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Global generation and very ampleness for adjoint linear series

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Let X be a smooth projective variety over an algebraically closed field $\mathbb{K}$ with arbitrary characteristic. Suppose L is an ample and globally generated line bundle. By Castelnuovo–Mumford regularity, we show that $K_X \otimes L^{\otimes \dim X} \otimes A$ is globally generated and $K_X \otimes L^{\otimes(\dim X+1)} \otimes A$ is very ample, provided the line bundle A is nef but not numerically trivial. On complex projective varieties, by investigating Kawamata-Viehweg-Nadel type vanishing theorems for vector bundles, we also obtain the global generation for adjoint vector bundles. In particular, for a holomorphic submersion $f : X \to Y$ with L ample and globally generated, and A nef but not numerically trivial, we prove the global generation of $f_*(K_{X/Y})^{\otimes s} \otimes K_Y \otimes L^{\otimes \dim Y} \otimes A$ for any positive integer s.

1. Introduction

In [20], Fujita proposed the following conjecture

Conjecture. Let X be a smooth projective variety and L be an ample line bundle. Then

1) $K_X \otimes L^{\otimes(\dim X+1)}$ is globally generated;
2) $K_X \otimes L^{\otimes(\dim X+2)}$ is very ample.

Fujita's conjecture is a deceptively simple open question in classical algebraic geometry. Up to dimension 4, the global generation conjecture has been proved ([13, 31, 47]). Recently, Fei Ye and Zhixian Zhu proved it in dimension 5 ([56]). Also, there are many other "Fujita Conjecture type" theorems have been proved, and we refer the reader to [1–3, 9, 18, 19, 23, 28, 29, 32, 34, 44, 45, 48, 50–52, 55]) and the references therein.

In this paper, we prove Fujita Conjecture type theorems by using analytic methods in complex geometry when the twisted line (resp. vector) bundle is nef (resp. Nakano semi-positive).

1.1. Fujita Conjecture type theorems on projective varieties over arbitrary fields

Let $\mathbb{K}$ be an algebraically closed field with arbitrary characteristic. By using characteristic p methods, Keeler proved in [32, Theorem 1.1] the following interesting result.

Theorem 1.1 (Keeler). *Let X be a smooth projective variety over $\mathbb{K}$ with dimension n. Suppose L is an ample and globally generated line bundle, and A is an ample line bundle. Then*

1) *$K_X \otimes L^{\otimes n} \otimes A$ is globally generated;*
2) *$K_X \otimes L^{\otimes (n+1)} \otimes A$ is very ample.*

In the case $\mathbb{K} = \mathbb{C}$, Angehrn and Siu proved the very ampleness part of Theorem 1.1 by analytic methods ([1, Lemma 11.1]). In [48, Corollary 4.8], Schwede generalized the global generation part of Theorem 1.1 to the case when A is nef and big, and $ch(\mathbb{K}) = p > 0$. The first result of our paper deals with a more general case when A is nef but not numerically trivial.

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{K}$ with dimension n. Suppose L is an ample and globally generated line bundle. If A is a nef but not numerically trivial line bundle, then*

1) *$K_X \otimes L^{\otimes n} \otimes A$ is globally generated;*
2) *$K_X \otimes L^{\otimes (n+1)} \otimes A$ is very ample.*

As far as the authors know, this is the greatest generality in which Fujita Conjecture type theorem has been proved in any characteristic. Theorem 1.2 is also optimal in the sense that one can not drop the non-triviality condition on A, which can be seen from the example $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$. One may want to know the limit case when A is indeed a trivial line bundle. For the global generation part, Smith proved in [52, Theorem 2] that, this example is the only exceptional case, i.e., if $(X, L) \neq (\mathbb{P}^n, \mathcal{O}(1))$, then $K_X \otimes L^{\otimes n}$ is globally generated. Her proof relies on the "tight closure" methods in the frame of commutative algebra.

1.2. Fujita Conjecture type theorems on complex projective varieties

In this subsection, we focus on the cases in complex geometry, i.e. X is defined over the complex number field $\mathbb{C}$. At first, we obtain a general version of the global generation part in Theorem 1.2:

Theorem 1.3. *Let X be a compact Kähler manifold of dimension n and L be an ample and globally generated line bundle. Suppose $(A, e^{-2\varphi})$ be a pseudo-effective line bundle and $\mathcal{I}(\varphi)$ is the multiplier ideal sheaf. If the numerical dimension of (A, φ) is not zero, i.e.* $\mathrm{nd}(A, \varphi) \neq 0$, *then*

$$K_X \otimes L^{\otimes n} \otimes A \otimes \mathcal{I}(\varphi) \tag{1.1}$$

is globally generated.

A key ingredient in the proof of Theorem 1.3 relies on vanishing theorems for pseudo-effective line bundles ([26, Corollary 1.7], [11, Theorem 0.15], [25, Corollary 3.2] or a weaker version [5, Theorem 1.3]).

By using analytic methods, we also investigate the globally generated property for adjoint vector bundles.

Theorem 1.4. *Let X be a compact Kähler manifold of dimension n and L be an ample and globally generated line bundle. Let (E, h) be a Hermitian holomorphic vector bundle with Nakano semi-positive curvature. Suppose A is a nef but not numerically trivial line bundle, then the adjoint vector bundle*

$$K_X \otimes L^{\otimes n} \otimes (E \otimes A) \tag{1.2}$$

is globally generated.

Theorem 1.4 is derived from the following vanishing theorem for vector bundles, building on ideas in [26], [10] and [5].

Theorem 1.5. *Let X be a complex projective variety with* $\dim X = n$. *If $(A, e^{-2\varphi})$ is a pseudo-effective line bundle and (E, h) a Nakano semi-positive vector bundle, then*

$$H^q(X, K_X \otimes E \otimes A \otimes \mathcal{I}(\varphi)) = 0 \quad \text{for} \quad q > n - \mathrm{nd}(A, \varphi). \tag{1.3}$$

Remark. According to [27], it is not hard to see that Theorem 1.5 is also true on compact Kähler manifolds. Hence, there is a version of Theorem 1.4

for pseudo-effective line bundle $(A, e^{-2\varphi})$ (e.g. Theorem 4.3). For simplicity, we only formulate applications for *nef line bundles* (see Theorem 4.4 for general cases).

As an application of Theorem 1.4, we obtain the following result in pure algebraic language.

Theorem 1.6. *Let $f : X \to Y$ be a holomorphic submersion between two complex projective varieties and $\dim Y = n$. Suppose $L \to Y$ is an ample and globally generated line bundle, and $A \to Y$ is a nef but not numerically trivial line bundle. Then*

$$f_*(K_{X/Y})^{\otimes s} \otimes K_Y \otimes L^{\otimes n} \otimes A \tag{1.4}$$

is globally generated for any $s \geq 1$.

As a special case of Theorem 1.6, we obtain the following well-known result of Kollár ([33, Theorem 3.5, Theorem 3.6]):

Corollary 1.7 (Kollár). *Let $f : X \to Y$ be a holomorphic submersion between two smooth projective varieties and $\dim_{\mathbb{C}} Y = n$. Suppose $L \to Y$ is an ample and globally generated line bundle, then*

$$f_*(K_{X/Y})^{\otimes s} \otimes K_Y \otimes L^{\otimes (n+1)}$$

is globally generated for $s \geq 1$.

As another application of Theorem 1.4, we also get global generation of pluricanonical adjoint bundles of canonically polarized families.

Theorem 1.8. *Let $f : X \to S$ be a holomorphic family of canonically polarized compact Kähler manifolds effectively parameterized by a smooth projective variety S with dimension n. Suppose $L \to S$ is an ample and globally generated line bundle, and $A \to S$ is a nef line bundle. Then*

$$f_*(K_X^{\otimes s}) \otimes L^{\otimes n} \otimes A \tag{1.5}$$

is globally generated for $s > 1$.

2. Fujita Conjecture type theorems on projective varieties over arbitrary fields

In this section, we investigate Fujita Conjecture type theorems on projective varieties over algebraically closed fields and prove Theorem 1.2. Let $\mathbb{K}$ be an algebraically closed field with arbitrary characteristic, and X be a smooth projective variety over $\mathbb{K}$. Firstly we introduce the theory of Castelnuovo–Mumford regularity.

2.1. Castelnuovo–Mumford regularity

Suppose L is an ample and globally generated line bundle over X.

Definition 2.1. A coherent sheaf $\mathcal{F}$ on X is m-regular with respect to L if

$$H^q(X, \mathcal{F} \otimes L^{\otimes(m-q)}) = 0 \quad \text{for} \quad q > 0. \tag{2.1}$$

The following results are well-known (e.g. [35, Section 1.8], or [14, Section 5.2]), and for the sake of completeness we include a proof here.

Lemma 2.2 (Mumford). *Let $\mathcal{F}$ be a 0-regular coherent sheaf on X with respect to L, then $\mathcal{F}$ is generated by its global sections.*

Proof. Suppose $\dim X = n$. We shall use standard hyperplane induction method to prove it. For simplicity, we write $L = \mathcal{O}_X(1)$. Since the coherent sheaf $\mathcal{F}$ has finitely many associated points, we can choose $s \in H^0(X, L)$ such that the corresponding divisor B does not contain any associated point of $\mathcal{F}$. Hence, for any $i \geq 0$, we have the exact sequence

$$0 \to \mathcal{F}(-i-1) \stackrel{\otimes s}{\to} \mathcal{F}(-i) \to \mathcal{F}_B(-i) \to 0. \tag{2.2}$$

By using the associated long exact sequence

$$\cdots \to H^i(X, \mathcal{F}(-i)) \to H^i(X, \mathcal{F}_B(-i)) \to H^{i+1}(X, \mathcal{F}(-i-1)) \to \cdots \tag{2.3}$$

and 0-regularity of $\mathcal{F}$, we see $H^i(X, \mathcal{F}_B(-i)) = 0$ for $i > 0$, i.e. $\mathcal{F}_B$ is 0-regular with respect to L. Similarly, for any $q \geq 0$, we have

$$\cdots \to H^q(X, \mathcal{F}(-q)) \to H^q(X, \mathcal{F}(1)(-q)) \to H^q(X, \mathcal{F}_B(1)(-q)) \to \cdots . \tag{2.4}$$

We show $\mathcal{F}(1)$ is 0-regular if $\mathcal{F}$ is 0-regular. Indeed, we have $\mathcal{F}_B$ is 0-regular and by hyperplane induction hypothesis, $\mathcal{F}_B(1)$ is 0-regular. By (2.4), $\mathcal{F}(1)$ is 0-regular. Hence, we know $\mathcal{F}(k)$ is 0-regular for all $k \geq 0$. In the commutative diagram

$$\begin{tikzcd} & H^0(X,\mathcal{F}(k))\otimes H^0(X,\mathcal{O}(1)) \arrow[r, "r_k\otimes 1"] \arrow[d, "\iota_X"] & H^0(X,\mathcal{F}_B(k))\otimes H^0(X,\mathcal{O}_X(1)) \arrow[d, "\iota_B"'] \\ H^0(X,\mathcal{F}(k)) \arrow[r, "\otimes s"] & H^0(X,\mathcal{F}(k+1)) \arrow[r, "r_{k+1}"] & H^0(X,\mathcal{F}_B(k+1)), \end{tikzcd}$$

$r_k\otimes 1$ and r_{k+1} are surjective according to long exact sequence associated to (2.2). By diagram chasing, it is obvious that ι_X is surjective if and only if ι_B is surjective. Note that in the above commutative diagram we can replace B by intersections of suitable divisors in $|L|$. Hence, we can show ι_X is surjective by induction on $\dim B$. When $\dim B = 0$, it is easy to see ι_B is surjective. Hence, by induction, we know ι_X is surjective. For any $x\in X$ and large N, in the commutative diagram

$$\begin{tikzcd} H^0(X,\mathcal{F})\otimes H^0(X,\mathcal{O}(1))^{\otimes N} \arrow[r, "\iota"] \arrow[d, "1\otimes ev_1"] & H^0(X,\mathcal{F}(N)) \arrow[d, "ev_2"'] \\ H^0(X,\mathcal{F})\otimes (\mathcal{O}(1)|_x)^{\otimes N} \arrow[r, "f"] & (\mathcal{F}(N))|_x, \end{tikzcd}$$

ι is surjective since ι_X is surjective. On the other hand, $1\otimes ev_1$ and ev_2 are both surjective and so f is surjective, and we deduce $H^0(X,\mathcal{F})\to\mathcal{F}|_x$ is surjective. □

2.2. The proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we present a more general result.

Theorem 2.3. *Let X be a smooth projective variety over $\mathbb{K}$ with dimension n. Suppose L is an ample and globally generated line bundle. If A is a line bundle such that $L\otimes A$ is ample and the Kodaira-Iitaka dimension $\kappa(A^*) = -\infty$, then*

1) *$K_X\otimes L^{\otimes n}\otimes A$ is globally generated;*
2) *$K_X\otimes L^{\otimes(n+1)}\otimes A$ is very ample.*

Proof. Suppose $ch(\mathbb{K}) = p > 0$. We shall show that $F^k_*(K_X)\otimes L^{\otimes n}\otimes A$ is 0-regular for all $k\geq k_0$ where $F: X\to X$ is the absolute Frobenius morphism. When $0<q<n$, i.e. $n-q\geq 1$, $L^{\otimes(n-q)}\otimes A$ is ample since both L and $L\otimes A$ are ample. Hence, by the Serre vanishing theorem, for each q with

$0 < q < n$, there exists a positive constant $k_q = k(q) > 0$ such that

$$H^q(X, K_X \otimes L^{\otimes p^k(n-q)} \otimes A^{\otimes p^k}) = 0, \tag{2.5}$$

for $k \geq k_q$. By the projection formula, one has

$$F_*^k(K_X \otimes L^{\otimes p^k(n-q)} \otimes A^{\otimes p^k}) = F_*^k(K_X) \otimes L^{\otimes(n-q)} \otimes A. \tag{2.6}$$

Since the Frobenius morphism F is a finite morphism, we get

$$\begin{aligned} H^q(X, F_*^k(K_X) \otimes L^{\otimes(n-q)} \otimes A) &\cong H^q(X, F_*^k(K_X \otimes L^{\otimes p^k(n-q)} \otimes A^{\otimes p^k})) \\ &\cong H^q(X, K_X \otimes L^{\otimes p^k(n-q)} \otimes A^{\otimes p^k}) \\ &= 0. \end{aligned}$$

When $q = n$, we want to show

$$H^n(X, F_*^k(K_X) \otimes A) = 0. \tag{2.7}$$

By the projection formula again, we have $F_*^k(K_X) \otimes A = F_*^k\left(K_X \otimes A^{\otimes p^k}\right)$ and

$$\begin{aligned} H^n(X, F_*^k(K_X) \otimes A) &\cong H^n\left(X, F_*^k(K_X \otimes A^{\otimes p^k})\right) \\ &\cong H^n(X, K_X \otimes A^{\otimes p^k}) \\ &\cong H^0(X, (A^{\otimes p^k})^*)^*. \end{aligned}$$

Since the Kodaira-Iitaka dimension $\kappa(A^*) = -\infty$, we have $H^0(X, (A^*)^{\otimes \ell}) = 0$ for any $\ell > 0$. Hence we get (2.7). By Definition 2.1, $F_*^k(K_X) \otimes A \otimes L^{\otimes n}$ is 0-regular for all $k \geq k_0$ where $k_0 = \max\{k_1, \ldots, k_{n-1}\}$. According to Lemma 2.2, $F_*^k(K_X) \otimes L^{\otimes n} \otimes A$ is globally generated. Thanks to [32, Lemma 3.3], $K_X \otimes L^{\otimes n} \otimes A$ is a quotient of

$$F_*^k(K_X) \otimes A \otimes L^{\otimes n}$$

for all $k \geq k_0$ and so $K_X \otimes L^{\otimes n} \otimes A$ is globally generated.

It is well-known that $K_X \otimes L^{\otimes(n+1)} \otimes A$ is very ample if and only if for every $x \in X$, $\mathfrak{m}_x \otimes K_X \otimes L^{\otimes(n+1)} \otimes A$ is globally generated. Since $F_*^k(K_X) \otimes L^{\otimes n} \otimes A$ is 0-regular with respect to L, it is proved in [32] that for every $x \in X$, $\mathfrak{m}_x \otimes F_*^k(K_X) \otimes L^{\otimes(n+1)} \otimes A$ is also 0-regular and so it is globally generated. By [32, Lemma 3.3], $\mathfrak{m}_x \otimes K_X \otimes L^{\otimes(n+1)} \otimes A$ is a quotient of $\mathfrak{m}_x \otimes F_*^k(K_X) \otimes L^{\otimes(n+1)} \otimes A$. Hence $\mathfrak{m}_x \otimes K_X \otimes L^{\otimes(n+1)} \otimes A$ is globally generated.

When $ch(\mathbb{K}) = 0$, $L^{\otimes k} \otimes A$ is ample for every $k > 0$. Hence by Kodaira vanishing theorem, we have

$$H^q(X, K_X \otimes L^{\otimes(n-q)} \otimes A) = 0 \tag{2.8}$$

for $0 < q < n$. Since $\kappa(A^*) = -\infty$, we also have

$$H^n(X, K_X \otimes A) \cong H^0(X, A^*) = 0. \tag{2.9}$$

Hence, by Lemma 2.2 again, we know $K_X \otimes L^{\otimes n} \otimes A$ is 0-regular with respect to L and hence it is globally generated. The very ampleness of $K_X \otimes L^{\otimes(n+1)} \otimes A$ can be proved similarly. □

Lemma 2.4. *Let A be a nef but not numerically trivial line bundle over a smooth projective variety X over $\mathbb{K}$. Then the Kodaira-Iitaka dimension $\kappa(A^*) = -\infty$, i.e. $H^0(X, (A^*)^{\otimes \ell}) = 0$ for all $\ell > 0$.*

Proof. We also use A to denote the divisor class of A. Since A is nef but not numerically trivial, it is well-known that (e.g. [6, Section 3.8]) that there exists an ample divisor H, such that $A \cdot H^{n-1} > 0$. We show $\kappa(A^*) = -\infty$. Suppose $H^0(X, \mathcal{O}(-\ell A)) \neq 0$ for some $\ell > 0$. Then $-\ell A$ is effective. Let $-\ell A = \sum \nu_i D_i$ where D_i are irreducible divisors and $\nu_i \geq 0$. Since $-\ell A$ is not numerically trivial, there is at least one $\nu_i > 0$. By Nakai-Moishezon criterion for ampleness, for ample divisor H in X, we have $H^{n-1} \cdot (-\ell A) > 0$, i.e. $H^{n-1} \cdot A < 0$ which contradicts $A \cdot H^{n-1} > 0$. □

The proof of Theorem 1.2. It follows from Lemma 2.4 and Theorem 2.3.

Remark 2.5. Theorem 2.3 is more general than Theorem 1.2. In Theorem 2.3, A can be certain numerically trivial line bundles. It is easy to see that, if A is a non-torsion point in $Pic^0(X)$, Theorem 2.3 also works. For instance, let $E = \mathbb{C}/(\mathbb{Z} \oplus \sqrt{-1}\mathbb{Z})$ be an elliptic curve. Suppose $A = \mathcal{O}(P - Q)$ where P is a rational point and Q is an irrational point on E. Then A is numerically trivial and $\kappa(A^*) = -\infty$. Indeed, $(A^*)^{\otimes \ell}$ has no nonzero section for any $\ell > 0$. Otherwise, the divisor $\ell(P - Q)$ is linearly equivalent to the zero divisor, which is absurd.

3. Vanishing theorems for vector bundles on compact Kähler manifolds

In this section, we investigate various vanishing theorems for vector bundles, which are the key ingredients in the proof of Fujita Conjecture type theorems. In particular, we give the proof of Theorem 1.5.

Let E be a holomorphic vector bundle over a compact complex manifold X and h be a smooth Hermitian metric on E. There exists a unique connection ∇ which is compatible with the Hermitian metric h and the complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}_{i=1}^n$ be local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $\Theta^E \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has components

$$R_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}. \tag{3.1}$$

Here and henceforth we sometimes adopt the Einstein convention for summation.

Definition 3.1. A Hermitian holomorphic vector bundle (E, h) is called Nakano positive (resp. Nakano semi-positive) if

$$R_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha} \bar{u}^{j\beta} > 0 \quad (\text{resp. } \geq 0)$$

for nonzero vector $u = (u^{i\alpha}) \in \mathbb{C}^{nr}$.

Let's describe some elementary properties on positive vector bundles.

Lemma 3.2 (Nakano vanishing theorem). *Let X be a compact Kähler manifold. Suppose $(E, h) \to X$ is a Hermitian holomorphic vector bundle with Nakano positive curvature, then*

$$H^q(X, K_X \otimes E) = 0, \quad q \geq 1. \tag{3.2}$$

Lemma 3.3. *Let (X, ω_g) be a compact Kähler manifold.*

1) *Let (E, h) be a Nakano positive vector bundle and A a nef line bundle. Then $E \otimes A$ admits a Hermitian metric with Nakano positive curvature.*

2) *Let (E, h) be a Nakano semi-positive vector bundle and A be an ample line bundle. Then $E \otimes A$ admits a Hermitian metric with Nakano positive curvature.*

3) *Let* (E, h^E) *and* $(\widetilde{E}, h^{\widetilde{E}})$ *be two Nakano semi-positive vector bundles, then* $(E \otimes \widetilde{E}, h \otimes \widetilde{h})$ *is also Nakano semi-positive.*

Proof. (1). For the fixed Kähler metric ω_g on X, there exists a constant $\varepsilon > 0$ such that

$$\sqrt{-1}\Theta^E(u(x), u(x)) \geq 2\varepsilon |u(x)|^2_{g\otimes h}$$

for any $u \in \Gamma(X, T^{1,0}X \otimes E)$. On the other hand, by analytic definition of nefness (e.g. [10]), there exists a smooth metric h_0 on the nef line bundle A such that

$$\sqrt{-1}\Theta^A \geq -\varepsilon\omega_g. \tag{3.3}$$

The curvature of $h \otimes h_0$ on $E \otimes A$ is $\Theta^{E\otimes A} = \Theta^E \cdot id_A + id_E \cdot \Theta^A$. Hence, for any $u \in \Gamma(X, T^{1,0}X \otimes E)$ and $v \in \Gamma(X, A)$

$$\begin{aligned} \sqrt{-1}\Theta^{E\otimes A}(u \otimes v, u \otimes v) &\geq \left(\sqrt{-1}\Theta^E(u, u) - \varepsilon |u|^2_{g\otimes h}\right) |v|^2_{h_0} \\ &\geq \varepsilon |u|^2_{g\otimes h} |v|^2_{h_0}. \end{aligned} \tag{3.4}$$

Therefore, $E \otimes A$ is Nakano positive. The proof of (2) is similar to that of (1).

(3). By using curvature formula of $h \otimes \widetilde{h}$ on $E \otimes \widetilde{E}$,

$$\Theta^{E\otimes A} = \Theta^E \cdot id_{\widetilde{E}} + id_E \cdot \Theta^{\widetilde{E}},$$

for any (local) vectors $u \in \Gamma(X, T^{1,0}X \otimes E \otimes \widetilde{E})$ with the form $u = u^{i\alpha A} dz^i \otimes e^\alpha \otimes e^A$ in the local holomorphic frames $\{z^i, e^\alpha, e^A\}$ of $\{X, E, \widetilde{E}\}$, we obtain

$$\Theta^{E\otimes\widetilde{E}}(u, u) = \Theta^E_{i\bar{j}\alpha\bar{\beta}} u^{i\alpha A} \overline{u^{j\beta B}} \cdot h^{\widetilde{E}}_{A\overline{B}} + \Theta^{\widetilde{E}}_{i\bar{j}A\overline{B}} u^{i\alpha A} \overline{u^{j\beta B}} \cdot h^E_{\alpha\bar{\beta}}. \tag{3.5}$$

It is nonnegative and one can see that by choosing normal coordinates for h^E and $h^{\widetilde{E}}$ at a fixed point. □

We need the following fundamental result in [12, Proposition 1.16].

Lemma 3.4. *Let* $E \to X$ *be a nef vector bundle over a compact complex manifold* X*. Suppose* $\sigma \in H^0(X, E^*)$ *is a nonzero section, then* σ *does not vanish anywhere.*

By refining the Bochner technique, we obtain the following vanishing theorem for vector bundles with “degenerate” curvature tensors.

Theorem 3.5. *Let (X, ω) be a compact Kähler manifold of dimension n. Let (E, h) be a holomorphic vector bundle and A be a line bundle. Suppose either*

1) *E is nef and the second Ricci curvature $tr_\omega \Theta^E$ is semi-positive, and A is semi-ample but non-trivial; or*
2) *the second Ricci curvature $tr_\omega \Theta^E$ is strictly positive and A is nef.*

Then

$$H^0(X, E^* \otimes A^*) = 0. \tag{3.6}$$

Proof. (1). Since A is semi-ample, A^k is generated by its global sections for large k. Hence, there is an induced smooth Hermitian metric h^A on A such that the curvature Θ^A is semi-positive, i.e.

$$\sqrt{-1}\Theta^A = -\sqrt{-1}\partial\bar{\partial} \log h^A \geq 0.$$

On the other hand, since A is not trivial, for the fixed Kähler metric ω on X, the scalar curvature function

$$tr_\omega \Theta^A$$

is non-negative, but not identically zero. Indeed, if it is identically zero, we deduce that Θ^A is identically zero, and so A is trivial. The curvature tensor of $E \otimes A$ can be written as

$$\Theta^{E\otimes A} = \Theta^E \otimes id_A + id_E \otimes \Theta^A.$$

In order to prove (3.5), we argue by contradiction. Suppose $H^0(X, E^* \otimes A^*) \neq 0$, i.e. there exists a nonzero section $\sigma \in H^0(X, E^* \otimes A^*)$. By Lemma 3.4, σ is nowhere vanishing. By using standard Bochner identity over $E^* \otimes A^*$,

$$\Delta_{\bar{\partial}} = \Delta_\partial + [\sqrt{-1}\Theta^{E^*\otimes A^*}, \Lambda_\omega], \tag{3.7}$$

we obtain

$$0 = \|\nabla\sigma\|^2 - (tr_\omega \Theta^{E^*\otimes A^*}\sigma, \sigma). \tag{3.8}$$

Note that $\Theta^{E^*} = -(\Theta^E)^T$ as $(1,1)$-form valued $r \times r$ matrices. Hence,

$$0 = \|\nabla\sigma\|^2 + \left(\left(\left[tr_\omega \Theta^E\right]^T \otimes id_A + id_E \otimes tr_\omega \Theta^A\right)\sigma, \sigma\right). \tag{3.9}$$

By using local holomorphic frames $\{z^i\}$, $\{e_1, \ldots, e_r\}$, $\{e\}$ of X, E and A respectively, we write

$$\begin{aligned}
\omega &= \sqrt{-1} g_{i\overline{j}} dz^i \wedge d\overline{z}^j, \\
\Theta^E &= R^{\beta}_{i\overline{j}\alpha} dz^i \wedge d\overline{z}^j \otimes e^\alpha \otimes e_\beta, \\
\Theta^A &= R^A_{i\overline{j}} dz^i \wedge d\overline{z}^j, \\
\sigma &= \sigma^\alpha e_\alpha \otimes e.
\end{aligned}$$

We obtain

$$\int_X h^A \left(g^{i\overline{j}} R_{i\overline{j}\beta\overline{\alpha}} \sigma^\alpha \overline{\sigma}^\beta + g^{i\overline{j}} R^A_{i\overline{j}} h_{\alpha\overline{\beta}} \sigma^\alpha \overline{\sigma}^\beta \right) \omega^n = 0. \tag{3.10}$$

By assumption, the (transposed) second Ricci curvature

$$\left[tr_\omega \Theta^E \right]^T = \left(\sum_{i,j} g^{i\overline{j}} R_{i\overline{j}\beta\overline{\alpha}} \right)$$

is Hermitian semi-positive as a $(r \times r)$ matrix and so

$$g^{i\overline{j}} R_{i\overline{j}\beta\overline{\alpha}} \sigma^\alpha \overline{\sigma}^\beta + g^{i\overline{j}} R^A_{i\overline{j}} h_{\alpha\overline{\beta}} \sigma^\alpha \overline{\sigma}^\beta \equiv 0. \tag{3.11}$$

It implies

$$g^{i\overline{j}} R^A_{i\overline{j}} (h_{\alpha\overline{\beta}} \sigma^\alpha \overline{\sigma}^\beta) \equiv 0.$$

Since σ is nowhere vanishing, we obtain $h_{\alpha\overline{\beta}} \sigma^\alpha \overline{\sigma}^\beta > 0$ at each point. Therefore

$$tr_\omega \Theta^A = g^{i\overline{j}} R^A_{i\overline{j}} \equiv 0.$$

This is a contradiction.

(2). For nef A, we use similar ideas as described in the first part of Lemma 3.3. Since $tr_\omega \Theta^E$ is strictly positive, there exists $\varepsilon > 0$ such that

$$(tr_\omega \Theta^E)^T (u(x), u(x)) \geq (n+1)\varepsilon |u(x)|^2_h$$

for any $u \in \Gamma(X, E)$ since X is compact. On the other hand, since A is nef, there exists a smooth metric h_0 on the nef line bundle A such that

$\sqrt{-1}\Theta^A \geq -\varepsilon\omega_g$. Hence, for any $u \in \Gamma(X, E)$ and $v \in \Gamma(X, A)$

$$\begin{aligned}
&\left(\left[tr_\omega\Theta^E\right]^T \otimes id_A + id_E \otimes tr_\omega\Theta^A\right)(u \otimes v, u \otimes v) \\
&\geq \left((tr_\omega\Theta^E)^T(u,u) - n\varepsilon|u|_h^2\right)|v|_{h_0}^2 \\
&\geq \varepsilon|u|_h^2|v|_{h_0}^2.
\end{aligned}$$

Therefore in (3.11), $\sigma \equiv 0$, i.e. $H^0(X, E^* \otimes A^*) = 0$. □

Remark 3.6. The semi-positivity of the second Ricci curvature $tr_\omega\Theta^E$ can be replaced by the semi-stability of E with respect to ω following Donaldson-Uhlenbeck-Yau's theorem. Moreover, the Griffiths (or Nakano, or dual Nakano) semi-positivity of E can also imply the semi-positivity of the second Ricci curvature $tr_\omega\Theta^E$.

The following Kawamata-Viehweg-Nadel type vanishing theorem for a semi-positive vector bundle twisted by a big line bundle is essentially known to experts (e.g., [7, Theorem 4.2.4], [10, Theorem 5.11], [15, Theorem 1.2], [16, Theorem 1.1], [46, Theorem 1.1], [17], [39]), although the statement is not written down precisely. For the sake of completeness, we include a short sketch here, following the approach in [10, Theorem 5.11] for line bundles.

Lemma 3.7. *Let (X, ω) be a Kähler weakly pseudo-convex manifold, and let A be a holomorphic line bundle over X equipped with a (possibly) singular Hermitian metric $h = e^{-2\varphi}$. Assume that*

$$\sqrt{-1}\Theta^A \geq \varepsilon\omega$$

for some continuous positive function ε on X. If (E, h^E) is a Nakano semi-positive vector bundle, then

$$H^q(X, K_X \otimes E \otimes A \otimes \mathcal{I}(\varphi)) = 0 \tag{3.12}$$

for all $q \geq 1$.

Proof. Let $\mathscr{L}^q$ be the sheaf of germs of (n,q)-forms u with values in $E \otimes A$ and with measurable coefficients such that both $|u|^2_{h^E} \cdot e^{-2\varphi}$ and $|\overline{\partial}^{E\otimes A}u|_{g\otimes h^E} \cdot e^{-2\varphi}$ are locally integrable. The $\overline{\partial}^{E\otimes A}$ operator defines a complex of sheaves $(\mathscr{L}^\bullet, \overline{\partial}^{E\otimes A})$ which is a fine resolution of the sheaf $\mathcal{O}(K_X \otimes$

$E \otimes A) \otimes \mathcal{I}(\varphi)$, i.e. we have the following exact sequence

$$0 \to \mathcal{O}(K_X \otimes E \otimes A) \otimes \mathcal{I}(\varphi) \to \mathscr{L}^0 \to \mathscr{L}^1 \to \cdots \to \mathscr{L}^n \to 0. \tag{3.13}$$

Indeed, it follows from a vector bundle version of Hörmander L^2-estimate ([10, Corollary 5.3]) since the vector bundle $E \otimes A$ has a singular metric which is Nakano positive in the sense of current. By using the L^2 estimate again (e.g. [10, Theorem 5.11]), one can show $H^q(\Gamma(X, \mathscr{L}^\bullet)) = 0$ for $q \geq 1$ and we obtain the desired vanishing cohomologies. □

Next, we introduce two different concepts on numerical dimension for nef and pseudo-effective line bundles.

Definition 3.8. Let N be a *nef line bundle* over a compact Kähler manifold X with $\dim_{\mathbb{C}} X = n$. The numerical dimension $\nu(N)$ of N is defined as ([10, Definition 6.20])

$$\nu(N) = \max\{k = 0, \ldots, n \mid c_1^k(N) \neq 0 \in H^{2k}(X, \mathbb{R}).\} \tag{3.14}$$

Definition 3.9. Let $(A, e^{-2\varphi})$ be a *pseudo-effective line bundle* over a compact Kähler manifold X of dimension n. The *numerical dimension* $\mathrm{nd}(A, \varphi)$ of A is defined in [5, Definition 3.1] as the largest number such that the cohomological product $\langle(\sqrt{-1}\partial\bar{\partial}\varphi)^k\rangle \neq 0$.

Note that, in general these two definitions do not coincide even for nef line bundles ([5, Remark 7], or [11, Remark 4.3.5]). For a discussion of the relationship between various definitions of numerical dimensions, we refer to the paper [11, Section 4.3].

Based on their solution to Demailly's strong openness conjecture ([25]), Guan-Zhou achieved in [26, Corollary 1.7] (see also [25, Corollary 3.2]) a celebrated Kawamata-Viehweg-Nadel type vanishing theorem which generalizes a theorem in [5, Theorem 1.3] (see also Demailly's survey paper [11] for a variety of vanishing theorems).

Lemma 3.10. *Let $(A, e^{-2\varphi})$ be a pseudo-effective line bundle over a compact Kähler manifold X of dimension n. Then for any $q > n - \mathrm{nd}(A, \varphi)$,*

$$H^q(X, K_X \otimes A \otimes \mathcal{I}(\varphi)) = 0. \tag{3.15}$$

We shall use the ideas in the proof of Lemma 3.10 (e.g. [5] and [25]) and Lemma 3.7 to prove Theorem 1.5.

The proof of Theorem 1.5. Suppose $\mathrm{nd}(A,\varphi)=n$. By using Guan-Zhou's solution to Demailly's strong openness conjecture ([25]) and the construction in [5, Lemma 5.5], there exists a singular metric $e^{-2\varphi_1}$ on the pseudo-effective line bundle A such that

$$\mathcal{I}(\varphi_1)=\mathcal{I}(\varphi)$$

and it is curvature current

$$\sqrt{-1}\partial\overline{\partial}\varphi_1>c\omega$$

for some smooth positive $(1,1)$ form ω and constant $c>0$. By applying Lemma 3.7 to $(X,E,A,\mathcal{I}(\varphi_1))$, we get

$$H^q(X,K_X\otimes E\otimes A\otimes\mathcal{I}(\varphi))=0,\quad\text{for}\quad q>0. \tag{3.16}$$

Now we assume that $\mathrm{nd}(A,\varphi)<n$. We use similar ideas as in [10, Theorem 6.25], [5, Proposition 5.6]. Let B be a very ample divisor such that $B\otimes A$ is ample and $\mathcal{I}(\varphi|_B)=\mathcal{I}(\varphi)|_B$ ([17, Theorem 1.10]). We consider the exact sequence

$$0\to\mathcal{O}_X(-B)\to\mathcal{O}_X\to(i_B)_*\mathcal{O}_B\to 0.$$

By tensoring with $K_X\otimes E\otimes A\otimes B\otimes\mathcal{I}(\varphi)$ and using adjunction formula, one gets

$$\begin{aligned}\cdots&\to H^q(X,K_X\otimes E\otimes A\otimes\mathcal{I}(\varphi))\\&\to H^q(X,K_X\otimes E\otimes A\otimes B\otimes\mathcal{I}(\varphi))\\&\to H^q(B,K_B\otimes(E\otimes A)|_B\otimes\mathcal{I}(\varphi|_B))\\&\to H^{q+1}(X,K_X\otimes E\otimes A\otimes\mathcal{I}(\varphi))\to\cdots.\end{aligned}$$

Since E is Nakano semi-positive, and $A\otimes B$ is ample, by Lemma 3.7, we have

$$H^q(X,K_X\otimes E\otimes A\otimes B\otimes\mathcal{I}(\varphi))=0\quad\text{for}\quad q>0. \tag{3.17}$$

Therefore

$$H^q(B,K_B\otimes(E\otimes A)|_B\otimes\mathcal{I}(\varphi|_B))\cong H^{q+1}(X,K_X\otimes E\otimes A\otimes\mathcal{I}(\varphi)) \tag{3.18}$$

for every $0<q<n$. Moreover, $E|_B$ is also Nakano semi-positive over B. Hence the induction hypothesis implies that the cohomology group on B on the right hand side of (3.18) is zero when $q>n-\mathrm{nd}(A,\varphi)$. $\square$

Similarly, we have the following variant of Theorem 1.5 (see also [10, Theorem 6.17] for the line bundle case.)

Proposition 3.11. *Let X be a smooth projective variety. Let A be a line bundle over X such that some positive multiple mA can be written as $mA = N + D$ where N is a nef line bundle and D is an effective divisor. If (E,h) is a Nakano semi-positive vector bundle, then*

$$H^q(X, K_X \otimes E \otimes A \otimes \mathcal{I}(m^{-1}D)) = 0 \quad \textit{for} \quad q > n - \nu(N), \tag{3.19}$$

where $\nu(N)$ is the numerical dimension of the nef line bundle N.

As a special case, one has

Corollary 3.12. *If (E, h^E) is a Nakano semi-positive vector bundle and A is a nef line bundle, then*

$$H^q(X, E^* \otimes A^*) = 0 \quad \textit{if} \quad q < \nu(A). \tag{3.20}$$

In particular, if in addition, A is not numerically trivial, then

$$H^0(X, E^* \otimes A^*) = 0. \tag{3.21}$$

Proof. We can take $m = 1$, $D = 0$ and $A = N$ in Proposition 3.11. By Serre duality, we obtain (3.20). Hence, when $\nu(A) \geq 1$, or equivalently, $\nu(A) \neq 0$, $H^0(X, E^* \otimes A^*) = 0$. It is well-known that for a nef line bundle A, $\nu(A) = 0$ if and only if A is numerically trivial. $\square$

As an application of Theorem 1.5, we obtain general vanishing theorems on the relative setting which generalize Kollár's vanishing theorems.

Theorem 3.13. *Let $f : X \to Y$ be a holomorphic submersion between two smooth complex projective varieties and $\dim_{\mathbb{C}} Y = n$. Let $(A, e^{-2\varphi})$ be a pseudo-effective line bundle over X. If both E_1 and E_2 are Nakano semi-positive vector bundles, then for any $s \geq 1$ and $q > n - \mathrm{nd}(A,\varphi)$, we have*

$$H^q\left(Y, f_*(K_{X/Y} \otimes E_1)^{\otimes s} \otimes K_Y \otimes E_2 \otimes A \otimes \mathcal{I}(\varphi)\right) = 0,$$

as long as $f_*(K_{X/Y} \otimes E_1)$ is locally free.

Proof. By using Theorem 1.5, we only need to show that

$$f_*(K_{X/Y} \otimes E_1)^{\otimes s} \otimes E_2$$

is Nakano semi-positive for any $s \geq 1$. Indeed by a result of [41] (see also [38]), if $f_*(K_{X/Y} \otimes E_1)$ is locally free, then $f_*(K_{X/Y} \otimes E_1)$ has a Nakano semi-positive metric. By part (3) of Lemma 3.3, we deduce

$$f_*(K_{X/Y} \otimes E_1)^{\otimes s} \otimes E_2$$

is Nakano semi-positive for any $s \geq 1$. □

Remark 3.14. Theorem 3.13 also holds when X and Y are compact Kähler manifolds.

4. Fujita Conjecture type theorems on complex projective varieties

In this section, we derive Fujita Conjecture type theorems on complex projective varieties and prove Theorem 1.3, Theorem 1.4, Theorem 1.6 and Theorem 1.8.

The proof of Theorem 1.3. We show the coherent sheaf $K_X \otimes L^{\otimes n} \otimes A \otimes \mathcal{I}(\varphi)$ is 0-regular, and so by Lemma 2.2, it is globally generated. When $0 < q < n$, $L^{\otimes(n-q)} \otimes A$ is indeed a big line bundle, and we can apply Nadel vanishing theorem ([42] or [11, Theorem 0.3] or Lemma 3.10),

$$H^q(X, K_X \otimes L^{\otimes(n-q)} \otimes A \otimes \mathcal{I}(\varphi)) = 0. \tag{4.1}$$

When $q = n$, we need to show

$$H^n(X, K_X \otimes A \otimes \mathcal{I}(\varphi)) = 0, \tag{4.2}$$

which follows from Lemma 3.10 and the assumption $\mathrm{nd}(A, \varphi) \geq 1$. □

The proof of Theorem 1.4. By Castelnuovo-Mumford regularity (e.g. Lemma 2.2), we only need to prove $K_X \otimes L^{\otimes n} \otimes (E \otimes A)$ is 0-regular with respect to L. Hence, it suffices to show

$$H^q(X, K_X \otimes L^{\otimes(n-q)} \otimes (E \otimes A)) = 0 \quad \text{for all} \quad q > 0. \tag{4.3}$$

For $0 < q < n$, we claim that the vector bundle $L^{\otimes(n-q)} \otimes (E \otimes A)$ has a smooth metric with strictly positive curvature in the sense of Nakano, and by Nakano vanishing theorem(Lemma 3.2), we have the desired vanishing cohomologies. Indeed, since $n-q \geq 1$, $L^{\otimes(n-q)} \otimes A$ is ample. By Lemma 3.3, $E \otimes L^{\otimes(n-q)} \otimes A$ is strictly positive in the sense of Nakano.

When $q = n$, we need to show $H^n(X, K_X \otimes E \otimes A) = 0$ or equivalently $H^0(X, E^* \otimes A^*) = 0$ when E is Nakano semi-positive and A is nef but not numerically trivial. This is assured by Corollary 3.12.

The case when $q > n$ is obvious and we complete the proof of Theorem 1.4. □

We have the following variant of Theorem 1.4.

Theorem 4.1. *Let X be a compact Kähler manifold and L be an ample and globally generated line bundle. Let (E, h) be a Hermitian holomorphic vector bundle with Nakano positive curvature. Suppose A to be a nef line bundle, then the vector bundle*

$$K_X \otimes L^{\otimes n} \otimes E \otimes A \tag{4.4}$$

is globally generated.

Proof. We use similar ideas as described in the proof of Theorem 1.4. Suppose E is Nakano positive and A is nef, then by Lemma 3.3, the vector bundle $E \otimes A$ admits a smooth Hermitian metric whose curvature is strictly positive in the sense of Nakano. In particular, the second Ricci curvature $tr_\omega \Theta^{E\otimes A}$ is strictly positive. We obtain the vanishing cohomology in Theorem 3.5. Finally, one can follow the steps in the proof of Theorem 1.4. □

It is proved in [4, Theorem 1.2] that if E is a semi-ample (resp. ample) vector bundle, then $E \otimes \det E$ is Nakano semi-positive (resp. Nakano positive). Hence, by Theorem 1.4 and Theorem 4.1, we get

Corollary 4.2. *Let X be a smooth complex projective variety and L be an ample and globally generated line bundle. Let E be a vector bundle and A be a line bundle. Suppose either*

1) *E is semi-ample, A is nef but not numerically trivial; or*
2) *E is ample and A is nef.*

Then the vector bundle

$$K_X \otimes L^{\otimes n} \otimes (E \otimes \det E) \otimes A \tag{4.5}$$

is globally generated.

Thanks to Theorem 1.5, one can also get the following variant of Theorem 1.4.

Theorem 4.3. *Let (X,ω) be a compact Kähler manifold and $L \to X$ be an ample and globally generated line bundle. Suppose $(A, e^{-2\varphi})$ is a pseudo-effective line bundle and $\mathcal{I}(\varphi)$ is the multiplier ideal sheaf. Let E be a Nakano semi-positive vector bundle. If the numerical dimension* $\mathrm{nd}(A,\varphi) \neq 0$*, then*

$$K_X \otimes L^{\otimes n} \otimes E \otimes A \otimes \mathcal{I}(\varphi)$$

is globally generated.

The proof of Theorem 1.6. By the assumption, the direct image sheaf of the relative canonical line bundle $f_*(K_{X/Y})$ is indeed a holomorphic vector bundle. In the literatures, it is known that the vector bundle $f_*(K_{X/Y})$ is weakly positive in suitable sense (e.g. [54, Theorem III], [24, Corollary 5], [33, Corollary 3.7]). Here, we use a recent fact [4, Theorem 1.2] of Berndtsson that $f_*(K_{X/Y})$ is actually semi-positive in the sense of Nakano. Let $E = f_*(K_{X/Y})^{\otimes s}$ and so

$$f_*(K_{X/Y})^{\otimes s} \otimes K_Y \otimes L^{\otimes n} \otimes A = K_Y \otimes L^{\otimes n} \otimes (E \otimes A). \tag{4.6}$$

According to Lemma 3.3, E is also semi-positive in the sense of Nakano. By Theorem 1.4, $K_Y \otimes L^{\otimes n} \otimes (E \otimes A)$ is globally generated as long as A is nef but not numerically trivial. The proof of Theorem 1.6 is completed. □

As an application of Theorem 3.13, we have the following slightly general version of Theorem 1.6 .

Theorem 4.4. *Let $f : X \to Y$ be a holomorphic submersion between two smooth complex projective varieties and* $\dim_{\mathbb{C}} Y = n$*. Suppose that $L \to Y$ is an ample and globally generated line bundle, and $(A, e^{-2\varphi}) \to Y$ be a pseudo-effective line bundle with* $\mathrm{nd}(A,\varphi) \neq 0$*. Suppose both E_1 and E_2 are Nakano semi-positive, then for any $s \geq 1$*

$$f_*(K_{X/Y} \otimes E_1)^{\otimes s} \otimes K_Y \otimes L^{\otimes n} \otimes E_2 \otimes A \otimes \mathcal{I}(\varphi) \tag{4.7}$$

is globally generated as long as $f_*(K_{X/Y} \otimes E_1)$ is locally free.

The proof of Theorem 1.8. When the family $\mathcal{X} \to S$ is effectively parameterized, Schumacher proved in [49, Theorem 1] that the naturally induced

Hermitian metric on the relative canonical line bundle $K_{\mathcal{X}/S}$ is strictly positive. By [49, Corollary 2] or [4, Theorem 1.2], we know

$$f_*(K_{X/S}^{\otimes s}) \tag{4.8}$$

is strictly Nakano positive for all $s > 1$. One can write

$$f_*(K_X^{\otimes s}) \otimes L^{\otimes n} \otimes A = f_*(K_{X/S}^{\otimes s}) \otimes K_S \otimes L^{\otimes n} \otimes (K_S^{\otimes (s-1)} \otimes A) \tag{4.9}$$

We first observe that the canonical bundle K_S is nef. Indeed, by a recent result in [53, Theorem 1], the compact complex base S is actually (Kobayashi) hyperbolic. Hence, it contains no rational curve. By using Mori's cone theorem [40], we deduce K_S is nef since it is a projective manifold without rational curve. The global generation of the vector bundle $f_*(K_{X/S}^{\otimes s}) \otimes K_S \otimes L^{\otimes n} \otimes (K_S^{\otimes(s-1)} \otimes A)$ follows from Theorem 4.1 since $f_*(K_{X/S}^{\otimes s})$ is Nakano positive and $K_S^{\otimes(s-1)} \otimes A$ is nef. □

Remark 4.5. On smooth complex projective varieties, we can also derive globally generation for symmetric powers and wedge powers of vector bundles by using the corresponding vanishing theorems (e.g. [38]).

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