A topological characterization of toroidally alternating knots

Seungwon Kim

We extend Howie's characterization of alternating knots to give a topological characterization of toroidally alternating knots, which were defined by Adams. We provide necessary and sufficient conditions for a knot to be toroidally alternating. We also give a topological characterization of almost-alternating knots which is different from Ito's recent characterization.

1. Introduction

Recently, Greene [7] and Howie [8, 9] independently gave a topological characterization of alternating knots, which answered a long-standing question of Ralph Fox. Below is Howie's characterization:

Theorem. *[8, 9] A non-trivial knot is alternating if and only if there exists a pair of connected spanning surfaces Σ and Σ' in the knot exterior such that*

$$(1) \qquad \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial\Sigma, \partial\Sigma') = 2,$$

where $i(\partial\Sigma, \partial\Sigma')$ is the minimal intersection number of $\partial\Sigma$ and $\partial\Sigma'$.

In [3], Adams defined a *toroidally alternating knot* as a knot which has an alternating diagram on an unknotted torus in S^3, such that the diagram divides the torus into a disjoint union of discs; i.e., the diagram is cellularly embedded. Toroidally alternating knots include almost-alternating knots and Turaev genus one knots. Adams showed that non-trivial prime toroidally alternating knots which are not torus knots are hyperbolic. In this paper, we give a topological characterization of toroidally alternating knots, extending Howie's characterization of alternating knots.

Several other generalizations of alternating knots have recently been topologically characterized. In [10], Ito gave a topological characterization of almost-alternating knots, which were defined by Adams in [4]. In [9],

Howie defined weakly generalized alternating knots and gave a topological characterization of these knots on the torus. Furthermore, in [11], Kalfagianni gave a characterization of adequate knots in terms of the degree of their colored Jones polynomial.

In this paper, we consider a pair of spanning surfaces satisfying an equation similar to equation (1). Theorem 1 shows that in this case, the knot has a "non-trivial" alternating diagram on the torus. Non-triviality is important because every knot has an alternating diagram on the torus boundary of its regular neighborhood. See Figure 1.
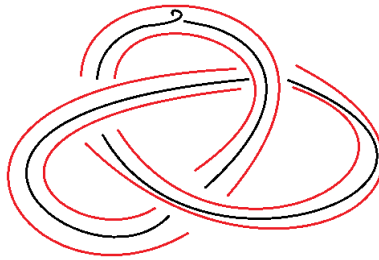

Figure 1. Every knot has an alternating diagram on the torus boundary of its regular neighborhood.

Theorem 1 also says that if one of the spanning surfaces is free, then we can find an alternating diagram of a knot on an unknotted torus. When the torus is unknotted, it is a Heegaard surface, and this condition plays an important role in defining *alternating distances,* which measure topologically how far a knot is from being alternating (see [13] for more details.). For example, the *alternating genus of a knot* is the minimal genus of a Heegaard surface such that the knot has a cellularly embedded alternating diagram on it. The *Turaev genus* is another interesting alternating distance, which is the minimal genus of a Heegaard surface with a Morse function condition, such that the knot has a cellularly embedded alternating diagram on it (see [6] for more details.). Alternating genus and Turaev genus are both defined for an alternating diagram that is cellularly embedded on the surface. The conditions in Theorem 1 are not enough to find a cellularly embedded diagram: The alternating diagrams on the torus that we get from Theorem 1 may have an annular region and they might not be checkerboard colorable. Note that every cellularly embedded alternating diagram on a closed orientable surface is checkerboard colorable.
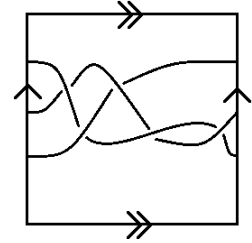
In Theorem 2, we give additional conditions – that the spanning surfaces are *relatively separable*, and a detachable curve is *incident to a bigon* (which
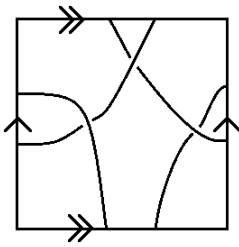
(a) An example of a cellularly embedded alternating diagram on a torus.
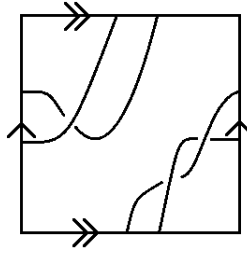
(b) An example of a non-cellularly embedded, checkerboard-colorable alternating diagram on a torus.

(c) An example of an alternating diagram on a torus which is not checkerboard-colorable.

are defined in Definition 4 below) – to find a cellularly embedded alternating diagram on a torus. These conditions give a trichotomy for a pair of spanning surfaces:

1) A pair of spanning surfaces is not relatively separable.

2) A pair of spanning surfaces is relatively separable, and every detachable curve on both spanning surfaces is incident to a bigon.

3) A pair of spanning surfaces is relatively separable, but there exists a detachable curve which is not incident to a bigon.

Theorem 2 shows that a knot is toroidally alternating if and only if there exists a pair of spanning surfaces that satisfies certain conditions and either condition (1) or (2). If every pair of spanning surfaces satisfies condition (3), then we can still find some non-trivial alternating diagram on an unknotted torus by Theorem 1, but it may or may not be checkerboard colorable.

Finally, a knot is *almost-alternating* if it has an almost-alternating diagram that can be changed to an alternating diagram by one crossing change. In Theorem 3, we show that for any knot as in Theorem 2, condition (2) is equivalent to a knot being almost-alternating. In [10], Ito gave a topological characterization of almost-alternating knots, but our characterization is different. He used all-A and all-B state surfaces of an almost-alternating diagram, which are the checkerboard surfaces of the Turaev surface of the almost-alternating diagram. We use a different pair of spanning surfaces to obtain a checkerboard-colorable alternating diagram on an unknotted torus, which is not cellularly embedded. It is an interesting question how the two

checkerboard surfaces of this diagram are related to the spanning surfaces used in [10].

2. Alternating knots on a torus

Throughout this paper, we use the following proposition that every alternating knot is both almost-alternating and toroidally alternating.

Proposition 1. *Let K be an alternating knot. Then K has an almost-alternating diagram and a toroidally alternating diagram.*

Proof. By [4], every alternating knot has an almost-alternating diagram. By [3], we can find a toroidally alternating diagram from an almost-alternating diagram. □

Definition 1. A **spanning surface** $\bar{\Sigma}$ of a knot K in S^3 is a surface embedded in S^3 such that $\partial\bar{\Sigma} = K$. For $\bar{\Sigma}$, we define a **spanning surface** Σ **in a knot exterior** $E(K) = S^3 - int(N(K))$ by $\Sigma = \bar{\Sigma} \cap E(K)$. A spanning surface $\bar{\Sigma}$ of a knot K in S^3 is **free** if $\pi_1(S^3 - \bar{\Sigma})$ is a free group.

Note that a spanning surface is free if and only if the closure of $S^3 - \bar{\Sigma}$ is a handlebody.

For every pair of spanning surfaces Σ and Σ', we can isotope them so that their boundaries realize the minimal intersection number, and each such isotopy can be extended to an isotopy of S^3. Then we have the following lemma from [8, 9].

Lemma 1 ([8, 9]). *If two spanning surfaces in a knot exterior are isotoped so that their boundaries realize the minimal intersection number, then every intersection arc is standard, as shown in Figure 2.*
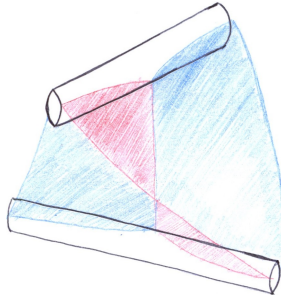
Figure 2. A neighborhood of a standard intersection arc of two spanning surfaces in a knot exterior.

Consider $\bar{\Sigma} \cup \bar{\Sigma}'$ in S^3 as above. If we contract all standard arcs, as in Figure 3, then we get an immersed surface in S^3 such that every self-intersection is a simple closed curve. We also get a connected 4-valent graph G_K on this immersed surface, coming from K, which is away from every self-intersection loop. We will call this immersed surface an **almost-projection surface** of Σ and Σ'.
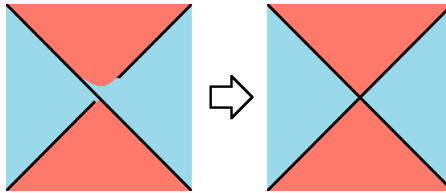

Figure 3. Contracting a standard arc intersection.

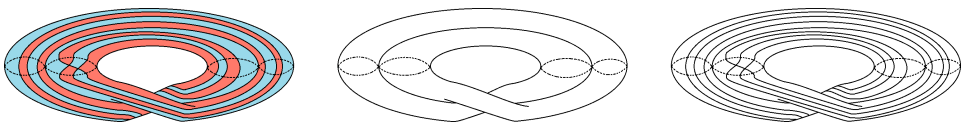Figure 4. An example of an almost-projection surface (left). It is homeomorphic to a Klein bottle which is immersed as in the middle figure, and a 4-valent graph G_K is on the surface as in the right figure.

Lastly, we define the complexity of an alternating diagram D_K of a knot K on an embedded surface. Below, let G_K denote the 4-valent graph obtained from projecting D_K on the surface naturally.

Definition 2. Let K be a knot which has a diagram D_K on some embedded surface T.

$r(D_K, T) = min\{|\gamma \cap G_K| \mid \gamma$ is a boundary of a compressing disc of T

(2)   which intersects G_K transversely only in edges of G_K.$\}$

Theorem 1. *Let Σ and Σ' be connected spanning surfaces in the knot exterior $E(K)$, such that*

$$(3) \qquad\qquad \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 0,$$

where $i(\partial\Sigma, \partial\Sigma')$ is the minimal intersection number of $\partial\Sigma$ and $\partial\Sigma'$. Then there exists a torus T embedded in S^3 such that K has an alternating diagram D_K on T with $r(D_K, T) \geq 2$. Furthermore, if Σ or Σ' is free, then T is an unknotted torus (i.e., a Heegaard torus).

Remark 1. Howie [9] considered an alternating diagram on the torus which is checkerboard colorable. To get his characterization of weakly generalized alternating knots, he added certain other conditions. In Theorem 1, we show that without additional conditions, we can still find a non-trivial alternating diagram of the knot on the torus.

Proof. First, we will prove the existence of D_K and T. Consider an almost-projection surface of Σ and Σ'. By equation (3), this almost-projection surface can be either an immersed torus or an immersed Klein bottle.

Case 1: The almost-projection surface is an immersed torus.

For each self-intersection curve, we have two possibilities. First, a self-intersection curve can bound a disc on the immersed torus. Then we can find an innermost self-intersection curve inside the disc. If we surger along the disc bounded by the innermost self-intersection curve as in Figure 5, then the resulting surface can be disconnected, or it is an immersed sphere.

If the resulting surface is an immersed sphere, then by [8, 9], K is alternating. Hence, by Proposition 1, K is toroidally alternating. Also in this case, by Corollary 4.6 in [12], $r(D_K, T) = 2$. If the resulting surface is disconnected, then one component is a torus, and the other component is a sphere. If G_K is on the sphere component, then again, K is alternating. Otherwise, we have reduced the number of self-intersections. We continue until all such inessential self-intersections are eliminated.
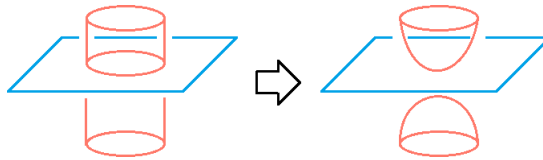
Figure 5. Sugery along a disc.

On the other hand, the self-intersection curve can be essential on the immersed torus. But we now prove by contradiction that this cannot occur. Let $f : T' \to S^3$ be an immersion map, and let σ be a self-intersection loop on $f(T')$. Then $f^{-1}(\sigma)$ consists of two essential simple closed curves, so they bound an annulus on the torus as in Figure 6. Since G_K does not intersect with any self-intersection curves, $f^{-1}(G_K)$ is a connected 4-valent graph on one of the annuli bounded by $f^{-1}(\sigma)$. Hence, both components of $f^{-1}(\sigma)$ are in the same region of $f^{-1}(G_K)$, which implies that σ is a self-intersection of either $\bar{\Sigma}$ or $\bar{\Sigma}'$, which contradicts the fact that each of them is an embedded surface. Hence, there are no essential self-intersection curves.

Figure 6. Preimages of a self-intersection loop bound an annulus on the torus.

Therefore, G_K is a 4-valent graph on an embedded torus T in S^3. We can recover the diagram D_K from G_K by replacing each vertex of G_K with a neighborhood of a standard arc. If the resulting diagram D_K is not alternating, there exists a bigon between $\partial\Sigma$ and $\partial\Sigma'$, which contradicts the minimality of the intersection number of boundaries. Hence, K has an alternating diagram on T. Also, from the construction, D_K is checkerboard colorable.

We claim that either $r(D_K, T) \geq 2$ or K is alternating. Suppose that for the resulting alternating diagram D_K on the torus T, $r(D_K, T) < 2$. Since D_K is checkerboard colorable, every simple closed curve on T intersects G_K transversely in an even number of points. Therefore, $r(D_K, T) = 0$, so we can find a compressing disc of T which does not intersect D_K. Then compressing T along this disc yields an embedded S^2, so K is an alternating

knot. Then, as above, by [12, Corollary 4.6], we can find an alternating diagram D_K of K on some embedded torus T'' such that $r(D_K, T'') = 2$. This concludes the proof of Case 1.

**Case 2:** The almost-projection surface is an immersed Klein bottle.

Let $f : B \to S^3$ be an immersion of a Klein bottle B. If a simple closed self-intersection curve of $f(B)$ bounds a disc on $f(B)$, then we can surger along this curve as in Figure 5 to reduce all such inessential intersections. If G_K is on an immersed sphere, then K is alternating. Hence, we can assume that all preimages of the remaining self-intersections are essential simple loops of $f(B)$. Let s_1, $s_2 \subset B$ be the preimages of an essential self-intersection σ of $f(B)$. For $i = 1, 2$, we call a regular neighborhood of s_i 2-sided if it is homeomorphic to an annulus, or 1-sided if it is homeomorphic to a Möbius band. Furthermore, the two regular neighborhoods of s_1 and s_2 in B are homeomorphic, because an annulus and Möbius band embedded in S^3 cannot intersect only in the core loop. Then we have three subcases to consider, depending on the topology of s_1 on B:
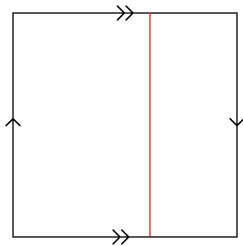
(a) An example of a 2-sided non-separating curve on the Klein bottle.
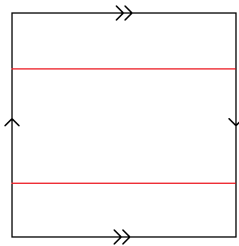
(b) An example of a 2-sided separating curve on the Klein bottle.
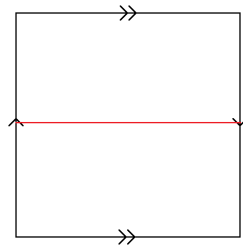
(c) An example of a 1-sided curve on the Klein bottle.

<u>Subcase 1:</u> s_1 is a non-separating, 2-sided curve on B.

We prove by contradiction that this subcase cannot occur. The complement of a regular neighborhood of s_1 in B is an annulus. Hence, s_2 is the core of the annulus. Then s_1 and s_2 cut B into two annuli, and $f^{-1}(G_K)$ is on one of them. Hence, s_1 and s_2 are in the same region of $f^{-1}(G_K)$ on B. See Figure 7. This implies that σ is a self-intersection of Σ or Σ', which contradicts the assumption that Σ and Σ' are embedded.
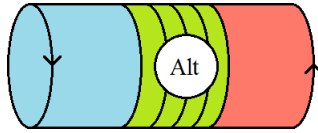
Figure 7. Two pre-images cut a Klein bottle into two annuli and $f^{-1}(G_K)$ is on one of them.

Subcase 2: s_1 is a separating, 2-sided curve on B.

In this case, s_1 cuts B into two Möbius bands. Then s_2 is on one of the Möbius bands, and cuts it into one Möbius band and one annulus. Hence, s_1 and s_2 cut B into two Möbius bands and one annulus.
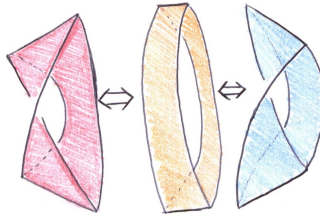

Figure 8. Two pre-images cut a Klein bottle into two Möbius bands and one annulus.

Furthermore, $f^{-1}(G_K)$ is contained in one of the components. If $f^{-1}(G_K)$ is on the Möbius band, then σ is a self-intersection of $\bar{\Sigma}$ or $\bar{\Sigma}'$ which is impossible. Hence, $f^{-1}(G_K)$ is on the annulus. In this annulus, every preimage of an essential self-intersection is isotopic to a core of the annulus. Let A be the annulus which contains $f^{-1}(G_K)$ and does not contain any preimages of self-intersections. Then we can recover D_K from G_K, as in the torus case, so that K is alternating on $f(A)$ as follows.

Now, to construct the torus, we consider $B - A$, which consists of two disjoint Möbius bands M and M'. The image of each Möbius band under f is a subset of either Σ or Σ'. Furthermore, both Möbius bands cannot be contained in the same spanning surface. Consider $M \cup A$, which is homeomorphic to a Möbius band. Now, $f(M \cup A)$ is embedded in S^3 because every self-intersection of $f(B)$ is an intersection of $f(M)$ and $f(M')$. Consider a thickening of the Möbius band $f(M \cup A)$ in S^3, which is homeomorphic to a solid torus. Let T be its boundary. Then using the natural projection, we can think of $T - \partial f(M \cup A)$ as a two fold cover of the Möbius band $f(M \cup A)$.

Then $f(A)$ is lifted to two annuli with disjoint interiors on the torus. Since K is alternating on $f(A)$, we can choose a lift of the alternating diagram D_K to one of the annuli. Hence, D_K is alternating on T.
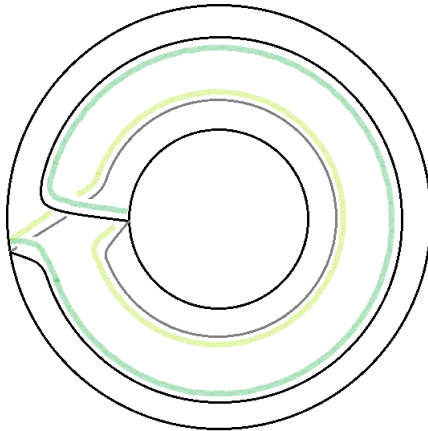

Figure 9. Torus T, coming from thickening the Möbius band $f(M \cup A)$, and an example of one of the lifts of $f(A)$ on T, denoted by a shaded band.

Now we show that $r(D_K, T) \geq 2$ or K is alternating. By construction, T bounds a solid torus for which the boundary of every compressing disc intersects each lift of $f(A)$ twice. If this boundary curve intersects the diagram less than twice, then this implies that G_K is contained in a disc in $f(A)$. But then, this implies that M and M' are in the same region of $f^{-1}(G_K)$, which cannot occur because these Möbius bands are not contained in the same spanning surface. Hence, the boundary of every compressing disc of this solid torus intersects G_K at least twice.

Finally, $r(D_K, T) < 2$ may occur for a compressing disc on the other side of T. If the 3-manifold on the other side of T has a compressing disc Ω, then it is a solid torus, hence, T is an unknotted torus. Note that $\partial\Omega \cap \partial f(M \cup A) \neq \emptyset$. If $\partial\Omega$ does not intersect the diagram, then just as above, D_K is contained in a disc. Suppose that $\partial\Omega$ intersects the diagram once. This implies that $f(M \cup A)$ is an embedded Möbius band in S^3 such that the core is the unknot and its boundary is also the unknot. This implies that we can find an essential arc on $f(A)$ which intersects G_K transversely once. Now, we need the following lemma.

Lemma 2. *Let K be a knot with an alternating diagram D_K on an annulus A embedded in S^3. If there exists a properly embedded simple arc $\tau \subset A$ which*

intersects D_K transversely once, then K is a connected sum of an alternating knot and a knot isotopic to the core of A.

Proof. Every crossing of D_K is in the complement of τ in A, which is a disc. Hence we can find a decomposing sphere from the boundary of the thickened disc which contains every crossing of D_K. This implies that we have an alternating 1-tangle in the decomposing sphere and another 1-tangle outside. Then K is a connected sum of an alternating knot, which is obtained by taking the trivial closure of the alternating 1-tangle, and the trivial closure the other 1-tangle, which is isotopic to the core of A. $\qquad\square$

Then, by Lemma 2, K is a connected sum of an alternating knot and a knot which is isotopic to the core of $f(A)$, which is an unknot. Hence, K is alternating.

Subcase 3: s_1 is a 1-sided curve on B.

The complement of s_1 and s_2 in B is an annulus. Hence, $f^{-1}(G_K)$ is on the annulus. If there is no other self-intersection, then G_K is on the embedded Möbius band $B - s_1$. If there exists another essential self-intersection, then its pre-images are separating 2-sided curves, so G_K is still on the embedded Möbius band $f(M \cup A)$. Hence, the claim follows by the same argument as in the previous subcase.

This completes the proof of Case 2.

To show that the torus T is unknotted, we need the following lemmas.

Lemma 3. *Let D_K be a knot diagram on the torus T with $r(D_K, T) \geq 2$. Then every region of $T - G_K$ is homeomorphic to a disc, except possibly one region which is homeomorphic to an annulus.*

Proof. Let R be a region of D_K. Since D_K is connected, $|T - R| = 1$. Hence, $\chi(R) \geq -1$. If $\chi(R) = -1$, then D_K is contained in a disc, hence we always can find a compressing disc of T which does not intersect D_K. But this violates the condition $r(D_K, T) \geq 2$. Lastly, if there exist two annular regions R_1 and R_2 of D_K, then $|T - (R_1 \cup R_2)| = 2$. Again, since D_K is connected, this is not possible. $\qquad\square$

Lemma 4. *Suppose that a link L has a checkerboard-colorable, connected diagram D_L on a torus T in S^3 such that $r(D_L, T) \geq 2$. Then T is unknotted if and only if one of the checkerboard surfaces is free.*

Proof. From Lemma 3, every region is homeomorphic to a disc except possibly one region, which is homeomorphic to an annulus. Let Σ and Σ' be two checkerboard surfaces of D_L. Suppose that Σ' is a checkerboard surface which consists of only disc regions. Since $\Sigma' - (\Sigma \cap \Sigma')$ is a set of disjoint discs and $S^3 - T$ has two connected components, $S^3 - \Sigma$ is homeomorphic to a 3-manifold obtained from connecting the two components of $S^3 - T$ with 3-dimensional 1-handles, each corresponding to a disc of $\Sigma' - (\Sigma \cap \Sigma')$ (see Figure 10). If Σ contains an annular region of D_L, then $S^3 - \Sigma'$ can be obtained similarly, except we connect two components with a thickened annulus.
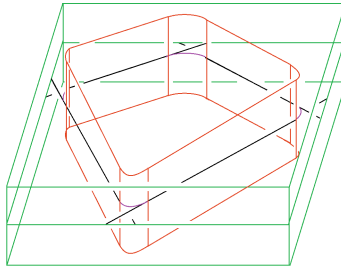

Figure 10. A 1-handle correspond to a disc region of D_L.

If T is an unknotted torus, then $S^3 - T$ is a disjoint union of two solid tori. If we connect two solid tori with several 3-dimensional 1-handles, then it is still a handlebody. Hence, Σ is free.

Conversely, suppose that T is knotted. We show that both checkerboard surfaces are not free. We use the fact that compressing a handlebody with a disjoint set of compressing discs yields a disjoint union of handlebodies. First, we show that Σ is not free. We can obtain $S^3 - \Sigma$ from $S^3 - T$ as above. If we compress $S^3 - \Sigma$ along all compressing discs, each corresponding to a disc of $\Sigma' - (\Sigma \cap \Sigma')$, then we get a solid torus and a 3-manifold with boundary, which is not a solid torus, because T is knotted. Hence Σ is not free. Lastly, we show that Σ' is not free. Consider $S^3 - \Sigma'$ and compress this manifold along all compressing discs each corresponding to a disc of $\Sigma - (\Sigma \cap \Sigma')$. Then we get a 3-manifold which is homeomorphic to a knot exterior, such that the knot is isotopic to a core of the annular region of D_L. T is knotted, so the core of the annular region is a non-trivial knot. So, the resulting 3-manifold is not a handlebody. Hence, Σ' is not free.   □

Now we can complete the proof of Theorem 1. Consider the almost-projection surface of Σ and Σ'. Suppose that the almost-projection surface is an immersed torus. If we surger the almost-projection surface along a

disc, the surface might become disconnected or its genus will decrease. If the surgery reduces the genus, then we get an alternating knot by [8, 9]. As mentioned above, using the Turaev surface, such a knot is toroidally alternating. We continue performing the surgery, cutting off spheres until we get an alternating diagram D_K on an embedded torus. As discussed above, since D_K is checkerboard colorable, $r(D_K, T)$ is even. If $r(D_K, T) = 0$, then K is an alternating knot. Suppose $r(D_K, T) \geq 2$. During the surgery, we cut off spheres, so the resulting checkerboard surfaces of D_K on T are isotopic to Σ and Σ'. By assumption, one of them is free, hence, T is unknotted by Lemma 4.

On the other hand, suppose that the almost-projection surface is an immersed Klein bottle. From the proof above, we can find an embedded Möbius band, $f(M \cup A)$.

Lemma 5. *The core of $f(M \cup A)$ is unknotted.*

Proof. Every region of D_K on $f(M \cup A)$ is a disc except one annular region in Σ' and one Möbius band region in Σ. Consider the regular neighborhood of Σ in S^3 as the following:

We first thicken $f(M \cup A)$, and remove every thickened region of D_K on $f(M \cup A)$ that is a subset of Σ'. The resulting manifold is homeomorphic to a regular neighborhood of Σ. Now, we compress the complement of Σ by filling each thickened disc region of Σ'. Then, under the assumption that Σ is free, the resulting complementary region is still a handlebody. We recover the complement of $f(M \cup A)$, so this handlebody is a solid torus. This implies that the core of the Möbius band is unknotted in S^3. □

Hence, by Lemma 5, the solid torus that we obtained in Case 2 is unknotted. This completes the proof of Theorem 1. □

3. Toroidally alternating knots

Definition 3. [9] Let Σ and Σ' be properly embedded surfaces in general position in $E(K)$. A **bigon** is a disc $\mathcal{B}$ embedded in $E(K)$ such that $\partial \mathcal{B} = \beta \cup \beta'$, where $\beta \subset \Sigma$ and $\beta' \subset \Sigma'$ are connected arcs, $\beta \cap \beta'$ consists of two distinct points of $\Sigma \cap \Sigma'$ and $\mathcal{B} \cap (\Sigma \cup \Sigma') = \partial \mathcal{B}$. The arcs β and β' are called **edges** of $\mathcal{B}$, and $\beta \cap \beta'$ are called **vertices** of $\mathcal{B}$. A bigon is **inessential** if it can be homotoped to an intersection arc or an intersection loop of Σ and Σ'. Otherwise, it is **essential**.

Here, the homotopy must be such that restricted to the boundary of $\mathcal{B}$, β and β' must remain in Σ and Σ', respectively, throughout the homotopy.

Let Σ and Σ' be a pair of spanning surfaces in $E(K)$. A *minimal representative* of a simple loop γ in Σ is a simple loop in Σ which is isotopic to γ and intersects Σ' minimally. We can define a minimal representative of a simple loop in Σ' in the same manner.

Definition 4. Let Σ and Σ' be a pair of spanning surfaces in $E(K)$. Then Σ and Σ' are **relatively separable** if there exists an essential 2-sided simple loop γ in Σ or Σ' such that its push-off γ' does not intersect the other spanning surface. We say such γ is **detachable**. In this case, γ is **incident to a bigon** if for every minimal representative of γ, there exists an essential bigon whose boundary intersects γ transversely in one point.
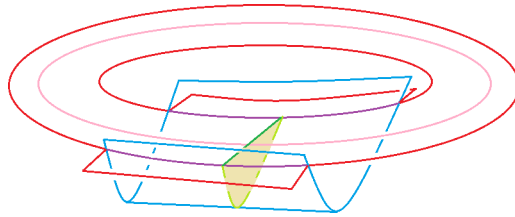

Figure 11. A detachable curve which is incident to a bigon.

Definition 5. Let Σ and Σ' be spanning surfaces in $E(K)$. We say that Σ and Σ' are **essentially intersecting** if their boundaries intersect minimally on $\partial E(K)$ and every intersection loop is essential on both surfaces.

Lemma 6. *Let Σ and Σ' be essentially intersecting spanning surfaces in $E(K)$. Then the almost projection surface F has no self-intersection loop that bounds a disc on F.*

Proof. Suppose that F has a simple loop intersection which bounds a disc in F. Take the innermost loop intersection σ and consider a disc bounded by σ. If this disc does not contain G_K, then Σ and Σ' are not essentially intersecting. Suppose that this disc contains G_K. Let $f : F' \to S^3$ be an immersion map such that $f(F') = F$. Consider the $f^{-1}(\sigma)$, which consists of two simple loops s_1 and s_2 on F'. Then without loss of generality, we can assume that s_1 bounds a disc on F' which contains $f^{-1}(G_K)$, but does not contain other pre-images of self-intersections of F. Then both pre-images

are contained in the same region of $f^{-1}(G_K)$, which implies that σ is a self-intersection of either Σ or Σ', which contradicts the assumption that Σ and Σ' are embedded. Hence, there is no self-intersection loop which bounds a disc on F. $\qquad\square$

Since by Proposition 1, every alternating knot is toroidally alternating, we only consider non-alternating knots below.

Theorem 2. *A non-alternating knot K is toroidally alternating if and only if there exists a pair of essentially intersecting connected, free spanning surfaces Σ and Σ' in the knot exterior which satisfy the following:*

1) $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 0.$

2) *If Σ and Σ' are relatively separable, then every detachable curve is incident to a bigon.*

Remark 2. In [9, Figure 3.18], Howie gave an example of a weakly generalized alternating projection of the knot 10_{139}, for which one of the regions is homeomorphic to an annulus. He showed that there is no essential bigon between the two checkerboard surfaces Σ and Σ'. Hence, this pair Σ and Σ' is an example of a pair of essentially intersecting free spanning surfaces which are relatively separable, but not every detachable curve is incident to a bigon.

Proof. First, we show that if the two checkerboard surfaces Σ and Σ' of a toroidally alternating diagram are relatively separable, then every detachable loop is incident to a bigon. Let γ be a minimal representative of a detachable loop on Σ. Consider the push-off γ' of γ. Let $\mathcal{A}$ be the annulus bounded by γ and γ' such that $\mathcal{A} \cap \Sigma = \gamma$. Every essential loop of Σ intersects Σ', so $\mathcal{A}$ intersects with Σ'. Then every intersection of Σ' and $\mathcal{A}$ is either an arc which has its endpoints on γ or a simple loop isotopic to γ. We can modify γ' so that $\mathcal{A}$ only intersects Σ' in arcs. Consider an innermost bigon $\mathcal{B}$ in $\mathcal{A}$ bounded by γ and one of the intersection arcs. Then $\mathcal{B}$ is an essential bigon, because if $\mathcal{B}$ is inessential, we can isotope γ and $\mathcal{A}$ to remove the intersection arc, which contradicts our hypothesis, γ is a minimal representative. Then we can slightly isotope this bigon to intersect γ in one point. Hence, every detachable curve is incident to a bigon. This completes the proof of the "only if" part of Theorem 2.

Now to show the "if" part, since Σ and Σ' are essentially intersecting, we can contract every standard arc to get an almost-projection surface. Below, let F denote the almost-projection surface of Σ and Σ'. By Lemma 6 and the

proof of Theorem 1, F is either an unknotted torus or an immersed Klein bottle with no 2-sided, non-separating self-intersection loop.

First, suppose that Σ and Σ' are not relatively separable. We will show that F is an unknotted torus and the alternating diagram on the almost-projection surface is cellularly embedded.

Lemma 7. *Let Σ and Σ' be essentially intersecting spanning surfaces of a knot K which are not relatively separable. Let F be the almost-projection surface of Σ and Σ'. Then F cannot intersect itself in an essential simple loop.*

Proof. Suppose that there exists an essential simple loop intersection ϕ. Then ϕ is either 1-sided or 2-sided. Consider one of the components ψ of the boundary of a regular neighborhood of ϕ on Σ.
Claim 1: ψ is detachable, and it is a minimal representative.

From the construction, ψ does not intersect Σ', so it is a minimal representative.

If ϕ is 2-sided, then ψ is isotopic to ϕ, hence essential. Furthermore, ψ has a push-off which does not intersect with Σ'. Hence, ψ is detachable.

If ϕ is 1-sided, then ψ bounds a Möbius band on Σ, which is a regular neighborhood of ϕ on Σ. If ψ bounds a disc on the other side, then we get a closed component, which is homeomorphic to a real projective plane. A real projective plane cannot be embedded in S^3, so the boundary does not bound a disc on Σ, which implies that ψ is essential. Furthermore, ψ has a push-off which does not intersect Σ', so, it is detachable.

The existence of a detachable curve contradicts the assumption that Σ and Σ' are not relatively separable. Hence, there cannot exist an essential simple loop intersection of F. $\qquad\square$

Lemma 8. *Let Σ and Σ' be essentially intersecting spanning surfaces of a knot K which are not relatively separable. Then Σ and Σ' are checkerboard surfaces of a cellularly embedded alternating diagram on a closed orientable surface F with Euler characteristic $\chi(F) = \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma')$.*

Proof. If F is non-orientable, then it must have a self-intersection. However, by Lemma 7, Σ and Σ' are relatively separable. Since this contradicts our hypothesis, F is orientable.

We now show that the alternating diagram on F is cellularly embedded. Suppose that there exists a region which is not homeomorphic to a disc. Without loss of generality, we can assume that this region is a subset of Σ. Consider a graph which is a deformation retract of this region. Any loop of

this graph is an essential loop of Σ because we can find an arc on Σ which has both of its endpoints on K and intersects this loop transversely once. Furthermore, we can find a push-off of this loop which does not intersect Σ'. Hence, it is a detachable curve, so Σ and Σ' are relatively separable, which contradicts our hypothesis. Therefore, the alternating diagram on F is cellularly embedded. This completes the proof of Lemma 8. $\qquad\square$

By Lemma 8, if Σ and Σ' are not relatively separable, then the almost-projection surface F is an unknotted torus with a cellularly embedded alternating diagram.

Now, suppose that Σ and Σ' are relatively separable. If F is an unknotted torus with a cellularly embedded alternating diagram, then we are done. Otherwise, by the proof of Theorem 1, we have two cases: either F is an unknotted torus with a non-cellularly embedded alternating diagram, or F is an immersed Klein bottle with no 2-sided, non-separating self-intersection loop. We will show in the first case, K is almost-alternating hence toroidally alternating, and that the second case is not possible.

Case 1: F is an unknotted torus with a non-cellularly embedded alternating diagram.

Let γ be a core of the annular region, which is a minimal representative of itself. We showed above that γ is detachable. We can assume that γ is on Σ. Now, we show that if γ is incident to a bigon, then K is almost-alternating. To show this, we need the following lemma.

Lemma 9. *Suppose Σ and Σ' are relatively separable and every detachable curve is incident to a bigon $\mathcal{B}$. Let F denote the almost-projection surface, and suppose F is an unknotted torus with a non-cellularly embedded diagram G_K. Then $\mathcal{B}$ is isotopic to a compressing disk of F which intersects the set of edges of G_K transversely twice.*

Proof. By assumption, there is an essential bigon $\mathcal{B}$ which intersects γ transversely once on its boundary $\partial\mathcal{B}$. After contracting standard arc intersections to get F, bigon $\mathcal{B}$ becomes a disc whose interior is embedded in the complement of F, and $\partial\mathcal{B}$ is a loop on F which intersects G_K only in its vertices. The loop $\partial\mathcal{B}$ is simple, whenever both vertices of $\partial\mathcal{B}$ are on different standard arc intersections of Σ and Σ'. If $\partial\mathcal{B}$ is simple, we can modify $\partial\mathcal{B}$ to intersect G_K transversely twice on edges of G_K. Otherwise, $\partial\mathcal{B}$ on F is a loop which has one self-intersection on some vertex of G_K. However, the interior of $\mathcal{B}$ does not intersect itself, so the self-intersection of $\partial\mathcal{B}$ is not transverse. Therefore, we can modify $\partial\mathcal{B}$ to be a simple loop, and intersect

G_K transversely twice on its edges.(See Figure 12.) Since $\partial \mathcal{B}$ intersects γ transversely once, $\partial \mathcal{B}$ on F is essential, so $\mathcal{B}$ is a compressing disc of F. □
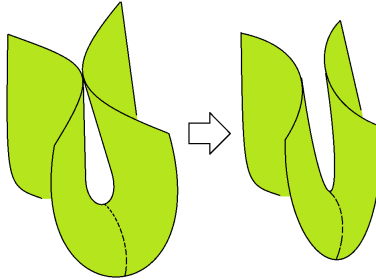

Figure 12. If $\mathcal{B}$ intersect itself in one point on $\partial \mathcal{B}$, then we can modify $\partial \mathcal{B}$ to remove the self-intersection.

With the assumption that F is unknotted, the fact that $\mathcal{B}$ is a compressing disc of F implies that a regular neighborhood of γ on F is an annulus whose core is unknotted in S^3. Then G_K is also on the annulus with an unknotted core, since G_K is on the complementary region of γ on F. Now, $\partial \mathcal{B}$ intersects G_K transversely twice, hence we can find an essential simple arc on the annulus which contains G_K such that it intersects G_K transversely twice. If we cut G_K along this arc, we get an alternating 2-tangle on a disc as in Figure 13.
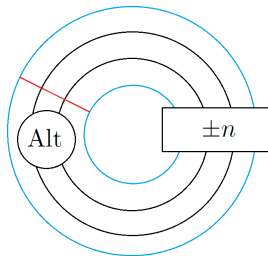

Figure 13. If we cut G_K along the red arc, we get an alternating 2-tangle on a disc.

Since the core of the annulus which contains G_K is unknotted, K can be obtained by taking n-full twists on two strands of some alternating knot diagram. This operation yields either an alternating knot diagram or a cycle of two alternating 2-tangles, as defined in [12](see Figure 14.). By assumption, K is not alternating, so it is a cycle of two alternating 2-tangles. Then

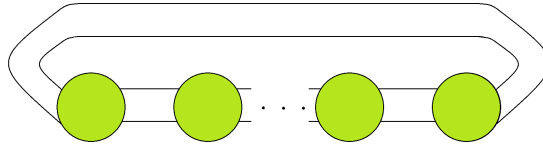by [12], its Turaev genus is one, so K is toroidally alternating.

Figure 14. A cycle of alternating 2-tangles. Green discs represent alternating 2-tangles, and black arcs are non-alternating edges.

Case 2: F is a Klein bottle.

By Lemma 8, Σ and Σ' are relatively separable. We will show that if every detachable curve is incident to a bigon, then K is alternating.

As we discussed above, every self-intersection loop of F is either a 2-sided separating loop or a 1-sided loop. Suppose that there exists a 2-sided self-intersection loop. Consider one of the 2-sided self-intersection loops γ' on Σ, which is adjacent to G_K on the almost-projection surface. Consider a simple loop γ on Σ which is on the region between γ' and G_K on the almost-projection surface, and isotopic to γ'. By Claim 1 in Lemma 7, γ is detachable and it is a minimal representative of itself. Consider a bigon $\mathcal{B}$ which is incident to γ. Let β be the edge of $\mathcal{B}$ which is on Σ and intersects γ transversely once. Since γ is adjacent to G_K, one of the vertices of β is on γ' and the other is on some standard arc intersection as in Figure 15b. This implies that there exists a properly embedded essential arc τ on the annulus $f(A)$, which intersects G_K once. (Recall that G_K is contained in the annulus $f(A)$ as in the proof of Theorem 1, Case 2, subcase 2.)

Hence, by Lemma 2, K is a connected sum of an alternating knot and the other knot, which is isotopic to a core of the annulus which contains G_K. By Lemma 5, the core of $f(M \cup A)$ is unknotted, so the core of $f(A)$ is a torus knot type $(2, 2q + 1)$, $q \geq 0$ in S^3, which is alternating. Therefore, K is alternating.

Instead, suppose that there is no 2-sided intersection loop. Then there exists a 1-sided self-intersection loop, η'. Note that there is no other 1-sided self-intersection loop because the complement of η' is an annulus, so every other loop is 2-sided. By Claim 1 in Lemma 7, the boundary η of a regular neighborhood of η' on Σ is detachable and a minimal representative of itself. Consider a bigon $\mathcal{B}$ incident to η. By the same argument, $\partial \mathcal{B}$ has one vertex on η', and the other vertex on the standard arc intersection. By the same argument as above, K is alternating.
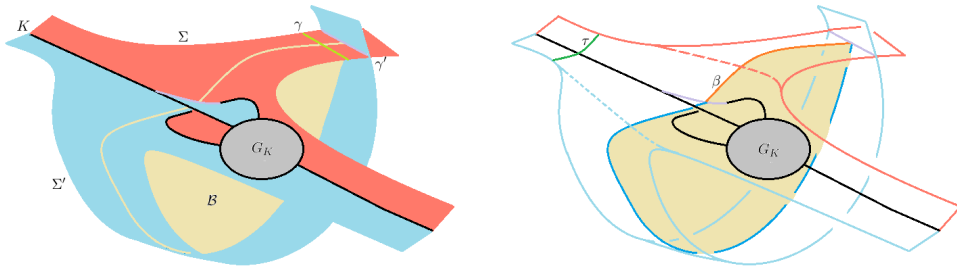
(a) Σ and Σ' near K and a detachable curve γ, which is incident to a bigon $\mathcal{B}$.

(b) One of the vertices of $\partial\mathcal{B}$ is on γ' and the other is on a standard arc. Then we can find τ on the annulus contains G_K, which intersects G_K once.

Figure 15.

This completes the proof of Theorem 2.   □

4. Almost-alternating knots

Theorem 3. *A non-alternating knot K is almost-alternating if and only if there exists a pair of essentially intersecting connected, free spanning surfaces Σ and Σ' in the knot exterior which satisfy the following:*

1) $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 0.$

2) *Σ and Σ' are relatively separable and every detachable curve is incident to a bigon.*

Proof. We show the "if" part first. In Case 2 of the proof of Theorem 2, we showed that if Σ and Σ' are relatively separable and their almost-projection surface is a Klein bottle, then K is alternating. Assuming that K is non-alternating, the almost-projection surface of Σ and Σ' is an unknotted torus. Below, let F denote the almost-projection surface of Σ and Σ'. The alternating diagram D_K on F may be cellularly embedded or not, which we consider in separate cases.

Suppose that D_K on F is not cellulary embedded. Then by Lemma 9, we can find an essential simple loop on F which intersects D_K twice and bounds a compressing disc. Hence, K has a cycle of two alternating 2-tangles, as defined in [12]. Then by [2, Proposition A.6], K can be transformed into an almost alternating diagram.

On the other hand, suppose that the diagram D_K is cellularly embedded. Let γ be a detachable curve on Σ. We will show that we can isotope Σ and Σ' so that after the isotopy, the new almost-projection surface of Σ and Σ' has an alternating diagram with an annular region whose core is γ. Let γ' be a push-off of γ, and $\mathcal{A}$ be an annulus bounded by γ and γ'.

Since γ is on Σ and D_K is cellularly embedded, $\mathcal{A} \cap \Sigma' \neq \emptyset$. Furthermore, by the relatively separable condition, every intersection of $\mathcal{A}$ and Σ' is an arc which has both its endpoints on γ. The innermost intersection arc bounds a bigon $\mathcal{B}$. If $\mathcal{B}$ is inessential, we can isotope γ and $\mathcal{A}$ to remove such an intersection. If $\mathcal{B}$ is essential, then we can isotope Σ' along $\mathcal{B}$ to remove the intersection, as in Figure 16. After the isotopy, we get a new bigon $\mathcal{B}'$ as in Figure 16(right).
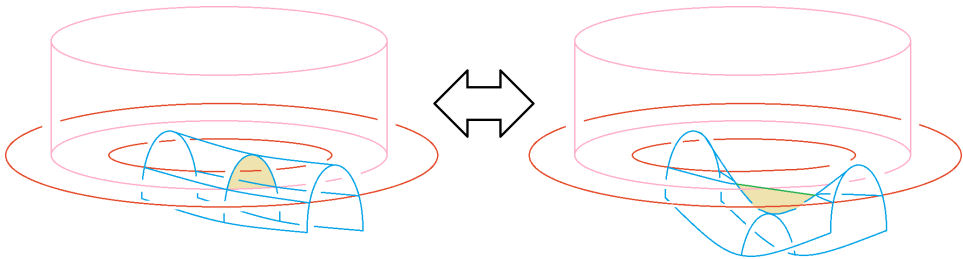

Figure 16. A surface isotoped along a bigon.

Now, we show that after the isotopy, Σ and Σ' are still essentially intersecting, the new almost-projection surface of Σ and Σ' is still an embedded unknotted torus. Below, let F' denote the new almost-projection surface of Σ and Σ' after the isotopy along $\mathcal{B}$.

First, we show that after the isotopy, Σ and Σ' are still essentially intersecting. Since this isotopy does not change Σ and Σ' near their boundaries, the number of arc intersections remains minimal. Therefore, if Σ and Σ' are not essentially intersecting after the isotopy, then there exists an inessential intersection loop. Furthermore, each isotopy can change the number of intersecting components at most once, so there is only one inessential intersection loop μ.

We will show that μ bounds a disc on both spanning surfaces. Since μ is inessential, it bounds a disc in one of the spanning surfaces, say Σ. If μ does not bound a disc in the other spanning surface, Σ', then we can surger Σ' along a disc bounded by μ on Σ. Let Σ^* be the resulting spanning surface. Then the first condition implies that $\chi(\Sigma) + \chi(\Sigma^*) + \frac{1}{2}i(\partial\Sigma, \partial\Sigma^*) = 2$, so K is alternating. As this contradicts the hypothesis that K is non-alternating, μ bounds discs in both spanning surfaces.

If we undo the isotopy, then the intersection pattern of Σ and Σ' changes as in Figure 17. More specifically, it changes from the right picture to the left picture. Then the edge of $\mathcal{B}$ is an arc on the left picture, which cobounds a disc with a subarc of $\Sigma \cap \Sigma'$. Then each edge can be isotoped onto $\Sigma \cap \Sigma'$, hence $\mathcal{B}$ is inessential. This contradicts the assumption that $\mathcal{B}$ is essential. Therefore, Σ and Σ' are essentially intersecting after the isotopy along $\mathcal{B}$.
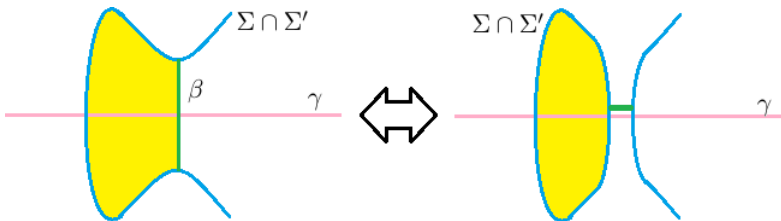

Figure 17. Intersection pattern changes whenever we isotope a surface along a bigon.

Now, we show that F' is an embedded torus. Suppose that F' is a Klein bottle. A Klein bottle cannot be embedded in S^3 so there exists at least one self-intersection loop. Since Σ and Σ' intersect only in standard arcs before the isotopy, the isotopy along $\mathcal{B}$ divides one standard arc into a standard arc and an essential intersection loop. Consider a new bigon $\mathcal{B}'$ after the isotopy as in Figure 16. Then one of the vertices of $\mathcal{B}'$ is on the standard arc intersection and the other is on the essential intersection loop, so $\mathcal{B}'$ is essential. This essential simple intersection loop is a self-intersection loop of F', and F' is a Klein bottle, so the self-intersection loop is either 1-sided or 2-sided and separating. Hence, the boundary of a regular neighborhood of the essential intersection loop on each spanning surface is incident to $\mathcal{B}'$ as in Figure 15a. Then, as in Case 2 of the proof of Theorem 2, we can show that K is alternating. By hypothesis, K is non-alternating, so F' cannot be a Klein bottle. Hence, we still have an alternating diagram on an embedded torus.

We can continue these isotopies to remove all intersections between $\mathcal{A}$ and Σ'. Then the almost-projection surface obtained from Σ and Σ' after isotopy is another unknotted torus, such that the alternating diagram on the torus has an annular region whose core is γ. We will show that γ is incident to a bigon after isotopies. Then by the same argument as in the previous case (non-cellularly embedded diagram), it follows that K is almost-alternating.

First, we will show that the new bigon $\mathcal{B}'$ after the isotopy (as in Figure 16) is an essential bigon. Suppose $\mathcal{B}'$ is inessential. Then by definition,

it can be homotoped to a standard arc intersection. This implies that two vertices of $\mathcal{B}'$ are on the same standard arc intersection and both edges of $\mathcal{B}'$ are homotopic to a subarc of the standard arc intersection on each spanning surface. Let β be an edge of $\mathcal{B}'$ on Σ. Then β and the subarc of the standard arc intersection cobound a disc on Σ as in Figure 17(left). If we undo the isotopy that we performed, we have an inessential loop intersection, which contradicts the assumption that Σ and Σ' are essentially intersecting. Hence, $\mathcal{B}'$ is essential.

Furthermore, by the argument similar to [14, Proposition 2.3], $\partial\mathcal{B}'$ is not an inessential curve on F'. Hence, $\mathcal{B}'$ is a compressing disc of F'.

Lastly, we will show that γ is incident to $\mathcal{B}'$. If γ is an inessential simple loop on F', then the diagram is disconnected, which contradicts the assumption that K is a knot. If γ is isotopic to $\partial\mathcal{B}'$, then we can compress F' along γ to get an alternating diagram on a sphere. Hence, γ intersects $\partial\mathcal{B}'$ at least once. Suppose that γ intersects $\partial\mathcal{B}'$ more than once. By Lemma 9, $\partial\mathcal{B}'$ intersects the diagram transversely twice. Hence, γ intersects $\partial\mathcal{B}'$ transversely once or twice.

We will show that γ intersects $\partial\mathcal{B}'$ once, which is equivalent to saying that γ is incident to $\mathcal{B}'$. To show this, we first assume that γ intersects $\partial\mathcal{B}'$ twice and show that K is alternating, which contradicts the assumption that K is non-alternating. If γ intersects $\partial\mathcal{B}'$ twice, then the annulus $F' - int(N(\gamma))$, which contains the diagram, intersects $\partial\mathcal{B}'$ in two essential arcs. The diagram intersects $\partial\mathcal{B}'$ in two points, so either two essential arcs intersect the diagram in one point or one of the essential arcs intersects the diagram in two points. In first case, by Lemma 2, K is a connected sum of an alternating knot and a core of the annulus. The core is a $(2, q)$ curve on the torus F', because it intersects $\partial\mathcal{B}'$ twice. F' is unknotted, hence, the core is a $(2, q)$ torus knot, which is alternating. Hence, K is alternating. In latter case, the diagram is in a disc, so K is also alternating. Since we assumed that K is non-alternating, γ intersects $\partial\mathcal{B}'$ once, so γ is incident to $\mathcal{B}'$

Now, to show the "only if" part, suppose that the knot K is almost-alternating. Consider almost-alternating diagram of K as in Figure 18(left). Then we can do a Reidemeister II move as in Figure 18(middle) to make the diagram as in Figure 18(right). Then K has a checkerboard-colorable, non-cellularly embedded, alternating diagram D'_K on an unknotted torus as in Figure 19(left). By Lemma 3, D'_K has a unique annular region. The core of the annular region is detachable, so the two checkerboard surfaces of D'_K are relatively separable. Now we need to show that every detachable curve is incident to a bigon. The core of the annular region is incident to the bigon shown in Figure 19(right). This bigon is essential because the two
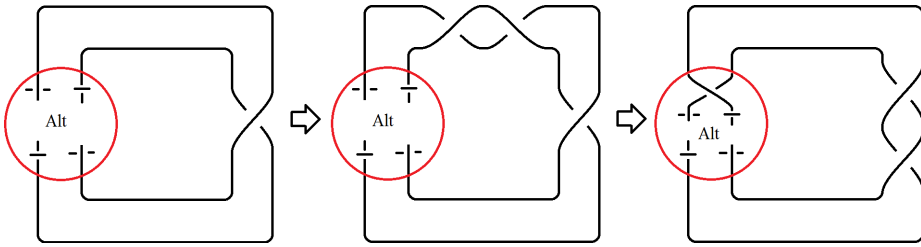
Figure 18.

vertices of this bigon are contained in different standard arc intersections. If there exists another detachable curve, then it must intersect a standard arc intersection. Hence, as in the proof of the "if" part of this theorem, we can find an essential bigon using the annulus bounded by the detachable curve and its push-off.
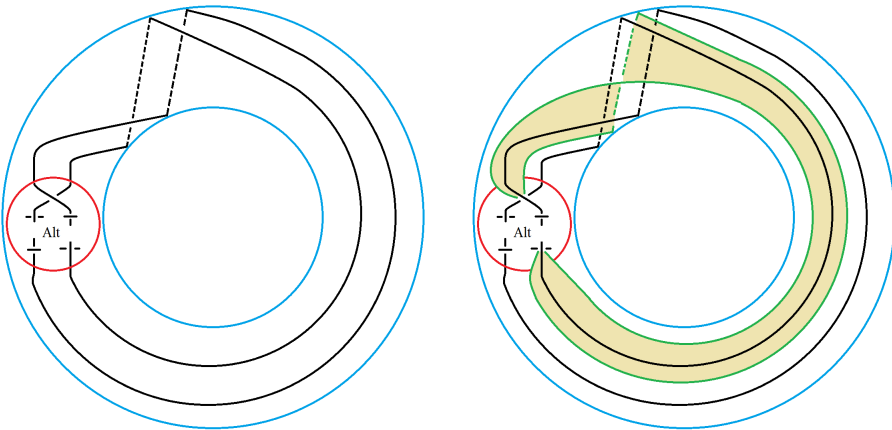
This completes the proof of Theorem 3.

Figure 19. □

References

[1] Tetsuya Abe, *The Turaev genus of an adequate knot*, Topology and its Applications **156** (2009), no. 17, 2704–2712.

[2] Tetsuya Abe and Kengo Kishimoto, *The dealternating number and the alternation number of a closed 3-braid* (appendix with In Dae Jong), Journal of Knot Theory and its Ramifications **19** (2010), no. 9, 1157–1181.

[3] Colin C. Adams, *Toroidally alternating knots and links*, Topology **33** (1994), no. 2, 353–369.

[4] Colin C. Adams, Jeffrey F. Brock, John Bugbee, Timothy D. Comar, Keith A. Faigin, Amy M. Huston, Anne M. Joseph, and David Pesikoff, *Almost alternating links*, Topology and its Applications **46** (1992), no. 2, 151–165.

[5] Cody W. Armond and Adam M. Lowrance, *Turaev genus and alternating decompositions*, Algebraic & Geometric Topology **17** (2017), no. 2, 793–830.

[6] Abhijit Champanerkar and Ilya Kofman, *A survey on the Turaev genus of knots*, Acta Mathematica Vietnamica **39** (2014), no. 4, 497–514.

[7] Joshua Evan Greene, *Alternating links and definite surfaces*, Duke Mathematical Journal **166** (2017), no. 11, 2133–2151.

[8] Joshua Howie, *A characterisation of alternating knot exteriors*, Geometry & Topology **21** (2017), no. 4, 2353–2371.

[9] Joshua Howie, *Surface-alternating knots and links*, Ph.D Thesis, The University of Melbourne (2015).

[10] Tetsuya Ito, *A characterization of almost alternating knot*, Journal of Knot Theory and Its Ramifications **27** (2018), no. 1, 1850009.

[11] Efstratia Kalfagianni, *A Jones slopes characterization of adequate knots*, Indiana Univ. Math. J. **67** (2018), 205–219.

[12] Seungwon Kim, *Link diagrams with low Turaev genus*, Proceedings of the American Mathematical Society 146 (2018), no. 2, 875–890.

[13] Adam M. Lowrance, *Alternating distances of knots and links*, Topology and its Applications **182** (2015), 53–70.

[14] William Menasco and Morwen Thistlethwaite, *The classification of alternating links*, Annals of Mathematics **138** (1993), no. 1, 113–171.

[15] Morwen B. Thistlethwaite, *An upper bound for the breadth of the Jones polynomial*, Math. Proc. Cambridge Philos. Soc **103** (1988), no. 3, 451–456.

Center for Geometry and Physics
Institute for Basic Science (IBS)
Nam-gu, Pohang 37673, South Korea
E-mail address: math751@gmail.com