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Non-existence of solutions for a mean field equation on flat tori at critical parameter 16π

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It is known from [17] that the solvability of the mean field equation $\Delta u + e^u = 8n\pi\delta_0$ with $n \in \mathbb{N}_{\geq 1}$ on a flat torus E_τ essentially depends on the geometry of E_τ. A conjecture is the non-existence of solutions for this equation if E_τ is a rectangular torus, which was proved for $n = 1$ in [17]. For any $n \in \mathbb{N}_{\geq 2}$, this conjecture seems challenging from the viewpoint of PDE theory. In this paper, we prove this conjecture for $n = 2$ (i.e. at critical parameter 16π).

1. Introduction

Let $E_\tau := \mathbb{C}/\Lambda_\tau$ be a flat torus on the plane, where $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, $\tau \in \mathbb{H} = \{\tau \mid \operatorname{Im}\tau > 0\}$. Consider the following mean field equation with a parameter $\rho > 0$:

$$\Delta u + e^u = \rho \cdot \delta_0 \quad \text{on } E_\tau, \tag{1.1}$$

where δ_0 is the Dirac measure at the origin 0. Equation (1.1) has a geometric origin (cf. [6]). In conformal geometry, for a solution $u(x)$, the new metric $ds^2 = e^{u(x)}|dx|^2$ has positive constant curvature. Since the RHS has singularities, ds^2 is a metric with *conic singularity*. The existence problem of such metrics with finitely many conical singularities on compact Riemann surfaces has been widely studied in the last several decades; see e.g. [2, 7, 8, 10, 13, 20, 22, 23] and references therein. Equation (1.1) also appears in statistical physics as the equation for the mean field limit of the Euler flow in Onsager's vortex model (cf. [5]), hence its name. Recently equation (1.1) was shown to be related to the self-dual condensates of the Chern-Simons-Higgs equation in superconductivity. We refer the readers to [9, 12, 14, 15, 19, 21] and references therein for recent developments of related subjects of equation (1.1).

When $\rho \notin 8\pi\mathbb{N}$, it can be proved that solutions of (1.1) have *uniform a priori* bounds in $C^2_{loc}(E_\tau\backslash\{0\})$ and hence the topological Leray-Schauder degree d_ρ is well-defined; see [3, 7]. Recently, Chen and the third author [8] proved that $d_\rho = m$ for any $m \in \mathbb{N}_{\geq 1}$ and $\rho \in (8\pi(m-1), 8\pi m)$. Consequently, *equation (1.1) always has solutions when $\rho \notin 8\pi\mathbb{N}$, no matter with the geometry of the torus E_τ.*

However when $\rho \in 8\pi\mathbb{N}_{\geq 1}$, *a priori* bounds for solutions of (1.1) must not exist (see [6] or Section 2 below for details), and the existence of solutions becomes an intricate question. In this paper, we consider this mean field equation at critical parameters $\rho = 8n\pi$ ([6, 17, 18]):

$$\Delta u + e^u = 8n\pi\delta_0 \quad \text{on } E_\tau, \tag{1.2}$$

where $n \in \mathbb{N}_{\geq 1}$. The case $n = 1$ was first studied by Wang and the third author [17], where they discovered that *the solvability of equation (1.2) essentially depends on the moduli τ of the torus E_τ*, a surprising phenomena which does not appear for non-critical parameter ρ's. For example, they proved that when $\tau \in i\mathbb{R}^+$ (i.e. E_τ is a rectangular torus), equation (1.2) with $n = 1$ has *no* solution; while for $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ (i.e. E_τ is a rhombus torus), equation (1.2) with $n = 1$ has solutions. Later, the case $n = 1$ was thoroughly investigated in [11].

To settle this challenging problem for $n \geq 2$, Chai-Lin-Wang [6] and subsequently Lin-Wang [18] studied it from the viewpoint of algebraic geometry. They developed a theory to connect this PDE problem with hyper-elliptic curves and modular forms. Among other things, they proposed the following conjecture.

Conjecture. [18] *When $\tau \in i\mathbb{R}^+$, i.e. E_τ is a rectangular torus, equation (1.2) has no solutions for any $n \geq 2$.*

In conformal geometry, this conjecture is equivalent to assert that the rectangular torus admits no conformal metric with constant curvature 1 and a conical singularity with angle $2\pi(1 + 2n)$. It is also related to the non-existence of certain meromorphic 1-forms on E_τ; see [10] for details.

This paper is the first in our project devoted to studying the existence (or non-existence) problem of equation (1.2) for $n \geq 2$. The purpose of this paper is to confirm the conjecture for $n = 2$.

Theorem 1.1. *Suppose $\tau \in i\mathbb{R}^+$, i.e. E_τ is a rectangular torus. Then equation (1.2) with $n = 2$ on E_τ has no solutions.*

Theorem 1.1 has important applications. In a forthcoming paper, we will apply Theorem 1.1 (together with the modular form theory established in [18]) to prove the following existence result on rhombus tori.

Theorem A. *Let $\tau = \frac{1}{2} + ib$ with $b > 0$. Then there exists $b^* \in (\frac{\sqrt{3}}{2}, \frac{6}{5})$ such that for any $b > b^*$, equation (1.2) with $n = 2$ on E_τ has a solution.*

Remark that Theorem A is almost optimal in the sense that if $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, then equation (1.2) with $n = 2$ on E_τ has *no* solutions (as mentioned before, (1.2) with $n = 1$ on this E_τ has solutions. This shows why we need to discuss different n's separately). See Theorem 3.1 in Section 3.

In PDE theory, a standard method of proving non-existence results is to apply the Pohozaev identity; see [4] for example. Obviously, this method by Pohozaev identity does not work here. Our proof is based on the fact that equation (1.2) can be viewed as an integrable system [6].

The paper is organized as follows. In Section 2, we give a short review of equation (1.2) from the aspect of integrable system. This point of view can reduce our existence problem to a couple equations involving with Weierstrass elliptic functions. In Section 3, we prove this couple equations have no solutions if $\tau \in i\mathbb{R}^+$. Our proof is elementary in the sense that only the basic theory of Weierstrass elliptic functions covered by the standard textbook (cf. [1]) are used. This gives the proof of Theorem 1.1.

2. Overview of (1.2) as an integrable system

In this section, we provide some basic facts about equation (1.2) from the viewpoint of integrable system; see [6] for a complete discussion. Throughout the paper, we use the notations: $\omega_0 = 0$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$.

The Liouville theorem says that for any solution $u(z)$ to (1.2), there is a meromorphic function $f(z)$ defined in $\mathbb{C}$ such that

$$u(z) = \log \frac{8|f'(z)|^2}{(1 + |f(z)|^2)^2}. \tag{2.1}$$

This $f(z)$ is called a developing map. Although u is a doubly periodic function, $f(z)$ is not an elliptic function. By differentiating (2.1), we have

$$u_{zz} - \frac{1}{2}u_z^2 = \{f; z\} := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.2}$$

Conventionally, the RHS of this identity is called the Schwarzian derivative of $f(z)$, denoted by $\{f; z\}$. By the classical Schwarzian theory, any two developing maps f_1 and f_2 of the same solution u must satisfy

$$f_2(z) = \gamma \cdot f_1(z) := \frac{af_1(z)+b}{cf_1(z)+d} \tag{2.3}$$

for some $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. Furthermore, by substituting (2.3) into (2.1), a direct computation shows $\gamma \in SU(2)$, i.e.

$$d = \bar{a}, \quad c = -\bar{b} \quad \text{and} \quad |a|^2 + |b|^2 = 1. \tag{2.4}$$

As we mentioned above, $f(z)$ is not doubly periodic. But $f(z+w_1)$ and $f(z+w_2)$ are also developing maps of the same $u(z)$ and then (2.3) implies the existence of $\gamma_i \in SU(2)$ such that

$$f(z+\omega_1) = \gamma_1 \cdot f(z) \quad \text{and} \quad f(z+\omega_2) = \gamma_2 \cdot f(z). \tag{2.5}$$

After normalizing $f(z)$ by the action of some $\gamma \in SU(2)$, (2.5) can be simplified by

$$f(z+\omega_j) = e^{2\pi i\theta_j} f(z), \quad j = 1, 2, \tag{2.6}$$

for some $\theta_j \in \mathbb{R}$. We call a developing map f satisfying (2.6) a *normalized developing map.*

A simple observation is that once f satisfies (2.6), then for any $\beta \in \mathbb{R}$, $e^\beta f(z)$ also satisfies (2.6). Therefore, once we have a solution $u(z)$, then we get a 1-parameter family of solutions:

$$u_\beta(z) := \log \frac{8e^{2\beta}|f'(z)|^2}{(1+e^{2\beta}|f(z)|^2)^2}.$$

Clearly $u_\beta(z)$ blow up as $\beta \to \pm\infty$. More precisely, $u_\beta(z)$ blow up at and only at any zeros of $f(z)$ as $\beta \to +\infty$, and $u_\beta(z)$ blow up at and only at any poles of $f(z)$ as $\beta \to -\infty$. For (1.2), the blowup set of a sequence of solutions u_β consists of n distinct points in E_τ. Hence $f(z)$ has zeros at $z = a_i \in E_\tau$, $i = 1, \dots, n$, and poles at $z = b_i \in E_\tau$, $i = 1, \dots, n$. Furthermore, $\{a_1, \dots, a_n\} = \{-b_1, \dots, -b_n\}$ in E_τ; see [6]. Since $\{a_i\}$ and $\{b_i\}$ are the zeros and poles of a meromorphic function, we have

$$a_i \neq a_j \text{ for any } i \neq j \ ; \quad a_i \neq -a_j \text{ for any } i, j. \tag{2.7}$$

In the sequel, we always assume $n = 2$ in (1.2). So u_β has exactly two blowup points as $\beta \to +\infty$, say a and b. Then (2.7) and the well known

Pohozaev identity imply that a and b satisfy

$$2G_z(a) = G_z(a-b), \quad 2G_z(b) = G_z(b-a), \quad a \notin \{-a, \pm b\}. \tag{2.8}$$

where $G(z) = G(z|\tau)$ is the Green function of $-\Delta$ on the torus E_τ. See [7, 8] for the Pohozaev identity. Since the Green function $G(z)$ is even, $G_z(z)$ is odd and (2.8) is equivalent to

$$G_z(a) + G_z(b) = 0,\ G_z(a) - G_z(b) - G_z(a-b) = 0,\ a \notin \{-a, \pm b\}. \tag{2.9}$$

On the other hand, the Green function $G(z)$ can be written in terms of Weierstrass elliptic functions, see [17]. In particular, we have

$$\begin{aligned} -4\pi G_z(z) &= \zeta(z|\tau) - \eta_1(\tau)z + \frac{2\pi i \operatorname{Im} z}{\operatorname{Im}\tau} \\ &= \zeta(z|\tau) - r\eta_1(\tau) - s\eta_2(\tau), \end{aligned} \tag{2.10}$$

where $z = r + s\tau$ with $r, s \in \mathbb{R}$. Here we recall that $\wp(z) = \wp(z|\tau)$ is the Weierstrass elliptic function with periods $\omega_1 = 1$ and $\omega_2 = \tau$, defined by

$$\wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

and $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ is the Weierstrass zeta function, which is an odd meromorphic function with two quasi-periods $\eta_j(z)$ (cf. [16]):

$$\zeta(z+1|\tau) = \zeta(z|\tau) + \eta_1(\tau), \quad \zeta(z+\tau|\tau) = \zeta(z|\tau) + \eta_2(\tau). \tag{2.11}$$

In view of (2.10), the second equation in (2.9) can be changed to

$$\zeta(a) - \zeta(b) - \zeta(a-b) = 0. \tag{2.12}$$

Next, we should apply the classical addition formula (cf. [16]):

$$\zeta(u+v) - \zeta(u) - \zeta(v) = \frac{1}{2}\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}$$

by taking $(u, v) = (a, -b)$. Then (2.12) becomes

$$\wp'(a) + \wp'(b) = 0.$$

Therefore, the Pohozaev identity (2.9) is equivalent to

$$G_z(a) + G_z(b) = 0 \quad \text{and} \quad \wp'(a) + \wp'(b) = 0, \quad a \notin \{-a, \pm b\}. \tag{2.13}$$

Thus, we summarize the main result in this short overview as follows: *Suppose the mean field equation (1.2) with $n = 2$ has a solution u, then there exist $a, b \in E_\tau$ such that (2.13) holds true.*

3. Non-existence for $\tau \in i\mathbb{R}^+$

In this section, we want to prove the non-existence of solutions to

$$\Delta u + e^u = 16\pi\delta_0 \quad \text{on } E_\tau, \tag{3.1}$$

if $\tau \in i\mathbb{R}^+$, i.e. E_τ is a rectangular torus. In the sequel, we always use notations $\omega_1 = 1$, $\omega_2 = \tau$ and $\omega_3 = 1 + \tau$.

As discussed in Section 2, to prove this non-existence result, it suffices to show that there are no pair (a, b) in E_τ such that (2.13) holds. The proof for $\tau \in i\mathbb{R}^+$ is really non-trivial, however, it is much simpler if $\tau = e^{\frac{\pi i}{3}}$.

Theorem 3.1. *Let $\rho := e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then equation*

$$\Delta u + e^u = 16\pi\delta_0 \quad \textit{on} \quad E_\rho \tag{3.2}$$

has no solutions.

Proof. Assume by contradiction that (3.2) has a solution. Then there exist $a, b \in E_\rho$ such that (2.13) holds, i.e.

$$G_z(a|\rho) + G_z(b|\rho) = 0, \quad \wp'(a|\rho) + \wp'(b|\rho) = 0 \quad \text{and} \quad a \notin \{-a, \pm b\}.$$

It is known (cf. [17]) that $g_2(\rho) = 0$ (see (3.4) for g_2) and $\wp(z|\rho) = \rho^2\wp(\rho z|\rho)$. Then by $\wp'(a|\rho)^2 = \wp'(b|\rho)^2$ and (3.4) below, we obtain $\wp(a|\rho)^3 = \wp(b|\rho)^3$, which implies

$$\text{either} \quad b = \pm\rho a \quad \text{or} \quad b = \pm\rho^2 a.$$

On the other hand, $G(\rho z|\rho) = G(z|\rho)$ gives $\rho G_z(\rho z|\rho) = G_z(z|\rho)$. Hence,

$$0 = G_z(a|\rho) + G_z(b|\rho) = \left(1 \pm \rho^{-j}\right) G_z(a|\rho) \text{ for some } j \in \{1, 2\},$$

which implies that a is a critical point of $G(z|\rho)$ and so does b. Recall from [17] that $G(z|\rho)$ has exactly five critical points $\{\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3, \pm\frac{1}{3}\omega_3\}$. So $a, b \in \{\frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3, \pm\frac{1}{3}\omega_3\}$, a contradiction with $\wp'(a|\rho) + \wp'(b|\rho) = 0$ and $a \notin \{-a, \pm b\}$. Therefore, (3.2) has no solutions. □

From now on, we assume that $\tau \in i\mathbb{R}^+$, i.e. E_τ is a rectangular torus. Under this assumption, we will prove Theorem 1.1. By making abuse of the notation, we also use the same notation E_τ to denote its fundamental parallelogram centered at 0, i.e. *E_τ is a rectangle centered at the origin and so ∂E_τ is well-defined in this sense.*

To prove Theorem 1.1, we will show that if (a, b) is a solution of (2.13), then both a and b lie in the same half plane, and then we exclude this possibility by using the elementary properties of the Green function G.

Our proof is elementary in the sense that only the basic theory of $\wp(z|\tau)$ covered by the standard textbook (cf. [1]) are used. For example, the following lemma only uses some properties of $\wp(z|\tau)$ on rectangles.

Lemma 3.2. *Let $\omega_2 = \tau \in i\mathbb{R}^+$. Then $\wp$ is one to one from $(0, \frac{1}{2}\omega_1] \cup [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3] \cup [\frac{1}{2}\omega_3, \frac{1}{2}\omega_2] \cup [\frac{1}{2}\omega_2, 0)$ onto $(-\infty, +\infty)$. Here $[z_1, z_2] = \{z : z = tz_2 + (1-t)z_1,\ 0 \le t \le 1\}$, and $[z_1, z_2)$, $(z_1, z_2]$, (z_1, z_2) are defined similarly.*

Proof. By $\tau \in i\mathbb{R}^+$ and the definition of $\wp(z)$:

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z-m-n\tau)^2} - \frac{1}{(m+n\tau)^2} \right), \tag{3.3}$$

it is easy to see that $\overline{\wp(z)} = \wp(\bar{z})$. Since $\bar{z} = z$ if $z \in (0, \frac{1}{2}]$, $\bar{z} = -z$ if $z \in (0, \frac{\tau}{2}]$, $\bar{z} = 1 - z$ if $z \in [\frac{1}{2}, \frac{1+\tau}{2}]$, $\bar{z} = z - \tau$ if $z \in [\frac{1+\tau}{2}, \frac{\tau}{2}]$, so $\wp$ is real-valued in $(0, \frac{\tau}{2}] \cup [\frac{\tau}{2}, \frac{1+\tau}{2}] \cup [\frac{1+\tau}{2}, \frac{1}{2}] \cup [\frac{1}{2}, 0)$.

On the other hand, since $\wp(z) = \wp(-z)$ and the degree of $\wp(z)$ is two, we conclude that $\wp(z)$ is one to one in $(0, \frac{\tau}{2}] \cup [\frac{\tau}{2}, \frac{1+\tau}{2}] \cup [\frac{1+\tau}{2}, \frac{1}{2}] \cup [\frac{1}{2}, 0)$. Moreover, since the second term in the RHS of (3.3) is bounded as $z \to 0$, we conclude

$$\lim_{[\frac{1}{2},0)\ni z\to 0} \wp(z) = +\infty, \qquad \lim_{(0,\frac{\tau}{2}]\ni z\to 0} \wp(z) = -\infty.$$

The proof is complete. □

Remark 3.3. In this paper, we always write $z = x_1 + ix_2$ with $x_1, x_2 \in \mathbb{R}$. Let $e_k = \wp(\frac{\omega_k}{2})$, $k = 1, 2, 3$. We recall that $\wp(z)$ satisfies the cubic equation:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 = 4\prod_{k=1}^{3}(\wp(z) - e_k), \tag{3.4}$$
$$\text{and } \wp''(z) = 6\wp(z)^2 - g_2/2.$$

Thus $e_1+e_2+e_3=0$. From Lemma 3.2, we have $e_j\in\mathbb{R}$, $e_2<e_3<e_1$ and $e_2<0<e_1$, also $\wp'(z)=\frac{\partial\wp(z)}{\partial x_1}\in\mathbb{R}$ if $z\in(0,\frac{1}{2}\omega_1]\cup(\frac{1}{2}\omega_2,\frac{1}{2}\omega_3]$, and $\wp'(z)=-i\frac{\partial\wp(z)}{\partial x_2}\in i\mathbb{R}$ if $z\in(0,\frac{1}{2}\omega_2]\cup(\frac{1}{2}\omega_1,\frac{1}{2}\omega_3]$. Lemma 3.2 also implies $\zeta(z)\in\mathbb{R}$ for $z\in(0,\frac{1}{2}\omega_1]$ and so $\eta_1\in\mathbb{R}$. In the following, we use $q_\pm$ to denote the solution of $\wp(q_\pm)=\pm\sqrt{g_2/12}$, i.e. $\wp''(q_\pm)=0$.

Recall our assumption that E_τ is a rectangle centered at the origin. We first discuss (2.13) by assuming $a\in\partial E_\tau$. To prove Theorem 1.1 in this case, we will solve the second equation in (2.13) to obtain a branch $b=b(a)$, and then insert $b=b(a)$ in the first equation of (2.13) to find a contradiction. For this purpose, we now discuss the second equation in (2.13) with $a\neq -b$. We have the following lemma.

Lemma 3.4. *The equation $\wp''(a)=0$ has exactly four distinct solutions $\pm q_\pm$, which all belong to ∂E_τ with $q_+\in(\frac{1}{2}\omega_1,\frac{1}{2}\omega_3)$ and $q_-\in(\frac{1}{2}\omega_2,\frac{1}{2}\omega_3)$. Moreover, for any $a\in E_\tau\setminus\{\pm q_\pm,\pm 2q_\pm\}$, there are two distinct solutions b's to the equation*

$$\wp'(a)+\wp'(b)=0,\quad a\neq -b.$$

Proof. From (3.4), $\wp''(z)=0$ has 4 zeros at $\pm q_+,\pm q_-$, where $\wp(q_\pm)=\pm\sqrt{g_2/12}$, and

$$e_1+e_2+e_3=0,\quad e_1e_2+e_1e_3+e_2e_3=-\frac{g_2}{4},\quad e_1e_2e_3=\frac{g_3}{4}, \tag{3.5}$$

which implies $g_2=2(e_1^2+e_2^2+e_3^2)>0$. So $\wp(q_\pm)\in\mathbb{R}$. We claim

$$e_2<-\sqrt{g_2/12}<e_3<\sqrt{g_2/12}<e_1. \tag{3.6}$$

Then it follows that $q_+\in(\frac{1}{2}\omega_1,\frac{1}{2}\omega_3)$ and $q_-\in(\frac{1}{2}\omega_2,\frac{1}{2}\omega_3)$, i.e. $\pm q_\pm\in\partial E_\tau$.

Since $e_1+e_2+e_3=0$ and $e_2<e_3<e_1$ by Remark 3.3, we have $e_2<0$, $e_1>0$ and $|e_3|<\min\{|e_2|,e_1\}$. Thus, for $i=1$ or $i=2$,

$$g_2=2(e_1^2+e_2^2+e_3^2)=4(e_i^2+e_ie_3+e_3^2)<12e_i^2,$$

namely $e_2<-\sqrt{g_2/12}$ and $e_1>\sqrt{g_2/12}$. If $e_3\leq 0$, then $g_2=4(e_2^2+e_2e_3+e_3^2)>12e_3^2$; if $e_3>0$, then $g_2=4(e_1^2+e_1e_3+e_3^2)>12e_3^2$. Therefore, $|e_3|<\sqrt{g_2/12}$, namely (3.6) holds.

For any $a\in E_\tau$, $\wp'(z)=-\wp'(a)$ has three solutions, because the degree of the map $\wp'$ from E_τ to $\mathbb{C}\cup\{\infty\}$ is three. Note that $\wp''(z)=0$ if and only if $z=\pm q_\pm$. Thus $\wp'(a)+\wp'(b)=0$ has three distinct solutions b's except for those a's such that $\wp'(a) + \wp'(\pm q_\pm) = 0$ for some $\pm q_\pm$. To find such a, we note that

$$\wp'(a)^2 = \wp'(b)^2, \quad \text{for some } b \in \{\pm q_+, \pm q_-\}.$$

It suffices to consider the case $a \notin \{\pm q_\pm\}$. Then $\wp(a) \neq \wp(b)$. By using

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{3.7}$$

at $z = a$ and $z = b$, we have

$$\wp(a)^2 + \wp(a)\wp(b) + \wp(b)^2 - \frac{g_2}{4} = 0. \tag{3.8}$$

Recalling $\wp(b) = \pm\sqrt{g_2/12}$ for $b \in \{\pm q_\pm\}$, we get

$$\wp(a) = \frac{-\wp(b) \pm \sqrt{g_2 - 3\wp(b)^2}}{2} = \frac{-\wp(b) \pm 3\wp(b)}{2}. \tag{3.9}$$

This, together with $\wp(a) \neq \wp(b)$, gives $\wp(a) = -2\wp(b)$. From the addition formula $\wp(2z) = \frac{1}{4}(\frac{\wp''(z)}{\wp'(z)})^2 - 2\wp(z)$ and $\wp''(b) = 0$ for $b \in \{\pm q_\pm\}$, we get $\wp(a) = \wp(2b)$. Therefore, $a \in \{\pm 2q_\pm\}$. This completes the proof. □

Remark 3.5. We have proved $q_+ \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$ and $q_- \in (\frac{1}{2}\omega_2, \frac{1}{2}\omega_3)$. From $e_1 + e_2 + e_3 = 0$, we have $g_2 = 2(e_1^2 + e_2^2 + e_3^2) > 3\max\{e_1^2, e_2^2\}$, which implies $\wp(2q_+) = -2\wp(q_+) = -\sqrt{g_2/3} < e_2$ and $\wp(2q_-) = -2\wp(q_-) = \sqrt{g_2/3} > e_1$. Hence $2q_+ \in (0, \frac{\omega_2}{2}) \cup (-\frac{\omega_2}{2}, 0)$ and $2q_- \in (0, \frac{\omega_1}{2}) \cup (-\frac{\omega_1}{2}, 0)$. We will prove in Lemma 3.7 that $2q_+ \in (0, \frac{\omega_2}{2})$.

Lemma 3.6. *There is no pair (a, b) with a or $b \in \partial E_\tau$, such that (2.13) holds.*

Proof. Assume by contradiction that such (a, b) exists. Since the degree of $\wp(z)$ is two and $\wp(-z) = \wp(z)$, we know that $\wp(a) \neq \wp(b)$ because of $a \neq \pm b$. Then just as in Lemma 3.4, it follows from $\wp'(a) + \wp'(b) = 0$ that (3.8) holds for $(\wp(a), \wp(b))$.

Without loss of generality, we assume $a \in \partial E_\tau$. From (3.8), we find

$$\wp(b) = \frac{-\wp(a) \pm \sqrt{g_2 - 3\wp(a)^2}}{2}. \tag{3.10}$$

We claim

$$g_2 - 3\wp(a)^2 > 0 \quad \text{for any } a \in \partial E_\tau. \tag{3.11}$$

From $\wp(-z) = \wp(z)$ and $\wp(z + \omega_j) = \wp(z)$, $j = 1, 2$, we only need to prove the claim for $a \in [\frac{1}{2}\omega_2, \frac{1}{2}\omega_3] \cup [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$. Let us assume $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$. Then $e_3 \le \wp(a) \le e_1$. If $\wp(a) \le 0$, then from (3.5) and $e_1 e_2 < 0$, we have

$$g_2 = -4(e_1e_2 + e_3(e_1 + e_2)) = 4(e_3^2 - e_1e_2) > 4e_3^2 > 3\wp(a)^2. \tag{3.12}$$

On the other hand, if $\wp(a) > 0$, by $e_1^2 - 4e_2e_3 = (e_2 - e_3)^2 > 0$, we have

$$g_2 = 4(e_1^2 - e_2e_3) > 3e_1^2 \ge 3\wp(a)^2. \tag{3.13}$$

Suppose now $a \in [\frac{1}{2}\omega_2, \frac{1}{2}\omega_3]$. Then $e_2 \le \wp(a) \le e_3$. If $\wp(a) > 0$, then (3.12) gives $g_2 > 3\wp(a)^2$. If $\wp(a) \le 0$, then similar to (3.13), we have

$$g_2 = 4(e_2^2 - e_1e_3) > 3e_2^2 \ge 3\wp(a)^2.$$

So, the claim (3.11) follows. Since $\wp(a) \in \mathbb{R}$, by the claim and (3.10) we also have $\wp(b) \in \mathbb{R}$.

To prove Lemma 3.7, let us argue for the case $a \in \left(\frac{1}{2}(\omega_1 - \omega_2), \frac{1}{2}\omega_1\right) \cup \left(\frac{1}{2}\omega_1, \frac{1}{2}\omega_3\right)$, which are two intervals on the line $\frac{1}{2}\omega_1 + i\mathbb{R}$. Without loss of generality, we may assume $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$. Lemma 3.4 and Remark 3.6 tell us that there are three branch solutions $b_i(a)$, $i = 1, 2, 3$, of $\wp'(a) + \wp'(b) = 0$ for $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3] \backslash \{q_+\}$, where we assign $b_1(a) = -a$ for any a. To continue our proof, we need two lemmas to study the basic properties of the other two branches.

Lemma 3.7. *For $a \in [\frac{1}{2}w_1, \frac{1}{2}w_3]$, there are two analytic branches $b_2(a)$ and $b_3(a)$ of solutions to $\wp'(a) + \wp'(b) = 0$ such that $b_2(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_3$, $b_2(q_+) = -q_+$ and $b_2(\frac{1}{2}\omega_3) = -\frac{1}{2}\omega_1$, $b_3(\frac{1}{2}\omega_1) = \frac{1}{2}\omega_2$, $b_3(q_+) = 2q_+$ and $b_3(\frac{1}{2}\omega_3) = \frac{1}{2}\omega_2$. Furthermore, $b_2(a) \in [-\frac{1}{2}\omega_3, -\frac{1}{2}\omega_1]$ and $b_3(a) \in [2q_+, \frac{1}{2}\omega_2]$, $2q_+ \in (0, \frac{1}{2}\omega_2)$.*

Proof. For $a \in [\frac{1}{2}\omega_1, q_+)$, there exist two analytic branch solutions $b_2(a)$ and $b_3(a)$ for $\wp'(a) + \wp'(b) = 0$. Since $\wp'(\frac{1}{2}\omega_1) = 0$, we have $\wp'(b(\frac{1}{2}\omega_1)) = 0$. Hence, $b(\frac{1}{2}\omega_1) = \frac{1}{2}\omega_2$ or $b(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_3$ since $a \ne \pm b$. Here, we assume $b_2(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_3$ and $b_3(\frac{1}{2}\omega_1) = \frac{1}{2}\omega_2$. By Lemma 3.2, $\wp(a)$ is decreasing in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, we find $\wp'(a) \in i\mathbb{R}^+$ for $a \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$, which gives $\wp'(b_i(a)) \in i\mathbb{R}^-$. By (3.10), $g_2 - 3\wp(a)^2 > 0$ and Lemma 3.2, we find $\wp(b_i(a)) \in \mathbb{R}$. Together with Remark 3.3, we conclude that

$$b_2 : [\tfrac{1}{2}\omega_1, q_+) \to -\tfrac{1}{2}\omega_3 + i\mathbb{R}^+,$$

and

$$b_3 : [\tfrac{1}{2}\omega_1, q_+) \to [\tfrac{1}{2}\omega_2, 0).$$

First, we note that b_2 is one-to-one for $a \in [\frac{1}{2}\omega_1, q_+)$, because if $b_2(a) = b_2(\tilde{a})$ for some $a, \tilde{a} \in [\frac{1}{2}\omega_1, q_+)$, then $\wp'(a) = -\wp'(b_2(a)) = -\wp'(b_2(\tilde{a})) = \wp'(\tilde{a})$, which implies $a = \tilde{a}$, since $\wp'' \neq 0$ on $[\frac{1}{2}\omega_1, q_+)$. Similarly, b_3 is one-to-one for $a \in [\frac{1}{2}\omega_1, q_+)$. By one-to-one, $b_2(a)$ is increasing from $b_2(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_3$ to $b_2(q_+) = \lim_{a \to q_+} b_2(a)$ as a varies from $\frac{1}{2}\omega_1$ to q_+. The previous proof of Lemma 3.6 shows that (3.8) holds for $a \in [\frac{1}{2}\omega_1, q_+)$ and $b_2(a)$. By letting $a \to q_+$, we also have that $(q_+, b_2(q_+))$ satisfies (3.8). Then similarly to (3.9), we obtain

$$\wp(b_2(q_+)) = \frac{-\wp(q_+) \pm 3\wp(q_+)}{2},$$

namely either $\wp(b_2(q_+)) = \wp(q_+)$ or $\wp(b_2(q_+)) = -2\wp(q_+) = \wp(2q_+)$ because $\wp''(q_+) = 0$. Since $b_2(q_+) \in -\frac{1}{2}\omega_3 + i\mathbb{R}^+$ and $2q_+ \in \omega_1 + i\mathbb{R} = i\mathbb{R}$ in the torus E_τ, we conclude that $b_2(q_+) = -q_+$.

The above argument also shows $\wp(b_3(q_+)) = -2\wp(q_+) = \wp(2q_+)$. So we have either $b_3(q_+) = 2q_+$ or $b_3(q_+) = -2q_+$. We claim

$$b_3(q_+) = 2q_+. \tag{3.14}$$

Recalling $b_3(\frac{1}{2}\omega_1) = \frac{1}{2}\omega_2$, (3.14) is equivalent to $2q_+ \in (0, \frac{1}{2}\omega_2)$. So it suffices to prove $q_+ \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_1 + \frac{1}{4}\omega_2)$ or equivalently, to show $\wp(q_+) > \wp(\frac{1}{2}\omega_1 + \frac{1}{4}\omega_2)$. We use the following addition formula to prove this inequality:

$$\wp(2z) + 2\wp(z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2. \tag{3.15}$$

Because $0 \neq \wp'(\frac{1}{2}\omega_1 + \frac{1}{4}\omega_2) \in i\mathbb{R}$ and $\wp''(\frac{1}{2}\omega_1 + \frac{1}{4}\omega_2) \in \mathbb{R}$, (3.15) gives

$$2\wp(\tfrac{1}{2}\omega_1 + \tfrac{1}{4}\omega_2) \leq -\wp(\tfrac{1}{2}\omega_2) = -e_2 < 2\wp(q_+),$$

where the last inequality follows from Remark 3.5. Hence (3.14) is proved.

It is easy to see that these two branches $b_2(a)$ and $b_3(a)$ can be extended from $[\frac{1}{2}\omega_1, q_+]$ to $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$ such that for $a \in (q_+, \frac{1}{2}\omega_3]$, $b_2(a) \in (-q_+, -\frac{1}{2}\omega_1]$ and $b_3(a) \in (2q_+, \frac{1}{2}\omega_2]$. This completes the proof. □

Lemma 3.8. *For $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, the following statements hold:*

(i) $-e_2 \leq \wp(a) + \wp(b_2(a)) \leq 2\wp(q_+)$;

(ii) $-e_1 \leq \wp(a) + \wp(b_3(a)) \leq -e_3$.

Proof. We define $f_i(a) := \wp(a) + \wp(b_i(a))$, $i = 2, 3$. Then for $a \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$,

$$f_i'(a) = \wp'(a) + \wp'(b_i(a))b_i'(a) = \wp'(a)(1 - b_i'(a)).$$

Note that $\wp'(a) \neq 0$ for $a \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$. Assume that $\bar{a} \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$ is a critical point of f_i. Then $b_i'(\bar{a}) = 1$. By the arguments in the proof of Lemma 3.7, we know that (3.8) holds for $(\wp(a), \wp(b_i(a)))$. Differentiating over (3.8), we easily conclude that

$$[\wp(a) + 2\wp(b_i(a))]b_i'(a) = 2\wp(a) + \wp(b_i(a)).$$

Recalling (3.10), we have $\wp(a) + 2\wp(b_i(a)) = \pm\sqrt{g_2 - 3\wp(a)^2} \neq 0$. Thus,

$$b_i'(a) = \frac{2\wp(a) + \wp(b_i(a))}{\wp(a) + 2\wp(b_i(a))}. \tag{3.16}$$

Letting $a = \bar{a}$ in (3.16), we obtain $\wp(b_i(\bar{a})) = \wp(\bar{a})$. This, together with (3.10), gives

$$\wp(\bar{a}) = \wp(b_i(\bar{a})) = \frac{-\wp(\bar{a}) \pm \sqrt{g_2 - 3\wp^2(\bar{a})}}{2},$$

which implies $\wp(b_i(\bar{a})) = \wp(\bar{a}) = \pm\sqrt{g_2/12}$. Thus, $\bar{a} = q_+$ and so $b_i(q_+) = -q_+$. Therefore, q_+ is the only critical point of f_2 in $(\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$, while f_3 has no critical points in $(\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$, namely f_3 is strictly monotone in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$. By Lemma 3.7, $f_2(\frac{1}{2}\omega_1) = f_2(\frac{1}{2}\omega_3) = e_1 + e_3 = -e_2 < \sqrt{g_2/3} = 2\wp(q_+) = f_2(q_+)$, hence (i) holds. Besides, $f_3(\frac{1}{2}\omega_1) = e_1 + e_2 = -e_3$ and $f_3(\frac{1}{2}\omega_3) = e_3 + e_2 = -e_1$, we see that (ii) holds. □

Now we go back to the proof of Lemma 3.6. First let us consider $b_2(a)$. Since $b_2(q_+) = -q_+$, $\nabla G(q_+) + \nabla G(b_2(q_+)) = 0$ due to the anti-symmetry of ∇G. We will show that $\nabla G(a) + \nabla G(b_2(a)) \neq 0$ for all $a \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3) \setminus \{q_+\}$. For this purpose, we consider the following real-valued function on $a \in I = [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$:

$$H_2(a) := G_{x_2}(a) + G_{x_2}(b_2(a)).$$

Since $b_2(\frac{1}{2}\omega_1) = -\frac{1}{2}\omega_3$ and $b_2(\frac{1}{2}\omega_3) = -\frac{1}{2}\omega_1$, we have $H_2(a) = 0$ if $a \in \{\frac{1}{2}\omega_3, \frac{1}{2}\omega_1, q_+\}$. We want to show that there is no other zeros of $H_2(a) = 0$ in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$. Note that $H_2'(a) = 0$ has at least two solutions because $H_2(a) = 0$ at $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_3$ and q_+. If we can prove that $H_2'(a) = 0$ has only two solutions in $\left(\frac{1}{2}\omega_1, \frac{1}{2}\omega_3\right)$, then except the three points $\frac{1}{2}\omega_1$, $\frac{1}{2}\omega_3$ and q_+, $H_2(a)$ has no other zeros in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$.

Let us compute $H_2'(a)$. Note that $G_{x_2x_2}(a)$ and $G_{x_2x_2}(b_2(a))$ can be derived as follows. From (2.10), we have

$$\left(4\pi G_z(z) + \frac{2\pi i x_2}{\operatorname{Im}\tau}\right)' = \left(-\zeta(z) + \eta_1 z\right)' = \wp(z) + \eta_1.$$

But

$$\begin{aligned}\left(4\pi G_z(z) + \frac{2\pi i x_2}{\operatorname{Im}\tau}\right)' &= \frac{\partial}{\partial x_1}\left(4\pi G_z(z) + \frac{2\pi i x_2}{\operatorname{Im}\tau}\right) = 4\pi\frac{\partial G_z(z)}{\partial x_1}\\ &= 2\pi G_{x_1x_1} - 2\pi i G_{x_1x_2}\\ &= -2\pi G_{x_2x_2} - 2\pi i G_{x_1x_2} + \frac{2\pi}{\operatorname{Im}\tau}.\end{aligned}$$

Thus we obtain

$$2\pi G_{x_1x_1}(z) = \operatorname{Re}(\eta_1 + \wp(z)), \tag{3.17}$$

$$2\pi G_{x_1x_2}(z) = -\operatorname{Im}(\eta_1 + \wp(z)), \tag{3.18}$$

$$2\pi G_{x_2x_2}(z) = \frac{2\pi}{\operatorname{Im}\tau} - \operatorname{Re}(\eta_1 + \wp(z)). \tag{3.19}$$

Since $\wp(z)$ is real for $z = a$ or $b_2(a)$, we have

$$\begin{aligned}2\pi i H_2'(a) &= 2\pi G_{x_2x_2}(a) + 2\pi G_{x_2x_2}(b_2(a))b_2'(a)\\ &= \frac{2\pi}{\operatorname{Im}\tau} - \eta_1 - \wp(a) + \left(\frac{2\pi}{\operatorname{Im}\tau} - \eta_1 - \wp(b_2(a))\right)b_2'(a).\end{aligned} \tag{3.20}$$

For $a \in \frac{1}{2}\omega_1 + i\mathbb{R}$, $H_2'(a) \in i\mathbb{R}$. Recalling (3.16) and denoting $\tilde{\eta}_1 = \eta_1 - \frac{2\pi}{\operatorname{Im}\tau}$ for convenience, we see that $H_2'(a) = 0$ is equivalent to

$$\tilde{\eta}_1 + \wp(a) + (\tilde{\eta}_1 + \wp(b_2(a)))\frac{2\wp(a) + \wp(b_2(a))}{\wp(a) + 2\wp(b_2(a))} = 0.$$

By direct computations, we get

$$3\tilde{\eta}_1(\wp(a) + \wp(b_2(a))) + 2\wp(a)\wp(b_2(a)) + [\wp(a) + \wp(b_2(a))]^2 = 0. \tag{3.21}$$

By (3.8), $\wp(a)\wp(b_2(a)) = [\wp(a) + \wp(b_2(a))]^2 - g_2/4$. Insert this into (3.21), we obtain

$$[\wp(a) + \wp(b_2(a))]^2 + \tilde{\eta}_1(\wp(a) + \wp(b_2(a))) - \frac{g_2}{6} = 0.$$

Thus,

$$f_2(a) = \wp(a) + \wp(b_2(a)) = \frac{1}{2}\left(-\tilde{\eta}_1 \pm \sqrt{\tilde{\eta}_1^2 + 2g_2/3}\right) := B_\pm. \tag{3.22}$$

Clearly $B_+ > 0 > B_-$. By Lemma 3.8, we have

$$f_2(a) = B_+ \geq -e_2 > 0. \tag{3.23}$$

Combining this with (3.21), we conclude that

$$\wp(a) + \wp(b_2(a)) = B_+ \quad \text{and} \quad \wp(a)\wp(b_2(a)) = -\frac{B_+^2 + 3\tilde{\eta}_1 B_+}{2} =: A_+,$$

and so

$$\wp(a) = \frac{B_+ \pm \sqrt{B_+^2 - 4A_+}}{2},$$

whenever $H_2'(a) = 0$. Since $\wp$ is one-to-one on $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, there are two distinct points a_+ and a_- such that $\wp(a_\pm) = \frac{B_+ \pm \sqrt{B_+^2 - 4A_+}}{2}$. Hence, we have proved that $H_2'(a) = 0$ has exactly two zero points in $(\frac{1}{2}\omega_1, q_+) \cup (q_+, \frac{1}{2}\omega_3)$, which implies that $H(a) \neq 0$ for any $a \in (\frac{1}{2}\omega_1, q_+) \cup (q_+, \frac{1}{2}\omega_3)$. In conclusion, for any $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, $(a, b_2(a))$ can not satisfy (2.13).

Next we consider $b_3(a)$. We also define

$$H_3(a) = G_{x_2}(a) + G_{x_2}(b_3(a)).$$

The difference is that $H_3(q_+) \neq 0$ since $b_3(q_+) = 2q_+ \in (0, \frac{1}{2}\omega_2) \subset i\mathbb{R}^+$. Thus, we have to show that $H_3(a)$ has only two zeros at $\frac{1}{2}\omega_1$ and $\frac{1}{2}\omega_3$, namely we need to prove $H_3'(a) = 0$ has only one zero point. The computation of $H_3'(a)$ is completely the same as $H_2'(a)$. Hence, $H_3'(a) = 0$ implies (see (3.22))

$$f_3(a) = \wp(a) + \wp(b_3(a)) = \frac{1}{2}\left(-\tilde{\eta}_1 \pm \sqrt{\tilde{\eta}_1^2 + 2g_2/3}\right) = B_\pm.$$

We note that this $B_\pm$ is the same one in (3.22). Recall from Lemma 3.8 that f_3 is strict monotone in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$ and $-e_1 \leq f_3 \leq -e_3$. Since $B_+ \geq -e_2 > -e_3$ by (3.23), it follows that $f_3(a) = B_-$ whenever $H_3'(a) = 0$. By the monotonicity of f_3, the a satisfying $f_3(a) = B_-$ is unique. Thus $H_3'(a) = 0$ has only one solution in $[\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, and then $H_3(a) \neq 0$ for any $a \in (\frac{1}{2}\omega_1, \frac{1}{2}\omega_3)$. In conclusion, for any $a \in [\frac{1}{2}\omega_1, \frac{1}{2}\omega_3]$, $(a, b_3(a))$ can not satisfy (2.13). This completes the proof of Lemma 3.6. □

Lemma 3.9. *Let (a, b) be a solution of (2.13). Then neither a nor b can be on the coordinate axes.*

Proof. Suppose that a is on the x_1 axis. Note from $a \neq -a$ that $a \notin \{0, \pm\frac{\omega_1}{2}\}$. Without loss of generality, we may assume $a > 0$, i.e. $a \in (0, \frac{\omega_1}{2})$. Then (cf. [9, Lemma 2.1])

$$G_{x_1}(a) < 0, \quad G_{x_2}(a) = 0.$$

As a result, $G_{x_2}(b) = -G_{x_2}(a) = 0$. It is known (cf. [9, Lemma 2.1]) that G satisfies

$$G_{x_2}(z) \neq 0, \quad \text{if } z \in E_\tau \setminus \left(\mathbb{R} \cup \left(\pm\tfrac{1}{2}\omega_2 + \mathbb{R}\right)\right).$$

So $b \in \mathbb{R} \cup \left(\pm\frac{1}{2}\omega_2 + \mathbb{R}\right)$. By Lemma 3.6, $b \notin \pm\frac{1}{2}\omega_2 + \mathbb{R}$. Hence, $b \in \mathbb{R}$ and $\wp'(b) = -\wp'(a) > 0$. This gives $b \in (-\frac{\omega_1}{2}, 0) = (-\frac{1}{2}, 0)$.

Note that $\lim_{z \to 0, z<0} \wp''(z) = +\infty$ and $\wp''(z) = 0$ has solutions only on ∂E_τ. So $\wp''(x_1) > 0$ for $x_1 \in (-\frac{1}{2}, 0)$. This implies that $x_1 = -a$ is the only solution of $\wp'(x_1) = -\wp'(a)$ for $x_1 \in (-\frac{1}{2}, 0)$. Thus, $b = -a$, a contradiction. The other case that a is on the x_2 axis can be proved similarly. □

Lemma 3.10. *Let (a_0, b_0) be a solution of (2.13). Then either the x_1 coordinates or the x_2 coordinates of a_0 and b_0 take the same sign.*

Proof. By Lemma 3.6, both a_0 and $b_0 \neq \pm a_0$ are in the interior of E_τ. Suppose that this lemma fails. Define

$$T_t := \left\{a : \ |x_j(a)| \leq t|x_j(a_0)|, \ j = 1, 2\right\}, \quad t > 0.$$

Here we use $x_j(z)$ to denote the j^{th} coordinate of z. We say the sign condition holds for t if for any pair $(a, b(a))$, $a \in T_t$, either $x_1(a)x_1(b) \geq 0$ or $x_2(a)x_2(b) \geq 0$, where $b(a)$ is the branch of solutions of $\wp'(a) + \wp'(b) = 0$ satisfying $b(a_0) = b_0$. By our assumption, the sign condition fails for T_1.

On the other hand, if $|z|$ is small, then $\wp'(z) = -\frac{2}{z^3} + O(|z|)$. So if t is small and $a \in T_t$, then we can deduce from $\wp'(a) + \wp'(b(a)) = 0$ and $b(a) \neq -a$ that

$$b(a) = e^{\pm\pi i/3} a(1 + O(|a|)).$$

Thus, the sign condition holds for T_t provided that t is small.

Let $t_0 \in (0, 1]$ be the smallest t so that for any small $\varepsilon > 0$, the sign condition fails for $T_{t_0+\varepsilon}$. So there is $a_\varepsilon \in T_{t_0+\varepsilon}$ such that both $x_1(a_\varepsilon)x_1(b(a_\varepsilon)) < 0$ and $x_2(a_\varepsilon)x_2(b(a_\varepsilon)) < 0$. We may assume $(a_\varepsilon, b(a_\varepsilon)) \to (\bar{a}_0, \bar{b}_0)$ as $\varepsilon \to 0$ up to a subsequence. Clearly $\wp'(\bar{a}_0) + \wp'(\bar{b}_0) = 0$ and $x_j(\bar{a}_0)x_j(\bar{b}_0) \leq 0$ for $j = 1, 2$. By the choice of t_0, $\bar{a}_0 \in \partial T_{t_0}$. Since $\wp''(z) = 0$ implies $z \in \partial E_\tau$, we have

$\wp''(-\bar{a}_0) \neq 0$, which implies that $-\bar{a}_0$ is a simple root of $\wp'(\bar{a}_0) + \wp'(b) = 0$. This, together with $b(a_\varepsilon) \neq -a_\varepsilon$, gives $\bar{b}_0 = b(\bar{a}_0) \neq -\bar{a}_0$.

To yield a contradiction, we first show that one of $\bar{a}_0$ or $\bar{b}_0$ must lie on the coordinate axis. If not, then $x_j(\bar{a}_0)x_j(\bar{b}_0) < 0$ for $j = 1, 2$. We could choose $a_\delta := (1-\delta)\bar{a}_0$, $b_\delta := b(a_\delta)$, such that $(a_\delta, b_\delta) \to (\bar{a}_0, \bar{b}_0)$ as $\delta \to 0$ and $a_\delta \in T_{(1-\delta)t_0}$ for δ small. Clearly, the sign condition fails for (a_δ, b_δ) provided δ is small, which yields a contradiction to the smallness of t_0.

Without loss of generality, we assume that one of $\bar{a}_0$ and $\bar{b}_0$ is on the imaginary axis. Since $\wp'(\bar{a}_0) + \wp'(\bar{b}_0) = 0$, we have both $\wp'(\bar{a}_0)$ and $\wp'(\bar{b}_0)$ are pure imaginary. Without loss of generality, we assume $\wp'(\bar{a}_0) = i\xi$ for some real number $\xi > 0$. We can prove the following fact about the curve $\{z : \wp'(z) \in i\mathbb{R}^+\}$. For $|z|$ small, $\wp'(z) = -\frac{2}{z^3} + O(|z|) \in i\mathbb{R}^+$ if and only if $z = re^{i\theta_i}(1 + O(r))$, where $\theta_i = \frac{\pi}{6}, \frac{5\pi}{6}$, or $\frac{3\pi}{2}$. Hence, for small δ,

$$\{|z| \leq \delta\} \cap \{z : \wp'(z) \in i\mathbb{R}^+\} \setminus i\mathbb{R}^- \subset \{z : z = (x_1, x_2), x_2 > 0\}. \tag{3.24}$$

Since $\wp'(z) \in \mathbb{R}$ for $z \in \mathbb{R}$, (3.24) implies

$$\{z : \wp'(z) \in i\mathbb{R}^+\} \setminus i\mathbb{R}^- \subset \{z : z = (x_1, x_2), x_2 > 0\}. \tag{3.25}$$

Similarly, we have

$$\{\wp'(z) : \wp'(z) \in i\mathbb{R}^-\} \setminus i\mathbb{R}^+ \subset \{z : z = (x_1, x_2), x_2 < 0\}. \tag{3.26}$$

Now we go back to $(\bar{a}_0, \bar{b}_0)$. Recall that we have assumed that one of $\bar{a}_0$ and $\bar{b}_0$ is on the imaginary axis and $\wp'(\bar{a}_0) \in i\mathbb{R}^+$. Suppose that $\bar{a}_0$ is on the imaginary axis. Since $\wp(t\omega_2)$ is increasing for $t \in (0, \frac{1}{2}]$, we have $\wp'(z) \in i\mathbb{R}^-$ for $z \in (0, \frac{1}{2}\omega_2]$. By our assumption $\wp'(\bar{a}_0) \in i\mathbb{R}^+$, we find that $\bar{a}_0 \in i\mathbb{R}^-$. Since $x_2(\bar{a}_0)x_2(\bar{b}_0) \leq 0$, we find $x_2(\bar{b}_0) > 0$. From $\wp'(\bar{b}_0) = -\wp'(\bar{a}_0) \in i\mathbb{R}^-$ and (3.26), we have $\bar{b}_0 \in i\mathbb{R}^+$. But $\wp'(\bar{b}_0) = -\wp'(\bar{a}_0) = \wp'(-\bar{a}_0)$ and both $-\bar{a}_0$ and $\bar{b}_0$ are on the line $i\mathbb{R}^+$, which implies $\bar{b}_0 = -\bar{a}_0$ because $\wp''(z) \neq 0$ for $z \in (0, \frac{1}{2}\omega_2)$. This is a contradiction. Thus we have proved that $\bar{a}_0$ is not on the imaginary axis, which implies that $\bar{b}_0$ is on the imaginary axis. Since $\wp'(\bar{b}_0) \in i\mathbb{R}^-$, we have $\bar{b}_0 \in i\mathbb{R}^+$. Then (3.25) gives

$$\bar{a}_0 \in \{z : \wp'(z) \in i\mathbb{R}^+\} \subset i\mathbb{R}^- \cup \{z : x_2 > 0\}.$$

Since $\bar{a}_0 \notin i\mathbb{R}^-$, we have $x_2(\bar{a}_0) > 0$ and then $x_2(\bar{a}_0)x_2(\bar{b}_0) > 0$. This is a contradiction to $x_2(\bar{a}_0)x_2(\bar{b}_0) \leq 0$. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We just need to prove that (2.13) has no solutions for $\tau \in i\mathbb{R}^+$, i.e. E_τ is a rectangle.

Assume by contradiction that (a, b) is a solution of (2.13). By Lemmas 3.6 and 3.9, both a and b are in the interior of E_τ, and neither a nor b is on the coordinate axes. On the other hand, it is well known (cf. [9, Lemma 2.1]) that the Green function G in the rectangle E_τ satisfies

$$\begin{aligned}
&G_{x_1}(x_1, x_2) < 0 \text{ if } x_1 \in (0, \tfrac{1}{2}) \text{ and } x_2 \in \left(-\tfrac{|\tau|}{2}, \tfrac{|\tau|}{2}\right);\\
&G_{x_1}(x_1, x_2) > 0 \text{ if } x_1 \in (-\tfrac{1}{2}, 0) \text{ and } x_2 \in \left(-\tfrac{|\tau|}{2}, \tfrac{|\tau|}{2}\right);\\
&G_{x_2}(x_1, x_2) < 0 \text{ if } x_2 \in (0, \tfrac{|\tau|}{2}) \text{ and } x_1 \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right);\\
&G_{x_2}(x_1, x_2) > 0 \text{ if } x_2 \in (-\tfrac{|\tau|}{2}, 0) \text{ and } x_1 \in \left(-\tfrac{1}{2}, \tfrac{1}{2}\right).
\end{aligned}$$

Together with Lemma 3.10, we conclude that $G_z(a) + G_z(b) \neq 0$, which yields a contradiction with (2.13). This completes the proof. □

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