} \left(n + \frac{\text{scal}_{\bar{g}}}{2} \right) \int_M v^2 dV - \frac{1}{2} \int_M f' \cdot v dV \\ &\leq -C_1 \|v\|_{H^1}^2, \end{aligned}$$

and this estimate is uniform in a small $C^{2,\alpha}$ -neighbourhood of g_E . Here we have used that by (5.1), the L^2 -scalar product of f' and v is positive. Given any $\epsilon > 0$, the remaining terms of the second variation can be estimated by

$$\int_M \langle \text{Ric}_{\bar{g}} + \bar{g}, h \circ h \rangle_{\bar{g}} e^{-f_{\bar{g}}} dV_{\bar{g}} = (\text{scal}_{\bar{g}} + n) \int_M v^2 dV \leq \epsilon \|v\|_{L^2}^2$$

and

$$\begin{aligned} & -\frac{1}{2} \int_M \langle \text{Ric}_{\bar{g}} + \bar{g}, h \rangle \left(-f'_{\bar{g}} + \frac{1}{2} \text{tr}h \right) e^{-f_{\bar{g}}} dV_{\bar{g}} \\ &= -\frac{\text{scal}_{\bar{g}} + n}{2} \int_M v \left(-f'_{\bar{g}} + \frac{n}{2}v \right) dV \leq \epsilon \|v\|_{L^2}^2, \end{aligned}$$

provided that the neighbourhood is small enough. In the last inequality, we used $\|f'\|_{L^2} \leq C_2 \|v\|_{L^2}$ which holds because of (5.1) and elliptic regularity. Thus, we have a uniform estimate

$$\frac{d^2}{dt^2} \Big|_{t=0} \mu_+(\bar{g} + tv\bar{g}) \leq -C_3 \|v\|_{H^1}^2.$$

Let now g be an arbitrary metric in a small $C^{2,\alpha}$ -neighbourhood of g_E . By the above, it can be written as $g = \tilde{v} \cdot \tilde{g}$, where $(\tilde{v}, \tilde{g}) \in C^\infty(M) \times \mathcal{C}_{g_E}$ is

close to $(1, g_E)$. By substituting

$$v = \frac{\tilde{v} - f \tilde{v} dV_{\tilde{g}}}{f \tilde{v} dV_{\tilde{g}}}, \quad \tilde{g} = \left(\int \tilde{v} dV_{\tilde{g}} \right) \tilde{g},$$

we can write $g = (1 + v)\tilde{g}$, where $\tilde{g} \in \mathcal{C}$ is close to g_E and $v \in C^\infty(M)$ with $\int_M v dV_{\tilde{g}} = 0$ is close to 0. Thus by Taylor expansion and Proposition 4.5,

$$\begin{aligned} \mu_+(g) &= \mu_+(\tilde{g}) + \frac{1}{2} \frac{d^2}{dt^2} \Big|_{t=0} \mu_+(\tilde{g} + tv\tilde{g}) \\ &\quad + \int_0^1 \left(\frac{1}{2} - t + \frac{1}{2}t^2 \right) \frac{d^3}{dt^3} \mu_+(\tilde{g} + tv\tilde{g}) dt \\ &\leq \mu_+(g_E) - C_4 \|v\|_{H^1}^2 + C_5 \|v\|_{C^{2,\alpha}} \|v\|_{H^1}^2. \end{aligned}$$

Now if we choose the $C^{2,\alpha}$ -neighbourhood small enough, $\mu_+(g) \leq \mu_+(g_E)$ and equality holds if and only if $v \equiv 0$ and $\mu_+(g) = \mu_+(g_E)$. As discussed earlier in the proof, this implies that g is Einstein with constant -1 . \square

6. A Lojasiewicz-Simon inequality

For proving a gradient inequality for μ_+ , we need to know that μ_+ is analytic. To show this, we use the implicit function theorem for Banach manifolds in the analytic category mentioned in [Koi83, Section 13]. Such arguments were also used in [SW13, Lemma 2.2] which is a result similar to the below lemma.

Lemma 6.1. *There exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E such that the map $g \mapsto \mu_+(g)$ is analytic on \mathcal{U} .*

Proof. Let $H(g, f) = -\Delta_g f - \frac{1}{2}|\nabla f|^2 + \frac{1}{2}\text{scal}_g - f$ and consider the map

$$\begin{aligned} L: \mathcal{M}^{C^{2,\alpha}} \times C^{2,\alpha}(M) &\rightarrow C^{0,\alpha}(M) \times \mathbb{R}, \\ (g, f) &\mapsto \left(H(g, f) - \int_M H(g, f) dV_{g_E}, \int_M e^{-f} dV_g - 1 \right). \end{aligned}$$

Here, $\mathcal{M}^{C^{2,\alpha}}$, is the set of $C^{2,\alpha}$ -metrics and $C_{g_E}^{k,\alpha}(M) = \{f \in C^{k,\alpha}(M) \mid \int_M f dV_{g_E} = 0\}$. This is an analytic map between Banach manifolds. Observe that $L(g, f) = (0, 0)$ if and only if we have $H(f, g) = \text{const}$ and $\int_M e^{-f} dV_g = 1$. The differential of L at (g_E, f_{g_E}) restricted to its second

argument is equal to

$$dL_{g_E, f_{g_E}}(0, v) = \left(-(\Delta_{g_E} + 1)v + \int_M v \, dV, - \int_M v \, dV \right).$$

The map $dL_{g_E, f_{g_E}}|_{C^{2,\alpha}(M)} : C^{2,\alpha}(M) \rightarrow C^{0,\alpha}(M) \times \mathbb{R}$ is a linear isomorphism because it acts as $-(\Delta_{g_E} + 1)$ on $C^{2,\alpha}_{g_E}$ and as $-\text{id}$ on constant functions. By the implicit function theorem for Banach manifolds, there exists a neighbourhood $\mathcal{U} \subset \mathcal{M}^{C^{2,\alpha}}$ and an analytic map $P : \mathcal{U} \rightarrow C^{2,\alpha}(M)$ such that we have $L(g, P(g)) = (0, 0)$. Moreover, there exists a neighbourhood $\mathcal{V} \subset C^{2,\alpha}(M)$ of f_{g_E} such that if $L(g, f) = 0$ for some $g \in \mathcal{U}, f \in \mathcal{V}$, we have $f = P(g)$.

Next, we show that $f_g = P(g)$ for all $g \in \mathcal{U}$ (or possibly on a smaller neighbourhood). Suppose this is not the case. Then there exists a sequence g_i which converges to g in $C^{2,\alpha}$ and such that $f_i \neq P(g_i)$ for all i . By the proof of Lemma 4.1, $\|f_{g_i}\|_{C^{2,\alpha}}$ is bounded and for every $\alpha' < \alpha$, there is a subsequence, again denoted by f_{g_i} converging to f_{g_E} in $C^{2,\alpha'}$. We obviously have $L(g_i, f_{g_i}) = (0, 0)$ and for sufficiently large i we have, by the implicit function theorem, $f_{g_i} = P(g_i)$. This causes the contradiction.

We immediately get that $\mu_+(g) = H(g, P(g))$ is analytic on \mathcal{U} since H and P are analytic. □

Theorem 6.2 (Lojasiewicz-Simon inequality for μ_+). *Let (M, g_E) be an Einstein manifold with constant -1 . Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E in the space of metrics and constants $\sigma \in [1/2, 1), C > 0$ such that*

$$(6.1) \quad |\mu_+(g) - \mu_+(g_E)|^\sigma \leq C \|\text{Ric}_g + g + \nabla^2 f_g\|_{L^2}$$

for all $g \in \mathcal{U}$.

Proof. The proof is an application of a general Lojasiewicz-Simon inequality which was proven in [CM14]. Here the analyticity of μ_+ is crucial.

Since both sides are diffeomorphism invariant, it suffices to show the inequality on a slice to the action of the diffeomorphism group. Let

$$\mathcal{S}_{g_E} = \mathcal{U} \cap \{g_E + h \mid h \in \delta_{g_E}^{-1}(0)\},$$

and let $\tilde{\mu}_+$ be the μ_+ -functional restricted to \mathcal{S}_{g_E} . Obviously, $\tilde{\mu}_+$ is analytic since μ_+ is. The L^2 -gradient of μ_+ is given by $\nabla \mu_+(g) = -\frac{1}{2}(\text{Ric}_g + g + \nabla^2 f_g)e^{-f_g}$. It vanishes at g_E . On the neighbourhood \mathcal{U} , we have the uniform

estimate

$$(6.2) \quad \|\nabla\mu_+(g_1) - \nabla\mu_+(g_2)\|_{L^2} \leq C_1 \|g_1 - g_2\|_{H^2},$$

which holds by Taylor expansion and Lemma 4.2. The L^2 -gradient of $\tilde{\mu}_+$ is given by the projection of $\nabla\mu_+$ to $\delta_{g_E}^{-1}(0)$. Therefore, (6.2) also holds for $\nabla\tilde{\mu}_+$. The linearization of $\tilde{\mu}_+$ at g_E is (up to a constant factor) given by the Einstein operator, see Lemma 3.5 (iii). By ellipticity,

$$\Delta_E : (\delta_{g_E}^{-1}(0))^{C^{2,\alpha}} \rightarrow (\delta_{g_E}^{-1}(0))^{C^{0,\alpha}}$$

is Fredholm. It also satisfies the estimate $\|\Delta_E h\|_{L^2} \leq C_2 \|h\|_{H^2}$.

By [CM14, Theorem 7.3], there exists a constant $\sigma \in [1/2, 1)$ such that $|\mu_+(g) - \mu_+(g_E)|^\sigma \leq \|\nabla\tilde{\mu}_+(g)\|_{L^2}$ for any $g \in \mathcal{S}_{g_E}$. Since

$$\|\nabla\tilde{\mu}_+(g)\|_{L^2} \leq \|\nabla\mu_+(g)\|_{L^2} \leq C_3 \|\text{Ric}_g + h + \nabla^2 f_g\|_{L^2},$$

(6.1) holds for all $g \in \mathcal{S}_{g_E}$. By the slice theorem ([Ebi70, Theorem 7.1]), any metric in \mathcal{U} is isometric to some metric in \mathcal{S}_{g_E} . Thus by diffeomorphism invariance, (6.1) holds for all $g \in \mathcal{U}$. □

7. Dynamical stability and instability

With the characterization of the maximality of μ_+ and the Lojasiewicz-Simon inequality, we are nearly ready to prove the dynamical stability and instability theorems in the case of negative scalar curvature. In this section, a Ricci flow is always of the form (3.1). Two preparing lemmas are left:

Lemma 7.1 (Estimates for $t \leq 1$). *Let (M, g_E) be an Einstein manifold with constant -1 and let $k \geq 2$. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|g_0 - g_E\|_{C_{g_E}^{k+2}} < \delta$, the Ricci flow starting at g_0 exists on $[0, 1]$ and satisfies*

$$\|g(t) - g_E\|_{C_{g_E}^k} < \epsilon$$

for all $t \in [0, 1]$.

Proof. This follows from the evolution inequalities of the Riemann and the Ricci tensor under the Ricci flow (3.1) and the maximum principle for scalars exactly as in [Has12, Lemma 5.1]. □

Lemma 7.2. *Let $g(t)$, $t \in [0, T]$ be a solution of the Ricci flow and suppose that*

$$\sup_{p \in M} |R_{g(t)}|_{g(t)} \leq T^{-1} \quad \forall t \in [0, T].$$

Then for each $k \geq 1$, there exists a constant $C(k)$ such that

$$\sup_{p \in M} |\nabla^k R_{g(t)}|_{g(t)} \leq C(k) \cdot T^{-1} t^{-k/2} \quad \forall t \in (0, T].$$

Proof. This is a well known result for the standard Ricci flow [Ham95, Theorem 7.1]. The proof also works for the flow (3.1), because the evolution inequality of the Riemann tensor needed in the proof is also satisfied under the flow (3.1). □

Theorem 7.3 (Dynamical stability). *Let (M, g_E) be an Einstein manifold with constant -1 . Let $k \geq 3$. If g_E is a local maximizer of the Yamabe functional, then for every C^k -neighbourhood \mathcal{U} of g_E , there exists a C^{k+2} -neighbourhood \mathcal{V} such that the following holds:*

For any metric $g_0 \in \mathcal{V}$ there exists a 1-parameter family of diffeomorphisms φ_t such that for the Ricci flow $g(t)$ solving (3.1) which starts at g_0 , the modified flow $\varphi_t^ g(t)$ stays in \mathcal{U} for all time and converges to an Einstein metric g_∞ with constant -1 in \mathcal{U} as $t \rightarrow \infty$. The convergence is of polynomial rate, i.e. there exist constants $C, \alpha > 0$ such that*

$$\|\varphi_t^* g(t) - g_\infty\|_{C^k} \leq C(t + 1)^{-\alpha}.$$

Proof. We write \mathcal{B}_ϵ^k for the ϵ -ball around g_E with respect to the $C_{g_E}^k$ -norm. Without loss of generality, we may assume that $\mathcal{U} = \mathcal{B}_\epsilon^k$ and $\epsilon > 0$ is so small that Theorems 5.1 and 6.2 hold on \mathcal{U} .

By Lemma 7.1, we can choose a small neighbourhood \mathcal{V} such that the Ricci flow starting at any metric $g \in \mathcal{V}$ stays in $\mathcal{B}_{\epsilon/4}^k$ up to time 1. Let $T \geq 1$ be the maximal time such that for any Ricci flow $g(t)$ starting in \mathcal{V} , there exists a family of diffeomorphisms φ_t such that the modified flow $\varphi_t^* g(t)$ stays in \mathcal{U} . By definition of T and diffeomorphism invariance, we have uniform curvature bounds

$$\sup_{p \in M} |R_{g(t)}|_{g(t)} \leq C_1 \quad \forall t \in [0, T].$$

By Lemma 7.2, we have

$$(7.1) \quad \sup_{p \in M} |\nabla^l R_{g(t)}|_{g(t)} \leq C(l), \quad \forall t \in [1, T].$$

Because $f_{g(t)}$ satisfies the equation $-\Delta f_g - \frac{1}{2}|\nabla f_g|^2 + \frac{1}{2}\text{scal}_g - f_g = \mu_+(g)$, we also have

$$(7.2) \quad \sup_{p \in M} |\nabla^l f_{g(t)}|_{g(t)} \leq \tilde{C}(l), \quad \forall t \in [1, T].$$

Note that all these estimates are diffeomorphism invariant.

We now construct a modified Ricci flow as follows: Let $\varphi_t \in \text{Diff}(M)$, $t \geq 1$ be the family of diffeomorphisms generated by $X(t) = -\text{grad}_{g(t)} f_{g(t)}$ and define

$$(7.3) \quad \tilde{g}(t) = \begin{cases} g(t), & t \in [0, 1], \\ \varphi_t^* g(t), & t \geq 1. \end{cases}$$

The modified flow satisfies (3.1) for $t \in [0, 1]$ while for $t \geq 1$, we have

$$\frac{d}{dt} \tilde{g}(t) = -2(\text{Ric}_{\tilde{g}(t)} + \tilde{g}(t) + \nabla^2 f_{\tilde{g}(t)}).$$

Let $T' \in [0, T]$ be the maximal time such that the modified Ricci flow, starting at any metric $g_0 \in \mathcal{V}$, stays in \mathcal{U} up to time T' . Then

$$\begin{aligned} \|\tilde{g}(T') - g_E\|_{C_{g_E}^k} &\leq \|\tilde{g}(1) - g_E\|_{C_{g_E}^k} + \int_1^{T'} \|\dot{\tilde{g}}(t)\|_{C_{g_E}^k} dt \\ &\leq \frac{\epsilon}{4} + 2 \int_1^{T'} \|\dot{\tilde{g}}(t)\|_{C_{\tilde{g}(t)}^k} dt, \end{aligned}$$

provided that \mathcal{U} is small enough. By the interpolation inequality for tensors (see [Ham82, Corollary 12.7]), (7.1) and (7.2), we have

$$\|\dot{\tilde{g}}(t)\|_{C_{\tilde{g}(t)}^k} \leq C_2 \|\dot{\tilde{g}}(t)\|_{L_{\tilde{g}(t)}^2}^{1-\eta}$$

for η as small as we want. In particular, we can assume that $\theta := 1 - \sigma(1 + \eta) > 0$, where σ is the exponent appearing in Theorem 6.2. By the first

variation of μ_+ ,

$$\frac{d}{dt}\mu_+(\tilde{g}(t)) \geq C_3 \|\dot{\tilde{g}}(t)\|_{L^2_{\tilde{g}(t)}}^{1+\eta} \|\dot{\tilde{g}}(t)\|_{L^2_{\tilde{g}(t)}}^{1-\eta}.$$

By Theorem 5.1 and Theorem 6.2 again,

$$\begin{aligned} & -\frac{d}{dt}|\mu_+(\tilde{g}(t)) - \mu_+(g_E)|^\theta \\ &= \theta|\mu_+(\tilde{g}(t)) - \mu_+(g_E)|^{\theta-1} \frac{d}{dt}\mu_+(\tilde{g}(t)) \\ &\geq C_4|\mu_+(\tilde{g}(t)) - \mu_+(g_E)|^{-\sigma(1+\eta)} \|\dot{\tilde{g}}(t)\|_{L^2_{\tilde{g}(t)}}^{1+\eta} \|\dot{\tilde{g}}(t)\|_{L^2_{\tilde{g}(t)}}^{1-\eta} \\ &\geq C_5 \|\dot{\tilde{g}}(t)\|_{C^k_{\tilde{g}(t)}}. \end{aligned}$$

Hence by integration,

$$\begin{aligned} \int_1^{T'} \|\dot{\tilde{g}}(t)\|_{C^k_{\tilde{g}(t)}} dt &\leq \frac{1}{C_5} |\mu_+(\tilde{g}(1)) - \mu_+(g_E)|^\theta \\ &\leq \frac{1}{C_5} |\mu_+(\tilde{g}(0)) - \mu_+(g_E)|^\theta \leq \frac{\epsilon}{8}, \end{aligned}$$

provided that \mathcal{V} is small enough. This shows that $\|\tilde{g}(T') - g_E\|_{C^k_{g_E}} \leq \epsilon/2 < \epsilon$, so T' cannot be finite. Thus, $T = \infty$ and $\tilde{g}(t)$ converges to some limit metric $g_\infty \in \mathcal{U}$ as $t \rightarrow \infty$. By the Lojasiewicz-Simon inequality, we have

$$\frac{d}{dt}|\mu_+(\tilde{g}(t)) - \mu_+(g_E)|^{1-2\sigma} \geq C_6,$$

which implies

$$|\mu_+(\tilde{g}(t)) - \mu_+(g_E)| \leq C_7(t + 1)^{-\frac{1}{2\sigma-1}}.$$

Here, we may assume that $\sigma > \frac{1}{2}$ because the Lojasiewicz-Simon inequality also holds after enlarging the exponent. Therefore, $\mu_+(g_\infty) = \mu_+(g_E)$, so g_∞ is an Einstein metric with constant -1 . The convergence is of polynomial rate since for $t_1 < t_2$,

$$\|\tilde{g}(t_1) - \tilde{g}(t_2)\|_{C^k} \leq C_8|\mu_+(\tilde{g}(t_1)) - \mu_+(g_E)|^\theta \leq C_9(t_1 + 1)^{-\frac{\theta}{2\sigma-1}},$$

and the assertion follows from $t_2 \rightarrow \infty$. □

Theorem 7.4 (Dynamical instability). *Let (M, g_E) be an Einstein manifold with constant -1 which is not a local maximizer of the Yamabe functional. Then there exists a nontrivial ancient Ricci flow $g(t)$ solving (3.1), defined on $(-\infty, 0]$, and a 1-parameter family of diffeomorphisms φ_t , $t \in (-\infty, 0]$ such that $\varphi_t^* g(t) \rightarrow g_E$ as $t \rightarrow -\infty$.*

Proof. Since (M, g_E) is not a local maximum of the Yamabe functional, it cannot be a local maximum of μ_+ . Let $g_i \rightarrow g_E$ in C^k and suppose that we have $\mu_+(g_i) > \mu_+(g_E)$ for all i . Let $\tilde{g}_i(t)$ be the modified flow defined in (7.3), starting at g_i . Then by Lemma 7.1, $\tilde{g}_i = \tilde{g}_i(1)$ converges to g_E in C^{k-2} and by monotonicity, $\mu_+(\tilde{g}_i) > \mu_+(g_E)$ as well. Let $\epsilon > 0$ be so small that Theorem 6.2 holds on $\mathcal{B}_{2\epsilon}^{k-2}$. Theorem 6.2 yields the differential inequality

$$\frac{d}{dt}(\mu_+(\tilde{g}_i(t)) - \mu_+(g_E))^{1-2\sigma} \leq -C_1,$$

from which we obtain

$$\mu_+(\tilde{g}_i(t)) \geq [(\mu_+(\tilde{g}_i(1)) - \mu_+(g_E))^{1-2\sigma} - C_1(t-1)]^{-\frac{1}{2\sigma-1}} + \mu_+(g_E)$$

as long as $\tilde{g}_i(t)$ stays in $\mathcal{B}_{2\epsilon}^{k-2}$. Thus, there exists a t_i such that

$$\|\tilde{g}_i(t_i) - g_E\|_{C^{k-2}} = \epsilon,$$

and $t_i \rightarrow \infty$. If $\{t_i\}$ was bounded, $\tilde{g}_i(t_i) \rightarrow g_E$ in C^{k-2} . By interpolation,

$$\|\text{Ric}_{\tilde{g}_i(t)} + \nabla^2 f_{\tilde{g}_i(t)} + \tilde{g}_i(t)\|_{C^{k-2}} \leq C_2 \|\text{Ric}_{\tilde{g}_i(t)} + \nabla^2 f_{\tilde{g}_i(t)} + \tilde{g}_i(t)\|_{L^2}^{1-\eta}$$

for $\eta > 0$ as small as we want. We may assume that $\theta = 1 - \sigma(1 + \eta) > 0$. By Theorem 6.2, we have the differential inequality

$$\frac{d}{dt}(\mu_+(\tilde{g}_i(t)) - \mu_+(g_E))^\theta \geq C_3 \|\text{Ric}_{\tilde{g}_i(t)} + \tilde{g}_i(t)\|_{L^2}^{1-\eta},$$

if $\mu_+(\tilde{g}_i(t)) > \mu_+(g_E)$. Thus,

$$(7.4) \quad \epsilon = \|\tilde{g}_i(t_i) - g_E\|_{C^{k-2}} \leq \|\tilde{g}_i - g_E\|_{C^{k-2}} + C_4(\mu_+(\tilde{g}_i(t_i)) - \mu_+(g_E))^\theta.$$

Now put $\tilde{g}_i^s(t) := \tilde{g}_i(t + t_i)$, $t \in [T_i, 0]$, where $T_i = 1 - t_i \rightarrow -\infty$. We have

$$\begin{aligned} \|\tilde{g}_i^s(t) - g_E\|_{C^{k-2}} &\leq \epsilon & \forall t \in [T_i, 0], \\ \tilde{g}_i^s(T_i) &\rightarrow g_E & \text{in } C^{k-2}. \end{aligned}$$

Because the embedding $C^{k-2}(M) \subset C^{k-3}(M)$ is compact, we can choose a subsequence of the \tilde{g}_i^s , converging in $C_{loc}^{k-3}(M \times (-\infty, 0])$ to an ancient flow $\tilde{g}(t)$, $t \in (-\infty, 0]$, satisfying the differential equation

$$\dot{\tilde{g}}(t) = -2(\text{Ric}_{\tilde{g}(t)} + \tilde{g}(t) + \nabla^2 f_{\tilde{g}(t)}).$$

Let φ_t , $t \in (-\infty, 0]$ be the diffeomorphisms generated by $X(t) = \text{grad}_{\tilde{g}(t)} f_{\tilde{g}}$ where $\varphi_0 = \text{id}$. Then $g(t) = \varphi_t^* \tilde{g}(t)$ is a solution of (3.1). From taking the limit $i \rightarrow \infty$ in (7.4), we obtain $\epsilon \leq C_4(\mu_+(g(0)) - \mu_+(g_E))^{\beta/2}$ and therefore, the Ricci flow is nontrivial. For $T_i \leq t$, we have, by the Lojasiewicz-Simon inequality,

$$\begin{aligned} \|\tilde{g}_i^s(T_i) - \tilde{g}_i^s(t)\|_{C^{k-3}} &\leq C_4(\mu_+(\tilde{g}_i(t+t_i)) - \mu_+(g_E))^\theta \\ &\leq C_4[-C_1 t + (\mu_+(\tilde{g}_i(t_i)) - \mu_+(g_E))^{1-2\sigma}]^{-\frac{\theta}{2\sigma-1}} \\ &\leq [-C_5 t + C_6]^{-\frac{\theta}{2\sigma-1}}. \end{aligned}$$

Thus,

$$\begin{aligned} \|g_E - \tilde{g}(t)\|_{C^{k-3}} &\leq \|g_E - \tilde{g}_i^s(T_i)\|_{C^{k-3}} \\ &\quad + [-C_5 t + C_6]^{-\frac{\theta}{2\sigma-1}} + \|\tilde{g}_i^s(t) - \tilde{g}(t)\|_{C^{k-3}}. \end{aligned}$$

It follows that $\|g_E - \tilde{g}(t)\|_{C^{k-3}} \rightarrow 0$ as $t \rightarrow -\infty$. Therefore, $(\varphi_t^{-1})^* g(t) \rightarrow g_E$ in C^{k-3} as $t \rightarrow -\infty$. □

Remark 7.5. Dynamical stability and instability under the volume-normalized Ricci flow follow from these theorems by projecting the flows above to the space of metrics of fixed volume and by a suitable rescaling of the time parameter. In this way, we obtain the results as stated in the introduction.

8. Einstein metrics with positive scalar curvature

In this section, we state analogous stability/instability results for Einstein metrics with positive scalar curvature. Since the methods are very similar, we skip the details and we just explain the key steps and the main differences. For details, we refer the reader to [Krö13, Chapter 6]. We define the Ricci

shrinker entropy which was first introduced by G. Perelman in [Per02]. Let

$$\mathcal{W}_-(g, f, \tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_M [\tau(|\nabla f|_g^2 + \text{scal}_g) + f - n] e^{-f} dV.$$

For $\tau > 0$, let

$$\mu_-(g, \tau) = \inf \left\{ \mathcal{W}_-(g, f, \tau) \mid f \in C^\infty(M), \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-f} dV_g = 1 \right\}.$$

For any $\tau > 0$, the infimum is realized by a smooth function. We define the shrinker entropy as

$$\nu_-(g) = \inf \{ \mu_-(g, \tau) \mid \tau > 0 \}.$$

Observe that ν_- is scale and diffeomorphism invariant. If $\lambda(g) > 0$, then $\nu_-(g)$ is finite and realized by some $\tau_g > 0$ (see [CCG⁺07, Corollary 6.34]). In this case, a pair (f_g, τ_g) realizing $\nu_-(g)$ satisfies the equations

$$(8.1) \quad \tau(2\Delta f + |\nabla f|^2 - \text{scal}) - f + n + \nu_- = 0,$$

$$(8.2) \quad \frac{1}{(4\pi\tau)^{n/2}} \int_M f e^{-f} dV = \frac{n}{2} + \nu_-,$$

see e.g. [CZ12, p. 5]. The first variation of ν_- is

$$(8.3) \quad \nu_-(g)'(h) = -\frac{1}{(4\pi\tau_g)^{n/2}} \int_M \left\langle \tau_g(\text{Ric} + \nabla^2 f_g) - \frac{1}{2}g, h \right\rangle e^{-f_g} dV_g,$$

where (f_g, τ_g) realizes $\nu_-(g)$. Because of diffeomorphism invariance, ν_- is nondecreasing under the τ -flow

$$(8.4) \quad \dot{g}(t) = -2\text{Ric}_{g(t)} + \frac{1}{\tau_{g(t)}}g(t).$$

Remark 8.1. The critical metrics of ν_- are precisely the shrinking gradient Ricci solitons. These are the metrics satisfying $\text{Ric} + \nabla^2 f = cg$ for some $f \in C^\infty(M)$ and $c > 0$. This includes all positive Einstein metrics. If g_E is a positive Einstein metric with Einstein constant μ , the pair (f_{g_E}, τ_{g_E}) satisfies

$$(8.5) \quad \tau_{g_E} = \frac{1}{2\mu}, \quad f_{g_E} = \log(\text{vol}(M, g_E)) - \frac{n}{2}(\log(2\pi) - \log(\mu)).$$

Proposition 8.2 (Second variation of ν_-). *The second variation of ν_- on a positive Einstein metric (M, g_E) with constant μ is given by*

$$\nu_-(g_E)''(h) = \begin{cases} -\frac{1}{4\mu} \int_M \langle \Delta_E h, h \rangle dV, & \text{if } \delta h = 0 \text{ and } \int_M \text{tr} h dV = 0, \\ 0, & \text{if } h \in \mathbb{R} \cdot g_E \oplus \delta^*(\Omega^1(M)). \end{cases}$$

Proof. This is a simpler expression of the second variational formula in [CHI04, Theorem 2.1]. By scale and diffeomorphism invariance, $\nu_-(g_E)''$ vanishes on the subspace $\mathbb{R} \cdot g_E \oplus \delta^*(\Omega^1(M))$. If $\delta h = 0$ and $\int_M \text{tr} h dV = 0$, the formula follows from the one in [CHI04]. Since $\Delta_E = \Delta_L - 2\mu$, $\text{tr}(\Delta_L h) = \Delta(\text{tr} h)$ and $\delta(\Delta_L h) = \Delta_H(\delta h)$ [Lic61, pp. 28-29], Δ_E preserves the above subspace. Here, Δ_H is the Hodge-Laplacian acting on one-forms. Thus, the splitting of above is orthogonal with respect to $\nu_-(g_E)''$. Observe also that these two subspaces span the whole space $\Gamma(S^2M)$. □

Now, one has to check that on a small neighbourhood \mathcal{U} of an Einstein metric, f_g and τ_g are unique. Moreover, if \mathcal{U} is small enough, $\nu_-(g)$, f_g and τ_g depend analytically on the metric. These facts follows from the the implicit function theorem for Banach manifolds and a bootstrap argument using elliptic regularity and the Euler-Lagrange equations (8.1) and (8.2). Similar arguments were used in Lemma 4.1 and Lemma 6.1.

Furthermore, one has to prove bounds for $\nu_-(g)$, f_g , τ_g and their derivatives as in Section 4. These follow essentially from differentiating (8.1) and (8.2) and using elliptic regularity. Having developed these technical tools, one is able to prove

Theorem 8.3. *Let (M, g_E) be a positive Einstein manifold with constant μ and let λ be the smallest nonzero eigenvalue of the Laplacian. If g_E is a local maximum of ν_- , it is a local maximum of the Yamabe functional and we have $\lambda \geq 2\mu$. Conversely, if g_E is a local maximum of the Yamabe functional and $\lambda > 2\mu$, then g_E is a local maximum of ν_- . In this case, any other local maximum is also an Einstein metric.*

Sketch of proof. We use the same notation as in the proof of Theorem 5.1. Let \bar{g} be a metric of constant scalar curvature. Then $\nu_-(\bar{g})$ is explicitly given by

$$\nu_-(\bar{g}) = \log(\text{vol}(M, \bar{g})) + \frac{n}{2} \log(\text{scal}_{\bar{g}}) + \frac{n}{2}(1 - \log(2\pi n)).$$

This follows from (8.1) and (8.2) and the analytic dependence of f_g and τ_g on g . Recall that g_E is a local maximum of the Yamabe functional if and only if

it is a local maximum of the scalar curvature on \mathcal{C}_c . Therefore, g_E is a local maximum of the Yamabe functional if and only if it is a local maximum of ν_- in \mathcal{C}_c (where $c = \text{vol}(M, g_E)$). By scale invariance, it is also a local maximum of ν_- in \mathcal{C} in this case. Let now $\bar{g} \in \mathcal{C}$ and $v \in C^\infty(M)$ such that $\int_M v \, dV_{\bar{g}} = 0$. Then by the first variational formula, $\frac{d}{dt}|_{t=0} \nu_-((1 + tv)\bar{g}) = 0$. A long but straightforward calculation shows that

$$(8.6) \quad \frac{d^2}{dt^2} \Big|_{t=0} \nu_-((1 + tv)\bar{g}) = - \int_M Lv \cdot v \, dV,$$

where L is the linear operator given by

$$L = \frac{n + 1}{4} \left(\frac{n}{\text{scal}} \Delta - 1 \right)^{-1} \left(\frac{n}{\text{scal}} \Delta - 2 \right) \left(\frac{n}{\text{scal}} \Delta - \frac{n}{n - 1} \right).$$

This formula shows how the eigenvalue condition comes into play. Now the first assertion is clear, since $\lambda > \frac{\text{scal}}{n-1}$ for any Einstein metric except the standard sphere [Oba62, Theorem 1 and Theorem 2]. The second assertion follows from the local decomposition of the space of metrics and Taylor expansion as in the proof of Theorem 5.1. \square

Corollary 8.4. *Let (M, g_E) be a compact positive Einstein manifold with constant μ . If g_E is a local maximum of the Yamabe invariant and $\lambda > 2\mu$, any shrinking gradient Ricci soliton in a sufficiently small neighbourhood of g_E is necessarily Einstein.*

Proof. This follows from Theorem 8.3 and the fact that shrinking gradient Ricci solitons are precisely the critical points of ν_- . \square

The proof of the following theorem is analogous to the proof of Theorem 6.2.

Theorem 8.5 (Lojasiewicz-Simon inequality). *Let (M, g_E) be a positive Einstein manifold. Then there exists a $C^{2,\alpha}$ -neighbourhood \mathcal{U} of g_E and constants $\sigma \in [1/2, 1)$, $C > 0$ such that*

$$(8.7) \quad |\nu_-(g) - \nu_-(g_E)|^\sigma \leq C \left\| \tau(\text{Ric}_g + \nabla^2 f_g) - \frac{1}{2}g \right\|_{L^2}$$

for all $g \in \mathcal{U}$.

Now we have the tools to prove the stability/instability results for positively curved Einstein metrics. The proofs are the same as in the negative case.

Theorem 8.6 (Dynamical stability). *Let (M, g_E) be a compact positive Einstein manifold with constant μ and let $k \geq 3$. Suppose that g_E is a local maximizer of the Yamabe functional and the smallest nonzero eigenvalue of the Laplacian is larger than 2μ . Then for every C^k -neighbourhood \mathcal{U} of g_E , there exists a C^{k+2} -neighbourhood \mathcal{V} such that the following holds:*

*For any metric $g_0 \in \mathcal{V}$, there exists a 1-parameter family of diffeomorphisms φ_t and a positive function v such that for the τ -flow $g(t)$ starting at g_0 , the modified flow $\varphi_t^*g(t)$ stays in \mathcal{U} for all time and converges to an Einstein metric g_∞ in \mathcal{U} as $t \rightarrow \infty$. The convergence is of polynomial rate, i.e. there exist constants $C, \alpha > 0$ such that*

$$\|\varphi_t^*g(t) - g_\infty\|_{C^k} \leq C(t+1)^{-\alpha}.$$

Theorem 8.7 (Dynamical instability). *Let (M, g_E) be a positive Einstein manifold that is not a local maximizer of ν_- . Then there exists a non-trivial ancient τ -flow $g(t)$, $t \in (-\infty, 0]$ and a 1-parameter family of diffeomorphisms φ_t , $t \in (-\infty, 0]$ such that $\varphi_t^*g(t) \rightarrow g_E$ as $t \rightarrow \infty$.*

Remark 8.8. One gets stronger convergence statements if one replaces the assumption of local maximality of the Yamabe functional in the Theorems 7.3 and 8.6 by the assumption that the Einstein operator Δ_E is non-negative and all infinitesimal Einstein deformations are integrable, i.e. all elements in the kernel of Δ_E can be integrated to curves of Einstein metrics. In this case, the flow will converge exponentially and we do not have to pull back the flow by diffeomorphisms. For details, see [Krö13, Section 5.4 and 6.4].

9. Dynamical instability of the complex projective space

In order to prove Theorem 1.7, we show that the Einstein metric cannot be a local maximum of ν_- . Since the second variation of ν_- at g_E may be nonpositive, we have to compute a third variation of it.

Proposition 9.1. *Let (M, g_E) be a positive Einstein manifold with constant μ and suppose we have a function $v \in C^\infty(M)$ such that $\Delta v = 2\mu \cdot v$. Then the third variation of ν_- in the direction of $v \cdot g_E$ is given by*

$$\left. \frac{d^3}{dt^3} \right|_{t=0} \nu_-(g_E + tv \cdot g_E) = (n-2) \int_M v^3 dV.$$

Proof. Put $u = \frac{e^{-f}}{(4\pi\tau)^{n/2}}$. By the first variation, the negative of the $L^2(u \, dV)$ -gradient of ν_- is given by $\nabla\nu_- = \tau(\text{Ric} + \nabla^2 f) - \frac{g}{2}$, so

$$\frac{d}{dt} \Big|_{t=0} \nu_-(g_E + th) = - \int_M \langle \nabla\nu_-, h \rangle u \, dV.$$

Since (M, g_E) is a critical point of ν_- , we clearly have $\nabla\nu_- = 0$. Since v is a nonconstant eigenfunction, $\int_M v \, dV = 0$. Thus by [CZ12, Lemma 2.4], τ' vanishes. Recall from (8.5) that $\tau_{g_E} = \frac{1}{2\mu}$ and f_{g_E} is constant. Therefore, by the first variation of the Ricci tensor,

$$\begin{aligned} \nabla\nu'_- &= \tau' \mu g_E + \frac{1}{2\mu} (\text{Ric}' + \nabla^2(f')) - \frac{g'}{2} \\ &= \frac{1}{2\mu} \left(\frac{1}{2} \Delta_L(v \cdot g_E) - \delta^* \delta(v \cdot g_E) - \frac{1}{2} \nabla^2 \text{tr}(v \cdot g_E) + \nabla^2(f') \right) - \frac{v \cdot g_E}{2} \\ &= \frac{1}{2\mu} \left(\left(1 - \frac{n}{2}\right) \nabla^2 v + \nabla^2(f') \right). \end{aligned}$$

To compute f' , we consider the Euler-Lagrange equation

$$(9.1) \quad \tau(2\Delta f + |\nabla f|^2 - \text{scal}) - f + n + \nu_- = 0.$$

By differentiating once and using $\tau' = 0$ and $\nu'_- = 0$,

$$\frac{1}{2\mu} (2\Delta f' - \text{scal}') - f' = 0,$$

and by the first variation of the scalar curvature,

$$\left(\frac{1}{\mu} \Delta - 1 \right) f' = \frac{1}{2\mu} \text{scal}' = \frac{1}{2\mu} ((n-1)\Delta v - n\mu v).$$

By a well-known eigenvalue estimate for the Laplacian ([Oba62, Theorem 1]), $\frac{1}{\mu} \Delta - 1$ is invertible. By using the eigenvalue equation, we therefore obtain

$$(9.2) \quad f' = \left(\frac{n}{2} - 1 \right) v.$$

Thus,

$$(9.3) \quad \nabla\nu'_- = 0,$$

and therefore, the third variation equals

$$\frac{d^3}{dt^3} \Big|_{t=0} \nu_-(g_E + tv \cdot g_E) = - \int_M \langle \nabla \nu_-'', v \cdot g_E \rangle u \, dV.$$

Since $\tau_{g_E} = \frac{1}{2\mu}$ and $\tau' = 0$,

$$\nabla \nu_-'' = \tau'' \mu \cdot g_E + \frac{1}{2\mu} (\text{Ric} + \nabla^2 f)'.$$

The function u is constant since f is constant. Thus, the τ'' -term drops out after integration. We are left with

$$(9.4) \quad \frac{d^3}{dt^3} \Big|_{t=0} \nu_-(g_E + tv \cdot g_E) = - \frac{1}{2\mu} \int_M \langle (\text{Ric} + \nabla^2 f)'', v \cdot g_E \rangle u \, dV.$$

We first compute Ric'' . Let $g_t = (1 + tv)g_E$ and $v_t = \frac{v}{1+tv}$. Then $g_t' = v_t \cdot g_t$ and $\frac{d}{dt} \Big|_{t=0} v_t = -v^2$. By the first variation of the Ricci tensor,

$$\frac{d}{dt} \text{Ric}_{g_t} = \frac{1}{2} [(\Delta v_t)g_t - (n - 2)\nabla^2 v_t],$$

and the second variation at g_E is equal to

$$\begin{aligned} & \frac{d^2}{dt^2} \Big|_{t=0} \text{Ric}_{g_E + tv \cdot g_E} = \frac{d}{dt} \Big|_{t=0} \frac{1}{2} [(\Delta v_t)g_t - (n - 2)\nabla^2 v_t] \\ &= \frac{1}{2} [(\Delta' v + \Delta(v')) + \Delta v \cdot v]g_E - (n - 2)(\nabla^2)'v - (n - 2)\nabla^2(v') \\ &= \frac{1}{2} [(\langle v \cdot g_E, \nabla^2 v \rangle - \langle \delta(v \cdot g_E) + \frac{1}{2} \nabla \text{tr}(v \cdot g_E), \nabla v \rangle)g_E \\ & \quad + (-\Delta v \cdot v + 2|\nabla v|^2)g_E - (n - 2) \left(\frac{1}{2} |\nabla v|^2 g_E - \nabla v \otimes \nabla v \right) \\ & \quad + (n - 2)(2\nabla^2 v \cdot v + 2\nabla v \otimes \nabla v)] \\ &= - \left(\frac{n}{2} - 2 \right) |\nabla v|^2 g_E - 2\mu v^2 g_E + 3 \left(\frac{n}{2} - 1 \right) \nabla v \otimes \nabla v + (n - 2)\nabla^2 v \cdot v, \end{aligned}$$

where we used the first variational formulas of the Laplacian and the Hessian in Lemma A.2. Let us now compute the $(\nabla^2 f)''$ -term. Since f_{g_E} is constant,

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \nabla^2 f_{g_E + tv \cdot g_E} &= \nabla^2(f'') + 2(\nabla^2)'f' \\ &= \nabla^2(f'') - \nabla v \otimes \nabla f' - \nabla f' \otimes \nabla v + \langle \nabla f', \nabla v \rangle g_E. \end{aligned}$$

We already know that $f' = (\frac{n}{2} - 1)v$ by (9.2). To compute f'' , we differentiate (9.1) twice. By (9.3), $v'' = 0$. Since also $\tau' = 0$ as remarked above, we obtain

$$(9.5) \quad \begin{aligned} 0 &= -\tau'' \text{scal} + \tau(2\Delta f + |\nabla f|^2 - \text{scal})'' - f'' \\ &= -\tau'' n\mu + \frac{1}{\mu} \Delta f'' + \frac{2}{\mu} \Delta' f' + \frac{1}{\mu} |\nabla(f')|^2 - \frac{1}{2\mu} \text{scal}'' - f''. \end{aligned}$$

Because $\Delta v = 2\mu v$,

$$(9.6) \quad \begin{aligned} \Delta' f' &= \langle v \cdot g, \nabla^2 f' \rangle - \left\langle \delta(v \cdot g) + \frac{1}{2} \nabla \text{tr}(v \cdot g), \nabla f' \right\rangle \\ &\stackrel{(9.2)}{=} \left(\frac{n}{2} - 1\right) \left[-2\mu v^2 - \left(\frac{n}{2} - 1\right) |\nabla v|^2\right]. \end{aligned}$$

Next, we compute scal'' . As above, let $g_t = (1 + tv)g_E$ and $v_t = \frac{v}{1+tv}$. Then by the first variation of the scalar curvature,

$$\frac{d}{dt} \text{scal}_{g_t} = (n - 1)\Delta v_t - \text{scal}_{g_t} v_t.$$

The second variation of the scalar curvature at g_E is equal to

$$\begin{aligned} &\frac{d^2}{dt^2} \Big|_{t=0} \text{scal}_{g_E+tv \cdot g_E} = \frac{d}{dt} \Big|_{t=0} [(n - 1)\Delta v_t - \text{scal}_{g_t} v_t] \\ &= (n - 1)[\Delta' v + \Delta(v')] - n\mu \cdot v' - \text{scal}' v \\ &= (n - 1) \left[\langle v \cdot g_E, \nabla^2 v \rangle - \langle \delta(v \cdot g_E) + \frac{1}{2} \nabla \text{tr}(v \cdot g_E), \nabla v \rangle - \Delta(v^2) \right] \\ &\quad + n\mu \cdot v^2 - [\Delta \text{tr}(v \cdot g_E) + \delta\delta(v \cdot g_E) - \langle \text{Ric}, v \cdot g_E \rangle] v \\ &= -(n - 1) \left(\frac{n}{2} - 3\right) |\nabla v|^2 + 2\mu(4 - 3n) \cdot v^2. \end{aligned}$$

By (9.2), $|\nabla(f')|^2 = (\frac{n}{2} - 1)^2 |\nabla v|^2$. Thus, we can rewrite (9.5) as

$$\begin{aligned} &\left(\frac{1}{\mu} \Delta - 1\right) f'' = \tau'' n\mu - \frac{1}{\mu} \left(2\Delta' f' + |\nabla(f')|^2 - \frac{1}{2} \text{scal}''\right) \\ &\stackrel{(9.6)}{=} \tau'' n\mu - \frac{1}{\mu} \left[-2(n - 2)\mu v^2 - 2\left(\frac{n}{2} - 1\right)^2 |\nabla v|^2 + \left(\frac{n}{2} - 1\right)^2 |\nabla v|^2\right. \\ &\quad \left.+ \frac{n - 1}{2} \left(\frac{n}{2} - 3\right) |\nabla v|^2 + (3n - 4)\mu v^2\right] \\ &= \tau'' n\mu - \frac{1}{\mu} \left[n\mu v^2 + \left(-\frac{3}{4}n + \frac{1}{2}\right) |\nabla v|^2\right] =: (A). \end{aligned}$$

Since $\frac{1}{\mu}\Delta - 1$ is invertible, we can rewrite the above as

$$f'' = \left(\frac{1}{\mu}\Delta - 1\right)^{-1} (A).$$

By integrating,

$$\begin{aligned} & -\frac{1}{2\mu} \int_M \langle (\nabla^2 f)'', v \cdot g_E \rangle u \, dV = -\frac{1}{2\mu} \int_M \langle (\nabla^2 f)'', v \cdot g_E \rangle \, dV \\ & = -\frac{1}{2\mu} \int_M \langle \nabla^2(f'') - \nabla v \otimes \nabla f' - \nabla f' \otimes \nabla v + \langle \nabla f', \nabla v \rangle g_E, v \cdot g_E \rangle \, dV \\ & \stackrel{9.2}{=} -\frac{1}{2\mu} \int_M \left\langle \nabla^2(f'') - (n-2)\nabla v \otimes \nabla v + \left(\frac{n}{2} - 1\right) |\nabla v|^2 g_E, v \cdot g_E \right\rangle \, dV \\ & = -\frac{1}{2\mu} \int_M \left[-\Delta(f'')v + \frac{1}{2}(n-2)^2 |\nabla v|^2 v \right] \, dV \\ & = -\frac{1}{2\mu} \int_M \left[-(A) \left(\frac{1}{\mu}\Delta - 1\right)^{-1} \Delta v + \frac{1}{2}(n-2)^2 |\nabla v|^2 v \right] \, dV \\ & = -\frac{1}{2\mu} \int_M \left[-2\mu(A)v + \frac{1}{2}(n-2)^2 |\nabla v|^2 v \right] \, dV. \end{aligned}$$

Now we insert the definition of (A) . Since the term containing τ'' drops out after integration, we are left with

$$\begin{aligned} & -\frac{1}{2\mu} \int_M \langle (\nabla^2 f)'', v \cdot g_E \rangle u \, dV \\ & = -\frac{1}{2\mu} \int_M \left[(2n\mu v^3 + \frac{1}{2}(n^2 - 7n + 6) |\nabla v|^2 v) \right] \, dV. \end{aligned}$$

By the second variation of the Ricci tensor computed above,

$$\begin{aligned} & -\frac{1}{2\mu} \int_M \langle \text{Ric}'', v \cdot g_E \rangle u \, dV = -\frac{1}{2\mu} \int_M \langle \text{Ric}'', v \cdot g_E \rangle \, dV \\ & = -\frac{1}{2\mu} \int_M \left[-n \left(\frac{n}{2} - 2\right) |\nabla v|^2 v \right. \\ & \quad \left. - 2\mu n v^3 + 3 \left(\frac{n}{2} - 1\right) |\nabla v|^2 v - (n-2)\Delta v \cdot v^2 \right] \, dV \\ & = -\frac{1}{2\mu} \int_M \left[\left(-\frac{n^2}{2} + \frac{7n}{2} - 3\right) |\nabla v|^2 v - 4(n-1)\mu v^3 \right] \, dV. \end{aligned}$$

Adding up these two terms, we obtain

$$\left. \frac{d^3}{dt^3} \right|_{t=0} \nu_-(g + tv \cdot g) \stackrel{(9.4)}{=} -\frac{1}{2\mu} \int_M (4 - 2n)\mu v^3 dV.$$

and therefore, we finally have

$$\left. \frac{d^3}{dt^3} \right|_{t=0} \nu_-(g + tv \cdot g) = (n - 2) \int_M v^3 dV,$$

which finishes the proof. □

Corollary 9.2. *Let (M^n, g_E) , $n \geq 3$ be a positive Einstein manifold with constant μ . Suppose there exists a function $v \in C^\infty(M)$ such that $\Delta v = 2\mu v$ and $\int_M v^3 dV \neq 0$. Then g_E is not a local maximum of ν_- .*

Proof. Let $\varphi(t) = \nu_-(g_E + tv \cdot g_E)$. By the proof of the proposition above, $\varphi'(0) = 0$, $\varphi''(0) = 0$ and $\varphi'''(0) \neq 0$. Depending on the sign of the third variation, $\varphi(t) > \varphi(0)$ either for $t \in (-\epsilon, 0)$ or $t \in (0, \epsilon)$. This proves the assertion. □

Theorem 1.7 now follows from Corollary 9.2 and Theorem 8.7.

Proof of Corollary 1.8. Let μ be the Einstein constant. We prove the existence of a function $v \in C^\infty(\mathbb{C}P^n)$ satisfying $\Delta v = 2\mu v$ and $\int_{\mathbb{C}P^n} v^3 dV \neq 0$. First, we review the construction of eigenfunctions on $\mathbb{C}P^n$ as explained in [BGM71, Section III C]. Consider $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$ with coordinates $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1})$ and let $z_j = x_j + iy_j$, $\bar{z}_j = x_j - iy_j$ be the complex coordinates. Defining $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and $\partial_{\bar{z}_j} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$, we can rewrite the Laplace operator on \mathbb{C}^{n+1} as

$$\Delta = -4 \sum_{j=1}^{n+1} \partial_{z_j} \circ \partial_{\bar{z}_j}.$$

Let $P_{k,k}$ be the space of complex polynomials on \mathbb{C}^{n+1} which are homogeneous of degree k in z and \bar{z} and let $H_{k,k}$ the subspace of harmonic polynomials in $P_{k,k}$. We have

$$P_{k,k} = H_{k,k} \oplus r^2 P_{k-1,k-1}.$$

Elements in $P_{k,k}$ are S^1 -invariant and thus, they descend to functions on the quotient $\mathbb{C}P^n = S^{2n+1}/S^1$. The eigenfunctions to the k -th eigenvalue of the

Laplacian on $\mathbb{C}P^n$ (where 0 is meant to be the 0-th eigenvalue) are precisely the restrictions of functions in $H_{k,k}$. Since 2μ is the first nonzero eigenvalue, its eigenfunctions are restrictions of functions in $H_{1,1}$.

Let $h_1(z, \bar{z}) = z_1\bar{z}_2 + z_2\bar{z}_1$, $h_2(z, \bar{z}) = z_2\bar{z}_3 + z_3\bar{z}_2$, $h_3(z, \bar{z}) = z_3\bar{z}_1 + z_1\bar{z}_3$ and let v be the eigenfunction which is the restriction of $h = h_1 + h_2 + h_3 \in H_{1,1}$. Note that h is real-valued and so is v . Then v^3 is the restriction of

$$(9.7) \quad h^3 \in P_{3,3} = H_{3,3} \oplus r^2 H_{2,2} \oplus r^4 H_{1,1} \oplus r^6 H_{0,0}.$$

We show that $\int_{S^{2n+1}} h^3 dV \neq 0$. At first,

$$h^3 = \sum_{j=1}^3 h_j^3 + 3 \sum_{j \neq l} h_j \cdot h_l^2 + 6h_1 \cdot h_2 \cdot h_3.$$

Note that $\int_{S^{2n+1}} h_1^3 dV = 0$ because h_1 is antisymmetric with respect to the isometry $(z_1, \bar{z}_1) \mapsto (-z_1, -\bar{z}_1)$. For the same reason, $\int_{S^{2n+1}} h_1 \cdot h_2^2 dV = 0$. Similarly, we show that all other terms of this form vanish after integration so it remains to deal with the last term of above. We have

$$h_1 \cdot h_2 \cdot h_3(z, \bar{z}) = 2|z_1|^2|z_2|^2|z_3|^2 + \sum_{\sigma \in S_3} |z_{\sigma(1)}|^2 z_{\sigma(2)}^2 \bar{z}_{\sigma(3)}^2.$$

Consider $|z_1|^2 z_2^2 \bar{z}_3^2$. This polynomial is antisymmetric with respect to the isometry $(z_2, \bar{z}_2) \mapsto (i \cdot z_2, i \cdot \bar{z}_2)$ and therefore,

$$\int_{S^{2n+1}} |z_1|^2 z_2^2 \bar{z}_3^2 dV = 0.$$

Similarly, we deal with the other summands. In summary, we have

$$\int_{S^{2n+1}} h^3 dV = 6 \int_{S^{2n+1}} h_1 \cdot h_2 \cdot h_3 dV = 12 \int_{S^{2n+1}} |z_1|^2|z_2|^2|z_3|^2 dV > 0,$$

since the integrand on the right hand side is nonnegative and not identically zero. We decompose $h^3 = \sum_{j=0}^3 h_j$, where $h_j \in r^{6-2j} H_{j,j}$. Since the restrictions of the h_j to S^{2n+1} are eigenfunctions to the $2k$ -th eigenvalue of the Laplacian on S^{2n+1} (see [BGM71, Section III C]), we have that $h_0 \neq 0$ because the integral is nonvanishing. This decomposition induces a decomposition of $v^3 = \sum_{i=0}^3 v_i$ where v_i is an eigenfunction of the i -th eigenvalue of $\Delta_{\mathbb{C}P^n}$ and $v_0 \neq 0$. Therefore, $\int_{\mathbb{C}P^n} v^3 dV \neq 0$. The assertion follows from Theorem 1.7. □

Appendix A. Variational formulas

Here we prove some variational formulas needed throughout the text.

Lemma A.1. *Let (M, g) be Riemannian manifold and denote the first variation of the Levi-Civita connection in the direction of h by G . Then G is a $(1, 2)$ tensor field, given by*

$$g(G(X, Y), Z) = \frac{1}{2}(\nabla_X h(Y, Z) + \nabla_Y h(X, Z) - \nabla_Z h(X, Y)).$$

The first variation of the Riemann tensor, the Ricci tensor and the scalar curvature are given by

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} R_{g+th}(X, Y, Z, W) &= \frac{1}{2}(\nabla_{X,Z}^2 h(Y, W) + \nabla_{Y,W}^2 h(X, Z) - \nabla_{Y,Z}^2 h(X, W) \\ &\quad - \nabla_{X,W}^2 h(Y, Z) + h(R_{X,Y}Z, W) - h(Z, R_{X,Y}W)), \\ \left. \frac{d}{dt} \right|_{t=0} \text{Ric}_{g+th}(X, Y) &= \frac{1}{2}\Delta_L h(X, Y) - \delta^*(\delta h)(X, Y) - \frac{1}{2}\nabla_{X,Y}^2 \text{tr}h, \\ \left. \frac{d}{dt} \right|_{t=0} \text{scal}_{g+th} &= \Delta_g(\text{tr}gh) + \delta_g(\delta gh) - \langle \text{Ric}_g, h \rangle_g. \end{aligned}$$

Furthermore, the first variation of the volume element is given by

$$\left. \frac{d}{dt} \right|_{t=0} dV_{g+th} = \frac{1}{2}\text{tr}_g h \cdot dV_g.$$

Proof. See [Bes08, Theorem 1.174] and [Bes08, Proposition 1.186]. □

Lemma A.2. *The first variation of the Hessian and the Laplacian are given by*

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} {}^{g+th}\nabla_{X,Y}^2 f &= -\frac{1}{2}[\nabla_X h(Y, \text{grad}f) + \nabla_Y h(X, \text{grad}f) - \nabla_{\text{grad}f} h(X, Y)], \\ \left. \frac{d}{dt} \right|_{t=0} \Delta_{g+th} f &= \langle h, \nabla^2 f \rangle - \left\langle \delta h + \frac{1}{2}\nabla \text{tr}h, \nabla f \right\rangle. \end{aligned}$$

Proof. We use local coordinates. Let f be a smooth function. Then the first variation of the Hessian is

$$\left. \frac{d}{dt} \right|_{t=0} (\nabla_{ij}^2 f) = \left. \frac{d}{dt} \right|_{t=0} (\partial_{ij}^2 - \Gamma_{ij}^k \partial_k f) = -\frac{1}{2}g^{kl}(\nabla_i h_{jl} + \nabla_j h_{il} - \nabla_l h_{ij})\partial_k f$$

by the first variation of the Levi-Civita connection. The first variation of the Laplacian is

$$\frac{d}{dt} \Big|_{t=0} (\Delta f) = - \frac{d}{dt} \Big|_{t=0} (g^{ij} \nabla_{ij}^2 f) = h^{ij} \nabla_{ij}^2 f - g^{kl} (\delta h_l + \frac{1}{2} \nabla_l \text{tr} h) \partial_k f. \quad \square$$

Lemma A.3. *The second variations of the Hessian, the Laplacian, the Ricci tensor and the scalar curvature have the schematic expressions*

$$\begin{aligned} \frac{d}{ds} \frac{d}{dt} \Big|_{s,t=0} \nabla_{g+sk+th}^2 f &= k * \nabla h * \nabla f + \nabla k * h * \nabla f, \\ \frac{d}{ds} \frac{d}{dt} \Big|_{s,t=0} \Delta_{g+sk+th} f &= k * \nabla h * \nabla f + \nabla k * h * \nabla f, \\ \frac{d}{ds} \frac{d}{dt} \Big|_{s,t=0} \text{Ric}_{g+sk+th} &= k * \nabla^2 h + \nabla^2 k * h + \nabla k * \nabla h + R * k * h, \\ \frac{d}{ds} \frac{d}{dt} \Big|_{s,t=0} \text{scal}_{g+sk+th} &= k * \nabla^2 h + \nabla^2 k * h + \nabla k * \nabla h + R * k * h. \end{aligned}$$

Here, $*$ is Hamilton's notation for a combination of tensor products with contractions.

Proof. By the first variation of the Christoffel symbols it is not hard to see that the first two covariant derivatives of a $(0, 2)$ -tensor h can be written as

$$(A.1) \quad \frac{d}{dt} \Big|_{t=0} \nabla_{g+tk} h = k * \nabla h + \nabla k * h,$$

$$(A.2) \quad \frac{d}{dt} \Big|_{t=0} \nabla_{g+tk}^2 h = k * \nabla^2 h + \nabla^2 k * h + \nabla k * \nabla h.$$

The first variation of the Hessian is of the schematic form $\nabla h * \nabla f$ and therefore,

$$\begin{aligned} \frac{d}{ds} \frac{d}{dt} \Big|_{s,t=0} \nabla_{g+sk+th}^2 f &= \frac{d}{ds} \Big|_{s=0} (\nabla h * \nabla f) \\ &= k * \nabla h * \nabla f + \left(\frac{d}{ds} \Big|_{s=0} \nabla h \right) * \nabla f \\ &= k * \nabla h * \nabla f + \nabla k * h * \nabla f. \end{aligned}$$

The expression for the second variation of the Laplacian is shown similarly. By Lemma A.1, the first variational formulas for the Riemann curvature

tensor and the Ricci tensor are of the form

$$(A.3) \quad \left. \frac{d}{dt} \right|_{t=0} R = (\nabla^2 * h) + (R * h),$$

$$(A.4) \quad \left. \frac{d}{dt} \right|_{t=0} \text{Ric} = (\nabla^2 * h) + (R * h).$$

The second variational expression of the Ricci tensor now follows from differentiating (A.4) in the direction of k and using (A.1) and (A.3). The last expression for the scalar curvature is shown similarly. \square

References

- [BGM71] Marcel Berger, Paul Gauduchon, and Edmond Mazet, *Le Spectre d'une Variété Riemannienne (The Spectrum of a Riemannian Manifold)*. Lecture Notes in Mathematics **194**, Berlin-Heidelberg-New York: Springer-Verlag. VII, 251 p., (1971).
- [Bes08] Arthur L. Besse, *Einstein Manifolds*, reprint of the 1987 edition, Berlin: Springer, (2008).
- [Böh05] Christoph Böhm, *Unstable Einstein metrics*, Math. Z. **250** (2005), no. 2, 279–286.
- [BWZ04] Christoph Böhm, McKenzie Y. Wang, and Wolfgang Ziller, *A variational approach for compact homogeneous Einstein manifolds*, Geom. Funct. Anal. **14** (2004), no. 4, 681–733.
- [BS09] Simon Brendle and Richard Schoen, *Manifolds with 1/4-pinched curvature are space forms*, J. Am. Math. Soc. **22** (2009), no. 1, 287–307.
- [Cao10] Huai-Dong Cao, *Recent progress on Ricci solitons*, Yng-Ing Lee (ed.) et al., *Recent Advances in Geometric Analysis*, Proceeding of the International Conference on Geometric Analysis, Taipei, Taiwan, June 18–22, 2007. Somerville, MA: International Press; Beijing: Higher Education Press. *Advanced Lectures in Mathematics (ALM)* **11** (2010), 1–38.
- [CHI04] Huai-Dong Cao, Richard Hamilton, and Tom Ilmanen, *Gaussian densities and stability for some Ricci solitons*, arXiv:math/0404165, (2004).

- [CH15] Huai-Dong Cao and Chenxu He, *Linear Stability of Perelmans ν -entropy on Symmetric spaces of compact type*, J. Reine Angew. Math. **709** (2015), 229–246.
- [CZ12] Huai-Dong Cao and Meng Zhu, *On second variation of Perelman's Ricci shrinker entropy*, Math. Ann. **353** (2012), no. 3, 747–763.
- [CCG⁺07] Bennett Chow, Sun-Chin Chu, David Glickenstein, Christine Guenther, James Isenberg, Tom Ivey, Dan Knopf, Peng Lu, Feng Luo, and Lei Ni, *The Ricci Flow: Techniques and Applications. Part I: Geometric Aspects*, Mathematical Surveys and Monographs **135**, Providence, RI: American Mathematical Society (AMS). xxiii, 536 p., (2007).
- [CM14] Tobias H. Colding and William P. Minicozzi, *On uniqueness of tangent cones for Einstein manifolds*, Invent. Math. **196** (2014), no. 3, 515–588.
- [DWW05] Xianzhe Dai, Xiaodong Wang, and Guofang Wei, *On the stability of Riemannian manifold with parallel spinors*, Invent. Math. **161** (2005), no. 1, 151–176.
- [DWW07] Xianzhe Dai, Xiaodong Wang, and Guofang Wei, *On the variational stability of Kähler-Einstein metrics*, Commun. Anal. Geom. **15** (2007), no. 4, 669–693.
- [Ebi70] David G. Ebin, *The manifold of Riemannian metrics*, Proc. Symp. AMS Bd. **15** (1970), 11–40.
- [FIN05] Michael Feldman, Tom Ilmanen, and Lei Ni, *Entropy and reduced distance for Ricci expanders*, J. Geom. Anal. **15** (2005), no. 1, 49–62.
- [GIK02] Christine Guenther, James Isenberg, and Dan Knopf, *Stability of the Ricci flow at Ricci-flat metrics*, Commun. Anal. Geom. **10** (2002), no. 4, 741–777.
- [Ham82] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differ. Geom. **17** (1982), 255–306.
- [Ham95] Richard S. Hamilton, *The formation of singularities in the Ricci flow*, C. C. Hsiung (ed.) et al., Proceedings of the Conference

- on Geometry and Topology Held at Harvard University, Cambridge, MA, USA, April 23-25, 1993. Cambridge, MA: International Press. *Surv. Differ. Geom., Suppl. J. Differ. Geom.* **2** (1995), 7–136.
- [Has12] Robert Haslhofer, *Perelman's lambda-functional and the stability of Ricci-flat metrics*, *Calc. Var. Partial Differ. Equ.* **45** (2012), no. 3-4, 481–504.
- [HM14] Robert Haslhofer and Reto Müller, *Dynamical stability and instability of Ricci-flat metrics*, *Math. Ann.* **360** (2014), no. 1-2, 547–553.
- [Hit74] Nigel Hitchin, *Harmonic spinors*, *Adv. Math.* **14** (1974), no. 1, 1–55.
- [Hui85] Gerhard Huisken, *Ricci deformation of the metric on a Riemannian manifold*, *J. Differ. Geom.* **21** (1985), 47–62.
- [IN05] Mitsuhiro Itoh and Tomomi Nakagawa, *Variational stability and local rigidity of Einstein metrics*, *Yokohama Math. J.* **51** (2005), no. 2, 103–115.
- [Koi78] Norihito Koiso, *Non-deformability of Einstein metrics*, *Osaka J. Math.* **15** (1978), 419–433.
- [Koi79] Norihito Koiso, *A decomposition of the space \mathcal{M} of Riemannian metrics on a manifold*, *Osaka J. Math.* **16** (1979), 423–429.
- [Koi83] Norihito Koiso, *Einstein metrics and complex structures*, *Invent. Math.* **73** (1983), 71–106.
- [Krö13] Klaus Kröncke, *Stability of Einstein manifolds*, Universität Potsdam, PhD thesis, (2013).
- [Krö15] Klaus Kröncke, *On the stability of Einstein manifolds*, *Ann. Glob. Anal. Geom.* **47** (2015), no. 1, 81–98.
- [LeB99] Claude LeBrun, *Einstein metrics and the Yamabe problem*, Vasilios Alexiades (ed.) et al., *Trends in Mathematical Physics, Proceedings of the conference, University of Tennessee, Knoxville, TN, USA, October 14–17, 1998*. Providence, RI: American Mathematical Society, *AMS/IP Stud. Adv. Math.* **13** (1999), 353–376.
- [Lic61] André Lichnerowicz, *Propagateurs et commutateurs en relativité générale*, *Publications Mathématiques de l'IHÉS* **10** (1961), no. 1, 5–56.

- [Oba62] Morio Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan **14** (1962), 333–340.
- [Per02] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications*, [arXiv:math/0211159](https://arxiv.org/abs/math/0211159), (2002).
- [Per03] Grisha Perelman, *Ricci flow with surgery on three-manifolds*, [arXiv:math/0303109](https://arxiv.org/abs/math/0303109), (2003).
- [Sch84] Richard M. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differ. Geom. **20** (1984), 479–495.
- [Ses06] Natasa Sesum, *Linear and dynamical stability of Ricci-flat metrics*, Duke Math. J. **133** (2006), no. 1, 1–26.
- [SW13] Song Sun and Yuanqi Wang, *On the Kähler-Ricci flow near a Kähler-Einstein metric*, J. Reine Angew. Math. (2013).
- [Ye93] Rugang Ye, *Ricci flow, Einstein metrics and space forms*, Trans. Am. Math. Soc. **338** (1993), no. 2, 871–896.

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RECEIVED JANUARY 14, 2016

ACCEPTED NOVEMBER 15, 2017