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A fractional free boundary problem related to a plasma problem

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We study the solutions and the free boundary from a model that arises as a limit in a random homogenization of a fractional obstacle problem. This model also arises as a fractional analogue of a plasma problem from physics. Specifically, for a fixed bounded domain Ω we study solutions to the eigenfunction equation

$$(-\Delta)^s u = \lambda(u-\gamma)_+$$

with $u \equiv 0$ on $\partial\Omega$. Our main object of study is the free boundary $\{u = \gamma\}$.

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1. Introduction

In this paper we study an equation that arises as a limit of solutions to the fractional obstacle problem. The obstacle problem consists roughly of finding the least superharmonic solution u in a domain Ω subject to the constraint u lying above some obstacle ψ. Similarly, the fractional obstacle problem also consists of finding the least superharmonic solution u with $u \geq \psi$, but the operator is the fractional Laplacian $(-\Delta)^s$. An initial study of the fractional obstacle problem began in [19]. Optimal regularity of solutions to the fractional obstacle problem as well as regularity results of the free boundary were given in [9]. The two-phase fractional obstacle problem was studied in [3] where results were proven regarding when the positive and negative phases could touch. The fractional obstacle problem has various applications including in mathematical finance. When $s = 1/2$ the fractional obstacle problem corresponds to the scalar time-independent Signorini problem which has applications in physics and biology [18]. The work in [5] considered the fractional obstacle problem over a random collection of small balls. The authors in [5] showed that under appropriate assumptions a limiting function satisfies

$$(-\Delta)^s u = \lambda(u - \phi)_-. \tag{1.1}$$

In this paper we study (1.1) when $\phi \equiv \gamma$ a constant.

The existence of the limiting function satisfying (1.1) shown in [5] is actually an extension to the fractional values $s \in (0, 1)$ of a result in [7]. The authors in [7] showed the limit of a random homogenization of the obstacle problem converges to

$$-\Delta u = \lambda u_- + f$$

in Ω with $u = 0$ on $\partial\Omega$. The above equation bears resemblance to a problem arising from plasma physics. A mathematical model for the region inhabited by plasma in a Tokamak machine is given by the two dimensional equation

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{1}{x_1}\frac{\partial u}{\partial x_i}\right) = -u_+$$

in a bounded domain Ω where u_+ is the positive part of the function u. The region inhabited by the plasma is given by $\{u > 0\}$. Properties of solutions to this and similar problems was studied in [21] and [4]. The simplified model with solutions to

$$-\Delta u = \lambda u_+ \tag{1.2}$$

was studied in [22]. The physical applications of the simplified model (1.2) are for two dimensions; however, one may study (1.2) in higher dimensions. Of particular interest is the free boundary $\partial\{u > 0\}$, and regularity of the free boundary $\partial\{u > 0\}$ for all dimensions was studied in [17].

It is clear that the study of (1.2) is the same as the study of $\Delta u = \lambda u_-$ simply by negating the solution. It is not immediately clear, however, that for constants γ_1, γ_2 the study of the problem $(-\Delta)^s u = \lambda(u - \gamma_1)_-$ should be the same as the study of $(-\Delta)^s u = \lambda(u - \gamma_2)_+$. By means of the extension operator, see section 2, the nonlocal equation $(-\Delta)^s$ can be localized and by subtracting a constant the study of the two equations will be equivalent (under negation of both the solution and λ) to studying the local equation (2.5). In fact, the localization technique via the extension operator was utilized to locally prove the main result in [5]. The equation we study is the following: for a bounded domain Ω in $\mathbb{R}^n$ we consider solutions to the equation

$$\begin{aligned} (-\Delta)^s u &= \lambda(u-\gamma)_+ \qquad &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega \end{aligned} \tag{1.3}$$

for $0 < s < 1$ with $u \equiv 0$ on $\partial\Omega$ and γ a constant. One may consider (1.3) as the fractional analogue of the simplified plasma model given by (1.2). In this paper we are primarily interested with the local properties of solutions to (1.3). The main aim of this paper is to study properties of the free boundary $\partial\{u > \gamma\}$. Our main result is Theorem 7.1 which gives a Hausdorff dimensional bound on the set of singular points of the subset $\partial\{u < \gamma\}$ of the free boundary. In order to prove the main result, the set of points in $\partial\{u < \gamma\}$ where the gradient vanishes is classified. A modified Almgren-type frequency function aids in this classification.

1.1. Outline

The outline of the paper is as follows: In Section 2 we establish certain properties of the fractional Laplacian that will be needed in our paper. We also discuss the notion of the extension operator that allows one to “localize” the fractional Laplacian. In Section 3 we prove existence of solutions to (1.3). In Section 4 we prove interior regularity for solutions. In Section 5 we begin the study of the free boundary. We prove topological properties of the free boundary and show how they may differ from the free boundary of solutions to the original local plasma problem (1.2). In Section 6 we use an Almgren's type frequency function to classify so-called blowup solutions. The classification of blowups allows us a classification of the free boundary points. We then give a regularity result for the regular set of the free boundary. In Section 7 we define the singular set and prove a Hausdorff dimensional bound for the singular set which shows that the singular set is "small".

1.2. Notation

The notation for this paper will be as follows. Throughout the paper $2s = 1 - a$ and $-1 < a < 1$ and s will always refer to the order of the fractional Laplacian $(-\Delta)^s$. The set Ω will always be a smooth bounded domain in $\mathbb{R}^n$, and V will always be an open subset of Ω. Throughout the paper we will make use of an extension operator and will therefore often consider functions in $\mathbb{R}^{n+1}$. The variables x and y will be used to denote $(x, y) \in \mathbb{R}^{n+1}$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$. For a set $U \in \mathbb{R}^{n+1}$,

- $L^2(a, U) := \{f : f|y|^{a/2} \in L^2(U)\}$.
- $L^2(a, \partial U) := \{f : f|y|^{a/2} \in L^2(\partial U)\}$ with respect to $\mathcal{H}^n$ Hausdorff measure.
- $H^1(a, U) := \{f : f, \nabla f \in L^2(a, U)\}$.
- $U' := \{x \in \mathbb{R}^n : (x, 0) \in U\}$.
- $U^+ := \{(x, y) \in U : y > 0\}$.

We refer to $\mathbb{R}^n \times \{0\}$ as the thin space, and use the following notations for balls in $\mathbb{R}^{n+1}$

- $B_r := \{(x, y) \in \mathbb{R}^{n+1} : |(x, y)| < 1\}$.

Finally, $f_\pm$ denote the positive and negative parts of f respectively so that $f = f_+ - f_-$.

2. Fractional Laplacian

We define the fractional Laplacian through the spectral decomposition. For a bounded domain $\Omega \subseteq \mathbb{R}^n$ let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ and $\{\phi_k\}$ be the eigenvalues and corresponding orthonormalized eigenfunctions for the

Laplacian with dirichlet zero boundary data. For $f \in L^2(\Omega)$ we write

$$f = \sum_{k=1} a_k \phi_k. \tag{2.1}$$

Then the fractional Laplacian is given by

$$(-\Delta)^s f(x) = \sum_{k=1} \lambda_k^s a_k \phi_k(x). \tag{2.2}$$

We note that

$$\int_\Omega f(x)(-\Delta)^s f(x) = \int_\Omega \sum_{k=1} \lambda_k^s a_k^2 \phi_k^2(x) = \sum_{k=1} \lambda_k^s a_k^2,$$

and define the space

$$H_0^s(\Omega) := \left\{ f = \sum a_k \phi_k : \sum \lambda_k^s a_k^2 < \infty \right\}.$$

The fractional Laplacian can also be given as a Dirichlet to Neumann boundary data map by the use of an extension operator. In the case when $(-\Delta)^s$ is defined on all of $\mathbb{R}^n$ instead of on a bounded domain this equivalency was given in the paper [6]. For a bounded domain there is an analoguous extension operator [20] which we now explain. We look at the solution to the following weighted elliptic problem in an extra dimension

$$\begin{aligned} \operatorname{div}(y^a \nabla u) &= 0 \text{ in } \Omega \times \mathbb{R}^+ \\ u(x,0) &= f(x) \\ u(x,y) &= 0 \text{ for } (x,y) \in \partial\Omega \times \mathbb{R}^+ \\ u(x,y) &\to 0 \text{ as } y \to \infty \end{aligned}$$

where $a = 1 - 2s$. The fractional Laplacian is a Dirichlet to Neumann boundary data map:

$$c_a \lim_{y \to 0} y^a u_y(x,y) = (-\Delta)^s f(x). \tag{2.3}$$

where c_a is a negative constant depending on a and dimension n. For the remainder of the paper we will work with the unique extension, so that if $u \in L^2(\Omega)$, the function $u(x,y)$ is the unique a-harmonic extension of u to $\Omega \times (0,\infty)$. When referring to u as a function on Ω, we will write either $u(x,0)$ or $u(\,\cdot\,,0)$. If there is no need to reference the extension (for example in both the statement and proof of Proposition 5.2), then we simply write $u(x)$. The other exception is when integrating $u(x,0)$ on B_r' such as in Section 6.

If w is a solution to (1.3), and we subtract the constant γ from the extension to obtain $u(x,y) = w(x,y) - \gamma$, then $u(x,y)$ will satisfy

$$c_a \lim_{y\to 0} y^a u_y(x,y) = \lambda u_+(x,0) \text{ for every } x \in \Omega.$$

Absorbing the negative constant c_a into the right hand side, we have

$$\begin{aligned} \operatorname{div}(y^a \nabla u(x,y)) &= 0 && \text{in } \Omega \times (0,\infty) \\ \lim_{y\to 0} y^a u_y(x,y) &= -\lambda u_+(x,0) && \text{for every } x \in \Omega \\ u(x,y) &= -\gamma && \text{on } \partial\Omega \times (0,\infty) \\ u_y(x,y) &\le 0 && \text{in } \Omega \times (0,\infty). \end{aligned} \tag{2.4}$$

where $\lambda > 0$ is a new constant, and the last condition is proven below in Proposition 2.1. In this paper we will consider solutions to a more general equation. If $U \subseteq \mathbb{R}^{n+1}$, one may consider solutions to

$$\begin{aligned} \operatorname{div}(y^a \nabla u(x,y)) &= 0 && \text{in } U^+ \\ \lim_{y\to 0} y^a u_y(x,y) &= -\lambda u_+(x,0) && \text{for } x \in U'. \end{aligned} \tag{2.5}$$

Any solution to (2.4) is a solution to (2.5) if $U \subseteq \Omega \times (0,\infty)$ and $U' \neq \emptyset$. Many of the results in this paper will apply to the more general class of solutions to (2.5). For instance, using the extension operator, many of the results apply to solutions of

$$(-\Delta)^s_{\mathbb{R}^n} u = \lambda(u-c)_+,$$

where $(-\Delta)^s_{\mathbb{R}^n}$ is the fractional Laplacian defined on all of $\mathbb{R}^n$. Often it will be necessary to assume in addition to (2.5) the last condition in (2.4) that

$$u_y(x,y) \le 0 \text{ in } U^+. \tag{2.6}$$

We now show that (2.6) is true for the extension of a solution w to (1.3) and therefore also for the solution $u = w - \gamma$ to (2.4).

Proposition 2.1. *Let w be a solution to* (1.3). *Let $w(x,y)$ be the a-harmonic extension of w to $\Omega \times (0,\infty)$, then*

$$w_y \le 0 \textit{ in } \Omega \times (0,\infty).$$

Proof. Let w_M solve

$$\begin{cases} \operatorname{div}(y^a \nabla w_M) = 0 & \text{in } \Omega \times (0, M) \\ w_M(x,y) = 0 & \text{if } x \in \partial\Omega \\ w_M(x,M) = 0 & \text{if } x \in \Omega \\ w_M(x,0) = w(x,0) & \text{if } x \in \Omega. \end{cases}$$

We point out that $0 \leq w_M(x,y) \leq w(x,y)$ and $w_M \nearrow w$ uniformly on compact sets as $M \to \infty$. If $v = w - w_M$, then v is a-harmonic in $\Omega \times (0, M)$ and $v(x,0) = 0$. If v is reflected across the thin space by odd reflection, then v is a-harmonic in $\Omega \times (-M, M)$. From the power series representation of v (see [2]) and since $v(x,y) \geq 0$ for $y > 0$ it follows that

$$\lim_{y \to 0} y^a \partial_y v(x,y) \geq 0 \text{ for } x \in \Omega.$$

Then

$$0 \geq \lim_{y \to 0} y^a \partial_y w(x,y) \geq \lim_{y \to 0} y^a \partial_y w_M(x,y).$$

Since $y^a \partial_y w_M(x,y) \leq 0$ on $\partial\Omega \times [0, M]$ and $\Omega \times \{M\}$, then $y^a \partial_y w_M(x,y) \leq 0$ on $\partial(\Omega \times (0,M))$. Since $y^a \partial_y w_M$ is $-a$-harmonic on $\Omega \times (0,M)$, it follows from the maximum principle that $y^a \partial_y w_M < 0$ in $\Omega \times (0,M)$. Since $w_M \to w$ as $M \to \infty$ on compact subsets of $\Omega \times (0,\infty)$, it follows from uniform convergence of $-a$-harmonic functions that $y^a \partial_y w_M \to y^a \partial_y w$ uniformly on sets compactly contained in $\Omega \times (0,\infty)$, so that $y^a w_y \leq 0$ and in particular $w_y \leq 0$ in $\Omega \times (0,\infty)$. □

We will utilize the following notion of trace for the weight y^a (see [3]).

Proposition 2.2. *Let $U \subseteq \mathbb{R}^{n+1}$ an open Lipschitz domain. Then there exists two compact operators*

$$T_1 : H^1(a,U) \hookrightarrow L^2(a, \partial U)$$
$$T_2 : H^1(a,U) \hookrightarrow L^2(U'),$$

Such that

$$T_1(\psi) = \psi \mid_{\partial U}$$
$$T_2(\psi) = \psi \mid_{U'},$$

for ψ smooth on $\overline{U}$.

By utilizing rescaling and Proposition 2.2 on B_1 we obtain the following

Corollary 2.3. *Let $v \in H^1(a, B_r^+)$. Then there exists a constant $C = C(n,a)$ such that*

$$\int_{B_r'} u^2 d\mathcal{H}^n \leq C\left(r^{1-a}\int_{B_r^+} y^a|\nabla u|^2 + r^{-a}\int_{(\partial B_r)^+} y^a u^2 d\mathcal{H}^n\right) \tag{2.7}$$

$$\int_{(\partial B_r)^+} y^a u^2 d\mathcal{H}^n \leq C\left(r\int_{B_r^+} y^a|\nabla u|^2 + r^a\int_{B_1'} u^2 d\mathcal{H}^n\right) \tag{2.8}$$

Proof. We note that since $L^2(a, B_1^+), L^2(B_1'), L^2(a, (\partial B_1)^+)$ are all compactly contained in $H^1(a, B_1^+)$ (the first from the work in [16], see for instance Proposition 2.1 in [1], and the latter two from Proposition 2.2), then one may use a compactness argument as in the proof of Poincare's inequality in [11] to show there exists a constant C depending on n and s such that for any $v \in H^1(a, B_1^+)$, we have

$$\|v\|_{L^2(B_1')} + \|v\|_{L^2(a,B_1^+)} \leq C\left(\|\nabla v\|_{L^2(a,B_1^+)} + \|v\|_{L^2(a,(\partial B_1)^+)}\right) \tag{2.9}$$

$$\|v\|_{L^2(a,(\partial B_1)^+)} + \|v\|_{L^2(a,B_1^+)} \leq C\left(\|\nabla v\|_{L^2(a,B_1^+)} + \|v\|_{L^2(B_1')}\right). \tag{2.10}$$

Then (2.7, 2.8) follow from (2.9, 2.10) and scaling. □

We will also need the following Hopf type Lemma

Lemma 2.4. *Let v be non-constant and a-harmonic in B_r^+ for some $r > 0$. Assume v achieves its minimum at $(x_0, 0) \in B_r'$. Then*

$$\lim_{y \to 0} \frac{v(x_0, y)}{y^{1-a}} > 0.$$

Proof. By subtracting a constant we may assume $v(x_0, 0) = 0$ and therefore $v \geq 0$. Let w (not identically zero) be an a-harmonic function satisfying $w(x, 0) = 0$ and $0 \leq w \leq v$ on $(\partial B_r)^+$ (and hence also on B_r^+). By the Boundary Harnack Principle for a-harmonic functions stated in [8] we have for $\rho <$dist$(x_0, \partial B_r)/2$ that

$$\sup_{B_\rho(x_0,0)} \frac{w}{y^{1-a}} \leq C(r) \inf_{B_\rho(x_0,0)} \frac{w}{y^{1-a}}.$$

Then

$$\inf_{B_\rho(x_0,0)} \frac{w(x,y)}{y^{1-a}} > 0,$$

and so

$$\lim_{y\to 0}\frac{v(x_0,y)}{y^{1-a}} \geq \lim_{y\to 0}\frac{w(x_0,y)}{y^{1-a}} > 0.$$

□

Theorem 2.5. *Assume u is not identically zero and obtains a minimum (maximum) at $x_0 \in \Omega$. If $0 < s < 1$ and $(-\Delta)^s u$ is continuous in a neighborhood of x_0, then $(-\Delta)^s u < 0$ in a sufficiently small neighborhood of x_0.*

Proof. We utilize the a-harmonic extension $u(x,y)$. Now $y^a u(x,y)$ is $-a$-harmonic and satisfies

$$\lim_{y\to 0} y^a u(x,y) = c_a^{-1}(-\Delta)^s u(x,0).$$

By assumption there exists a neighborhood $V_1 \subseteq \Omega$ of x_0 in which $c_a^{-1}(-\Delta)^s u(x,0)$ is continuous. We take a smooth cut-off function ψ with values 1 in a neighborhood of x_0 and which vanishes before reaching the boundary of V_1. Then $c_a^{-1}\psi(x)(-\Delta)^s\ u(x,0)$ is a bounded continuous function on $\mathbb{R}^n$. We take $v(x,y)$ to be the $-a$-harmonic extension in $\mathbb{R}^n \times \mathbb{R}^+$ with boundary data $v(x,0) = c_a^{-1}\psi(x)(-\Delta)^s u(x,0)$. By using the Poisson kernel for a-harmonic functions [6], one may use the same basic proof as for harmonic functions (see for instance Theorem 14 in [11]) to show that $v(x,y)$ is continuous up to the boundary $\mathbb{R}^n \times \{0\}$. Then $v(x,y) - y^a u(x,y)$ is $-a$-harmonic and has zero boundary data in a neighborhood $V_2 \subseteq V_1$ containing x_0. By using odd reflection $v(x,y) - y^a u(x,y)$ is $-a$-harmonic in $V_2 \times \mathbb{R}$ and therefore continuous. Since both $v(x,y)$ and $v(x,y) - y^a u(x,y)$ are continuous up to the boundary $V_2 \times \{0\}$, it follows that $y^a u(x,y)$ is continuous up to the boundary $V_2 \times \{0\}$. From Lemma 2.4

$$0 < \lim_{y\to 0}\frac{u(x_0,y)-u(x_0,0)}{y^{1-a}} = \lim_{y\to 0} y^a u(x_0,y) = c_a^{-1}(-\Delta)^s u(x_0,0).$$

Recalling that $c_a < 0$, we have that $(-\Delta)^s u(x_0) < 0$. Since $(-\Delta)^s u(x)$ is continuous, we conclude that there exists a neighborhood V_3 of x_0 such that $(-\Delta)^s u(x) < 0$ for $x \in V_3$.

If x_0 is a maximum for u we simply apply the result to $-u$. □

We obtain as an immediate consequence a maximum principle for the fractional Laplacian.

Corollary 2.6. *Assume $u(x_0)$ is a minimum (maximum) for u in Ω. If $(-\Delta)^s u$ is continuous in a neighborhood of x_0 and $(-\Delta)^s u \geq 0$ (≤ 0), then $u \equiv 0$ in Ω.*

This last result will also be useful.

Lemma 2.7. *Let u be a solution to* $\operatorname{div}(y^a \nabla u) = 0$ *in* U^+ *with*

$$\begin{cases} u(x,0) = c & \text{on } B_r' \\ \lim_{y\to 0} y^a u_y(x,y) = 0 & \text{for } x \in B_r' \end{cases}$$

where c is a constant and $B_r^+ \subseteq U^+$. Then $u \equiv c$ on all of U^+.

Proof. This result is well known to be true for harmonic functions when $a = 0$. We assume $a \neq 0$ and let $v = u - c$. We suppose by way of contradiction that v is not identically zero. Although the exact statement of Lemma 2.7 is not contained in [2], the proof of Lemma 2.7 is a consequence of the ideas of a blowup v_r and Almgren's frequency function $N(r, v)$ used to prove a power series-type representation for a-harmonic functions in [2]. The reader is also referred to (6.1) and (6.6) in Section 6 where blowups and Almgren's frequency function are defined and used for solutions to (2.5). All of the following claims are shown in [2].

If we take a blowup v_r at $(0,0)$ to obtain v_0, then v_0 will be a-harmonic and satisfy for any $x \in \mathbb{R}^n$

$$\begin{aligned} v_0(x,0) &= 0 \\ \lim_{y\to 0} y^a \partial_y v_0(x,y) &= 0. \end{aligned}$$

Since we supposed v is not identically zero, v_0 is not identically zero and homogeneous of degree $N(0,v) > 0$. From the first boundary condition for v_0 above, we may use odd reflection across the thin space $\mathbb{R}^n \times \{0\}$ so that v_0 is a-harmonic in all of $\mathbb{R}^{n+1}$ and odd in the y variable. From Proposition 2.3.1 in [2], the degree of homogeneity satisfies $N(0,v) = k - a$ for some $k \in \mathbb{N}$. From the second boundary condition we may reflect v_0 evenly across the thin space, so that from Proposition 2.3.1 in [2], we have $N(0,v) \in \mathbb{N}$. Then $v_0 \equiv 0$. This is a contradiction, so we conclude that $v \equiv 0$ in B_r^+. Since v is real analytic away from the thin space $\mathbb{R}^n \times \{0\}$, we conclude by unique continuation that $v \equiv 0$ in U^+, so that $u \equiv c$ in U^+. □

3. Existence

In order to show that the set of solutions to (1.3) is nonempty, we give a short proof of existence of solutions to (1.3) for certain values of λ. In [22] existence for the local problem $-\Delta u = \lambda u_+$ is shown for any $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the Laplacian for the domain Ω. It would be of interest to show existence for any $\lambda > \lambda_1^s$ for this fractional problem as well. However, since the focus of this paper is on properties of the free boundary, we have chosen to only give a quick proof for some values $\lambda > 0$ to demonstrate that our class of solutions we study is nonempty.

To obtain an eigenfunction we consider minimizing the fractional energy

$$D(v) := \int_\Omega v(-\Delta)^s v \tag{3.1}$$

for $v \in H_0^s(\Omega)$ subject to the constraint

$$G(v) := \int_\Omega (v-\gamma)_+^2 = c \tag{3.2}$$

where $c, \gamma > 0$ are two fixed constants. Using the extension mentioned in Section 2, this is equivalent to minimizing

$$\iint_{\Omega\times\mathbb{R}^+} y^a |\nabla g|^2 \, dx \, dy, \tag{3.3}$$

for $g \in H^1(a, \Omega \times (0,\infty))$ with $g(x,y) = 0$ on $\partial\Omega \times (0,\infty)$ and $g(x,0)$ subject to the constraint (3.2). This is because for any function ϕ on Ω, the unique a-harmonic extension of ϕ minimizes (3.3) subject to the constraint $\phi(x) = g(x,0)$. Furthermore, we have the identity

$$\begin{aligned}\int_\Omega v(x,0)(-\Delta)^s v(x,0) \, dx &= c_a \int_\Omega v(x,0) \lim_{y\to 0} y^a v_y(x,y) \, dx \\ &= -c_a \int_{\Omega\times(0,\infty)} y^a |\nabla v(x,y)|^2 \, dx \, dy.\end{aligned}$$

We now show the existence.

Lemma 3.1. *There exists a minimizer of* (3.1) *subject to the constraint* (3.2).

Proof. Minimizing (3.1) subject to the constraint (3.2) is equivalent to minimizing

$$\iint_{\Omega\times\mathbb{R}^+} y^a |\nabla v|^2 \, dx \, dy$$

for $v \in H^1(a, \Omega \times (0,\infty))$ with $v(x,y) = 0$ for $x \in \partial\Omega$, and $v(x,0)$ satisfying (3.2). Corollary 2.3 states that $L^2(\Omega)$ is compactly contained in $H^1(a, \Omega \times \mathbb{R}^+)$. The existence of a minimizer then immediately follows. □

We note that the extension is not necessary to prove a compactness theorem. Using only the spectral decomposition there is an elementary proof using power series that if a sequence u_k, is bounded in $H_0^s(\Omega)$, then there exists u_0 and a subsequence such that $u_k \rightharpoonup u_0$ in $H_0^s(\Omega)$ and $u_k \to u_0$ in $L^2(\Omega)$.

Theorem 3.2. *For a domain Ω and fixed constant γ there exists a solution to* (1.3) *for some $\lambda > 0$.*

Proof. Since the functionals $D : H_0^s \to \mathbb{R}$ and $G : H_0^s \to \mathbb{R}$ have the Frechet derivatives

$$D'(u) = 2(-\Delta)^s u \qquad G'(u) := 2(u-\gamma)_+$$

then there exists $\lambda > 0$ such that for a minimizer of $D(u)$ subject to the constraint $G(u) = c$ satisfies

$$D'(u) = \lambda G'(u).$$

See [23] for a discussion of Lagrange Multipliers with Frechet derivatives. □

4. Interior regularity of solutions

In this section we obtain the interior regularity of solutions which will enable us to obtain regularity of the free boundary where the gradient does not vanish. Since we are dealing with an eigenvalue equation, we use a bootstrap technique. Obtaining regularity for $u(x,0)$ passes the same regularity to $\lambda u_+(x,0)$ up to Lipschitz regularity. Regularity for $\lambda u_+(x,0) = (-\Delta)^s u(x,0)$ allows us to obtain higher regularity for $u(x,0)$. To use the bootstrap technique we utilize the following Proposition from [19] which is proven for the fractional Laplacian $(-\Delta)^s_{\mathbb{R}^n}$ defined on all of $\mathbb{R}^n$.

Proposition 4.1. *Let $w = (-\Delta)^s_{\mathbb{R}^n} u$. Assume $w \in C^{0,\alpha}(\mathbb{R}^n)$ and $u \in L^\infty$ for $\alpha \in (0,1)$ and $s > 0$. Then*

$$\begin{aligned} &\text{If } \alpha + 2s \le 1 \quad \text{then } u \in C^{0,\alpha+2s}(\mathbb{R}^n), \\ &\text{If } \alpha + 2s > 1 \quad \text{then } u \in C^{1,\alpha+2s-1}(\mathbb{R}^n). \end{aligned}$$

We will use a smooth cut-off function $\psi : B_1' \mapsto [0,\infty)$ with $\psi \equiv 1$ on $B_{1/2}$ and $\psi \equiv 0$ on $B_{3/4}$. We let the variable $z = (z', z_{n+1}) \in \mathbb{R}^n \times [0,\infty)$ and define

$$\Phi(z) := c_{n,s} \int_{\mathbb{R}^n} \frac{\lambda \psi(x) u_+(x,0)}{|x-z|^{n-2s}}\, dx. \tag{4.1}$$

Proposition 4.2. *For Φ as defined in (4.1) we have the following properties*

(I) $\Phi(z)$ *is a-harmonic in* $\mathbb{R}^n \times (0,\infty)$.

(II) $u(z) - \Phi(z)$ *is a-harmonic in* $B_{1/2}$ *if reflected evenly in the y variable.*

(III) $0 \le \Phi(z) \le \Phi((z',0))$.

(IV) $\|\Phi(z',0)\|_{L^q(\mathbb{R}^n)} \le C(n,s,p,q) \|\lambda u_+(x,0)\|_{L^p(B_1')}$ *if* $1/p - 1/q < 2s/n$.

(V) $\|\Phi\|_{L^2(a,B_R^+)} \le C(R,n,s) \|\lambda u_+(x,0)\|_{L^2(B_1')}$.

(VI) $\|u - \Phi\|_{C_x^{k,\alpha}(B_{1/4})} \le C(n,s,k) \left(\|u\|_{L^2(a,B_{1/2})} + \|\lambda u_+(x,0)\|_{L^2(B_1')} \right)$.

Proof. Property (I) and (II) are from the definition of Φ. Property (III) is immediate. Property (IV) is a consequence of the Hardy-Littlewood-Sobolev inequality. We now show property (V).

$$\begin{aligned}
\|\Phi\|^2_{L^2(a,B_R^+)} &\le \int_0^R \int_{B_R'} |\Phi(z',0)|^2 dz' z_{n+1}^a dz_{n+1} && \text{by property } (III) \\
&\le C \int_0^R \int_{B_R'} |\lambda u_+(z',0)|^2 dz' z_{n+1}^a dz_{n+1} && \text{by property } (IV) \\
&= C(R,n,s) \|\lambda u_+(x,0)\|_{L^2(B_1')}.
\end{aligned}$$

Property (VI) is a consequence of property (II), Corollary 2.5 in [9], and property (V). □

Using Propositions 4.1 and 4.2 we are able to obtain the following interior regularity result.

Theorem 4.3. *Let u be a solution to* (2.5) *in* $U \subseteq \mathbb{R}^{n+1}$. *Then for any* $K \Subset U'$

If $s < 1/2$, *then* $u(x,0) \in C^{1,2s}(K)$,
If $s = 1/2$, *then* $u(x,0) \in C^{1,\alpha}(K)$ *for every* $\alpha < 1$,
If $s > 1/2$, *then* $u(x,0) \in C^{2,2s-1}(K)$.

Furthermore, there exists a constant C depending on n, s such that if $B_1 \Subset U$, then the norms of $u(x,0)$ in the above spaces on $B'_{1/2}$ are bounded by

$$C\left(\|u\|_{L^2(a,B_1^+)} + \|\lambda u_+(x,0)\|_{L^2(B_1')}\right). \tag{4.2}$$

Proof. In this proof $C(n,s,k)$ is allowed to change line by line in the proof and is a constant depending only on dimension n, s, and k which is the order of the derivative in the x direction. We take Φ as defined in (4.1), and evenly reflect $u - \Phi$ across the thin space. From property (VI) we have

$$\begin{aligned}\|u-\Phi\|_{L^p(B'_{1/4})} &\le \|u-\Phi\|_{C_x^{k,\alpha}(B_{1/4})}\\ &\le C(n,s,k)\left(\|u\|_{L^2(a,B_{1/2})} + \|\lambda u_+(x,0)\|_{L^2(B_1')}\right).\end{aligned}$$

Now using property (IV) we have

$$\|u\|_{L^p(B'_{1/4})} \le C(n,s,k)\left(\|u\|_{L^2(a,B^+_{1/2})} + \|\lambda u_+(x,0)\|_{L^2(B_1')}\right),$$

as long as $1/2 - 1/p < 2s/n$. By using a standard rescaling and covering argument we obtain

$$\|u\|_{L^p(B'_{3/4})} \le C(n,s,k)\left(\|u\|_{L^2(a,B_1^+} + \|\lambda u_+(x,0)\|_{L^2(B_1')}\right),$$

with a new constant C. Using again properties (VI) and (VI), after finitely many iterations (depending on s), we obtain

$$\|u\|_{L^\infty(B'_{3/4})} \le C(n,s,k)\left(\|u\|_{L^2(a,B_1^+} + \|\lambda u_+(x,0)\|_{L^2(B_1')}\right).$$

Since the same inequality holds for $\lambda u_+(x,0)$, then from [19] we obtain that

$$\begin{aligned}&\Phi(x,0) \in C^{0,\alpha}(\mathbb{R}^n) \quad \text{if } \alpha < 2s \le 1\\ &\Phi(x,0) \in C^{1,\alpha}(\mathbb{R}^n) \quad \text{if } \alpha < 2s-1 \text{ and } 2s > 1\end{aligned}$$

The regularity of Φ is passed to u as before. After finitely many iterations (again depending on s) we obtain the conclusion of the theorem. Lipschitz regularity is the most we can know for $u_+(x,0)$, and this is when the iteration process stops. To obtain the critical exponent $2s$ and $2s-1$ respectively when $s \neq 1/2$, we use Theorem 6.4 from [3] which shows that for $s \neq 1/2$, if both $u(x,0)$, $(-\Delta)^s u(x,0) \in L^\infty$, then on a compact subdomain $u(x,0) \in C^{0,2s}$ ($C^{1,2s-1}$) for $s < 1/2$ ($s > 1/2$).

To obtain the result for any $K \Subset \Omega$ a standard rescaling and covering argument applies. □

Remark 4.4. In general it is not true that if $\phi \in L^\infty(\Omega)$ and $(-\Delta)^{1/2}\phi \in C^{0,1}(\Omega)$, then $\phi \in C^{1,1}(\Omega)$. However, it would be interesting to answer if solutions to (1.3) are in $C^{1,1}$ when $s = 1/2$.

Remark 4.5. If $\lambda \le 1$, then by using (2.9) all of the norms in the statement of Theorem 4.3 are bounded by

$$C\left(\|\nabla v\|_{L^2(a,B_1^+)} + \|v\|_{L^2(a,(\partial B_1)^+)}\right).$$

where C is a constant depending on n and s.

Since a-harmonic functions are real analytic off the thin space, one can expect that for a solution $u(x,y)$ to (2.5), to transfer the local regularity of $u(x,0)$ to $u(x,y)$ for $y > 0$. For the purposes of this paper we will not need such a result; however, we will need at least uniform Hölder continuity up to the thin space as given in the following Theorem.

Theorem 4.6. *Let u be a solution to* (2.5) *in B_1. Then there exists $0 < \alpha < 1$ depending on s and a constant C depending on n and s such that*

$$\|u\|_{C^{0,\alpha}(\overline{B_{1/2}^+})} \le C\left(\|\nabla v\|_{L^2(a,B_1^+)} + \|v\|_{L^2(a,(\partial B_1)^+)}\right). \tag{4.3}$$

Proof. From Theorem 4.3 and Remark 4.5, $u(x,0) \in C^{0,1}(B'_{3/4})$ with the right hand side bound as given in (4.3). Then $u_+(x,0) \in C^{0,1}(B'_{3/4})$. From the Poisson kernel representation of Φ in [6], one may use the same proof as for harmonic functions to show

$$\|\Phi\|_{C^{0,\alpha_1}(B_{1/2}^+)} \le C\|\lambda\psi u_+(x,0)\|_{C^{0,1}(B'_{3/4})},$$

provided that $\alpha_1 < 1 - a$. From Proposition 2.1 in [9] which is a result from [13], there exists $0 < \alpha_2 < 1$ such that

$$\|u - \Phi\|_{C^{0,\alpha_2}(B_{1/4})} \le \|u - \Phi\|_{L^2(a,B_{1/2})}.$$

Choosing $\alpha = \min\{\alpha_1, \alpha_2\}$, combining the above two inequalities, applying (2.9), and using a standard covering and rescaling argument, we obtain (4.3). □

5. Topology of the free boundary

For solutions to (1.2), the original local plasma problem studied in [17], the free boundary is exactly

$$\partial\{u > 0\} = \partial\{u < 0\} = \{u = 0\}.$$

That these two boundaries are the same follows from the maximum and minimum principle since $\Delta u = 0$ in $\{u < 0\}$ and $\Delta u \leq 0$ everywhere. The situation is different in the nonlocal/fractional case: we cannot apply the same local minimum principle to a solution u of

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 \\ \lim_{y\to 0} y^a u_y(x,y) \leq 0 \end{cases}$$

since it is possible for $u(x,0)$ to have a local minimum at $x_0 \in \Omega$ and still satisfy the above equation at x_0. It may be possible to construct solutions to (2.5) that satisfy

$$\partial\{u(\,\cdot\,,0) < 0\} \subsetneq \partial\{u(\,\cdot\,,0) > 0\}.$$

We are mostly interested in the portion of the free boundary $\partial\{u(\,\cdot\,,0) < 0\}$. This next proposition gives an inclusion when we assume additionally (2.6).

Proposition 5.1. *Let u be a solution to* (2.5) *and assume also* (2.6)*, then*

$$\partial\{u(\,\cdot\,,0) < 0\} \subseteq \partial\{u(\,\cdot\,,0) > 0\}.$$

In particular, if w is a solution to (1.3)*, then $\partial\{w < \gamma\} \subseteq \partial\{w > \gamma\}$.*

Proof. If u is constant, then both $\partial\{u(\,\cdot\,,0) < 0\}$ and $\partial\{u(\,\cdot\,,0) > 0\}$ are empty and the inclusion is vacuously true. Assume therefore that u is not constant. Suppose there exists $(x_0,0) \in (\partial\{u(\,\cdot\,,0) < 0\} \setminus \partial\{u(\,\cdot\,,0) > 0\})$. Then there exists V an open subset compactly contained in U' and such that $x_0 \in V \subseteq \{u(\cdot\,,0) \leq 0\}$. We note that $(x_0,0)$ is a maximum for u on $\overline{V}$. From (2.6), the point $(x_0,0)$ is then also a maximum for u on $\overline{V} \times [0,\epsilon]$ for $\epsilon > 0$ and small. Now $\lim_{y\to 0} y^a u_y(x,y) = 0$ for $x \in V$. If we evenly reflect in the y-variable, then u is a-harmonic on the domain $V \times (-\epsilon,\epsilon)$ for $\epsilon > 0$. Since u achieves an interior maximum at $(x_0,0)$, it follows from the maximum principle for a-harmonic functions [12] that u is constant in $V \times (0,\epsilon)$. Since u is real analytic away from the thin space, it follows that u is constant in U^+. However, this contradicts our assumption that u is not constant. □

This next Proposition shows that if u is a solution to (1.3) and $1/2 < s < 1$, then u is strictly subharmonic, in the classical sense for the Laplacian (not fractional Laplacian) on the thin space $\mathbb{R}^n$, across the free boundary $\partial\{u > \gamma\}$. This is not true when $s = 1$. This illustrates why one cannot hope for a strong minimum principle in $\{u > \gamma\}$. The assumption $s > 1/2$ is necessary to insure that $u \in C^2$ and hence Δu is a continuous function.

Proposition 5.2. *Let u be a solution to* (1.3) *with $1/2 < s < 1$. Then there exists an open set $V \subseteq \Omega$ containing $\{u = \gamma\}$ such that*

$$\Delta u(x) > 0 \text{ for every } x \in U,$$

where Δ is the classical n-dimensional Laplacian on the thin space $\mathbb{R}^n$.

Proof. Since $s > 1/2$, from Theorem 4.3 we know u is C^2 on the thin space $\mathbb{R}^n$, so Δu exists in the classical sense. Now

$$(-\Delta)u = (-\Delta)^{1-s}(-\Delta)^s u = (-\Delta)^{1-s}\lambda(u-\gamma)_+. \tag{5.1}$$

Now $\lambda(u-\gamma)_+ \geq 0$. If $u(x_0) = \gamma$, then $\lambda(u(x_0) - \gamma) = 0$. We apply Theorem 2.5 to $\lambda(u-\gamma)_+$ to conclude that there exists an open set V of x_0 such that $(-\Delta)^{1-s}\lambda(u-\gamma)_+(x) < 0$ for any $x \in V$. Then by (5.1) we have $-\Delta u(x) < 0$ for any $x \in V$. □

In the next Lemma, we show that certain symmetries of Ω are inherited by solutions of (1.3) that arise as minimizers of (3.1) subject to the constraint (3.2).

Lemma 5.3. *Let u be a solution to* (1.3) *that arises as a minimizer of* (3.1) *subject to the constraint* (3.2)*. Assume Ω is invariant under Steiner symmetrization in the direction ν. Then the level sets of u are also invariant under Steiner symmetrization in the same coordinate direction.*

Proof. We utilize the a-harmonic extension $u(x,y)$ in $\Omega \times (0,\infty)$ with $u - \gamma$ satisfying (2.4). We pick our coordinates so that $\nu = (1, 0, \ldots, 0)$. We recall that $u(x, 0)$ minimizing (3.1) subject to the constraint (3.2) is equivalent to the extension $u(x,y)$ minimizing the energy functional (3.3) for $v \in H^1(a, \Omega \times (0,\infty))$ with $v(x,y) = 0$ for $x \in \partial\Omega$ and subject to the constraint (3.2). We also recall from the discussion immediately following (3.3) that a competitor v does not need to be a-harmonic in $\Omega \times (0,\infty)$.

The basic idea of the proof is that if u is not Steiner symmetric in the x_1 variable, then we Steiner symmetrize u to obtain a competitor u^* with less energy for (3.3) and with u^* still satsifying (3.2). (See Lemma B.4 in [1] where this idea was used for a ball.) Now since Ω is invariant under Steiner symmetrization in the x_1 variable, and since $u(x,y) = 0$ for $x \in \partial\Omega \times (0,\infty)$, if we Steiner symmetrize [15] in the x_1 direction (which is orthogonal to y) to obtain u^*, and if $0 < T_1 < T_2 < \infty$, then

$$\iint\limits_{\Omega\times[T_1,T_2]} y^a|\nabla u^*|^2 \leq \iint\limits_{\Omega\times[T_1,T_2]} y^a|\nabla u|^2. \tag{5.2}$$

Equality is achieved if and only if u is already Steiner symmetric in the x_1 direction. The case of equality holds from the result in [15] since u is a-harmonic and therefore real analytic away from $\Omega \times \{0\}$. That is why initially we restrict ourselves to $T_1 > 0$. Now we let $T_1 \to 0$ and $T_2 \to \infty$ to obtain

$$\iint\limits_{\Omega\times\mathbb{R}^+} y^a|\nabla u^*|^2 \leq \iint\limits_{\Omega\times\mathbb{R}^+} y^a|\nabla u|^2$$

If u is not already Steiner symmetric in the x_1 direction when $y = 0$, then by continuity u will not be Steiner symmetric at some time $T > 0$, and so by (5.2) the energy of u^* will be less on an interval $\Omega \times [T_1, T_2]$ and hence also on $\Omega \times \mathbb{R}^+$. Notice that the constraint (3.2) is preserved for u^*, so u^* is a valid competitor. Then if u is a minimizer of (3.1), it must be symmetric in the x_1 direction. □

This next Theorem gives a sufficient condition on the shape of the domain Ω under which $\partial\{w > \gamma\} = \partial\{w < \gamma\}$ for solutions of (1.3) that arise as minimizers. A good question would be what conditions are necessary on Ω in order to ensure $\partial\{w > \gamma\} = \partial\{w < \gamma\}$ for solutions of (1.3).

Theorem 5.4. *Let u be as in Lemma 5.3, and assume Ω is invariant under Steiner symmetrization with respect to x_i for $1 \leq i \leq n$. Then*

$$\partial\{u(\,\cdot\,,0) < \gamma\} = \partial\{u(\,\cdot\,,0) > \gamma\}.$$

Proof. According to the constraint (3.2), we have $\{u(\,\cdot\,,0) > \gamma\} \neq \emptyset$. From Lemma 5.3, the solution u is Steiner symmetric in each x_i variable. Suppose there exists $z = (z_1, z_2, \ldots, z_n, 0) \in \partial\{u(\,\cdot\,,0) > \gamma\}$ and $z \notin \partial\{u(\,\cdot\,,0) < \gamma\}$. By the symmetry of u we can assume $z_i > 0$ for each i. Also, from the symmetry of u we have

$$\frac{\partial u}{\partial x_i}(z) \leq 0.$$

If $z \notin \partial\{u(\,\cdot\,,0) < \gamma\}$, then by continuity there exists an open region $V \subseteq \{u(\,\cdot\,,0) = \gamma\}$. Then since u satisfies

$$\begin{aligned} u(x,0) &= \gamma \\ \lim_{y\to 0} y^a u_y(x,y) &= 0, \end{aligned}$$

for every $x \in V$, by Lemma 2.7, we conclude $u \equiv \gamma$ in $\Omega \times [0,\infty)$ which contradicts the fact that $u(x,0)$ satisfies the constraint (3.2). □

6. Regularity of the free boundary

In this section we look at the regularity of the free boundary. All of the results in this section except for Theorem 6.7 apply to solutions of (2.5) without assuming (2.6). Theorem 6.7 does, however, have the assumption (2.6). We now state a Lemma that will allow us to utilize Almgren's frequency function. For solutions of (2.5) Almgren's frequency function will not be monotone. However, we will prove that the limit at the origin exists and use this result to put a bound on the dimension of the singular set of free boundary points. We define

$$\begin{aligned} D(r,u) &:= \int_{B_r^+} y^a |\nabla u|^2, \quad H(r,u) := \int_{(\partial B_r)^+} y^a u^2, \\ N(r,u) &:= r\frac{D(r,u)}{H(r,u)}. \end{aligned} \tag{6.1}$$

When the function u is understood we will simply write $N(r), D(r), H(r)$. Since $r>0$, the operation $\lim_{r\to 0}$ will always be understood as the limit as r goes to 0 from the right.

Lemma 6.1. *Let u be a nonconstant solution to* (2.5) *in B_1^+, and assume $u(0,0)=0$. Then $\lim_{r\to 0} N(r)$ exists and*

$$0 < \lim_{r\to 0} N(r) < \infty$$

Proof. For solutions of (2.5), the frequency $N(r)$ will not necessarily be monotone. We therefore begin by considering the modified function

$$\tilde{N}(r) := r\frac{D(r) - \lambda \int_{B_r'} u_+^2}{H(r)}.$$

Both $N(r), \tilde{N}(r)$ are absolutely continuous and hence differentiable for almost every r. By the same computations as in [6] and using that u is a solution to (2.5) (along with the accompanying $C^{1,\alpha}$ regularity) we obtain the following Rellich-type identity

$$D'(r) = \frac{n-1+a}{r}D(r) + 2\int_{(\partial B_r)^+} y^a u_\nu^2 + \frac{\lambda}{r}\int_{B_r'} \langle x, \nabla u_+^2 \rangle. \tag{6.2}$$

We also have by routine computations

$$\begin{gathered} H'(r) = \frac{n+a}{r}H(r) + 2\int_{(\partial B_r)^+} y^a u u_\nu \\ \frac{d}{dr}\left[\int_{B_r'} u_+^2\right] = \int_{\partial B_r'} u_+^2 \\ \int_{B_r'} \langle x, \nabla u_+^2 \rangle = r\int_{\partial B_r'} u_+^2 - n\int_{B_r'} u_+^2. \end{gathered} \tag{6.3}$$

Combining (6.2) and (6.3) we have

$$\begin{aligned} \frac{d}{dr}\log \tilde{N}(r) = \frac{1}{r} &+ \frac{n-1+a}{r}\frac{D(r)}{D(r) - \lambda \int_{B_r'} u_+^2} - \frac{n+a}{r} \\ &+ 2\frac{\int_{(\partial B_r)^+} y^a u_\nu^2}{D(r) - \lambda \int_{B_r'} u_+^2} - 2\frac{\int_{(\partial B_r)^+} y^a u u_\nu}{\int_{(\partial B_r)^+} y^a u^2} \\ &+ \frac{n\lambda}{r}\frac{\int_{B_r'} u_+^2}{D(r) - \lambda \int_{B_r'} u_+^2} \end{aligned}$$

Using integration by parts and that u is a solution to (2.5) we have

$$D(r) - \lambda\int_{B_r'} u_+^2 = \int_{(\partial B_r)^+} y^a u u_\nu.$$

Then

$$\frac{d}{dr}\log \tilde{N}(r) = \frac{1-a}{r}\left[1 - \frac{D(r)}{D(r) + \lambda \int_{B_r'} u_+^2}\right] + 2\left(\frac{\int_{(\partial B_r)^+} y^a u_\nu^2}{\int_{(\partial B_r)^+} y^a u u_\nu} - \frac{\int_{(\partial B_r)^+} y^a u u_\nu}{\int_{(\partial B_r)^+} y^a u^2}\right).$$

The first term is clearly nonnegative, and the second term is nonnegative by the Cauchy-Schwarz inequality. From (2.7) and the fact that $N(r) \geq 0$ we obtain for small enough r,

$$N(r) \geq \tilde{N}(r) \geq (1 - C\lambda r^{1-a})N(r) - Cr^{1-a} \geq -Cr^{1-a}. \tag{6.4}$$

Since $\tilde{N}(r)$ is monotone and bounded below, the limit as $r \to 0$ exists, and

$$0 \leq \lim_{r\to 0} \tilde{N}(r) < \infty. \tag{6.5}$$

From (6.4) it follows that

$$\limsup_{r\to 0} N(r) \leq \tilde{N}(0) \quad \text{and} \quad \liminf_{r\to 0} N(r) \geq \tilde{N}(0),$$

and therefore $\lim_{r\to 0} N(r)$ exists and equals $\tilde{N}(0)$. To show that $N(0) > 0$, we note the following rescaling property $N(r,u) = N(1,u_r)$ where

$$u_r := \frac{u(rx,ry)}{\left(\frac{1}{r^{n+a}}\int_{(\partial B_r)^+} y^a u^2\right)^{1/2}}. \tag{6.6}$$

From the rescaling and since $N(r)$ is bounded we have

$$\|u_r\|_{L^2(a,(\partial B_1)^+)} = 1 \quad \text{and} \quad \|u_r\|_{H^1(a,B_1^+)} \leq M$$

where M is a constant depending on u. Then from Proposition 2.2 and Theorem 4.6 we have that there exists a sequence $r_k \to 0$ and a function u_0 such that $u_{r_k} \to u_0$

$$\begin{aligned} &u_{r_k} \rightharpoonup u_0 \text{ in } H^1(a, B_1^+) \\ &u_{r_k} \to u_0 \text{ in } L^2(a,(\partial B_1)^+) \text{ and } C^{0,\alpha}(\overline{B_{1/2}^+}). \end{aligned}$$

We note also that

$$\int_{B_1^+} y^a |\nabla u_0|^2 = N(0),$$

and

$$\int_{(\partial B_1)^+} y^a u_0^2 = 1.$$

so u_0 is not identically zero. Since $u_0(0) = 0$ by the uniform convergence, the gradient is not identically zero, and hence we conclude that $N(0) > 0$. □

Corollary 6.2. *Let u be a solution to (2.5) and assume $u(0,0) = 0$. Let u_r be defined as in (6.6). Then for every sequence $r_k \to 0$, there exists a subsequence and a function u_0 not identically zero so that*

$$u_{r_k} \rightharpoonup u_0 \text{ in } H^1(a, B_1^+).$$

Furthermore, $N(u,0) = N(u_0, r)$ for every $0 < r < 1$ and if we evenly reflect u_0 in y, then u_0 is a-harmonic and homogeneous of degree $N(0,u)$.

Proof. The weak convergence of a subsequence was established in the proof of Lemma 6.1. Now for $t > 0$

$$N(t, u_0) = \lim_{r \to 0} N(t, u_r) = \lim_{r \to 0} N(tr, u) = N(u, 0),$$

so $N(r, u_0) \equiv N(0)$. From the rescaling we have the property

$$\lim_{y \to 0} y^a \partial_y u_r(x,y) = -r\lambda (u_r)_+(x,0).$$

If we let $\psi \in C_0^2(B_1)$, then

$$\lim_{r_k \to 0} \int_{B_1^+} y^a \langle \nabla \psi, \nabla u_{r_k} \rangle = r_k \lambda \int_{B_1'} (u_{r_k})_+ \psi \to 0.$$

The right hand side goes to zero by utilizing Proposition 2.2. Then if we extend u evenly in the y variable, u is a-harmonic in B_1. Since $N(r, u_0) = N(0,u)$ for every $r > 0$, then u_0 is homogeneous of degree $N(0,u)$ from Theorem 6.1 in [6]. □

From Corollary 6.2 if $r_k \to 0$ and $r_j \to 0$ are two different subsequences, and $u_{r_k} \to u_1$ and $u_{r_k} \to u_2$, then u_1 and u_2 will both be homogeneous of degree $N(0,u)$. It is not immediate however that $u_1 \equiv u_2$. In order to obtain a unique blowup solution we follow the ideas in [14] and consider blowups of the form

$$u_r^{(k)} := \frac{u(rx)}{r^k}. \tag{6.7}$$

The following three results as well as the proofs are analogous to those in [14].

Lemma 6.3. *Let u be a solution to (2.5) in B_1^+ with $u(0,0) = 0$ and such that $N(0,u) = k$. Then there exists C depending on u such that*

$$\sup_{B_r^+} u \leq C r^k.$$

for $r \leq 1/2$.

Proof. From (6.3) we have

$$\frac{H'(r)}{H(r)} = \frac{n + a + 2\tilde{N}(r)}{r}, \tag{6.8}$$

so that

$$\log\left(\frac{H(1)}{H(r)}\right) \geq (n + a + 2k)\log(1/r),$$

from which it follows that

$$H(r) \leq H(1) r^{n+a+2k}.$$

Now $N(1, u_r^{(k)}) = N(r,u) \leq C$, so that $D(1, u_r^{(k)}) \leq CH(1, u_r^{(k)}) \leq CH(1,u)$. The conclusion now follows from Theorem 4.6. □

For this next Lemma we define the Weiss energy for $k > 0$ as

$$W_k(r,u) := \frac{1}{r^{n-1+a+2k}} \int_{B_r^+} |\nabla u|^2 y^a - \frac{k}{r^{n+a+2k}} \int_{(\partial B_r)^+} u^2 y^a.$$

We have the following

Lemma 6.4. *Let p_k be a homogeneous a-harmonic polynomial of degree $k \in \mathbb{N}$, and let u be as in Lemma 6.3. If we define*

$$M_k(r,u,p_k) := \frac{1}{r^{n+a+2k}} \int_{(\partial B_r)^+} (u - p_k)^2 y^a$$

then $\lim_{r \to 0} M_k(r,u,p_k)$ exists.

Proof. Now

$$W_k(r,u) = \frac{H(r)}{r^{n+a+2k}}(N(r,u)-k) \geq 0.$$

We recall that since p_k is homogeneous of degree $k \in \mathbb{N}$, then from Proposition 2.3.1 in [2], p_k is even in the y variable, so that $W_k(r,p_k)=0$. We let $w = u - p_k$. Then

$$\begin{aligned}
&r^{n+a+2k}W_k(r,u) = r^{n+a+2k}\left(W_k(r,u) - W_k(r,p_k)\right)\\
&= \frac{1}{r}\int_{B_r^+}(|\nabla w|^2 + 2\langle \nabla w, \nabla p_k\rangle)y^a - k\int_{(\partial B_r)^+}(w^2 + 2wp_k)y^a\\
&= \frac{1}{r}\int_{B_r^+}|\nabla w|^2 y^a + \int_{(\partial B_r)^+} -kw^2y^a + 2w(\langle x, \nabla p_k\rangle - kp_k)y^a\\
&= \frac{1}{r}\int_{B_r^+}|\nabla w|^2 y^a - k\int_{(\partial B_r)^+} w^2 y^a\\
&= \frac{1}{r}\int_{B_r'}\lambda u_+(u-w) + \frac{1}{r}\int_{(\partial B_r)^+} ww_\nu y^a - k\int_{(\partial B_r)^+} w^2y^a.
\end{aligned}$$

We also have with the change of variable $rz = (rz', rz_{n+1}) = (x,y)$ that

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{r^{n+a+2k}}\int_{(\partial B_r)^+} w^2y^a\right) &= \frac{d}{dr}\int_{(\partial B_1)^+}\frac{w^2(rz)z_{n+1}^a}{r^{2k}}\\
&= \int_{(\partial B_1)^+}\frac{2w(rz)(\langle rz, \nabla w(rz) - kw(rz))}{r^{2k+1}z_{n+1}^a}\\
&= \frac{2}{r^{n+2k+a}}\int_{(\partial B_r)^+} w(\langle x, \nabla w\rangle - kw)y^a.
\end{aligned}$$

Then recalling that $W_k(r,u) \geq 0$, and $|u| \leq Cr^k$ from Lemma 6.3, and p_k is homogeneous of degree k, we have

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{r^{n+a+2k}}\int_{(\partial B_r)^+} w^2y^a\right) &= 2W_k(r,u) - \frac{2}{r^{n-1+a+2k}}\int_{B_r'}\lambda u_+(u-w)\\
&\geq -2Cr^{1-a}.
\end{aligned}$$

Then $\lim_{r\to 0} M_k(r,u,p_k)$ exists. ∎

Corollary 6.5. *Let u be as in Lemma 6.3. Then there exists a constant c depending on u such that*

$$\sup_{B_r^+}|u| \geq cr^k.$$

for $r < 1$.

Proof. We claim that $\lim_{r\to 0} M_k(r,u,0) > 0$. Suppose by way of contradiction that $\lim_{r\to 0} M_k(r,u,0) = 0$. Let u_{r_j} be a blowup of type (6.6) with $u_{r_j} \to p_k$ with p_k homogeneous of degree k, a-harmonic, and not identically zero by Corollary 6.2. We now put a bound on the derivative of $M_k(r,u_\rho,p_k)$. We note that u_ρ is a solution to (2.5) with eigenvalue $\lambda\rho$. Now for $r<\rho<1$ we have

$$\begin{aligned}
\frac{\lambda\rho}{r^{n-1+a+2k}}\int_{B'_r}(u_\rho)_+^2 &= \frac{\lambda\rho^{1+a}}{r^{n-1+a+2k}}\left(\frac{\int_{B'_{r\rho}} u_+^2}{H(r\rho)}\right)\cdot\left(\frac{H(r\rho)}{H(\rho)}\right)\\
&\le \lambda\rho^{1+a} r\left(\frac{\int_{B'_{r\rho}} u_+^2}{H(r\rho)}\right) \qquad \text{from (6.8)}\\
&\le \lambda\rho^{1+a} rC(r\rho)^{-a}[N(r\rho)+1] \qquad \text{from (2.7)}\\
&\le \lambda\rho r^{1-a}C[N(1)+1] \qquad \text{for } r\rho \text{ small enough.}
\end{aligned}$$

Then recalling the derivative for $M_k(r,u_\rho,p_k)$ we have

$$\begin{aligned}
\frac{d}{dr}M_k(r,u_\rho,p_k) &= -\frac{2}{r^{n-1+a+2k}}\int_{B'_r}\lambda(u_\rho)_+p_k \\
&\ge -\frac{1}{r^{n-1+a+2k}}\left(\int_{B'_r}\lambda(u_\rho)_+^2\right)^{1/2}\left(\int_{B'_r}p_k^2\right)^{1/2}\\
&\ge -\sqrt{\lambda\rho r^{1-a}C[N(1)+1]},
\end{aligned} \tag{6.9}$$

where C also depends on C_1 with $|p_k(x)|\le C_1|x|^k$, but C is independent of ρ and r. For fixed $0<\rho<1$, by our assumption we have that $M_k(r,u_\rho^{(k)}) = M_k(r\rho,u,0)\to 0$ as $r\to 0$. Then $M_k(r,u_\rho,p_k)\to M_k(1,0,p_k)>0$ as $r\to 0$ since the polynomial p_k will dominate. For the subsequence $\rho_j\to 0$ we have that $M_k(1,u_{\rho_j},p_k)\to 0$ since $u_{\rho_j}\to p_k$ in $L^2(a,(\partial B_1)^+)$. Now from (6.9) we have that

$$\begin{aligned}
M_k(1,0,p_k) &= \lim_{r\to 0} M_k(r,u_{\rho_j},p_k)\\
&\le M_k(1,u_{\rho_j},p_k) - \int_0^1 \frac{d}{dr}M_k(r,u_{\rho_j},p_k)\,dr\\
&\le M_k(1,u_{\rho_j},p_k) + C\sqrt{\rho}.
\end{aligned}$$

Letting $\rho\to 0$ we obtain that $M_k(1,0,p_k)=0$ which is a contradiction. Then our claim that $\lim_{r\to 0} M_k(r,u,0)>0$ is true, and as in the proof of Lemma 6.3 the conclusion follows from Theorem 4.6. □

Corollary 6.6. *Let u be as in 6.3. Then there exists a unique a-harmonic polynomial p_k homogeneous of degree k such that if*

$$u_{r_j}^{(k)} := \frac{u(r_j x, r_j y)}{r_j^k}$$

then $u_{r_j}^{(k)} \rightharpoonup p_k$ in $H^1(a, B_\rho^+)$ for $\rho < 1$ and any sequence $r_j \to 0$. Furthermore,

$$u(x,y) = p_k(x,y) + o(r^k). \tag{6.10}$$

Proof. From Lemma 6.3 and Corollary 6.5 we have that there exist two constants c, C such that

$$c \le \frac{1}{r^{n+a+2k}} \int_{(\partial B_r)^+} u^2 y^a \le C \tag{6.11}$$

We choose $r_j = 2^{-j}$ and perform a blowup of type (6.7) such that $u_{r_j}^{(k)} \rightharpoonup u_0$ in $H^1(a, B_1^+)$. Since $\lim_{y\to 0} \partial_y u_0(x,y) = 0$, we may reflect u_0 evenly across the thin space. Then u_0 is a-harmonic in $\mathbb{R}^{n+1}$, even in the y variable, and satisfies (6.11) for any $r \in (0, \infty)$. Then from Lemma 2.7 in [9], u_0 is a polynomial of degree k. From the bound below in (6.11) it follows that u_0 is homogeneous of degree k. Now $\lim_{r_j \to 0} M_k(r_j, u, u_0) = 0$. But then $\lim_{r_i \to 0} M_k(r_i, u, u_0) = 0$ for any subsequence $r_i \to 0$ and

$$0 = \lim_{r_i \to 0} M_k(r_i, u, u_0) = \lim_{r_i \to 0} M_k(1, u_{r_i}^{(k)}, u_0).$$

From the uniform convergence that follows from Theorem 4.6, we obtain (6.10). □

We now show that by assuming (2.6) we may classify what types of homogeneous a-harmonic polynomials may arise as blowups.

Theorem 6.7. *Let u be a solution to (2.5) in B_1^+ with $u(x_0, 0) = 0$. Assume also $u_y \le 0$ in B_1^+. Let $u_r \to v$ be a blow-up of u at x_0. If $\nabla_x u(x_0) \neq 0$, then v is a linear function. If $\nabla_x u(x_0) = 0$ we have the following alternative, either (I): v is homogeneous of degree 2 and of the form*

$$v(x,y) = p(x) - cy^2 \tag{6.12}$$

where $p(x)$ is homogeneous of degree 2 in the x-variable and $c \ge 0$, or (II): $v_y \equiv 0$ and $\Delta v(\,\cdot\,, 0) = 0$ and u is homogeneous of degree $k \in \mathbb{N}$ with $k > 2$.

Proof. By translation in the x variable we may assume that $(x_0, 0) = (0, 0)$. If $\nabla_x u(0,0) \neq 0$, then $N(0,u) = 1$, and $u_r^{(1)} \to \nabla_x u(0,0) \cdot x$ by the uniform convergence of derivatives that follows from Theorem 4.3.

Assume now that $\nabla u(0,0) = 0$. Since $y^a \partial_y u_r^{(k)}(x,y) \leq 0$ for all (x,y), this inequality is preserved in the limit for v. Then $y^a v_y$ is $-a$-harmonic, non-positive for $y > 0$, and has zero dirichlet data when $y = 0$ (since v is even). From the Boundary Harnack Principle [9], it follows that $y^a v_y$ is comparable to y^{1+a}, or identically zero. Assume first that v_y is not identically zero. Since v is homogeneous, then $y^a v_y$ is also homogeneous, and since $y^a v_y$ is comparable to y^{1+a}, then $y^a v_y$ is homogeneous of degree $1 + a$. Consequently, v must be homogeneous of degree 2, and $v = p(x) + l(x)y - cy^2$ where $l(x)$ is a linear function of x. Since $l(x) - cy = v_y(x,y) \leq 0$ for $y > 0$ and all x, it follows that $l(x) \equiv 0$, so that v must be of the form (6.12). Thus, if v_y is not identically zero, v is in alternative (I). If $v_y \equiv 0$, then since v satisfies $\mathrm{div}(y^a \nabla v) = 0$, it follows that $\Delta_x v = 0$. Since $\nabla u(0,0) = 0$ we also have in the limit (from the interior $C^{1,\alpha}$ convergence that follows from Theorem 4.3) that $\nabla_x v(0,0) = 0$. Since v is also homogeneous it follows that v is a homogeneous harmonic polynomial of order $k \geq 2$. If $k = 2$, then v falls in alternative (I) with $c = 0$. If $k > 2$, then v falls in alternative (II). □

We now define the regular set of the free boundary. Let u be a solution to (2.5).

$$R_u := \{x : u(x,0) = 0 \text{ and } \nabla_x u(x,0) \neq 0\}.$$

From the implicit function theorem and Theorem 4.3 we have the following regularity result for R_u.

Theorem 6.8. *Let u be a solution to* (2.5)*. If $u(x_0,0) = 0$ and $(x_0,0) \in R_u$, then in a neighborhood V of x_0, $\{u(x,0) = 0\}$ is a $C^{1,\alpha}$ $(C^{2,\alpha})$ graph for $s \leq 1/2$ $(> 1/2)$ and α as in Theorem 4.3.*

It is common in free boundary problems for the free boundary to be more regular than the solution. For instance around regular points for the original local plasma problem (1.2) the free boundary is real analytic [17]. A question of interest would be to prove higher regularity of the free boundary for solutions of (2.5).

7. The singular set

In this section we consider solutions to (2.5) with the additional assumption (2.6). This is the case for instance for the unique extension of solutions to

(1.3). The main result of this section is to give a Hausdorff dimensional bound for the singular set of solutions u to (2.5) defined as

$$S_u := \{(x,0) \in \partial\{u(\,\cdot\,,0) < 0\} : \nabla_x u(x,0) = 0\}.$$

Notice that we do not consider the set $(x,0) \in (\partial\{u(\,\cdot\,,0) > 0\} \setminus \partial\{u(\,\cdot\,,0) < 0\})$. The main result of this section is

Theorem 7.1. *Let u be a solution to* (2.5) *in Ω. Then the Hausdorff dimension*

$$\mathcal{H}^{\tau}(S_u) = 0$$

for $\tau > n-2$.

To prove Theorem 7.1 we follow Federer's method of dimension reduction utilized in [10]. The Theorem is an immediate consequence of the following two Lemmas which are analogous to Lemmas 5.7 and 5.8 in [10].

Lemma 7.2. *Assume that $\mathcal{H}^{\tau}(S_v) = 0$ for all functions v which arise as blowups of type* (6.7) *of solutions to* (2.5) *also satisfying* (2.6)*. Then $\mathcal{H}^{\tau}(S_u \cap V) = 0$ for all solutions u to* (2.5) *and also satisfying* (2.6) *with $V \Subset U'$.*

Proof. We first show the following Property (P): for every $x \in S_u$ there exists $d_x > 0$ such that for all $\delta \le d_x$ any subset D of $S_u \cap B'_\delta(x)$ can be covered by a finite number of balls $B'_{r_i}(x_i)$ with $x_i \in D$ such that

$$\sum_i r_i^{\tau} \le \frac{\delta^{\tau}}{2}.$$

We show (P) using a compactness argument. Suppose by way of contradiction the conclusion is not true for a sequence $\delta_j \to 0$. By picking a subsequence if necessary we perform a blow-up $u_{\delta_j}^{(k)} \to u_0$ at x. By assumption there exists finitely many $B'_{r_i/4}(x_i)$ such that

$$S_{u_0} \subseteq \cup B'_{r_i/4} \quad \text{and} \quad \sum_i r_i^{\tau} \le \frac{1}{2}.$$

Since $u_{\delta_j}^{(k)} \to u_0$ in $C^{1,\beta}$ it follows that there exists j_0 such that if $j > j_0$, then

$$S_{u_{\delta_j}} \cap B'_1 \subseteq \cup B'_{r_i/4}.$$

Then rescaling backward u satisfies the conclusion for all large j in B'_{δ_j} and we reach a contradiction.

We now denote $D_j := \{y \in S_u : d_x \geq 1/j\}$. Fix $x_0 \in D_j$. By Property (P) we can cover $D_j \cap B'_{r_0}(x_0)$ where $r_0 = 1/j$ with a finite number of balls $B'_{r_i}(x_i)$ with $x_i \in D_j$ and

$$\sum_i r_i^\tau \leq \frac{r_0^\tau}{2}.$$

Now we repeat the same argument for each $B'_{r_i}(x_i)$ and cover it with balls $B'_{r_{ij}}(x_{ij})$ to obtain

$$\sum_j r_{ij}^\tau \leq \frac{r_i^\tau}{2}.$$

Repeating this argument m times we obtain $\mathcal{H}^\tau(D_j \cap B'_{r_0}(x_0)) = 0$. Then $\mathcal{H}^\tau(D_j) = 0$. We then let $j \to \infty$ to conclude the Lemma. □

Lemma 7.3. *Let u be a solution to* (2.5) *satisfying* (2.6) *and $x_0 \in S_u$. Let u_0 be a blowup of u of type* (6.7) *at x_0. Then*

$$\mathcal{H}^\tau(S_{u_0}) = 0 \text{ for } \tau > n-2.$$

Proof. We begin by considering the problem in dimension $n = 1$, and suppose there exists $x_0 \in S_u$. From Theorem 6.7, we have either $u_0 = bx^2 - cy^2$ for two constants b and c, or $\Delta_x u_0(x, 0) = 0$ with $u_0(x, 0)$ homogeneous of degree greater than 2. Since $n = 1$, all harmonic functions are linear, so that the latter option is not possible. Since $x_0 \in \partial\{u(x, 0) < 0\}$, it follows from (6.10) that $0 \in \partial\{u_0(x, 0) < 0\}$. Since $c \leq 0$ we have $a \geq 0$, and we obtain a contradiction since then $x_0 \notin \partial\{u(x, 0) < 0\}$. Then $S_u = \emptyset$ if $n = 1$. The result now follows by using the standard argument of Federer for homogeneous solutions. In dimension $n = 2$ if $0 \neq x_0 \in S_{u_0}$, then performing a blow-up at x_0 and using the homogeneity of u_0 we obtain a blow-up limit u_{00} in $n = 1$ with $0 \in S_{u_{00}}$ which is a contradiction. Then for $n = 2$, S_u consists of at most a single point, so that $\mathcal{H}^\tau(S_{u_0}) = 0$ for $\tau > 0$. To proceed in Federer's dimension reduction argument we claim that if for all blowup solutions in dimension n we have $\mathcal{H}^\tau(S_{u_0}) = 0$, then for all blowup solutions v_0 in dimension $n+1$ we have $\mathcal{H}^{\tau+1}(S_{v_0}) = 0$. Since v_0 is homogeneous, it is only necessary to show

$$\mathcal{H}^\tau(S_{v_0} \cap \partial(B'_1)) = 0. \tag{7.1}$$

Since we assume $\mathcal{H}^\tau(S_{u_0}) = 0$ for all blowup solutions in dimension n, we can deduce as in Lemma 7.2 that $S_{v_0} \cap \partial(B'_1)$ satisfies Property (P) as in Lemma 7.2, so that using the same argument as in Lemma 7.2 we conclude (7.1). □

We first remark that using the same arguments one can show $\mathcal{H}^{\tau}(\partial\{u(x,0)>0\})=0$ for $\tau>n-1$, so that the full free boundary is of Hausdorff dimension $n-1$.

When $\lambda=0$ the Hausdorff dimension of the singular set can be exactly $n-2$ as shown by the homogeneous solution $u(x_1,x_2,\ldots,x_n,y)=x_1^2-4x_2^2-3/(a+1)y^2$. More interesting would be to construct a solution u to (2.5) or (1.3) with $\lambda>0$ such that S_u has Hausdorff dimension $n-2$.

References

[1] M. Allen, *Separation of a lower dimensional free boundary in a two-phase problem*, Math. Res. Lett. **19** (2012), no. 5, 1055–1074.

[2] M. Allen, *Thin free boundary problems*, Ph.D. thesis, Purdue University (2013).

[3] M. Allen, E. Lindgren, and A. Petrosyan, *The two-phase fractional obstacle problem*, SIAM J. Math. Anal. **47** (2015), no. 3, 1879–1905.

[4] H. Berestycki and H. Brézis, *On a free boundary problem arising in plasma physics*, Nonlinear Anal. **4** (1980), no. 3, 415–436.

[5] L. Caffarelli and A. Mellet, *Random homogenization of fractional obstacle problems*, Netw. Heterog. Media **3** (2008), no. 3, 523–554.

[6] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[7] L. A. Caffarelli and A. Mellet, *Random homogenization of an obstacle problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 2, 375–395.

[8] L. A. Caffarelli, J.-M. Roquejoffre, and Y. Sire, *Variational problems for free boundaries for the fractional Laplacian*, J. Eur. Math. Soc. (JEMS) **12** (2010), no. 5, 1151–1179.

[9] L. A. Caffarelli, S. Salsa, and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461.

[10] D. De Silva and O. Savin, *Regularity of Lipschitz free boundaries for the thin one-phase problem*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 6, 1293–1326.

[11] L. C. Evans, Partial Differential Equations, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI (1998),ISBN 0-8218-0772-2.

[12] E. B. Fabes, C. E. Kenig, and D. Jerison, *Boundary behavior of solutions to degenerate elliptic equations*, in: Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., 577–589, Wadsworth, Belmont, CA (1983).

[13] E. B. Fabes, C. E. Kenig, and R. P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116.

[14] N. Garofalo and A. Petrosyan, *Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem*, Invent. Math. **177** (2009), no. 2, 415–461.

[15] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Vol. 1150 of Lecture Notes in Mathematics, Springer-Verlag, Berlin (1985), ISBN 3-540-15693-3.

[16] T. Kilpeläinen, *Smooth approximation in weighted Sobolev spaces*, Comment. Math. Univ. Carolin. **38** (1997), no. 1, 29–35.

[17] D. Kinderlehrer and J. Spruck, *The shape and smoothness of stable plasma configurations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **5** (1978), no. 1, 131–148.

[18] A. Petrosyan, H. Shahgholian, and N. Uraltseva, Regularity of Free Boundaries in Obstacle-Type Problems, Vol. 136 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI (2012), ISBN 978-0-8218-8794-3.

[19] L. Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

[20] P. R. Stinga and J. L. Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Comm. Partial Differential Equations **35** (2010), no. 11, 2092–2122.

[21] R. Temam, *A non-linear eigenvalue problem: the shape at equilibrium of a confined plasma*, Arch. Rational Mech. Anal. **60** (1975/76), no. 1, 51–73.

[22] R. Temam, *Remarks on a free boundary value problem arising in plasma physics*, Comm. Partial Differential Equations **2** (1977), no. 6, 563–585.

[23] E. Zeidler, Applied Functional Analysis, Vol. 109 of Applied Mathematical Sciences, Springer-Verlag, New York (1995), ISBN 0-387-94422-2. Main principles and their applications.

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