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Splitting theorems for hypersurfaces in Lorentzian manifolds

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The splitting problem for spacetimes with timelike Ricci curvature bounded below by zero has been discussed extensively in the past (most notably by Eschenburg, Galloway and Newman), in particular there exist versions for both spacetimes containing a complete timelike line and spacetimes containing a maximal hypersurface Σ and a (future) complete Σ-ray. For timelike Ricci curvature bounded below by some $\kappa > 0$ only an analogue to the first case has been shown explicitly (see [AGH96]).

In this paper we employ their methods (a geometric maximum principle for the level sets of the Busemann function) to study analogues of the second case for hypersurfaces with mean curvature bounded from above by β. We show that given a Σ-ray of maximal length $J^+(\Sigma)$ is isometric to a warped product if either $\kappa > 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$. Additionally we present an elementary proof for such a splitting if one assumes that the volume of (future) distance balls over subsets of this hypersurface is maximal.

1. Introduction

Over the past 50 years the study of comparison and rigidity theorems has been an important part of Riemannian geometry and, as so often the case, this interest soon carried over to Lorentzian geometry. In the Riemannian context important results for manifolds with a bound on the Ricci curvature (instead of the sectional curvatures) include Myers's theorem, the maximal diameter theorem ([Che75, Thm. 3.1]) and the Cheeger-Gromoll splitting theorem ([CG71, Thm. 2]), which is already very similar to the most interesting Lorentzian case from a physics point of view:

Theorem (Cheeger-Gromoll splitting theorem). *Let (M, g) be a complete Riemannian manifold of dimension ≥ 2 which satisfies*

$$\mathbf{Ric}(v, v) \geq 0 \text{ for all } v \in TM$$

and which contains a complete geodesic line (i.e., a complete geodesic that is minimizing between each of its points). Then (M, g) can be decomposed uniquely as an isometric product $N \times \mathbb{R}^k$, where N contains no lines, $k \geq 1$ and $\mathbb{R}^k$ is equipped with the standard euclidean metric.

In Lorentzian geometry one usually assumes only a bound on the timelike Ricci curvature, i.e., we want to look at spacetimes (M, g) where

$$\mathbf{Ric}(v, v) \geq -(n-1)\kappa\, g(v, v) \text{ for all timelike } v \in TM$$

for some $\kappa \in \mathbb{R}$.

So far most results have been focused on spacetimes having non-negative timelike Ricci curvature (i.e., satisfying the *strong energy condition*), the exception being [AGH96] who looked at the case $\kappa > 0$. A nice overview of past work can also be found in [BEE96, Ch. 14].

The first Lorentzian splitting theorem for spacetimes using a bound on the Ricci curvature instead of the sectional curvatures (for non-positive timelike sectional curvatures the first such result was obtained by Beem, Ehrlich, Markvorsen and Galloway in 1985, [BEMG85]) was due to Eschenburg in 1988 ([Esc88]) who additionally assumed both global hyperbolicity and timelike geodesic completeness. Shortly thereafter Galloway showed that the assumption of only global hyperbolicity is sufficient ([Gal89b]) and a year later Newman gave a proof assuming timelike geodesic completeness but not global hyperbolicity ([New90]). These three results are summarized as follows:

Theorem (Lorentzian splitting theorem). *Let (M, g) be a spacetime of dimension $n \geq 2$ that*

1) *is either globally hyperbolic or timelike geodesically complete*
2) *satisfies the strong energy condition and*
3) *contains a complete timelike line (i.e., a curve maximizing the distance between any of its points).*

Then (M, g) splits isometrically as a product $(\mathbb{R} \times V, -dt^2 \oplus h)$, where (V, h) is a complete Riemannian manifold.

While $\kappa = 0$ certainly is the most important case from a physical point of view, it nevertheless seems to be interesting to give a complete description under which curvature assumptions similar results hold. To allow for spacetimes that behave differently in one time direction than in the other (e.g., ones that are incomplete to the future but not to the past) we assume the existence of a smooth, acausal spacelike hypersurface Σ that is future causally complete (cf. Def. 2.2) and look at both a lower bound κ on the timelike Ricci curvature and an upper bound β for the mean curvature H_Σ of Σ. This combination has so far mainly been studied for $\kappa = \beta = 0$ (though there are also some recent results for $\kappa > 0$ and $\beta = -(n-1)\sqrt{|\kappa|}$, see [GV16]). This case is again of exceptional physical interest because for $\kappa = 0$ and $\beta < 0$ these are exactly the curvature assumptions in the Hawking singularity theorem. Here [Gal89a, Thm. C] showed

Theorem. *Let M be a space-time which obeys the strong energy condition containing a smooth acausal maximal (i.e., zero mean curvature) spacelike hypersurface Σ, which is either geodesically complete or future causally complete. Assume $J^+(\Sigma)$ is future timelike geodesically complete. If γ is a future complete Σ-ray such that $I^-(\gamma) \cap J^+(\Sigma)$ is globally hyperbolic then $J^+(\Sigma)$ is isometric to $([0, \infty) \times \Sigma, -dt^2 \oplus h)$, where h is the induced metric on Σ.*

For general κ, β there is recent work by Treude and Grant ([TG13]) using Riccati comparison theorems from [EH90] to derive comparison results regarding the time evolution of the area and volume of subsets of Σ, comparing them to the evolution in fixed Lorentzian warped product manifolds. Similar comparison techniques have been used in the past with the Raychaudhuri equation to show the Hawking singularity theorem, or more precisely that no timelike geodesic starting at Σ can have length greater than $-\frac{n-1}{\beta}$ if $\kappa = 0$ and $\beta < 0$ (see, e.g., [Sen98] for an overview). Those same techniques can be used to show that this length is bounded from above by a constant $b_{\kappa,\beta} \leq \infty$ for arbitrary κ, β. Concrete values for $b_{\kappa,\beta}$ can be found in Table 1. Our first goal is to investigate under which conditions the existence of an inextendible geodesic maximizing the distance to Σ of length exactly $b_{\kappa,\beta}$ already implies that $I^+(\Sigma)$ is isometric to the warped product $(0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} (\Sigma, \frac{1}{f_{\kappa,\beta}(0)} g|_\Sigma)$ (with $f_{\kappa,\beta}$ from Table 1).

For $\kappa = \beta = 0$ this question is basically answered positively by [Gal89a, Thm. C] (see above), using the Lorentzian Busemann function and that the value of $b_{\kappa,\beta}$ going to infinity from below as $\beta \nearrow 0$ (and remains infinity for all $\beta \geq 0$). For $\kappa < 0$ the same transition happens at $\beta = -(n-1)\sqrt{|\kappa|}$, hence the methods used in [Gal89a] for $\kappa = \beta = 0$ would carry over to $\kappa < 0$, $\beta = -(n-1)\sqrt{|\kappa|}$. We refer to [GV14, GV16] for a more modern treatment of these cases based on a low regularity maximum principle shown in [AGH96] and using generalized horospheres instead of Busemann functions. For other values κ, β with $b_{\kappa,\beta} = \infty$, i.e., $\kappa \leq 0$ and $\beta > -(n-1)\sqrt{|\kappa|}$, it is easy to see that similar results are false (see Example 4.4, the spacetime containing an inextendible maximizing geodesic is nothing "special" in that case). For the remaining variations of κ, β (with $b_{\kappa,\beta} < \infty$, i.e., $\kappa > 0$ or $\beta < -(n-1)\sqrt{|\kappa|}$) analogues remain true, but the proof now requires the aforementioned low regularity version of the maximum principle by [AGH96].

At this point one should also briefly mention recent results of Bernal and Sánchez ([BS05]), who showed that actually any globally hyperbolic spacetime (M, g) admits a smooth time function $\mathcal{T}$ with smooth Cauchy hypersurfaces $\Sigma_{\mathcal{T}}$ as level sets and thus splits isometrically as $M \cong \mathbb{R} \times \Sigma$ with $g = -\beta d\mathcal{T}^2 + h_{\mathcal{T}}$, where Σ is a smooth Cauchy hypersurface for M, $\beta : \mathbb{R} \times \Sigma \to \mathbb{R}_+$ is smooth and $h_{\mathcal{T}}$ is a Riemannian metric on $\Sigma_{\mathcal{T}}$. Their work improves upon a classical topological splitting result obtained by Geroch in 1970 ([Ger70]). They refined their arguments further to also show that given any spacelike Cauchy hypersurface Σ there exists a Cauchy temporal function $\mathcal{T} : M \to \mathbb{R}$ such that $\Sigma = \mathcal{T}^{-1}(0)$ (see [BS06]). One should note, however, that these results require neither curvature nor any maximality assumptions and thus there is no additional information on β or the time evolution of $h_{\mathcal{T}}$ and the product structure obtained this way will in general not be a warped product.

The outline of the paper is as follows. In Sections 2 and 3 we review basic definitions and the comparison results presented in [TG13]. We also include a table (Table 1) giving a detailed description of the comparison spaces (introduced by [TG13]) that we will use.

In Section 4 we show that maximality in the injectivity radius already implies that M is (isometric to) a warped product: While this seems to be a somewhat well-known fact a detailed proof is hard to find and it ties in nicely with the following results.

In Section 5 we use a combination of arguments from [Esc88], [Gal89b], [Gal89a] and [AGH96] to show our main result, which is that for $\kappa < 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$ the existence of an inextendible geodesic maximizing the distance to Σ of length exactly $b_{\kappa,\beta}$ already implies that $I^+(\Sigma) \cong (0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} (\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2} g|_\Sigma)$ (with $f_{\kappa,\beta}$ from Table 1).

Then in Section 6 we give an elementary proof (that requires neither the Busemann function nor the maximum principle) of the same result under the assumption of maximality in certain volumes instead of the existence of a ray of maximal length.

Notation

Throughout, M will always be a connected, Hausdorff and second countable smooth manifold of dimension $n \geq 2$ with a Lorentzian metric g and a time orientation. We also always assume that (M, g) is globally hyperbolic. The curvature tensor of the metric is defined with the convention $\mathbf{R}(X,Y)Z = \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z$ and we denote the Ricci tensor of g by **Ric**. Given a spacelike, acausal hypersurface $\Sigma \subset M$ with future pointing unit normal $\mathbf{n}_\Sigma$ we define the shape operator with sign convention $\mathbf{S}_\Sigma = \nabla \mathbf{n}_\Sigma$ and the mean curvature as $H_\Sigma = \operatorname{tr} \mathbf{S}_\Sigma$.

2. Definitions

As in [Chr11] we define causal (timelike) curves to be locally Lipschitz continuous maps $\gamma : I \to M$ (I being an interval) with $\dot\gamma \neq 0$ and $g(\dot\gamma, \dot\gamma) \leq 0$ (< 0) a.e. and a causal curve is called future (past) directed if $\dot\gamma$ is future (past) pointing almost everywhere. For $p, q \in M$ we write $p \ll q$ if there is a future directed (f.d.) timelike curve from p to q and $p \leq q$ if either $p = q$ or there exists a f.d. causal curve from p to q and we set

$$I^+(p) := \{q \in M : p \ll q\}$$
$$J^+(p) := \{q \in M : p \leq q\}.$$

Definition 2.1 (Signed time separation). Let $p \in M$. Then for $q \in M$ the *future time separation* to p is defined by

$$\tau_p(q) := \sup(\{L(\gamma) : \gamma \text{ is a f.d. causal curve from } p \text{ to } q\} \cup \{0\}), \tag{2.1}$$

where $L(\gamma)$ denotes the Lorentzian arc-length of γ, i.e., for a curve $\gamma : (t_1, t_2) \to M$ one has $L(\gamma) := \int_{t_1}^{t_2} \sqrt{|g(\dot{\gamma(t)}, \dot{\gamma(t)})|} dt$.

Similarly one defines the *signed time separation* to an acausal subset Σ by

$$\tau_\Sigma(p) : = \begin{cases} \sup_{q\in\Sigma} \tau(q,p) & p \in I^+(\Sigma) \\ -\sup_{q\in\Sigma} \tau(p,q) & p \in I^-(\Sigma) \\ 0 & \text{otherwise} \end{cases}. \tag{2.2}$$

It is easy to see that both the time separation to a point and to an acausal subset satisfy the reverse triangle inequality

$$\tau_p(q) + \tau_q(r) \leq \tau_p(r) \text{ and } \tau_\Sigma(q) + \tau_q(r) \leq \tau_\Sigma(r) \tag{2.3}$$

for $p \leq q \leq r$ and $r \geq q \in I^+(\Sigma)$, respectively.

If (M, g) is globally hyperbolic (i.e., $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$ and M contains no closed causal curves[1]) then any two points $p, q \in M$ with $p \leq q$ can be connected by a maximizing curve ([O'N83, Prop. 14.19]). If an acausal subset has the following property, one also gets the existence of maximizing curves to this subset.

Definition 2.2 (Future causally complete). A subset $\Sigma \subset M$ is called *future causally complete* (*FCC*) if for any $p \in J^+(\Sigma)$ the set $J^-(p) \cap \Sigma$ has compact relative closure in Σ.

Remark 2.3. Note that any smooth spacelike Cauchy hypersurface for M is also a smooth, spacelike, acausal, FCC hypersurface (see [O'N83, Lem. 14.40, 14.42 and 14.43]). If Σ is also past causally complete and (M, g) is globally hyperbolic then Σ is a (smooth, spacelike) Cauchy hypersurface.

The following Proposition sums up some common knowledge about the (future) time-separation to an acausal ($p \leq q$ implies $p = q$ for any $p, q \in \Sigma$), FCC subset (see [TG13, Thm. 2]).

[1]Note that this definition of global hyperbolicity is equivalent to the one using strong causality instead of causality, see [BS07].

Proposition 2.4. *Let $\Sigma \subset M$ be an acausal, FCC subset. Then the future time-separation $\tau_\Sigma : M \to \mathbb{R}$ to Σ is finite-valued and continuous and for any $p \in J^+(\Sigma) \setminus \Sigma$ there exists $q \in \Sigma$ and a causal curve γ from q to p with $\tau_\Sigma(p) = \tau(q,p) = L(\gamma)$. Any such maximizing curve γ has to be a (reparametrization of) a geodesic, which is timelike for $p \in I^+(\Sigma)$ and null otherwise. If $\Sigma \subset M$ is, additionally, a spacelike hypersurface, then any maximizing geodesic has to start orthogonally to Σ (so in particular $I^+(\Sigma) = J^+(\Sigma)$).*

An important tool will be the normal exponential map to Σ.

Definition 2.5 (Normal exponential map). Let $\mathcal{D}^N \subset T\Sigma^\perp$ be the set of all $w \in T\Sigma^\perp$ such that $w \in \mathrm{dom}(\exp_{\pi(w)})$. The normal exponential map $\exp^N : \mathcal{D}^N \to M$ to Σ is defined by

$$\exp^N(w) := \exp_{\pi(w)}(w) = \gamma_v(t)$$

for $t \in \mathbb{R}$ and $v \in S^+N\Sigma$ such that $w = tv$.

For any $v \in TM$ we denote by γ_v the unique inextendible geodesic starting at $\pi(v)$ with initial velocity v. For $v \in T\Sigma^\perp$ each $\gamma_v|_{[0,t]}$ maximizes the distance to Σ for small t, but it may not remain maximizing for larger t. We write $S^+N\Sigma$ for the (future) unit normal bundle to Σ, i.e.

$$\begin{aligned} S^+N\Sigma := \big\{ v \in TM|_\Sigma :\ & v \textit{ f.p.}, \ g(v,w) = 0 \\ & \forall w \in T_{\pi(v)}\Sigma \text{ and } g(v,v) = -1 \big\} \subset T\Sigma^\perp, \end{aligned}$$

and define

Definition 2.6 (Cut function). The function

$$\begin{aligned} s^+_\Sigma &: S^+N\Sigma \to [0,\infty] \\ & s^+_\Sigma(v) := \sup \big\{ t > 0 :\ \tau_\Sigma(\gamma_v(t)) = L(\gamma_v|_{[0,t]}) \big\} \end{aligned}$$

is called future cut function.

An easy adaptation (looking at a hypersurface instead of a point) of arguments from [BEE96, Prop. 9.7 and Thm. 9.8] (see also [Tre, 3.2.29]) shows

Lemma 2.7. *The cut function s^+_Σ is lower semi-continuous and continuous at points v where $s^+_\Sigma(v) = \infty$ or $s^+_\Sigma(v)v \in \mathcal{D}^N$.*

Definition 2.8 (Cut locus). The (future) cut locus of Σ is defined as the image of the tangential cut locus under the normal exponential map:

$$\mathrm{Cut}^+(\Sigma) := \left\{ \exp^N(s^+_\Sigma(v)v) : \, v \in S^+N\Sigma \text{ and } s^+_\Sigma(v)v \in \mathcal{D}^N \right\}.$$

An important fact is that $\mathrm{Cut}^+(\Sigma)$ has measure zero, is closed and $\exp^N|_{J_T(\Sigma)^\circ}$ (where $J_T(\Sigma) := \left\{ tv : \, v \in S^+N\Sigma \text{ and } t \in [0, s^+_\Sigma(v)) \right\}$) is a diffeomorphism onto $I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$ ([TG13, Thm. 3]).

3. Comparison results

In this section we will briefly review the comparison results from [TG13]. We will generally omit the proofs, but may give a sketch if it will be helpful later on. First, we need to define the sets of whose areas respectively volumes will be estimated.

Definition 3.1 (Future spheres and balls). For any $t > 0$ and $A \subset \Sigma$ we define the spheres $S^+_A(t)$ and balls $B^+_A(t)$ of time t above A by

$$\begin{aligned} S^+_A(t) : &= \{ p \in I^+(\Sigma) : \, \exists q \in A \text{ with } d(q,p) = \tau_\Sigma(p) = t \} \text{ and} \\ B^+_A(t) : &= \bigcup_{s \in (0,t)} S^+_A(s) \end{aligned}$$

We also set $\mathcal{I}^+(\Sigma) := I^+(\Sigma) \setminus \mathrm{Cut}^+(\Sigma)$ and

$$\begin{aligned} \mathcal{S}^+_A(t) : &= S^+_A(t) \setminus \mathrm{Cut}^+(\Sigma) \text{ and} \\ \mathcal{B}^+_A(t) : &= B^+_A(t) \setminus \mathrm{Cut}^+(\Sigma). \end{aligned}$$

Second, we need appropriate curvature conditions.

Definition 3.2 (Cosmological comparison condition). Let $\kappa, \beta \in \mathbb{R}$. We say that (M, g, Σ) satisfies the cosmological comparison condition $CCC(\kappa, \beta)$ if

1) (M, g) is a globally hyperbolic spacetime and $\Sigma \subset M$ is a smooth, connected, spacelike, acausal, FCC hypersurface,
2) the mean curvature H_Σ of Σ satisfies $H_\Sigma \leq \beta$ and
3) $\mathbf{Ric}(v,v) \geq -(n-1)\,\kappa\, g(v,v)$ for all timelike $v \in TM$.

Under these assumptions [TG13] showed various estimates for mean curvature, area and volume, comparing them to the respective quantities in

Table for $\kappa < 0$

β	$\Sigma_{\kappa,\beta}$	c	$f_{\kappa,\beta}(t)$	$\frac{1}{n-1}H_{\kappa,\beta}(t)$	$b_{\kappa,\beta}$
$\frac{\lvert\beta\rvert}{(n-1)\sqrt{\lvert\kappa\rvert}} < 1$	S^{n-1}	$\tanh^{-1}(\frac{\beta}{(n-1)\sqrt{\lvert\kappa\rvert}})$	$\frac{1}{\sqrt{\lvert\kappa\rvert}}\cosh(\sqrt{\lvert\kappa\rvert}t+c)$	$\sqrt{\lvert\kappa\rvert}\tanh(\sqrt{\lvert\kappa\rvert}t+c)$	∞
$\frac{\lvert\beta\rvert}{(n-1)\sqrt{\lvert\kappa\rvert}} = 1$	$\mathbb{R}^{n-1}$	0	$\exp(\operatorname{sgn}(\beta)\sqrt{\lvert\kappa\rvert}t)$	$\operatorname{sgn}(\beta)\sqrt{\lvert\kappa\rvert}$	∞
$\frac{\beta}{(n-1)\sqrt{\lvert\kappa\rvert}} > 1$	H^{n-1}	$\coth^{-1}(\frac{\beta}{(n-1)\sqrt{\lvert\kappa\rvert}})$	$\frac{1}{\sqrt{\lvert\kappa\rvert}}\sinh(\sqrt{\lvert\kappa\rvert}t+c)$	$\sqrt{\lvert\kappa\rvert}\coth(\sqrt{\lvert\kappa\rvert}t+c)$	∞
$\frac{\beta}{(n-1)\sqrt{\lvert\kappa\rvert}} < -1$	H^{n-1}	$\coth^{-1}(\frac{\beta}{(n-1)\sqrt{\lvert\kappa\rvert}})$	$\frac{1}{\sqrt{\lvert\kappa\rvert}}\sinh(\sqrt{\lvert\kappa\rvert}t+c)$	$\sqrt{\lvert\kappa\rvert}\coth(\sqrt{\lvert\kappa\rvert}t+c)$	$-\frac{c}{\sqrt{\lvert\kappa\rvert}}$

Table for $\kappa = 0$

β	$\Sigma_{\kappa,\beta}$	c	$f_{\kappa,\beta}(t)$	$\frac{1}{n-1}H_{\kappa,\beta}(t)$	$b_{\kappa,\beta}$
$\beta = 0$	$\mathbb{R}^{n-1}$	0	1	0	∞
$\beta > 0$	H^{n-1}	$\frac{n-1}{\beta}$	$t+c$	$\frac{1}{t+c}$	∞
$\beta < 0$	H^{n-1}	$\frac{n-1}{\beta}$	$t+c$	$\frac{1}{t+c}$	$-\frac{n-1}{\beta}$

Table for $\kappa > 0$

β	$\Sigma_{\kappa,\beta}$	c	$f_{\kappa,\beta}(t)$	$\frac{1}{n-1}H_{\kappa,\beta}(t)$	$b_{\kappa,\beta}$
$\beta > 0$	H^{n-1}	$\cot^{-1}(\frac{\beta}{(n-1)\sqrt{\kappa}})$	$\frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t+c)$	$\sqrt{\kappa}\cot(\sqrt{\kappa}t+c)$	$\frac{-c+\pi}{\sqrt{\kappa}}$
$\beta < 0$	H^{n-1}	$\cot^{-1}(\frac{\beta}{(n-1)\sqrt{\kappa}})$	$\frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t+c)$	$\sqrt{\kappa}\cot(\sqrt{\kappa}t+c)$	$\frac{-c}{\sqrt{\kappa}}$
$\beta = 0$	H^{n-1}	$\frac{\pi}{2}$	$\frac{1}{\sqrt{\kappa}}\cos(\sqrt{\kappa}t)$	$\sqrt{\kappa}\tan(\sqrt{\kappa}t)$	$\frac{\pi}{2\sqrt{\kappa}}$

Table 1: Warping functions for different values of κ, β. The mean curvature is given by $H_{\kappa,\beta} = (n-1)\frac{f'}{f}$ and $b_{\kappa,\beta}$ is the upper bound of the interval containing zero on which $f_{\kappa,\beta} \neq 0$ and $c = c(\kappa, \beta)$ is a (κ, β dependent) constant. This table is based on [TG13, Table 1].

certain warped products $M_{\kappa,\beta} = (a_{\kappa,\beta}, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \Sigma_{\kappa,\beta}$ where $a_{\kappa,\beta}, b_{\kappa,\beta} \in \mathbb{R}$, $f_{\kappa,\beta} : (a_{\kappa,\beta}, b_{\kappa,\beta}) \mapsto \mathbb{R} \setminus \{0\}$ is the warping function and $\Sigma_{\kappa,\beta}$ is the (unique) simply connected, complete $(n-1)$-dimensional Riemannian manifold with constant curvature $k_{\kappa,\beta} \in \{-1, 0, 1\}$ (i.e., $\Sigma_{\kappa,\beta}$ is either hyperbolic space H^{n-1}, euclidean space $\mathbb{R}^{n-1}$ or the sphere S^{n-1}). These comparison spaces are listed in Table 1.

Now we are ready to state some of the relevant results of [TG13]: Their arguments are based on a comparison result for Riccati equations from [EH90], which they then adapt to their needs [TG13, Thm. 6].

Theorem 3.3 (Riccati comparison). *Let $R : \mathbb{R} \to S(E)$ (self-adjoint operators from an n-dimensional vector space E into itself) be smooth and assume that $\operatorname{tr} R \geq n\kappa$ for some $\kappa \in \mathbb{R}$. Furthermore, let $S : (0, b) \to S(E)$ be a solution of $S' + S^2 + R = 0$, and $s_\kappa : (0, b_\kappa) \to \mathbb{R}$ a solution of $s'_\kappa + s^2_\kappa + \kappa = 0$ that can not be extended beyond b_κ. If $\lim_{t\searrow 0}(s_\kappa(t) - \frac{1}{n}\mathrm{tr}S)$ exists and is non-negative, then $b \leq b_\kappa$ and*

$$\operatorname{tr} S(t) \leq n s_\kappa(t)$$

for all $t \in (0, b)$. Moreover, if equality holds for some $t_0 \in (0, b)$, then equality holds for all $t < t_0$. In this case, we also have $S(t) = s_\kappa(t)\mathrm{Id}_E$ and $R(t) = \kappa\mathrm{Id}_E$ for all $t \in (0, t_0]$.

It is easy to see that the shape operators $\mathbf{S}_t = -\nabla\mathrm{grad}\tau_\Sigma$ to the level sets $\Sigma_t := \tau_\Sigma^{-1}(\{t\})$ satisfy such a Riccati equation along each timelike geodesic γ starting orthogonally to Σ with $R_t : \dot\gamma(t)^\perp \to \dot\gamma(t)^\perp$ given by $\mathbf{R}(., \dot\gamma(t))\dot\gamma(t)$. This leads to the following result about the mean curvatures $H_t := H_{\Sigma_t}$ of the level sets Σ_t [TG13, Thm. 7]:

Theorem 3.4 (Mean curvature comparison). *Let $\kappa, \beta \in \mathbb{R}$ and assume that M and $\Sigma \subset M$ satisfy $CCC(\kappa, \beta)$. Then*

1) *For any inextendible unit-speed geodesic $\gamma : [0, a) \to M$ maximizing the distance to Σ, one has $H_t(\gamma(t)) \leq H_{\kappa,H_\Sigma(\gamma(0))}(t)$ along γ, hence $a < b_{\kappa,H_\Sigma(\gamma(0))}$*
2) *For each $q \in \mathcal{I}^+(\Sigma)$, we have $\tau_\Sigma(q) < b_{\kappa,\beta}$ and $\operatorname{tr} \mathbf{S}_{\tau_\Sigma(q)}(q) = H_{\tau_\Sigma(q)}(q) \leq H_{\kappa,\beta}(\tau_\Sigma(q)) = \operatorname{tr} \mathbf{S}_{\kappa,\beta}(\tau_\Sigma(q))$.*
3) *If $H_{\tau_\Sigma(q)}(q) = H_{\kappa,\beta}(\tau_\Sigma(q))$ and $\gamma : [0, \tau_\Sigma(q)] \to M$ is the (unique unit-speed) geodesic maximizing the distance from q to Σ, then even $\mathbf{S}_t(\gamma(t)) = \frac{1}{n-1} H_{\kappa,\beta}(t) \operatorname{id}$ for all $t \in [0, \tau_\Sigma(q)]$.*

Not stated explicitly in [TG13] is an immediate corollary we will use later on.

Corollary 3.5. *Actually $\tau_\Sigma(q) < b_{\kappa,\beta}$ for all $q \in I^+(\Sigma)$.*

Proof. From $\tau_\Sigma(q) < b_{\kappa,\beta}$ for any $q \in \mathcal{I}^+(\Sigma)$ it follows from density of $\mathcal{I}^+(\Sigma)$ in $I^+(\Sigma)$ that $\tau_\Sigma(q) \leq b_{\kappa,\beta}$ for any $q \in I^+(\Sigma)$. Now assume there exists a $q \in I^+(\Sigma)$ with$\tau_\Sigma(q) = b_{\kappa,\beta}$ and let $\gamma : [0, b_{\kappa,\beta}] \to M$ be a geodesic maximizing the distance from Σ to q. By extending this geodesic we get a point $q' \in I^+(\Sigma)$ with $\tau_\Sigma(q') > b_{\kappa,\beta}$, arriving at a contradiction. ☐

Next [TG13] use a standard result on the variation of area (see [Sim83, Ch. 2]).

Proposition 3.6 (First variation of area). *Let $K \subset \mathcal{S}_t$ be compact and let $\varepsilon > 0$ be such that the flow, Φ, of $\mathbf{n}$ is defined on $[-\varepsilon, \varepsilon] \times K$. Set $K_s := \Phi_s(K) \subset \mathcal{S}_{t+s}$ for each $s \in [-\varepsilon, \varepsilon]$. Then*

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{area} K_s = \int_K \mathrm{tr}\, \mathbf{S}_t d\mu_t, \tag{3.1}$$

where μ_t denotes the Riemannian volume measure on $\mathcal{S}_t$ induced by g.

This allows them to proof the following area comparison theorem [TG13, Thm. 8].

Theorem 3.7 (Area comparison). *Let $\kappa, \beta \in \mathbb{R}$ and assume that (M, g, Σ) satisfies $CCC(\kappa, \beta)$. Then, for any $B \subset \Sigma_{\kappa,\beta}$ with finite, non-zero measure and any measurable $A \subset \Sigma$, the function*

$$t \mapsto \frac{\mathrm{area}\, \mathcal{S}_A^+(t)}{\mathrm{area}_{\kappa,\beta} S_B^+(t)},$$

is nonincreasing on $[0, b_{\kappa,\beta})$. Further, for $t \searrow 0$ this function converges to $\frac{\mathrm{area} A}{\mathrm{area}_{\kappa,\beta} B}$, so

$$\mathrm{area}\, \mathcal{S}_A^+(t) \leq \frac{\mathrm{area}_{\kappa,\beta} S_B^+(t)}{\mathrm{area}_{\kappa,\beta} B}\, \mathrm{area} A.$$

Proof. We will only give a sketch here. If A is compact and the flow Φ of the unit normal vector field $\mathbf{n}$ is defined on $[0, b_{\kappa,\beta})$, then $\mathcal{S}_A^+(t) = \Phi_t(A)$ and one can use Prop. 3.6 and Thm. 3.4 to calculate

$$\begin{aligned} \frac{d}{dt} \log(\mathrm{area}\, \mathcal{S}_A^+(t)) &= \frac{1}{\mathrm{area}\, \mathcal{S}_A^+(t)} \int_{\mathcal{S}_A^+(t)} H_t(q) d\mu_t(q) \\ &\leq H_{\kappa,\beta}(t) = \frac{d}{dt} \log(\mathrm{area}_{\kappa,\beta} S_B^+(t)), \end{aligned} \tag{3.2}$$

proving the assertion. If this is not the case, one looks at $0 < t_1 < t_2 < b_{\kappa,\beta}$ and a sequence of compact sets $K_i \subset \mathcal{S}_A^+(t_2)$ with $\mathrm{area} K_i \nearrow \mathrm{area}\, \mathcal{S}_A^+(t_2)$ and uses the sets $K_i(t) := \Phi_{t-t_2}(K_i)$ (for $t \in [0, t_2]$) instead of $\mathcal{S}_A^+(t)$ in (3.2). $\square$

The co-area formula (note that $\mathrm{Cut}^+(\Sigma)$ has measure zero) implies

$$\mathrm{vol} B_A^+(t) = \int_0^t \mathrm{area} \mathcal{S}_A^+(\tau) d\tau, \tag{3.3}$$

and some basic analysis regarding integrals of functions with a non-increasing quotient gives [TG13, Thm. 9]:

Theorem 3.8 (Volume comparison). *Let $\kappa, \beta \in \mathbb{R}$ and assume (M, g, Σ) satisfies $CCC(\kappa, \beta)$. Then, for any $B \subset \Sigma_{\kappa,\beta}$ with finite, non-zero measure and any measurable $A \subset \Sigma$, the function*

$$t \mapsto \frac{\mathrm{vol} B_A^+(t)}{\mathrm{vol}_{\kappa,\beta} B_B^+(t)},$$

is nonincreasing. Further, for $t \searrow 0$ this function converges to $\frac{\mathrm{area} A}{\mathrm{area}_{\kappa,\beta} B}$, so

$$\mathrm{vol} B_A^+(t) \le \frac{\mathrm{vol}_{\kappa,\beta} B_B^+(t)}{\mathrm{area}_{\kappa,\beta} B} \mathrm{area} A. \tag{3.4}$$

A similar result has also recently been shown for $\mathcal{C}^{1,1}$-metrics ([Gra16]).

Moving away from the hypersurface case for a moment we will also need a comparison theorem for the d'Alembertian of the distance function to a point. This seems to be a well known result (see, e.g., [BEE96, Eq. (14.29)] for $\kappa = 0$, the proof of [AGH96, Prop. 4.9] for $\kappa < 0$ or [Tre, Thm. 3.3.5]).

Theorem 3.9. *Assume M is globally hyperbolic and its timelike Ricci curvature is bounded from below by $\kappa \in \mathbb{R}$. Fix $p \in M$. Then for any $q \in I^+(p) \setminus \mathrm{Cut}^+(p)$ we have*

$$-\Box \tau_p(q) \le (n-1) s_\kappa(\tau_p(q)), \tag{3.5}$$

where

$$s_\kappa(t) := \begin{cases} \sqrt{\kappa} \cot(\sqrt{\kappa} t) & \kappa > 0 \\ \frac{1}{t} & \kappa = 0 \\ \sqrt{|\kappa|} \coth(\sqrt{|\kappa|} t) & \kappa < 0 \end{cases}. \tag{3.6}$$

Proof. As in Thm. 3.4 we have that along a maximizing, unit speed geodesic γ from p to q the function $f(t) := -\Box \tau_p(\gamma(t)) = \mathrm{tr}\, \mathbf{S}_{\tau_p^{-1}(t)}(\gamma(t))$ is smooth

$(q \notin \mathrm{Cut}^+(p))$ and satisfies

$$f' + \frac{f^2}{2} \leq (n-1)\kappa \text{ and } (s_\kappa(t) - \frac{1}{n-1} f(t)) \to 0 \,\mathrm{as}\, t \searrow 0,$$

where the limiting behavior is seen by looking at Minkowski space. This gives (3.5) by Thm. 3.3. □

4. Maximality in the injectivity radius

In the next three sections we will investigate manifolds (M, g, Σ) satisfying $CCC(\kappa, \beta)$ which are in a sense maximal with respect to the bounds on distance, area and volume from Thm. 3.4-3.8 implied by the curvature.

The first (and simplest) involves maximality in the Σ-injectivity radius of M and although this seems to be a somewhat well-known fact, we will nevertheless provide a detailed proof.

Definition 4.1. The future Σ-injectivity radius $\mathrm{inj}^+_\Sigma(M)$ is defined as the infimum over the future cut parameter of points in Σ, i.e.,

$$\mathrm{inj}^+_\Sigma(M) := \inf_{p \in \Sigma} s^+_\Sigma(p).$$

Note that $\exp^N |_{(0, \mathrm{inj}^+_\Sigma(M)) \cdot S^+ N\Sigma}$ will be a diffeomorphism onto

$$B^+_\Sigma(\mathrm{inj}^+_\Sigma(M)) \setminus \mathrm{Cut}^+(\Sigma).$$

If (M, g, Σ) satisfies $CCC(\kappa, \beta)$ for some κ, β, then Cor. 3.5 shows that $\tau_\Sigma(q) < b_{\kappa,\beta}$ for all $q \in I^+(\Sigma)$, which in turn implies $\mathrm{inj}^+_\Sigma(M) \leq b_{\kappa,\beta}$. We will now show

Theorem 4.2 (Maximal injectivity radius rigidity). *Let (M, g) be globally hyperbolic and assume that (M, g, Σ) satisfies $CCC(\kappa, \beta)$ with constants κ, β such that either $\kappa > 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$. If $\mathrm{inj}^+_\Sigma(M) = b_{\kappa,\beta}$, then $I^+(\Sigma)$ is isometric to the warped product*

$$I^+(\Sigma) \cong (0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \left(\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma\right). \tag{4.1}$$

Proof. Since $\mathrm{inj}^+_\Sigma(M) = b_{\kappa,\beta}$ and $\tau_\Sigma(p) < b_{\kappa,\beta}$ for all $p \in I^+(\Sigma)$ one has $\mathrm{Cut}^+(\Sigma) = \emptyset$ and hence the normal exponential map is a diffeomorphism

$$\exp^N : (0, b_{\kappa,\beta}) \cdot S^+N\Sigma \to I^+(\Sigma).$$

Identifying $(0, b_{\kappa,\beta}) \cdot S^+N\Sigma$ with $(0, b_{\kappa,\beta}) \times \Sigma$ in the usual way and pulling back the metric g on $I^+(\Sigma)$ we obtain a metric $\bar{g}$ on $(0, b_{\kappa,\beta}) \times \Sigma$ that is of the form $\bar{g} = -dt^2 + h(t,x)$, where $h(t,.)$ denotes the induced Riemannian metric on the time slice $\{t\} \times \Sigma$. It remains to show that $\bar{g} = -dt^2 + \frac{f_{\kappa,\beta}(t)^2}{f_{\kappa,\beta}(0)^2} h_{ij}(0,x)$.

Next we show that $S_t(q) = \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)} \mathrm{id}$ for all $t < b_{\kappa,\beta}$ and $q \in S^+_\Sigma(t) = \mathcal{S}^+_\Sigma(t)$. From Thm. 3.4 we know that it suffices to show that $\frac{1}{n-1} H_t(q) = \frac{1}{n-1} H_{\kappa,\beta}(t) = \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)}$. Assume to the contrary that there exists $q_0 \in S^+_\Sigma(t_0)$ with $\tilde{\beta} := H_t(\gamma(t_0)) < H_{\kappa,\beta}(t_0) =: \beta_{t_0}$ and let γ be the unique geodesic γ starting orthogonally to Σ with $\gamma(t_0) = q_0$ (any such curve maximizes the distance due to $\mathrm{Cut}^+(\Sigma) = \emptyset$). Then starting the Riccati comparison argument not at $\gamma(0)$ but at $\gamma(t_0)$ (note that Σ_{t_0} is again a smooth, acausal, spacelike, FCC hypersurface) we see that $H_{\Sigma_t}(\gamma(t-t_0)) \leq H_{\kappa,\tilde{\beta}}(t-t_0)$ for $t > t_0$. Looking at Table 1 (or Thm. 3.3 and 3.4) we see that $H_{\kappa,\tilde{\beta}}(t-t_0) \to -\infty$ for $t - t_0 \nearrow b_{\kappa,\tilde{\beta}}$ and that the map $\beta \to b_{\kappa,\beta}$ is strictly increasing on $\mathbb{R}$ for $\kappa < 0$ and on $(-\infty, -(n-1)\sqrt{|\kappa|}]$ for $\kappa \leq 0$, hence in all cases we are considering one has $b_{\kappa,\tilde{\beta}} < b_{\kappa,\beta_{t_0}}$. Using that $f_{\kappa,\beta_{t_0}}(t-t_0) = f_{\kappa,\beta}(t)$ by uniqueness of solutions of ODE we see that $b_{\kappa,\beta_{t_0}} = b_{\kappa,\beta} - t_0$. This gives $b_{\kappa,\tilde{\beta}} < b_{\kappa,\beta} - t_0$, i.e., $H_t(\gamma(t)) \to -\infty$ for $t \nearrow b_{\kappa,\tilde{\beta}} + t_0 < b_{\kappa,\beta}$, which contradicts γ not having a focal point before $b_{\kappa,\beta}$.

Now (4.1) follows from

$$\begin{aligned}
\frac{d}{dt} h_{ij}(t,x) &= \frac{d}{dt} g(\partial_{x_i}, \partial_{x_j}) \\
&= \nabla_{\partial_t}(g(\partial_{x_i}, \partial_{x_j})) \\
&= g(\nabla_{\partial_t}\partial_{x_i}, \partial_{x_j}) + g(\partial_{x_i}, \nabla_{\partial_t}\partial_{x_j}) \\
&= g(\nabla_{\partial_{x_i}}\partial_t, \partial_{x_j}) + g(\partial_{x_i}, \nabla_{\partial_{x_j}}\partial_t) \\
&= g(S_i^k(t,x)\partial_{x_k}, \partial_{x_j}) + g(\partial_{x_i}, S_j^k(t,x)\partial_{x_k}) \\
&= 2\frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)} h_{ij}(t,x),
\end{aligned} \tag{4.2}$$

as the solution of this equation is given by $h_{ij}(t,x) = \frac{h_{ij}(0,x)}{f_{\kappa,\beta}(0)^2} f_{\kappa,\beta}(t)^2$. $\square$

Remark 4.3. As mentioned above this result in itself is not surprising. One can find a related result in [AH98, Thm. 5.3] and similar calculations also appear in [Esc88]. In general, if Σ is a spacelike hypersurface in M the normal exponential map is defined on $(0, \mathrm{inj}^+_\Sigma(M)) \cdot S^+N\Sigma$, $\exp^N((0, \mathrm{inj}^+_\Sigma(M)) \cdot S^+N\Sigma) \cong (0, \mathrm{inj}^+_\Sigma(M)) \times \Sigma$ and the induced metric on $(0, \mathrm{inj}^+_\Sigma(M)) \times \Sigma$ is adapted to this product structure (as defined by [AH98, Def. 5.1]): For any $p \in \Sigma$ the curve $t \mapsto c_p(t) := (t, p) \cong \exp^N(t\mathbf{n}_p)$ is a unit speed geodesic which shows that g is locally of the form $-dt^2 + \sum_{i,j=1}^{n-1} g_{ij}(t,x)dx_i dx_j$. So in the case of a maximal Σ-injectivity radius, while it remains to actually calculate the $g_{ij}(t,x)$, one gets an "almost" warped product for free. This will no longer be the case if one looks at maximality in the volume as will be done in Section 6.

Example 4.4. For $\kappa \geq 0$ and $\beta > -(n-1)\sqrt{|\kappa|}$ the analogue of Thm. 4.2 is false: Obviously the warped product spacetimes $M_{\kappa,\tilde{\beta}} := (a_{\kappa,\beta}, \infty) \times_{f_{\kappa,\tilde{\beta}}} (\Sigma, \frac{1}{f_{\kappa,\tilde{\beta}}(0)} h)$ with $\tilde{\beta} \in [\beta, -(n-1)\sqrt{|\kappa|})$ satisfy $CCC(\kappa, \beta)$, $\mathrm{inj}^+_\Sigma(M_{\kappa,\tilde{\beta}}) = \infty$ and $g_{\kappa,\tilde{\beta}}|_\Sigma = h$ but they are not isometric to $M_{\kappa,\beta}$ unless $\tilde{\beta} = \beta$.

5. A splitting theorem for hypersurfaces with a maximal ray

The goal of this section is to show that one does not need $\mathrm{inj}^+_\Sigma(M) = b_{\kappa,\beta}$ to obtain a splitting result and that indeed the existence of only one Σ-ray of length $b_{\kappa,\beta}$ is sufficient. As mentioned in the introduction the proof will be a rather straightforward combination of arguments from [Esc88], [Gal89b], [Gal89a] and [AGH96].

Definition 5.1 ((Maximal length) Σ-rays). Let $\Sigma \subset M$ be an acausal subset. A timelike future inextendible unit-speed geodesic $\gamma : [0, a) \to M$ is called a Σ-ray if $\gamma(0) \in \Sigma$ and γ maximizes distance to Σ, i.e., $L(\gamma|_{[0,t]}) = \tau_\Sigma(\gamma(t))$ for all $t \in [0, a)$. If (M, g, Σ) satisfies $CCC(\kappa, \beta)$ we say a Σ-ray γ is *maximal* (or has *maximal length*) if $a = b_{\kappa,\beta}$.

To any ray one can define asymptotes:

Definition 5.2 (Asymptotes). For $p \in M$ we call an inextendible geodesic $\alpha_p : [0, \bar{a}) \to M$ an asymptote to the $(\Sigma$-$)$ray $\gamma : [0, a) \to M$ at p if $\alpha_p(0) = p$ and $\dot{\alpha}_p(0) = \lim_{n\to\infty} \dot{\alpha}_{p,s_n}(0)$ for some sequence $s_n \to a$, where $\alpha_{p,s}$ denotes the maximizing unit speed geodesic from p to $\gamma(s)$. So α_p arises as a limit curve of a sequence α_{p,s_n} of maximizing curves from p to $\gamma(s_n)$ as $s_n \to a$.

Given a ray $\gamma : [0, a) \to M$ we define the Busemann function b associated to this ray.

Definition 5.3 (Busemann function). Given a Σ-ray $\gamma : [0, a) \to M$ one defines its *Busemann function* b as the limit

$$b(x) := \lim_{r \to a} r - \tau_x(\gamma(r)) \tag{5.1}$$

for $x \in I^-(\gamma) \cap I^+(\Sigma)$.

Remark 5.4. That this limit actually exists is seen as follows: By the reverse triangle inequality (2.3) one has $\tau_x(\gamma(r)) \geq \tau_x(\gamma(s)) + r - s$ for $r \geq s \geq r_0$ with r_0 such that $x \in I^-(\gamma(r_0))$ so the map $r \mapsto r - \tau_x(\gamma(r))$ is monotonously decreasing and using $\tau_x(\gamma(r)) \leq \tau_{\gamma(0)}(\gamma(r)) - \tau_\Sigma(x)$ it is easy to see that $r - \tau_x(\gamma(r)) \geq \tau_\Sigma(x)$ for all $x \in I^-(\gamma) \cap I^+(\Sigma)$. This also shows

$$b(x) \geq \tau_\Sigma(x). \tag{5.2}$$

Before we summarize the most important facts about the Busemann function in the following Proposition we need one more definition.

We say that a set $N \subset M$ in a spacetime (M, g) is *edgeless* if for all $p \in N$ and all neighborhoods V of p in M any timelike curve from $I^-(p, V)$ to $I^+(p, V)$ must meet N. The following definition was introduced in [EG92].

Definition 5.5 ($\mathcal{C}^0$ spacelike hypersurface). A subset $N \subset M$ of a spacetime (M, g) is called $\mathcal{C}^0$ *spacelike hypersurface* if for each $p \in N$ there is a neighborhood U of p in M such that $N \cap U$ is acausal and edgeless in U. Note that this implies that N is a topological hypersurface by [O'N83, Prop. 14.25].

Proposition 5.6 (Properties of the Busemann function). *Let (M, g) be a globally hyperbolic spacetime, $\Sigma \subset M$ an acausal, FCC, spacelike hypersurface and $\gamma : [0, a) \to M$ a Σ-ray. Then for any $t \in (0, a)$ there is a neighborhood U of $\gamma(t)$ (called a* nice neighborhood*) such that the following holds:*

1) *The Busemann function b is continuous on U and if $q \in J^+(p)$ one has*

$$b(q) \geq b(p) + \tau_p(q) \tag{5.3}$$

2) *For any given Riemannian background metric h there exists a constants C and $t < T < a$ such that for any maximizing geodesic $\alpha_{p,s}$ from a point $p \in U$ to $\gamma(s)$ with $s \geq T$*

$$h(\dot{\alpha}_{p,s}(0), \dot{\alpha}_{p,s}(0)) \leq C, \tag{5.4}$$

i.e., the set $\{\dot{\alpha}_{p,s}(0) : p \in U, T \leq s < a\} \subset TM$ is contained in a compact set.

3) *For any $p \in U$ there exists a timelike, unit-speed asymptote $\alpha_p : [0, a - b(p)) \to M$ at p that is future inextendible, maximizing, and satisfies*

$$b(\alpha_p(t)) = t + b(p). \tag{5.5}$$

4) *The level set $N_t := \{x \in U : b(x) = t\}$ of b in U is edgeless and acausal, i.e., a $\mathcal{C}^0$ spacelike hypersurface in U.*

Proof. This was shown in [Esc88], [Gal89b] and [Gal89a], see specifically [Esc88, Lem. 3.3] for Lipschitz continuity, [Esc88, Lem. 3.2] for the estimate (5.4) and [Gal89b, Lem. 2.3] for the properties of N_t. The existence of timelike, unit speed asymptotes follows from (5.4). By a standard result about the length functional regarding limits of curves contained in a common compact set one then has$\limsup_{s\to a} L(\alpha_{p,s}|_{[0,t]}) \leq L(\alpha_p|_{[0,t]})$ (for any $t > 0$ such that there exists s_0 such that $\alpha_{p,s}$ is well defined on $[0, t]$ for $s_0 < s < a$). This shows that the asymptote is maximzing and has length at least

$$\limsup_{s\to a} L(\alpha_{p,s}) = \lim_{s\to a} \tau_p(\gamma(s)) = \lim_{s\to a} (s - (s - \tau_p(\gamma(s)))) = a - b(p).$$

Finally, because $\gamma(s) \to \infty$ (i.e., leaves every compact set) for $s \to a$ the asymptote $\alpha_p : [0, a - b(p)) \to M$ is inextendible. Equations (5.3) and (5.5) are immediate consequences of the reverse triangle inequality (2.3).

The statement is also included in [AGH96]. Note that all of this is independent of any curvature assumptions. □

The main argument we use from [AGH96] will be a theorem about $\mathcal{C}^0$ spacelike hypersurfaces with curvature bounds. Given two $\mathcal{C}^0$ spacelike hypersurfaces in (M, g) which meet at a point q we say that N_0 is locally to the future of N_1 near q if they meet at q and for some neighborhood U of q in which N_1 is acausal and edgeless, $N_0 \cap U \subset J^+(N_1, U)$. Now one can define mean curvature bounds of such a $\mathcal{C}^0$ spacelike hypersurface as follows

Definition 5.7. Let N be a $\mathcal{C}^0$ spacelike hypersurface in the spacetime (M, g) and H_0 a constant. Then

1) N has *mean curvature* $\leq H_0$ *in the sense of support hypersurfaces* if for all $q \in N$ and $\varepsilon > 0$ there is a $\mathcal{C}^2$ future support hypersurface $S_{q,\varepsilon}$ (i.e., $q \in S_{q,\varepsilon}$ and $S_{q,\varepsilon}$ is locally to the future of N near q) such that

$$H_{S_{q,\varepsilon}}(q) \leq H_0 + \varepsilon.$$

2) N has *mean curvature* $\geq H_0$ *in the sense of support hypersurfaces with one-sided Hessian bounds* if for all compact sets $K \subset N$ there exists a compact set $\hat{K} \subset TM$ and a constant $C > 0$ such that for all $q \in K$ there is a $\mathcal{C}^2$ past support hypersurface $P_{q,\varepsilon}$ (i.e., $q \in P_{q,\varepsilon}$ and $P_{q,\varepsilon}$ is locally to the past of N near q) such that the future pointing unit normal $\mathbf{n}_{P_{q,\varepsilon}}(q)$ is in $\hat{K}$, the second fundamental form $h_{P_{q,\varepsilon}}$ satisfies

$$h_{P_{q,\varepsilon}}(q) \geq -C_K g|_{P_{q,\varepsilon}}(q) \tag{5.6}$$

and

$$H_{P_{q,\varepsilon}}(q) \geq H_0 - \varepsilon.$$

This definition was introduced in [AGH96] and allows them to prove a Lorentzian geometric maximum principle for $\mathcal{C}^0$ spacelike hypersurfaces.

Theorem 5.8 (Lorentzian Geometric Maximum Principle). *Let N_0 and N_1 be $\mathcal{C}^0$ spacelike hypersurfaces in a spacetime (M, g) which meet at a point q_0, such that N_0 is locally to the future of N_1 near q_0. Assume for some constant H_0:*

1) *N_0 has mean curvature $\leq H_0$ in the sense of support hypersurfaces and*

2) *N_1 has mean curvature $\geq H_0$ in the sense of support hypersurfaces with one-sided Hessian bounds.*

Then $N_0 = N_1$ near q_0, i.e., there is a neighborhood U of q_0 such that $N_0 \cap U = N_1 \cap U$. Moreover, $N_0 \cap U = N_1 \cap U$ is a smooth spacelike hypersurface with mean curvature H_0.

Proof. See [AGH96, Thm. 3.6]. □

We are now going to show the analogue to [AGH96, Prop. 4.9, 4.] for our situation.

Proposition 5.9. *Let (M, g) be a globally hyperbolic spacetime, $\Sigma \subset M$ an acausal, FCC and spacelike hypersurface, $\gamma : [0, a) \to M$ a Σ-ray and let*

U be a nice neighborhood of $\gamma(t)$. If $\mathbf{Ric}(v,v) \geq -(n-1)\,\kappa\, g(v,v)$ for all timelike $v \in TM$, then

$$H_{N_t} \geq -(n-1)s_\kappa(a-t) = \begin{cases} -\sqrt{\kappa}\cot(\sqrt{\kappa}(a-t)) & \kappa > 0 \\ -\frac{1}{a-t} & \kappa = 0 \\ -\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}(a-t)) & \kappa < 0 \end{cases} \tag{5.7}$$

in the sense of support hypersurfaces with one-sided Hessian bounds. Note that by Thm. 3.4 $a = \infty$ can only happen if $\kappa = 0$ or $\kappa < 0$ (for $\kappa > 0$ one has $b_{\kappa,\beta} < \infty$ for any $\beta \in \mathbb{R}$) in which cases the functions behave nicely at infinity and we set $\frac{1}{a-t} := 0$ and $-\sqrt{|\kappa|}\coth(\sqrt{|\kappa|}(a-t)) := -\sqrt{|\kappa|}$, respectively.

Proof. The proof is completely analogous to [AGH96, Prop. 4.9, 4.]. Given any $p \in N_t$ there exists a timelike asymptote $\alpha_p : [0, a-t) \to M$ by Prop. 5.6. Now we look at $S^-_{\alpha_p(s)}(s) := \{x \in M : \tau_x(\alpha_p(s)) = s\}$. Clearly $S^-_{\alpha_p(s)}(s)$ is a smooth hypersurface for any $s \in (0, a-t)$, $p \in S^-_{\alpha_p(s)}(s)$, and by Thm. 3.9

$$H_{S^-_{\alpha_p(s)}(s)} \geq -(n-1)s_\kappa(s).$$

From (5.3) and (5.5) we get immediately that

$$b(x) \leq b(\alpha_p(s)) - \tau_x(\alpha_p(s)) = t$$

for all $x \in S^-_{\alpha_p(s)}(s)$ and invoking (5.3) again this shows $S^-_{\alpha_p(s)}(s) \cap I^+(N_t) = \emptyset$. Since N_t is an acausal topological hypersurface in U its Cauchy development D (in U) must be open ([O'N83, Lem. 14.43]), N_t is edgeless and acausal in D and $S^-_{\alpha_p(s)}(s) \cap D \subset J^-(N_t, D)$ (because as noted above $S^-_{\alpha_p(s)}(s) \cap I^+(N_t) = \emptyset$), hence the $S^-_{\alpha_p(s)}(s)$ lie locally to the past of N_t near p for any $s \in (0, a-t)$. But this means that they are past support hypersurfaces with the right curvature bounds. By (5.4) the unit normals $\alpha_p'(0)$ are contained in a compact set for all $p \in N_t \cap U$, so we can use [AGH96, Prop. 3.5] to see hat they also satisfy the estimate (5.6) on the second fundamental form. $\square$

Combining the above with the mean curvature comparison Thm. 3.4 and the geometric maximum principle (Thm. 5.8, [AGH96, Prop. 4.6]) yields the following analogue to [Gal89a, Lem. 3.2]:

Proposition 5.10. *Assume that (M, g, Σ) satisfies $CCC(\kappa, \beta)$ with either $\kappa > 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$. If $\gamma : [0, b_{\kappa,\beta}) \to M$ is a maximal Σ-ray, then* *there exists a neighborhood U of $\gamma(0)$ in Σ such that any inextendible (f.d., unit-speed) geodesic σ with $\sigma'(0) \in T\Sigma^{\perp}|_U$ is also a Σ-ray.*

Proof. Choose a neighborhood V of $\gamma(0)$ in Σ and $\delta > 0$ small enough such that $\exp^N$ is smooth on $V \times (-\delta, \delta)$ and denote by Σ_δ the hypersurface $\exp^N(V \times \{\delta\})$. Note that by shrinking V if necessary we can assume $\Sigma_\delta \subset U$ for a nice neighborhood U of $\gamma(\delta)$. From (5.2) it follows that $b(p) \geq \delta$ for all $p \in \Sigma_\delta$. Since b is strictly increasing along timelike curves (again (5.3)) this shows $\Sigma_\delta \cap I^-(N_\delta) = \emptyset$, so as before looking at the Cachy development D of $U \cap N_\delta$ we see that $\Sigma_\delta \subset J^+(N_\delta, D)$ and obviously $\gamma(\delta) \in \Sigma_\delta \cap N_\delta$, so Σ_δ lies locally to the future of N_δ. Now Thm. 3.4 shows $H_{\Sigma_\delta} \leq H_{\kappa,\beta}(\delta)$ and comparing the definition of s_κ in (3.6) with Table 1 shows $H_{\kappa,\beta}(\delta) = -(n-1)s_\kappa(b_{\kappa,\beta} - \delta)$ if $b_{\kappa,\beta} < \infty$ (i.e., $\kappa > 0$ or $\beta < -(n-1)\sqrt{|\kappa|}$) and $H_{\Sigma_\delta} \leq H_{\kappa,\beta}(\delta) \leq \beta = -(n-1)\sqrt{|\kappa|} = \lim_{r\to\infty} -(n-1)s_\kappa(r)$ if $b_{\kappa,\beta} = \infty$ (i.e., $\beta = -(n-1)\sqrt{|\kappa|}$). Thus, taking into acount the lower bound (5.7) on H_{N_δ}) we can apply Thm. 5.8 to obtain $N_\delta = H_{\Sigma_\delta}$.

Now for any $p \in V$ we look at the curve $\tilde{\alpha}_p : [0, b_{\kappa,\beta}) \to M$ given by

$$\tilde{\alpha}_p(t) := \begin{cases} \exp^N(t\mathbf{n}_p) & 0 \leq t \leq \delta \\ \alpha_{\exp^N(\delta \mathbf{n}_p)}(t - \delta) & \delta \leq t < b_{\kappa,\beta} \end{cases}.$$

This curve satisfies $\tau_\Sigma(\tilde{\alpha}_p(t)) = t$: By (5.5) one has

$$b(\tilde{\alpha}_p(t)) = t - \delta + b(\exp^N(\delta \mathbf{n}_p)) = t$$

and the claim follows from (5.2). Because $\tilde{\alpha}_p$ is parametrized by arc-length this shows that $\tilde{\alpha}_p$ always maximizes the distance to Σ so it has to be a geodesic starting orthogonally to Σ and a Σ-ray. □

The previous result allows us to prove a local splitting via Thm. 4.2. To extend this to a global one we need one more Lemma.

Lemma 5.11. *Let $\kappa, \beta \in \mathbb{R}$ with either $\kappa > 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$, let (Σ, h) be an $(n-1)$-dimensional Riemannian manifold, and let $M := [0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \Sigma$. Then for any $t \in [0, b_{\kappa,\beta})$ and any $r > 0$ there exists $\tilde{t} \in (t, b_{\kappa,\beta})$ such that $\{t\} \times B_p(r) \subset J^-((\tilde{t}, p))$ for all $p \in \Sigma$. Furthermore $M = J^-([0, b_{\kappa,\beta}) \times \{p\})$ for any $p \in \Sigma$.*

Proof. We look at (future directed) null curves $c = (c_0, \bar{c})$ starting at a point (t, p) such that the projection $\bar{c}$ is a unit-speed curve in (Σ, h). This yields the ODE $c_0'(s)^2 = f_{\kappa,\beta}^2(c_0(s))\, |\bar{c}'(s)|_h^2 = f_{\kappa,\beta}^2(c_0(s))$ with $c_0(0) = t$. Since we want c to be future directed, we need $c_0' > 0$, so the ODE becomes $c_0'(s) = |f_{\kappa,\beta}(c_0(s))|$. Noting that $f_{\kappa,\beta}^2$ is monotonously decreasing on $[r_{\kappa,\beta}, b_{\kappa,\beta})$ for some $r_{\kappa,\beta} < b_{\kappa,\beta}$(see Table 1) this gives that $c_0(s) \leq |f_{\kappa,\beta}(t)|\, s + t$ for $t \geq r_{\kappa,\beta}$. So given any radius r there exists t such that $c_0(r) \leq |f_{\kappa,\beta}(t)|\, r + t < b_{\kappa,\beta}$ Now let $p, q \in \Sigma$ with $q \in S_p(\bar{r})$ for $\bar{r} < r$ then there is a future directed null curve $c : [0, \tilde{r}] \to M$ (with $r > \tilde{r} \geq \bar{r}$ since there may not exist a curve from p to q in Σ of minimal length) from (t, q) to $(c_0(\tilde{r}), p)$, i.e., $(t, q) \in J^-((c_0(\tilde{r}), p)) \subset J^-((c_0(r), p))$. This finishes the proof. □

Now we are ready to prove the theorem.

Theorem 5.12. *Assume that (M, g, Σ) satisfies $CCC(\kappa, \beta)$ with constants κ, β such that $\kappa > 0$ or $\beta \leq -(n-1)\sqrt{|\kappa|}$. If M contains a maximal Σ-ray $\gamma : [0, b_{\kappa,\beta}) \to M$, then $I^+(\Sigma)$ is isometric to the warped product*

$$I^+(\Sigma) \cong (0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \left(\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma \right). \tag{5.8}$$

Proof. Let $U \subset \Sigma$ be as in Prop. 5.10 and let $j : \mathbb{R} \times \Sigma \to T\Sigma^\perp$denote the map $(t, p) \mapsto t\mathbf{n}_p$. Then $\exp^N \circ j : (0, b_{\kappa,\beta}) \times U \to M$ is a diffeomorphism onto its image and by Thm. 4.2 even an isometry if we equip $(0, b_{\kappa,\beta}) \times U$ with the metric $-dt^2 + \frac{f_{\kappa,\beta}(t)^2}{f_{\kappa,\beta}(0)^2} g|_U$. Now let $r > 0$ be such that $U = B_r(\gamma(0))$ is the largest open ball in Σ such that $\exp^N \circ j|_{(0,b_{\kappa,\beta})\times U}$ is a diffeomorphism. If $U = \Sigma$ we are done. Otherwise there exists a point $p \in \partial U$ such that $t \mapsto \exp^N(t\mathbf{n}_p) =: \sigma(t)$ either stops existing or being maximizing before $b_{\kappa,\beta}$.

If it stops being maximizing but not existing the cut function $s_\Sigma^+ : S^+N\Sigma \to (0, \infty]$ is continuous at $\dot{\sigma}(0)$ by Lem. 2.7, so we find a neighborhood V of p such that all f.d., unit-speed geodesics starting in V orthogonally to Σ also have a cut parameter less than $b_{\kappa,\beta}$, which contradicts $p \in \partial U$.

If it stops existing at $T < b_{\kappa,\beta}$, then $\sigma \subset \overline{\exp^N([0, T) \cdot S^+N\Sigma|_U)}$. Now by Lem. 5.11 there exists $\tilde{t} < b_{\kappa,\beta}$ such that

$$\{T\} \times B_r(\gamma(0)) \subset J^-((\tilde{t}, \gamma(0))),$$

hence $[0, T] \times B_r(\gamma(0)) \subset J^-((\tilde{t}, \gamma(0)))$. But this shows

$$\sigma \subset \overline{\exp^N([0, T) \cdot S^+N\Sigma|_U)} \subset J^-(\gamma(\tilde{t})),$$

so σ is contained in the compact set $J^+(p) \cap J^-(\gamma(\tilde{t}))$, contradicting its inextendibility. □

Remark 5.13. If Σ is compact it is sufficient that $\sup\{\tau_\Sigma(p) : p \in I^+(\Sigma)\} = b_{\kappa,\beta}$ for Thm. 5.12 to hold: Let $p_n \in M$ be such that $\tau_\Sigma(p_n) \to b_{\kappa,\beta}$. Since Σ is assumed to be FCC there exist $q_n \in \Sigma$ and unit speed timelike geodesics $\gamma_n : [0, \tau_\Sigma(p_n)] \to M$ from q_n to p_n that maximize the distance to Σ. Now if Σ is compact, a subsequence of these γ_n must converge to a geodesic $\gamma : [0, b_{\kappa,\beta}) \to M$ that also maximizes the distance to Σ. So we have constructed a maximal Σ-ray $\gamma : [0, b_{\kappa,\beta}) \to M$.

Looking at some special cases of Thm. 5.12 we see that for $\kappa = \beta = 0$ the warping function is given by $f_{0,0} \equiv 1$ so one recovers that $I^+(\Sigma)$ is isometric to $((0,\infty) \times \Sigma, -dt^2 \oplus h)$ (cf. [Gal89a]). For $\kappa = -1, \beta = -(n-1)$ the warping function is $f_{-1,-(n-1)}(t) = e^{-t}$ giving the splitting $I^+(\Sigma) \cong ((0,\infty) \times \Sigma, -dt^2 + e^{-2t}h)$ (cf. [GV16, Thm. 5.13]). In this case β is not quite negative enough to force incompleteness. If $\beta < -(n-1)$ (and $\kappa = -1$) one has $c_\beta = \coth^{-1}(\frac{\beta}{n-1}) < 0$ and the warping function $f^2_{-1,\beta} = \sinh^2(t + c_\beta) = \sinh^2(t - |c_\beta|)$. Finally, in the case of positive timelike Ricci curvature, we see that for e.g. $\kappa = 1$ and $\beta = n-1, 0, -(n-1)$ one has the warping functions $f^2_{1,n-1} = \sin^2(t + \frac{\pi}{4})$, $f^2_{1,0} = \sin^2(t + \frac{\pi}{2})$ and $f^2_{1,-(n-1)} = \sin^2(t + \frac{3\pi}{4})$, respectively.

6. A splitting theorem for maximal volume

In this section we are going to look at spacetimes that are in a sense maximal in volume, specifically we want the volume of distance balls $B^+_A(t)$ over a set $A \subset \Sigma$ to be maximal. Obviously this volume depends on the area of the base set, so we first introduce a function $v_{\kappa,\beta}$ on our comparison spaces giving the volume of future balls over a subset $A \subset \Sigma_{\kappa,\beta}$ in $M_{\kappa,\beta}$ relative to the area of A.

Definition 6.1. Given $\kappa, \beta \in \mathbb{R}$ and any measurable set $A \subset \Sigma_{\kappa,\beta}$ with non-zero measure we define

$$v_{\kappa,\beta}(t) := \frac{\mathrm{vol}_{\kappa,\beta} B^+_A(t)}{\mathrm{area}_{\kappa,\beta} A}.$$

Note that $\operatorname{tr} \mathbf{S}_{\kappa,\beta} = H_{\kappa,\beta} = (n-1)\frac{f'}{f}$ for warped products ([O'N83, Prop. 7.35]), so the variation of area formula (3.1) and the coarea formula (3.3) show

$$v_{\kappa,\beta}(t) = \frac{1}{f_{\kappa,\beta}(0)^{n-1}} \int_0^t f_{\kappa,\beta}(\tau)^{n-1} d\tau$$

for all $t \leq b_{\kappa,\beta}$. For $t \geq b_{\kappa,\beta}$ one obviously has $v_{\kappa,\beta}(t) = v_{\kappa,\beta}(b_{\kappa,\beta}) =: \bar{v}_{\kappa,\beta}$, so $v_{\kappa,\beta}$ really is independent of the choice of A. If $\kappa > 0$ or $\beta < -(n-1)\sqrt{|\kappa|}$ then $b_{\kappa,\beta} < \infty$ and hence $\bar{v}_{\kappa,\beta} < \infty$.

We are now ready to prove a splitting theorem if the volume of $B_K^+ := \bigcup_{s\in(0,\infty)} S_K^+(s)$ is finite and maximal (w.r.t. (3.4)) for compact $K \subset \Sigma$.

Theorem 6.2 (Maximal volume splitting). *If (M, g, Σ) satisfies $CCC(\kappa, \beta)$ with either $\kappa > 0$ or $\beta < -(n-1)\sqrt{|\kappa|}$ and there exists an exhaustion by compact sets $\{K_n\}_{n\in\mathbb{N}}$ for Σ such that*

$$\frac{\mathrm{vol} B_{K_n}^+}{\mathrm{area} K_n} = \bar{v}_{\kappa,\beta} \tag{6.1}$$

for all n, then

$$I^+(\Sigma) \cong (0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \left(\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma\right). \tag{6.2}$$

If furthermore Σ is past causally complete, $\kappa > 0$ and there exists an exhaustion of compact sets $\{K_n\}_{n\in\mathbb{N}}$ for Σ such that also

$$\frac{\mathrm{vol} B_{K_n}^-}{\mathrm{area} K_n} = \bar{v}_{\kappa,-\beta}$$

for all n, then

$$M \cong (a_{\kappa,\beta}, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \left(\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma\right). \tag{6.3}$$

Proof. By Cor. 3.5 we have $\tau_\Sigma(q) < b_{\kappa,\beta}$ for all $q \in I^+(\Sigma)$, so $\mathrm{vol} B_{K_n}^+(b_{\kappa,\beta}) = \mathrm{vol} B_{K_n}^+ = \bar{v}_{\kappa,\beta}\, \mathrm{area} K_n = v_{\kappa,\beta}(b_{\kappa,\beta})\, \mathrm{area} K_n$. Using this and the coarea formula (3.3) it follows that

$$\begin{aligned} 0 &= v_{\kappa,\beta}(b_{\kappa,\beta}) - \frac{\mathrm{vol} B_{K_n}^+(b_{\kappa,\beta})}{\mathrm{area} K_n} \\ &= \int_0^{b_{\kappa,\beta}} \frac{\mathrm{area}_{\kappa,\beta} S_A^+(\tau)}{\mathrm{area}_{\kappa,\beta} A} - \frac{\mathrm{area} \mathcal{S}_{K_n}^+(\tau)}{\mathrm{area} K_n} d\tau \end{aligned} \tag{6.4}$$

for any $A \subset \Sigma_{\kappa,\beta}$ with finite, non-zero measure. Now by the area comparison theorem Thm. 3.7 the integrand is always non-negative, so it has to be zero almost everywhere and we obtain

$$a_{\kappa,\beta}(t) := \frac{\mathrm{area}_{\kappa,\beta} S_A^+(t)}{\mathrm{area}_{\kappa,\beta} A} = \frac{\mathrm{area}\mathcal{S}_{K_n}^+(t)}{\mathrm{area} K_n} \tag{6.5}$$

for almost all $t \leq b_{\kappa,\beta}$. Since $t \mapsto \frac{\mathrm{area}\mathcal{S}_{K_n}^+(t)}{\mathrm{area}_{\kappa,\beta} S_A^+(t)}$ is non-increasing (again Thm. 3.7), the equality (6.5) follows for all $t < b_{\kappa,\beta}$.

Next we show that thus (6.5) holds for any compact set $K \subset \Sigma$: Given K choose $n \in \mathbb{N}$ such that $K \subset K_n$. Then it follows immediately from the definition of these spheres (see Def. 3.1) that $\mathcal{S}_{K_n}^+(t) = \mathcal{S}_K^+(t) \cup \mathcal{S}_{K_n \setminus K}^+(t)$ and the union is disjoint. But then

$$\begin{aligned} a_{\kappa,\beta}(t)\, \mathrm{area} K_n &= \mathrm{area}\mathcal{S}_{K_n}^+(t) = \mathrm{area}\mathcal{S}_K^+(t) + \mathrm{area}\mathcal{S}_{K_n \setminus K}^+(t) \\ &\leq a_{\kappa,\beta}(t)\, \mathrm{area} K + \mathrm{area}\mathcal{S}_{K_n \setminus K}^+(t) \\ &\leq a_{\kappa,\beta}(t)\, (\mathrm{area} K + \mathrm{area} K_n \setminus K) \\ &= a_{\kappa,\beta}(t)\, \mathrm{area} K_n, \end{aligned}$$

so all inequalities have to be equalities, showing (6.5) for K.

This allows us to prove that actually $\mathrm{Cut}^+(\Sigma) = \emptyset$: First note that it suffices to show $\mathrm{Cut}^+(\Sigma) \cap S_\Sigma^+(t) = \emptyset$ for all $t < b_{\kappa,\beta}$ since we know $\tau_\Sigma(q) < b_{\kappa,\beta}$ for all $q \in I^+(\Sigma)$ from Cor. 3.5. Assume $p \in \mathrm{Cut}^+(\Sigma) \cap S_\Sigma^+(t)$, then $p = \gamma(t)$ for some f.d., unit-speed geodesic γ starting orthogonally to Σ with $t = s_\Sigma^+(\dot\gamma(0))$. Since this γ is certainly defined on an open interval containing t the cut function $s_\Sigma^+ : S^+ N\Sigma \to (0, \infty]$ is continuous at $\dot\gamma(0)$ by Lem. 2.7. Let $\varepsilon > 0$ with $t + \varepsilon < b_{\kappa,\beta}$. We can choose a relatively compact neighborhood V in Σ of $\gamma(0)$ such that all f.d., unit-speed geodesics starting in $\bar V$ orthogonally to Σ have a cut parameter less than $t + \varepsilon < b_{\kappa,\beta}$. But then $\mathcal{S}_{\bar V}^+(t+\varepsilon) = \emptyset$, contradicting $0 \neq a_{\kappa,\beta}(t+\varepsilon) = \frac{\mathrm{area}\mathcal{S}_{\bar V}^+(t)}{\mathrm{area}\bar V}$.

Next we will show that $H_t(q) = H_{\kappa,\beta}(t)$ for all $q \in I^+(\Sigma)$ with $t = \tau_\Sigma(q)$. To see this, let γ be the unique geodesic from Σ to q realizing the distance and choose $K \subset \Sigma$ to be a compact neighborhood of $\gamma(0)$ such that the normal exponential map is defined on $[0, t') \times K$ for some $t' > t$. By (6.5) the map $t \mapsto \frac{\mathrm{area}\mathcal{S}_K^+(t)}{\mathrm{area}_{\kappa,\beta} S_A^+(t)}$ is constant on $[0, t')$ and since the set $\mathcal{S}_K^+(t) = S_K^+(t) =$

$\exp^N(\{t\} \times K)$ is compact we may proceed as in the proof of the area comparison theorem and use the first variation of area (Prop. 3.6) to obtain

$$\begin{aligned}
0 &= \frac{d}{ds}\bigg|_{s=t} \log \frac{\mathrm{area}_{\kappa,\beta} S_A^+(s)}{\mathrm{area}\, S_K^+(s)} \\
&= \frac{d}{ds}\bigg|_{s=t} \log \mathrm{area}_{\kappa,\beta} S_A^+(s) - \frac{d}{ds}\bigg|_{s=t} \log \mathrm{area}\, S_K^+(s) \\
&= \frac{1}{\mathrm{area}\, S_K^+(t)} \int_{S_K^+(t)} H_{\kappa,\beta}(t) - H_t(q) d\mu_t(q).
\end{aligned}$$

Now the integrand is non-negative (by the mean curvature comparison theorem, see Thm. 3.4) and smooth (in q) on $\mathcal{S}_\Sigma^+(t) = S_\Sigma^+(t)$ (because the normal exponential map is a diffeomorphism away from the cut locus), hence $H_t(q) = H_{\kappa,\beta}(t)$ for all $q \in S_K^+(t)$.

By Thm. 3.4 this already implies $\mathbf{S}_t = H_{\kappa,\beta}(t)\,\mathrm{id} = \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)}\,\mathrm{id}$ for all $t < b_{\kappa,\beta}$. Unfortunately, $\exp^N$ need a priori not be defined on all of $(0, b_{\kappa,\beta}) \cdot S^+N\Sigma$, so there is still some more work to do than in Thm. 4.2. We can, however, proceed similarly: Using the normal exponential map we obtain coordinates (t, x) on an open submanifold of M containing $I^+(\Sigma) \cup \Sigma$ (note again that $\mathrm{Cut}^+(\Sigma) = \emptyset$) in which $g = -dt^2 + h(t, x)$ where for any $0 \leq t < b_{\kappa,\beta}$ the expression $h(t, .)$ denotes the induced Riemannian metric on the spacelike hypersurface $\mathcal{S}_\Sigma^+(t) = S_\Sigma^+(t)$ (which is just the $\{t\}$-level set of the distance function τ_Σ). Calculating as in (4.2) we see that for $t > 0$ and $x \in S_\Sigma^+(t)$

$$\frac{d}{dt} h_{ij}(t, x) = 2 \frac{f'_{\kappa,\beta}(t)}{f_{\kappa,\beta}(t)} h_{ij}(t, x).$$

The solution of this equation is again given by $h_{ij}(t, x) = \frac{h_{ij}(0,x)}{f_{\kappa,\beta}(0)^2}\, f_{\kappa,\beta}(t)^2$.

This shows that

$$\begin{aligned}
I^+(\Sigma, M) &\cong \left(\mathcal{D} \cap \left((0, b_{\kappa,\beta}) \cdot S^+N\Sigma\right), -dt^2 + \frac{f_{\kappa,\beta}(t)^2}{f_{\kappa,\beta}(0)^2} g|_\Sigma\right) \\
&\subset [0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} \left(\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma\right) =: M_{\kappa,\beta},
\end{aligned}$$

so it is isometric to an open submanifold of the warped product.

It only remains to show that all f.d., unit-speed geodesics starting orthogonally to Σ are defined in M on $[0, b_{\kappa,\beta})$, i.e., they remain in the submanifold $I^+(\Sigma, M) \subset M_{\kappa,\beta}$. Assume to the contrary that there exists such a geodesic $\gamma : [0, T) \to I^+(\Sigma, M)$ with $T < b_{\kappa,\beta}$ that is inextendible in M.

Let $\varepsilon > 0$ such that $T + \varepsilon < b_{\kappa,\beta}$. Then area$S^+_U(T + \varepsilon)$ has to be maximal for any relatively compact neighborhood U of $q := \gamma(0)$ in Σ and hence non-zero, in particular $S^+_U(T + \varepsilon) \neq \emptyset$. Thus there exists a sequence of $q_n \in \Sigma$ with $d_\Sigma(q, q_n) = \frac{1}{n}$ (where d_Σ is the Riemannian distance on Σ induced by $g|_\Sigma$) such that the corresponding γ_n exist until at least $T + \varepsilon$. Set $p_n := \gamma_n(T + \varepsilon)$.

Let V be a relatively compact and geodesically convex neighborhood of q in Σ and choose N such that that $q_n \in V$ for all $n > N$. Now for any $0 < \delta \leq T$ let $\sigma_{n,\delta} : [0, s_{\max}) \to M_{\kappa,\beta}$ be defined by $\sigma_{n,\delta}(s) := (T + \varepsilon - s, c_n(s\,\frac{1}{n(\varepsilon+\delta)}))$ where $c_n : [0, b) \to \Sigma$ is the *unit-speed* geodesic in Σ starting at q_n in direction q. Note that because V was chosen to be geodesically convex the curve $\sigma_{n,\delta}$ is actually well-defined on $[0, \varepsilon + \delta]$, its projection to the second coordinate is contained in V and $\sigma_{n,\delta}(0) = (T + \varepsilon, c_n(0)) = \gamma_n(T + \varepsilon) = p_n$ and $\sigma_{n,\delta}(\varepsilon + \delta) = (T - \delta, c_n(d_\Sigma(q, q_n))) = (T - \delta, q) = \gamma(T - \delta)$. We have $\dot\sigma_{n,\delta}(s) = (-1, \frac{1}{n(\varepsilon+\delta)}\dot c_n(\frac{s}{n(\varepsilon+\delta)}))$, so for $n > \max_{s\in[0,T+\varepsilon]} \frac{f_{\kappa,\beta}(T+\varepsilon-s)^2}{f_{\kappa,\beta}(0)^2\varepsilon}$ we have

$$g(\dot\sigma_{n,\delta}(s), \dot\sigma_{n,\delta}(s)) = -1 + \frac{f_{\kappa,\beta}(T + \varepsilon - s)^2}{f_{\kappa,\beta}(0)^2 n(\varepsilon + \delta)} < 0$$

Note that this bound on n is independent of δ. So if we fix N large enough, we see that, at least in $M_{\kappa,\beta}$, the curves $\sigma_{N,\delta} : [0, \varepsilon + \delta] \to M_{\kappa,\beta}$ can be used to give a timelike connection from p_N to any point on γ.

Next we show that actually $\sigma_{N,\delta} : [0, \varepsilon + \delta] \to I^+(\Sigma, M) \subset M \subset M_{\kappa,\beta}$ for any $0 < \delta < T$. Fix δ. Since we chose $\sigma_{N,\delta}(0) = p_N \in I^+(\Sigma, M)$ and $I^+(\Sigma, M) \subset M_{\kappa,\beta}$ is open we get that $s_0 := \sup\{s \in [0, \varepsilon + \delta] : \sigma_{N,\delta}|_{[0,s)} \subset I^+(\Sigma, M)\} > 0$. If $s_0 = \varepsilon + \delta$ we are finished since then $\sigma_{N,\delta} = \sigma_{N,\delta}|_{[0,\varepsilon+\delta)} \cup \gamma(T - \delta) \subset I^+(\Sigma, M)$. So assume that $0 < s_0 < \varepsilon + \delta$. Then the curve $\sigma_{N,\delta}|_{[0,s_0)} \subset I^+(\Sigma, M)$ is a (past) inextendible, p.d., timelike curve in M and $\sigma_{N,\delta}([0, s_0)) \subset J^-(p_N, M) \cap J^+(\Sigma, M)$ which is compact (because Σ is FCC and M is globally hyperbolic). This contradicts global hyperbolicity of M.

This shows that $\sigma_{N,\delta}$ is a timelike curve from p_N to $\gamma(T - \delta)$ in M for any $0 < \delta < T$, so the original inextendible geodesic γ is contained in $J^-(p_N, M) \cap J^+(q, M)$, which again contradicts global hyperbolicity of M.

The second assertion follows by reversing the time orientation of M (note that while a bound from above on H^+_Σ will in general only translate to a bound from below for H^-_Σ, the previous calculations show that H^+_Σ and hence also H^-_Σ are constant anyways). $\square$

Contrary to the earlier two results (Thm. 4.2 and Thm. 5.12) this last theorem can easily be adapted to all remaining possible values of κ, β (and not only $\kappa \leq 0$ and $\beta = -(n-1)\sqrt{|\kappa|}$), by slightly tweaking the assumptions.

Proposition 6.3. *Let (M, g, Σ) satisfy $CCC(\kappa, \beta)$ with $\kappa \leq 0$ and $\beta > -(n-1)\sqrt{|\kappa|}$ and assume that there exists an exhaustion by compact sets $\{K_m\}_{m\in\mathbb{N}}$ for Σ and a sequence of times $t_n \to \infty$ such that*

$$\lim_{n\to\infty} \left(v_{\kappa,\beta}(t_n) - \frac{\mathrm{vol} B^+_{K_m}(t_n)}{\mathrm{area} K_m} \right) = 0$$

for all m, then

$$I^+(\Sigma) \cong (0, b_{\kappa,\beta}) \times_{f_{\kappa,\beta}} (\Sigma, \frac{1}{f_{\kappa,\beta}(0)^2}\, g|_\Sigma).$$

Proof. The proof remains largely same, only in (6.4) one uses that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(v_{\kappa,\beta}(t_n) - \frac{\mathrm{vol} B^+_{K_m}(t_n)}{\mathrm{area} K_m} \right) \\ &= \lim_{n\to\infty} \int_0^{t_n} \frac{\mathrm{area}_{\kappa,\beta} S^+_A(\tau)}{\mathrm{area}_{\kappa,\beta} A} - \frac{\mathrm{area} \mathcal{S}^+_{K_m}(\tau)}{\mathrm{area} K_m} d\tau \\ &= \int_0^{\infty} \frac{\mathrm{area}_{\kappa,\beta} S^+_A(\tau)}{\mathrm{area}_{\kappa,\beta} A} - \frac{\mathrm{area} \mathcal{S}^+_{K_m}(\tau)}{\mathrm{area} K_m} d\tau \end{aligned}$$

by positivity of the integrand to get (6.5) for almost all $t < \infty$. The rest follows exactly as above. □

To summarize, the above Thm. 6.2 and Prop. 6.3 complement the main splitting Theorem 5.12 nicely: Using a slightly stronger assumption leads to both a very natural and elementary proof and a natural generalization to all possible curvature bounds (whereas Thm. 5.12 only looks at ones that lead to a finite bound $b_{\kappa,\beta}$ on τ_Σ or that are boundary cases in the sense that $b_{\kappa,\beta} = \infty$ but $b_{\kappa,\bar\beta} < \infty$ for all $\bar\beta < \beta$).

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