

On hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$

FRANCISCO URBANO

We classify the homogeneous and isoparametric hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$. In the classification, besides the hypersurfaces $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1]$, it appears a family of hypersurfaces with three different constant principal curvatures and zero Gauss-Kronecker curvature. Also we classify the hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with at most two constant principal curvatures and, under certain conditions, with three constant principal curvatures.

1	Introduction	1381
2	Preliminaries	1384
3	Examples	1386
4	Characterizations of the examples.	
	Homogeneous hypersurfaces	1393
5	Isoparametric hypersurfaces	1401
6	Hypersurfaces with constant principal curvatures	1405
	References	1415

1. Introduction

Let (N, g) be a compact 4-dimensional Riemannian manifold and $\Phi : M \rightarrow N$ a two-sided hypersurface. We are interested in the following properties:

Research partially supported by a MINECO-FEDER grant no. MTM2014-52368-P.

- 1) M is (extrinsically) locally homogeneous, i.e., for any points $p, q \in M$ there exist neighbourhoods V and W of p and q in M and an isometry F of N such that $F(\Phi(V)) = \Phi(W)$.
- 2) M has constant principal curvatures.
- 3) M is isoparametric, i.e., there exists an isoparametric function $F : N \rightarrow \mathbb{R}$ such that $M = F^{-1}(t)$, for some regular value t of F . F is isoparametric if the gradient and the Laplacian of F satisfy

$$|\nabla F|^2 = f(F), \quad \Delta F = g(F),$$

where $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are smooth functions.

When N is the 4-dimensional sphere \mathbb{S}^4 or the complex projective plane $\mathbb{C}\mathbb{P}^2$, these properties have been studied and the corresponding classifications have been done (see [C], [K], [M] and [T]). In both cases, the above three properties are equivalent and the number of possible different principal curvatures are 1, 2 or 3 when $N = \mathbb{S}^4$ and 2 or 3 when $N = \mathbb{C}\mathbb{P}^2$.

Besides the above ambient spaces, $\mathbb{S}^2 \times \mathbb{S}^2$ is the most interesting compact 4-manifold to study its hypersurfaces. It is, together with $\mathbb{C}\mathbb{P}^2$, the only compact Hermitian symmetric 4-manifold.

In this paper we start the study of the above properties for the hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$. In section 3, we give a complete description of the most important examples, which appear in two families of isoparametric hypersurfaces. The first one, $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$, is a family of homogeneous and isoparametric hypersurfaces with 1 or 2 constant principal curvatures. The second one, $\{M_t, t \in (-1, 1)\}$, with

$$M_t = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, q \rangle = t\},$$

is also a family of homogeneous and isoparametric hypersurfaces but with three constant principal curvatures and with Gauss-Kronecker curvature $K = 0$. All these examples are tubes over distinguish totally geodesic surfaces of $\mathbb{S}^2 \times \mathbb{S}^2$, and, in contrast with the cases of \mathbb{S}^4 and $\mathbb{C}\mathbb{P}^2$, the geodesic spheres of $\mathbb{S}^2 \times \mathbb{S}^2$ do not belong to the above families of examples.

As it is well-known, the Gauss and Codazzi equations (and hence the curvature of $\mathbb{S}^2 \times \mathbb{S}^2$) play an important role in the study of the above properties. In our case, the curvature depends of the product structure of $\mathbb{S}^2 \times \mathbb{S}^2$ (see section 2) and so the Codazzi equation reflects the behaviour of the hypersurface with respect to the product structure. This behaviour is described by a function C (see (2.2)) defined on the hypersurface and satisfying

$-1 \leq C \leq 1$, and so, the properties of this function will be quite important throughout the paper. This function is constant in all the above examples ($C = 1$ for the first family and $C = 0$ for the second one).

The first important results in the paper, Theorem 1 and Corollary 1, provide a local characterization of the above examples among the family of hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ where the function C is constant. This characterization will be used along the paper.

In Corollary 2 and Corollary 3 we prove the following local result, which classifies the homogeneous and isoparametric hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$:

- 1) *Open subsets of $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$ and $\{M_t, t \in (-1, 1)\}$ are, up to congruences, the only locally homogeneous orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$.*
- 2) *$\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$ and $\{M_t, t \in (-1, 1)\}$ are, up to congruences, the only isoparametric orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$.*

In fact, in Theorem 2 we prove a stronger result than in (2): we characterize locally the above examples as the only orientable hypersurfaces whose parallel hypersurfaces have constant mean curvature. It is well-known that this property is satisfied by any isoparametric hypersurface.

Finally in section 6, we study the orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant principal curvatures. In Theorem 3 we locally classify them, when the number of constant principal curvatures is one or two, proving that

- 1) *Up to congruences, open subsets of $\mathbb{S}^1 \times \mathbb{S}^2$ are the only orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with one constant principal curvature.*
- 2) *Up to congruences, open subsets of $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1)\}$, are the only orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with two different constant principal curvatures.*

When the number of different principal curvatures is three, the classification problem is harder, and we have only got partial results. Using Theorem 4, where we study the critical points of the function C in such hypersurfaces, we prove in Corollary 4 the following result:

$\{M_t, t \in (-1, 1)\}$ are, up to congruences, the only orientable compact hypersurfaces with three different constant principal curvatures, with scalar curvature $\rho \neq 1/2$ and Gauss-Kronecker curvature $K = 0$.

It is interesting to remark that, in [BD], the authors classify the hypersurfaces of the complex hyperbolic plane with three different constant principal curvatures. Their proof is novel, but unfortunately we have not been able to adapt their ideas to get the complete classification of the hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with three different constant principal curvatures.

To finish it could be interesting to study hypersurfaces in the product of two spheres of different radii. In this case, although the ambient manifold loses important geometric properties and its curvature is more complicated, we think that the behaviour of its hypersurfaces could be similar to our case.

2. Preliminaries

Let \mathbb{S}^2 be the 2-dimensional unit sphere, \langle, \rangle its standard metric and J its complex structure defined by

$$J_p v = p \wedge v, \quad p \in \mathbb{S}^2, v \in T_p \mathbb{S}^2.$$

We endow $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric (also denoted by \langle, \rangle) and the complex structures

$$J_1 = (J, J), \quad J_2 = (J, -J)$$

which define two structures of Kähler surface on $\mathbb{S}^2 \times \mathbb{S}^2$. It is clear that, if $Id : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is the identity map and $F : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ is any anti-holomorphic isometry of \mathbb{S}^2 , then $Id \times F : (\mathbb{S}^2 \times \mathbb{S}^2, J_1) \rightarrow (\mathbb{S}^2 \times \mathbb{S}^2, J_2)$ is a holomorphic isometry.

The product structure P on $\mathbb{S}^2 \times \mathbb{S}^2$, defined by

$$P(v_1, v_2) = (v_1, -v_2), \quad v_1, v_2 \in T\mathbb{S}^2,$$

satisfies $P = -J_1 J_2 = -J_2 J_1$ and $\bar{\nabla} P = 0$, where $\bar{\nabla}$ is the Levi-Civita connection on $\mathbb{S}^2 \times \mathbb{S}^2$.

On the other hand, using that $\mathbb{S}^2 \times \mathbb{S}^2$ is a product manifold, its curvature tensor \bar{R} is given by

$$\begin{aligned} \bar{R}(v, w, x, y) = & \frac{1}{2} \{ \langle v, y \rangle \langle w, x \rangle - \langle v, x \rangle \langle w, y \rangle \\ & + \langle Pv, y \rangle \langle Pw, x \rangle - \langle Pv, x \rangle \langle Pw, y \rangle \}, \end{aligned}$$

where $v, w, x, y \in T(\mathbb{S}^2 \times \mathbb{S}^2)$, and hence $\mathbb{S}^2 \times \mathbb{S}^2$ is an Einstein manifold with scalar curvature 4 and non-negative sectional curvature.

Finally, the group of isometries of $\mathbb{S}^2 \times \mathbb{S}^2$ is the 6-dimensional subgroup of the orthogonal group $O(6)$ given by

$$(2.1) \quad \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right), \left(\begin{array}{cc} 0 & A \\ B & 0 \end{array} \right) / A, B \in O(3) \right\}.$$

Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ and N a unit normal vector field to Φ . The behaviour of M with respect to the product structure is given by the smooth function C and the vector field X tangent to M defined by

$$(2.2) \quad \begin{aligned} C : M &\rightarrow \mathbb{R}, & C &= \langle PN, N \rangle = \langle J_1 N, J_2 N \rangle, \\ X &= PN - CN. \end{aligned}$$

It is clear that $-1 \leq C \leq 1$, that X is the tangential component of PN and that $|X|^2 = 1 - C^2$.

From the Gauss equation it follows that the scalar curvature ρ of M is given by

$$\rho = 2 + 9H^2 - |\sigma|^2,$$

where σ is the second fundamental form and $H = \frac{1}{3} \text{tr } \sigma$ the mean curvature vector field of Φ . The Codazzi equation is given by

$$(\nabla\sigma)(v, w, x) - (\nabla\sigma)(w, v, x) = \frac{1}{2}(\langle X, v \rangle \langle Pw, x \rangle - \langle X, w \rangle \langle Pv, x \rangle),$$

where $\nabla\sigma$ is the covariant derivative of the second fundamental form.

In the following result we describe some properties of C and X which will be used along the paper.

Lemma 1. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface and A the shape operator associated to the unit normal field N . Then*

1) *The gradient of C and the covariant derivative of X are given by*

$$\nabla C = -2AX, \quad \nabla_V X = CAV - P^T AV,$$

2) *The Hessian of C is given by*

$$(\nabla^2 C)(V, W) = -2(\nabla\sigma)(V, X, W) - 2C\langle AV, AW \rangle + 2\langle PAV, AW \rangle.$$

3) *The Laplacian of C and the divergence of X are given by*

$$\Delta C = -6\langle X, \nabla H \rangle - 2C|\sigma|^2 + 2\text{tr}(P^T A^2), \quad \text{div } X = 3CH - \text{tr}(P^T A),$$

where $P^T : TM \rightarrow TM$ is the tangential component of the restriction of P to M , tr stands for the trace and V, W are vector fields on M .

Proof. Derivating the second equation of (2.2) and taking into account that P is parallel, we get easily (1). Now, (2) and (3) follow easily from (1) using the Codazzi's equation. \square

Finally, if $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ is an orientable hypersurface and $\{e_1, e_2, e_3\}$ is an orthonormal reference of M such that $\{e_1, e_2, e_3, N\}$ is positively oriented and $P_{ij} = \langle Pe_i, e_j \rangle$, $b_i = \langle Pe_i, N \rangle = \langle X, e_i \rangle$, then the product structure P , in the above reference, is written as follows

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & b_1 \\ P_{21} & P_{22} & P_{23} & b_2 \\ P_{31} & P_{32} & P_{33} & b_3 \\ b_1 & b_2 & b_3 & C \end{pmatrix}$$

It is clear that $P = P^t$ and that $\text{tr } P = 0$ because $P = -J_1 J_2$ and $J_i, i = 1, 2$ are complex structures. Also, as $\langle Pv, Pw \rangle = \langle v, w \rangle, \forall v, w \in T(\mathbb{S}^2 \times \mathbb{S}^2)$ and P preserves the orientation of $\mathbb{S}^2 \times \mathbb{S}^2$, it follows that, for any $p \in M$, the matrix $P(p) \in SO(4)$. Hence, for $i \neq j \neq k$

$$(2.3) \quad \begin{aligned} CP_{ii} - b_i^2 &= P_{jj}P_{kk} - P_{jk}^2, & CP_{ij} - b_i b_j &= P_{ik}P_{jk} - P_{ij}P_{kk}, \\ b_i P_{ij} - b_j P_{ii} &= -b_k P_{kj} + b_j P_{kk}. \end{aligned}$$

3. Examples

In this section we are going to give the most regular examples of hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$, some of them will be characterized in the paper.

3.1. Hypersurfaces with function C satisfying $C^2 = 1$.

Given $a \in \mathbb{S}^2$, let $G : \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}$ be the function defined by

$$G(p, q) = \langle p, a \rangle.$$

Then it is easy to check that the gradient and the Laplacian of G satisfy

$$|\bar{\nabla}G|^2 = 1 - G^2, \quad \bar{\Delta}G = -2G.$$

This means that G is an isoparametric function on $\mathbb{S}^2 \times \mathbb{S}^2$ and hence the level hypersurfaces of G define a one-parameter family of hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant mean curvature.

In this particular case, $G^{-1}(t) = \emptyset$ if $|t| > 1$, $G^{-1}(1) = \{a\} \times \mathbb{S}^2$ and $G^{-1}(-1) = \{-a\} \times \mathbb{S}^2$ are the focal sets, which are totally geodesic surfaces of $\mathbb{S}^2 \times \mathbb{S}^2$. Finally, for $t \in (-1, 1)$ we have that

$$G^{-1}(t) = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, a \rangle = t\}$$

is a hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant mean curvature. The isometry of $\mathbb{S}^2 \times \mathbb{S}^2$ given by $-Id \times Id$ transforms $G^{-1}(-t)$ onto $G^{-1}(t)$. Also, it is clear that, up to congruences, we can take $a = (0, 0, 1)$. So we have a family of hypersurfaces

$$G^{-1}(t) = \mathbb{S}^1(r) \times \mathbb{S}^2, \quad r^2 = 1 - t^2, \quad t \in [0, 1), \quad r \in (0, 1],$$

where $\mathbb{S}^1(r) = \{(x, y, \sqrt{1 - r^2}) \in \mathbb{S}^2\}$. It is trivial to check that $G^{-1}(0) = \mathbb{S}^1 \times \mathbb{S}^2$ is totally geodesic and that $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1)$, has two constant principal curvatures: 0 with multiplicity two and $\frac{\sqrt{1-r^2}}{r}$ with multiplicity one.

Also, $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$ are tubes of radius $\arcsin \sqrt{1 - r^2}$ over the focal surface $\{a\} \times \mathbb{S}^2$, with $a = (0, 0, 1)$.

Finally, the group of isometries of $\mathbb{S}^2 \times \mathbb{S}^2$ given by

$$\left\{ \left(\begin{array}{cc} A & 0 \\ 0 & B \end{array} \right) / A = \begin{pmatrix} \hat{A} & 0 \\ 0 & 1 \end{pmatrix}, \hat{A} \in \text{SO}(2), B \in \text{SO}(3) \right\}$$

acts transitively on $\mathbb{S}^1(r) \times \mathbb{S}^2$ and hence these hypersurfaces are homogeneous. Sumarizing, we have that

$\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$ is a family of homogeneous isoparametric hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with two constant principal curvatures when $r \in (0, 1)$ and totally geodesic when $r = 1$.

These hypersurfaces satisfy that $C = 1$, because the unit normal field has no component in the second factor. We remark that the isometry of $\mathbb{S}^2 \times \mathbb{S}^2$ given by $(p, q) \mapsto (q, p)$ transforms $\mathbb{S}^1(r) \times \mathbb{S}^2$ onto $\mathbb{S}^2 \times \mathbb{S}^1(r)$ whose function $C = -1$.

Now we are going to characterize locally the hypersurfaces satisfying $C^2 = 1$. Without loss of generality we can assume that $C = 1$.

If $\Phi = (\phi, \psi) : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ is an orientable hypersurface with $C = 1$, then $X = 0$, $PN = N$ and $J_1N = J_2N$. Hence, from Lemma 1 we get that

$A = PA$. Moreover the tangent bundle decomposes as $TM = \langle J_1N \rangle \oplus D$, where D is the two-dimensional distribution orthogonal to J_1N . As $P(J_1N) = J_1N$, it is clear that $P|_D = -Id$, and so $A|_D = 0$. Also, if V, W are vector fields on D , we have that

$$\langle \nabla_V W, J_1N \rangle = -\langle W, \bar{\nabla}_V J_1N \rangle = \langle W, J_1(AV) \rangle = 0,$$

which means that D is a totally geodesic foliation on M . If Σ is a leaf of D , it follows that $\psi : \Sigma \rightarrow \mathbb{S}^2$ is a local isometry, and hence

Any hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ with $C \equiv 1$ is locally the product of an integral curve of J_1N in \mathbb{S}^2 and an open subset of \mathbb{S}^2 .

3.2. Hypersurfaces with three constant principal curvatures

Let $F : \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}$ be the function defined by

$$F(p, q) = \langle p, q \rangle.$$

Then it is not difficult to check that

$$|\bar{\nabla} F|^2 = 2(1 - F^2), \quad \bar{\Delta} F = -4F,$$

and so F is an isoparametric function on $\mathbb{S}^2 \times \mathbb{S}^2$. Hence the level hypersurfaces of F have constant mean curvature. In this case, $F^{-1}(t)$ is empty if $|t| > 1$, and the diagonal surface $F^{-1}(1) = \{(p, p) \in \mathbb{S}^2 \times \mathbb{S}^2\}$ and the anti-diagonal surface $F^{-1}(-1) = \{(p, -p) \in \mathbb{S}^2 \times \mathbb{S}^2\}$ of $\mathbb{S}^2 \times \mathbb{S}^2$ are the focal sets of F .

For $t \in (-1, 1)$ we have that

$$M_t = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, q \rangle = t\}$$

is an hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant mean curvature.

The hypersurfaces M_t and M_{-t} are congruents because the isometry I of $\mathbb{S}^2 \times \mathbb{S}^2$ given by $I(p, q) = (p, -q)$ transforms M_t onto M_{-t} .

Moreover, the tube of radius $\arccos(t/\sqrt{2})$ over the diagonal surface $F^{-1}(1) = \{(p, p) \in \mathbb{S}^2 \times \mathbb{S}^2\}$ is given by the sets of points $\{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2\}$ such that

$$(x, y) = \left(\cos\left(\frac{t}{\sqrt{2}}\right) p + \sqrt{2} \sin\left(\frac{t}{\sqrt{2}}\right) v, \cos\left(\frac{t}{\sqrt{2}}\right) p - \sqrt{2} \sin\left(\frac{t}{\sqrt{2}}\right) v \right),$$

$$p \in \mathbb{S}^2, v \in T_p\mathbb{S}^2, |v| = 1/\sqrt{2}.$$

As $\langle x, y \rangle = \cos^2 \frac{t}{\sqrt{2}} - \sin^2 \frac{t}{\sqrt{2}} = \cos(\sqrt{2}t)$, we obtain that the hypersurface M_t is a tube of radius $\arccos(t/\sqrt{2})$ over the diagonal surface.

On the other hand, it is clear that $SO(3)$ acts transitively by isometries on M_t by

$$A(p, q) = (Ap, Aq), \quad A \in SO(3),$$

and hence $\{M_t, t \in (-1, 1)\}$ is a family of homogeneous hypersurfaces.

Also, the isotropy subgroup of the above action at any point of M_t is the identity. So M_t is diffeomorphic to $SO(3) \cong \mathbb{R}P^3$. Hence M_t is a homogeneous Riemannian manifold and $SO(3)$ is the group of isometries of M_t when $t \neq 0$ and that $SO(3)$ joint with the one-parameter group of isometries $\{h_t : M_0 \rightarrow M_0, t \in \mathbb{R}\}$ defined by

$$h_t(p, q) = (tp + \sqrt{1-t^2}q, \sqrt{1-t^2}p - tq),$$

is the group of isometries of M_0 . We remark that $\{h_t\}$ is only well-defined on M_0 and that they are the restriction to M_0 of isometries of $O(6)$, which no define isometries of $\mathbb{S}^2 \times \mathbb{S}^2$.

In [MP], the simply connected homogeneous Riemannian three-manifolds are described in detail. Following its notation, M_0 is the Berger projective space with $\kappa = 1$ and $\tau^2 = 1/2$. Also, $M_t, t \neq 0$ is the projective space with the metric given by the parameters $c_1 = 2 = c_2 + c_3$ with $c_2 = 1 + t$ and $c_3 = 1 - t$.

Now, we are going to study more properties of the hypersurfaces $\{M_t\}$. It is easy to check that

$$N_{(p,q)} = \frac{1}{\sqrt{2(1-t^2)}}(q - tp, p - tq)$$

is a unit normal vector field to M_t in $\mathbb{S}^2 \times \mathbb{S}^2$ and so we have that these hypersurfaces have the function C constantly zero. Hence, if $J_i, i = 1, 2$, and P are the complex structures and the product structure on $\mathbb{S}^2 \times \mathbb{S}^2$,

then

$$J_1 N_{(p,q)} = \frac{(p \wedge q, -p \wedge q)}{\sqrt{2(1-t^2)}}, \quad J_2 N_{(p,q)} = \frac{(p \wedge q, p \wedge q)}{\sqrt{2(1-t^2)}},$$

$$X = PN_{(p,q)} = \frac{1}{\sqrt{2(1-t^2)}}(q - tp, -p + tq),$$

is a trivialization of M_t by orthonormal vectors fields, where \wedge stands for the vectorial product in \mathbb{R}^3 . If A denotes the shape operator associated to N , then for any vector (v_1, v_2) tangent to M_t , we have that

$$A(v_1, v_2) = \frac{1}{\sqrt{2(1-t^2)}} \{t(v_1, v_2) - (v_2, v_1) + \langle p, v_2 \rangle (p, -q)\}.$$

From here we obtain that

$$A(J_1 N) = \frac{1}{\sqrt{2}} \sqrt{\frac{1+t}{1-t}} J_1 N, \quad A(J_2 N) = -\frac{1}{\sqrt{2}} \sqrt{\frac{1-t}{1+t}} J_2 N, \quad A X = 0.$$

So, $\{M_t, t \in (-1, 1)\}$ are hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with three constant principal curvatures, $\lambda_1 = 0$, $\lambda_2 \lambda_3 = -1/2$. So the Gauss-Krocnker curvature of M_t is zero. The lengths of the mean curvature vector field and the second fundamental form are given by

$$H = \frac{\sqrt{2}t}{3\sqrt{1-t^2}}, \quad |\sigma| = \sqrt{\frac{1+t^2}{1-t^2}}.$$

Among all the M_t , only M_0 is minimal. From the Gauss equation we obtain that the sectional curvature, the Ricci tensor and the scalar curvature of M_t satisfy the following properties:

$$K(J_1 N \wedge J_2 N) = -\frac{1}{2}, \quad K(J_1 N \wedge X) = K(J_2 N \wedge X) = \frac{1}{2},$$

$$Ric(v) = \langle v, X \rangle^2 \geq 0, \quad \rho = 1.$$

We remark that the curvatures of M_t do not depend of t . Sumarizing, we have that

$\{M_t = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle x, y \rangle = t\}, t \in (-1, 1)\}$ is a family of homogeneous isoparametric hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with three constant principal curvatures. Their Gauss-Krocnker curvatures vanish and only M_0 is a minimal hypersurface. Moreover, all these hypersurfaces have $C = 0$.

Remark 1. The above examples can be defined in higher dimension. In fact, if \mathbb{S}^n is the n -dimensional unit sphere with its canonical metric and in $\mathbb{S}^n \times \mathbb{S}^n$ we consider the product metric, then

$$M_t = \{(p, q) \in \mathbb{S}^n \times \mathbb{S}^n \mid \langle p, q \rangle = t\}, \quad t \in (-1, 1)$$

define a one-parameter family of homogeneous isoparametric hypersurfaces of $\mathbb{S}^n \times \mathbb{S}^n$ with three constant principal curvatures: 0 with multiplicity one and $\frac{\sqrt{1+t}}{\sqrt{2(1-t)}}$, $-\frac{\sqrt{1-t}}{\sqrt{2(1+t)}}$ with multiplicities $n - 1$.

3.3. Other interesting examples

1) Given $a, b \in \mathbb{S}^2$ we define

$$M_{a,b} = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, a \rangle + \langle q, b \rangle = 0\}.$$

Then it is easy to check that $M_{a,b}$ is a compact hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ with two isolated singularities: $(a, -b), (-a, b)$. Outside of these singularities, a unit normal vector field to $M_{a,b}$ is defined by

$$N(p, q) = \frac{1}{\sqrt{2(1 - \langle p, a \rangle^2)}}(a - \langle p, a \rangle p, b - \langle q, b \rangle q),$$

and hence the function $C = \langle PN, N \rangle = 0$.

Also it is straightforward to see that the orthonormal reference $\{X, E_2 = (J_1N + J_2N)/\sqrt{2}, E_3 = (J_1N - J_2N)/\sqrt{2}\}$ diagonalizes the second fundamental form with

$$AX = 0, \quad AE_2 = \frac{\langle p, a \rangle}{\sqrt{2(1 - \langle p, a \rangle^2)}}E_2, \quad AE_3 = \frac{-\langle p, a \rangle}{\sqrt{2(1 - \langle p, a \rangle^2)}}E_3.$$

Hence $M_{a,b}$ is a minimal hypersurface with Gauss-Kronecker curvature $K = 0$ and with scalar curvature

$$\rho(p, q) = \frac{2 - 3\langle p, a \rangle^2}{1 - \langle p, a \rangle^2}, \quad -\infty < \rho \leq 2.$$

A parametrization of $M_{a,b}$ when $a = (0, 0, 1), b = (0, 0, -1)$, is given by the triply periodic ramified immersion

$$\begin{aligned} \Phi : \mathbb{R}^3 &\rightarrow \mathbb{S}^2 \times \mathbb{S}^2 \\ \Phi(t, r, s) &= \cos\left(\frac{t}{\sqrt{2}}\right) ((\cos r, \sin r, 0), (\cos s, \sin s, 0)) + \sin\left(\frac{t}{\sqrt{2}}\right) (a, -b). \end{aligned}$$

2) Given $a, b \in \mathbb{S}^2$ we define

$$\hat{M}_{a,b} = \{(p, q) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, a \rangle^2 + \langle q, b \rangle^2 = 1\}.$$

Then it is easy to check that $\hat{M}_{a,b}$ is a compact hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$ with four curves of singularities

$$\{(p, \pm b) \mid \langle p, a \rangle = 0\}, \quad \{(\pm a, q) \mid \langle q, b \rangle = 0\}.$$

Outside of these singularities, a unit normal vector field to $\hat{M}_{a,b}$ is defined by

$$N(p, q) = \frac{1}{\sqrt{2} |\langle p, a \rangle \langle q, b \rangle|} (\langle p, a \rangle (a - \langle p, a \rangle p), \langle q, b \rangle (b - \langle q, b \rangle q)),$$

and hence the function $C = \langle PN, N \rangle = 0$.

Also it is straightforward to see that the orthonormal reference $\{X, E_2 = (J_1 N + J_2 N)/\sqrt{2}, E_3 = (J_1 N - J_2 N)/\sqrt{2}\}$ diagonalizes the second fundamental form with

$$AX = 0, \quad AE_2 = \frac{\langle p, a \rangle^2}{\sqrt{2} |\langle p, a \rangle \langle q, b \rangle|} E_2, \quad AE_3 = \frac{\langle q, b \rangle^2}{\sqrt{2} |\langle p, a \rangle \langle q, b \rangle|} E_3.$$

Hence $\hat{M}_{a,b}$ is a hypersurface with Gauss-Kronecker curvature $K = 0$, with constant curvature $1/2$ and the length of the mean curvature vector field is given by

$$H(p, q) = \frac{1}{3\sqrt{2} |\langle p, a \rangle \langle q, b \rangle|}.$$

A parametrization of $\hat{M}_{a,b}$, when $a = b = (0, 0, 1)$, is given by the triply periodic ramified immersion

$$\begin{aligned} \Phi &= (\phi, \psi) : \mathbb{R}^3 \rightarrow \mathbb{S}^2 \times \mathbb{S}^2 \\ \phi(t, r, s) &= \frac{\cos\left(\frac{t}{\sqrt{2}}\right) - \sin\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}} (\cos r, \sin r, 0) \\ &\quad + \left(0, 0, \frac{\cos\left(\frac{t}{\sqrt{2}}\right) + \sin\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}}\right), \end{aligned}$$

$$\psi(t, r, s) = \frac{\cos\left(\frac{t}{\sqrt{2}}\right) + \sin\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}}(\cos s, \sin s, 0) + \left(0, 0, \frac{\cos\left(\frac{t}{\sqrt{2}}\right) - \sin\left(\frac{t}{\sqrt{2}}\right)}{\sqrt{2}}\right).$$

4. Characterizations of the examples. Homogeneous hypersurfaces

In the next result we give two local characterizations of the examples defined in section 3.

Theorem 1. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface with $C = \langle PN, N \rangle$ constant, where N is a unit normal field to Φ . Then*

- 1) *M has constant mean curvature if and only if either*
 - a) *$C^2 = 1$ and $\Phi(M)$ is congruent to an open set of $\mathbb{S}^1(r) \times \mathbb{S}^2$ for some $r \in (0, 1]$,*
 - b) *or $C = 0$ and $\Phi(M)$ is congruent to an open set of M_t for some $t \in (-1, 1)$,*
 - c) *or $C = 0$ and M is a non-compact minimal hypersurface with non-constant scalar curvature.*
- 2) *M has constant scalar curvature if and only if either*
 - a) *$C^2 = 1$ and $\Phi(M)$ is congruent to an open set of $\mathbb{S}^1(r) \times \mathbb{S}^2$ for some $r \in (0, 1]$,*
 - b) *or $C = 0$ and $\Phi(M)$ is congruent to an open set of M_t for some $t \in (-1, 1)$,*
 - c) *or $C = 0$ and M is a non-complete hypersurface with constant curvature $1/2$ and non-constant mean curvature.*

Remark 2. 1) The family of minimal hypersurfaces in item (1c) is not empty, because the hypersurface $M_{a,b}$ given in section 3.3 is a non-complete minimal hypersurface with non-constant scalar curvature and with $C = 0$.

2) The family of hypersurfaces in item (2c) is also not empty, because the hypersurface $\tilde{M}_{a,b}$ given in section 3.3 is a non-complete hypersurface with constant curvature $1/2$, non-constant mean curvature and with $C = 0$.

- Corollary 1.** 1) $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$, $\{M_t, t \in (-1, 1)\}$ and their compact coverings are the only compact orientable constant mean curvature hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with C constant.
- 2) $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$, $\{M_t, t \in (-1, 1)\}$ and their compact coverings are the only complete orientable hypersurfaces with constant scalar curvature of $\mathbb{S}^2 \times \mathbb{S}^2$ with C constant.
- 3) Open subsets of $\{\mathbb{S}^1(r) \times \mathbb{S}^2, r \in (0, 1]\}$ and $\{M_t, t \in (-1, 1)\}$, are the only orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ which have the mean curvature, the scalar curvature and the function C constants.

Proof. Taking into account section 3, the sufficient conditions in (1) and (2) are clear.

In order to prove the necessary conditions, first we suppose that $C^2 = 1$, and without loss of generality we consider $C = 1$. In Section 3.1, we prove that M is locally congruent to the product of a curve in \mathbb{S}^2 and an open subset of \mathbb{S}^2 . If the mean curvature or the scalar curvature of the hypersurface is constant, then the curvature of the curve of \mathbb{S}^2 is also constant and so we obtain the case (1a) or (2a).

Now we suppose that $C = c_0 \in (-1, 1)$. Then from Lemma 1 (1), it follows that $AX = 0$ with $|X|^2 = 1 - c_0^2 > 0$. So, at any point of M , 0 is a principal curvature of the hypersurface with corresponding eigenvector X . Hence, on M we can consider the orthonormal reference $\{E_i, i = 1, 2, 3\}$ where

$$E_1 = \frac{X}{\sqrt{1 - c_0^2}}, \quad E_2 = \frac{J_1N + J_2N}{\sqrt{2(1 + c_0)}}, \quad E_3 = \frac{J_1N - J_2N}{\sqrt{2(1 - c_0)}}.$$

Using (2.2), the shape operator A and the tangential component of the product structure P^T are given, with respect to this reference, by

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix}, \quad P^T = \begin{pmatrix} -c_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Using Lemma 1 and that $J_i, i = 1, 2$, are Kähler structures on $\mathbb{S}^2 \times \mathbb{S}^2$, i.e. $\bar{\nabla} J_i = 0$, it is not difficult to check that the Levi-Civita connection ∇ of

the induced metric on M is given by

$$\begin{aligned} \nabla_{E_1} E_i &= 0, & \nabla_{E_2} E_3 &= -\sqrt{\frac{1+c_0}{1-c_0}} \sigma_{23} E_1, & \nabla_{E_3} E_2 &= -\sqrt{\frac{1-c_0}{1+c_0}} \sigma_{23} E_1, \\ \nabla_{E_2} E_1 &= -\sqrt{\frac{1-c_0}{1+c_0}} \sigma_{22} E_2 + \sqrt{\frac{1+c_0}{1-c_0}} \sigma_{23} E_3, \\ \nabla_{E_3} E_1 &= -\sqrt{\frac{1-c_0}{1+c_0}} \sigma_{23} E_2 + \sqrt{\frac{1+c_0}{1-c_0}} \sigma_{33} E_3, \\ \nabla_{E_2} E_2 &= \sqrt{\frac{1-c_0}{1+c_0}} \sigma_{22} E_1, & \nabla_{E_3} E_3 &= -\sqrt{\frac{1+c_0}{1-c_0}} \sigma_{33} E_1. \end{aligned}$$

The knowledge of the Levi-Civita connection and the Codazzi equation, joint with Lemma 1, allow us to get the derivatives of the second fundamental form, obtaining

$$\begin{aligned} X(\sigma_{22}) &= \frac{1-c_0^2}{2} + (1-c_0)\sigma_{22}^2 - (1+c_0)\sigma_{23}^2, \\ X(\sigma_{33}) &= \frac{c_0^2-1}{2} + (1-c_0)\sigma_{23}^2 - (1+c_0)\sigma_{33}^2, \\ X(\sigma_{23}) &= (1-c_0)\sigma_{22}\sigma_{23} - (1+c_0)\sigma_{33}\sigma_{23}, \\ E_2(\sigma_{33}) &= E_3(\sigma_{23}), \quad E_3(\sigma_{22}) = E_2(\sigma_{23}). \end{aligned} \tag{4.1}$$

Case (1): the mean curvature H is constant.

In this case, from Lemma 1,(3) it follows that $c_0|\sigma|^2 = \text{trace } P^T A^2$. Using the above reference, this equation becomes in

$$c_0|\sigma|^2 = 3H(\sigma_{22} - \sigma_{33}). \tag{4.2}$$

First we are going to prove that $c_0 = 0$. In fact, if $H = 0$, as the hypersurface cannot be totally geodesic because in such case $C^2 = 1$, the equation (4.2) says that $c_0 = 0$. If $H \neq 0$, derivating the equality $9H^2 - |\sigma|^2 = 2(\sigma_{22}\sigma_{33} - \sigma_{23}^2)$ with respect to X and using (4.1) and (4.2) it is not difficult to get that

$$X(|\sigma|^2) = \frac{c_0((1-c_0^2)|\sigma|^2 + (9H^2 - |\sigma|^2)^2)}{3H}.$$

On the other hand, using (4.1) and (4.2) it is straightforward to get that

$$X(\sigma_{22} - \sigma_{33}) = 1 - |\sigma|^2 - c_0^2(1 + |\sigma|^2) + 9H^2 + \frac{c_0^2|\sigma|^4}{9H^2}.$$

Now, derivating (4.2) with respect to X and taking into account the last two equations we obtain that

$$(4.3) \quad |\sigma|^2 = \frac{9H^2(1 + 9H^2)}{c_0^2 + 9H^2}.$$

In particular $|\sigma|^2$ is constant and from (4.2) , the function $\sigma_{22} - \sigma_{33}$ is also constant. This implies, taking into account that H and $|\sigma|^2$ are constant functions, that all the functions σ_{ij} are constants. Using in (4.1) that $X(\sigma_{23}) = 0$ and (4.2) we get

$$c_0\sigma_{23}(9H^2 - |\sigma|^2) = 0.$$

If $9H^2 - |\sigma|^2 = 0$, equation (4.3) says that $c_0^2 = 1$, which is imposible. If $\sigma_{23} = 0$, using that $X(\sigma_{22}) = 0$ in (4.1), it follows that $c_0 = 1$, which is imposible. Hence last equation says that $c_0 = 0$ again.

Hence we have proved that $c_0 = 0$ and so (4.2) says that either M is minimal, i.e. $H = 0$, or $\sigma_{22} = \sigma_{33}$. We are going to study these cases separately.

First case: $\sigma_{22} = \sigma_{33}$. In this case we have that $3H = 2\sigma_{22}$ and hence σ_{22} and σ_{33} are constant functions. Using this in (4.1) we obtain that $\sigma_{23}^2 = 1/2 + \sigma_{22}^2$ and so σ_{23} is also constant and the hypersurface has constant scalar curvature $\rho = 1$.

Now, taking into account that $c_0 = 0$, the second fundamental form, with respect to the orthonormal reference on M given by $\{X, J_1N, J_2N\}$, is given by

$$AX = 0, \quad A(J_1N) = (\sigma_{22} + \sigma_{23})J_1N, \quad A(J_2N) = (\sigma_{22} - \sigma_{23})J_2N,$$

with $(\sigma_{22} + \sigma_{23})(\sigma_{22} - \sigma_{23}) = -1/2$. As these principal curvatures are constant and their product is $-1/2$, these numbers can be written, without loss of generality, as

$$\sigma_{22} + \sigma_{23} = \frac{1}{\sqrt{2}}\sqrt{\frac{1+t}{1-t}}, \quad \sigma_{22} - \sigma_{23} = -\frac{1}{\sqrt{2}}\sqrt{\frac{1-t}{1+t}}$$

for certain $0 \leq t < 1$.

Now to find the focal set of M , we consider the parallel hypersurfaces to M . As $C = 0$, we have that $N = (N_1, N_2)$ with $|N_1|^2 = |N_2|^2 = 1/2$. Hence,

the parallel hypersurfaces to M are given by $\Phi_s : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$, $s \geq 0$, where

$$\Phi_s(p, q) = (\exp_p(sN_1), \exp_q(sN_2)) = \cos\left(\frac{s}{\sqrt{2}}\right)(p, q) + \sqrt{2} \sin\left(\frac{s}{\sqrt{2}}\right)N.$$

Then

$$\begin{aligned} (\Phi_s)_*(X) &= \cos\left(\frac{s}{\sqrt{2}}\right)X - \frac{1}{\sqrt{2}}\sin\left(\frac{s}{\sqrt{2}}\right)\hat{\Phi} \\ (\Phi_s)_*(J_1N) &= \left(\cos\left(\frac{s}{\sqrt{2}}\right) - \sqrt{\frac{1+t}{1-t}}\sin\left(\frac{s}{\sqrt{2}}\right)\right)J_1N, \\ (\Phi_s)_*(J_2N) &= \left(\cos\left(\frac{s}{\sqrt{2}}\right) + \sqrt{\frac{1-t}{1+t}}\sin\left(\frac{s}{\sqrt{2}}\right)\right)J_2N, \end{aligned}$$

where $\hat{\Phi} = S \circ \Phi$, S being the isometry of $\mathbb{S}^2 \times \mathbb{S}^2$ given by $S(p, q) = (p, -q)$.

Hence, the focal surface of M happens when $\cot\left(\frac{s}{\sqrt{2}}\right) = \sqrt{\frac{1+t}{1-t}}$, i.e., when

$$\cos\left(\frac{s}{\sqrt{2}}\right) = \sqrt{(1+t)/2}, \quad \sin\left(\frac{s}{\sqrt{2}}\right) = \sqrt{(1-t)/2}.$$

But this means that $\cos(\sqrt{2}s) = t$, and so $s = \frac{1}{\sqrt{2}} \arccos t$.

It Σ is the focal surface of M and we denote by Ψ the restriction of Φ_s (with $s = \frac{1}{\sqrt{2}} \arccos t$) to Σ , then the immersion $\Psi : \Sigma \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ is given by

$$\Psi = \frac{\sqrt{1+t}}{\sqrt{2}}\Phi + \sqrt{1-t}N.$$

As $(\Phi_s)_*(J_1N) = 0$, for $s = \frac{1}{\sqrt{2}} \arccos t$, it is clear that $\{X, J_2N\}$ is an orthonormal reference of the tangent bundle to Σ and that

$$\begin{aligned} (\Psi)_*(X) &= \frac{\sqrt{1+t}}{\sqrt{2}}X - \frac{\sqrt{1-t}}{2}\hat{\Psi}, \\ (\Psi)_*(J_2N) &= \frac{\sqrt{2}}{\sqrt{1+t}}J_2N. \end{aligned}$$

Hence, $\{J_1N, \frac{\sqrt{1-t}}{2}\Phi - \frac{\sqrt{1+t}}{\sqrt{2}}N\}$ is an orthonormal reference on the normal bundle of Ψ . Now, it is easy to check that the corresponding Weingarten endomorphisms associated to these two unit normal vector fields vanish, and so Ψ is a totally geodesic immersion. Moreover, as $J_1X = J_2N$, the immersion Ψ is a complex surface with respect to the complex structure J_1

and a Lagrangian surface with respect to the other complex structure J_2 . From [CU], we have that Ψ is congruent to an open subset of the diagonal surface $\{(p, p) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid p \in \mathbb{S}^2\}$ and our hypersurface M is an open subset of the tube of radius $s = \frac{1}{\sqrt{2}} \arccos t$ over the diagonal surface. Taking into account section 3.2, we get that M is locally congruent to some $\{M_t, t \in (-1, 1)\}$. Hence we have obtained (1b).

Second case: $H = 0$. In this case, equation (4.1) becomes in

$$(4.4) \quad \begin{aligned} X(\sigma_{22}) &= \frac{1}{2} + \sigma_{22}^2 - \sigma_{23}^2, & X(\sigma_{23}) &= 2\sigma_{22}\sigma_{23}, \\ \langle \nabla \sigma_{22}, E_2 \rangle &= -\langle \nabla \sigma_{23}, E_3 \rangle, & \langle \nabla \sigma_{22}, E_3 \rangle &= \langle \nabla \sigma_{23}, E_2 \rangle. \end{aligned}$$

Now, if $\Delta = \sum_{i=1}^3 (E_i E_i - \nabla_{E_i} E_i)$ is the Laplacian of the induced metric on M , from (4.4) we have that

$$\Delta \sigma_{22} = \Delta \sigma_{23} = 0,$$

that is σ_{22} and σ_{23} are harmonic functions on M .

If M is compact, then σ_{22} and σ_{23} are constant functions. Using (4.4) again we have two possibilities: $\sigma_{22} = 0$ or $\sigma_{23} = 0$. In the first case, $\sigma_{23}^2 = 1/2$ and hence we are again in the situation of the first case. So our hypersurface is congruent to M_0 and we obtain (1b). In the second case ($\sigma_{23} = 0$) and as σ_{22} is constant, we get from (4.4) that $0 = X(\sigma_{22}) = 1/2 + \sigma_{22}^2$, which is a contradiction. Hence the hypersurface cannot be compact.

Also, if the scalar curvature is constant, then $\sigma_{22}^2 + \sigma_{23}^2$ will be constant, and so, derivating with respect to X and using (4.4) we will obtain that

$$0 = \sigma_{22}(1 + 2\sigma_{22}^2 + 2\sigma_{23}^2),$$

which implies that $\sigma_{22} = 0$ and hence $\sigma_{23}^2 = 1/2$. This says that M is locally congruent to M_0 . Hence, in this case we get that either M is locally congruent to M_0 or $H = 0$, $C = 0$ and M is not compact with non-constant scalar curvature. This implies (1c).

Case (2): the scalar curvature ρ is constant.

In this case, as the scalar curvature $\rho = 2 + 2(\sigma_{22}\sigma_{33} - \sigma_{23}^2)$ is constant, from (4.1) and $X(\rho) = 0$ it follows that

$$(4.5) \quad (c_0 - 1)(c_0 - \rho + 3)\sigma_{22} = (c_0 + 1)(c_0 + \rho - 3)\sigma_{33}.$$

Derivating equation (4.5) with respect to X , using (4.1), (4.5) and the expression of ρ , it is straightforward to obtain

$$(\rho - 1)(\rho - 3) + c_0^2 = 0.$$

Hence there are two possible values of the scalar curvature: $\rho = 2 \pm \sqrt{1 - c_0^2}$.

First case: $\rho = 2 - \sqrt{1 - c_0^2}$. Putting this information in (4.5) it follows that

$$(4.6) \quad \sqrt{1 - c_0} \sigma_{22} = \sqrt{1 + c_0} \sigma_{33}.$$

As in this case $\sigma_{22}\sigma_{33} - \sigma_{23}^2 = -\frac{\sqrt{1-c_0^2}}{2}$, (4.6) becomes in

$$\frac{\sqrt{1 - c_0}}{\sqrt{1 + c_0}} \sigma_{22}^2 - \sigma_{23}^2 = -\frac{\sqrt{1 - c_0^2}}{2}.$$

Using the last equation of (4.1) in the above equation and taking into account (4.6) we get

$$\begin{aligned} \sigma_{23}E_2(\sigma_{23}) - \sigma_{22}E_3(\sigma_{23}) &= 0, \\ \frac{\sqrt{1 - c_0}}{\sqrt{1 + c_0}} \sigma_{22}E_2(\sigma_{23}) - \sigma_{23}E_3(\sigma_{23}) &= 0. \end{aligned}$$

Now, the only solution to this compatible homogeneous system is $E_2(\sigma_{23}) = E_3(\sigma_{23}) = 0$.

But

$$[E_2, E_3] = -\frac{2c_0}{\sqrt{1 - c_0^2}} \sigma_{23} E_1,$$

and so $0 = [E_2, E_3](\sigma_{23}) = \frac{-2c_0}{1-c_0^2} \sigma_{23} X(\sigma_{23}) = \frac{-2c_0(\sqrt{1-c_0}-\sqrt{1+c_0})}{\sqrt{1-c_0^2}} \sigma_{23}^2 \sigma_{22}$. Now, as $\sigma_{23}^2 = \frac{\sqrt{1-c_0}}{\sqrt{1+c_0}} \sigma_{22}^2 + \sqrt{1-c_0^2}/2 > 0$, we get that either $c_0 = 0$ or $\sigma_{22} = 0$ and so $\sigma_{33} = 0$. In the second case, going again to (4.1) we get that $c_0 = 0$. At any case $c_0 = 0$, and so $\rho = 1$. This situation has been studied in Case 1, and we obtain (2b).

Second case: $\rho = 2 + \sqrt{1 - c_0^2}$. As in the above case, putting the value of the scalar curvature in (4.5) it follows that

$$\sqrt{1 - c_0}(\sqrt{1 + c_0} - \sqrt{1 - c_0})\sigma_{22} = -\sqrt{1 + c_0}(\sqrt{1 + c_0} - \sqrt{1 - c_0})\sigma_{33}.$$

If $c_0 \neq 0$, from the above equation we get that $\sqrt{1-c_0} \sigma_{22} = -\sqrt{1+c_0} \sigma_{33}$ and using that $\sigma_{22}\sigma_{33} - \sigma_{23}^2 = \frac{\sqrt{1-c_0^2}}{2}$ we have that

$$-\left(\frac{\sqrt{1-c_0}}{\sqrt{1+c_0}}\sigma_{22}^2 + \sigma_{23}^2\right) = \frac{\sqrt{1-c_0^2}}{2}$$

which is impossible. Hence, in this second case, we get that $c_0 = 0$, $\rho = 3$ and $\sigma_{22}\sigma_{33} - \sigma_{23}^2 = \frac{1}{2}$.

If $\{v, w\}$ is an orthonormal basis of a plane $\Pi \subset T_pM$, then the Gauss equation says that the curvature K of Π is given by

$$K = \bar{R}(v, w, w, v) + \sigma(v, v)\sigma(w, w) - \sigma(v, w)^2.$$

Using the above information it is easy to check that

$$\begin{aligned} \bar{R}(v, w, w, v) &= 1/2 - 1/2(\langle v, E_2 \rangle \langle w, E_3 \rangle - \langle v, E_3 \rangle \langle w, E_2 \rangle)^2, \\ \sigma(v, v)\sigma(w, w) - \sigma(v, w)^2 &= (\sigma_{22}\sigma_{33} - \sigma_{23}^2)(\langle v, E_2 \rangle \langle w, E_3 \rangle - \langle v, E_3 \rangle \langle w, E_2 \rangle)^2, \end{aligned}$$

and so $K = 1/2$. This means that M has constant curvature $1/2$.

Now, we are going to see that M is not complete. In fact, if M is complete, as M has constant positive curvature, Myers' theorem says that M is compact. On the other hand, from (4.1) we have that

$$X(\sigma_{22} - \sigma_{33}) = 2 + (\sigma_{22} - \sigma_{33})^2.$$

But Lemma 1 says that $\operatorname{div}(X) = -(\sigma_{22} - \sigma_{33})$. So

$$\begin{aligned} \operatorname{div}((\operatorname{div}X)X) &= X(\operatorname{div}X) + (\operatorname{div}X)^2 \\ &= -2 - (\sigma_{22} - \sigma_{33})^2 + (\sigma_{22} - \sigma_{33})^2 = -2, \end{aligned}$$

and the divergence Theorem gives a contradiction. Hence M is not complete.

Also, we are going to see that the mean curvature H is not constant. In fact, if H is constant, from Lemma 1 we get that

$$0 = \operatorname{tr}(P^T A^2) = 3H(\sigma_{22} - \sigma_{33}).$$

As H cannot be zero, because $\sigma_{22}\sigma_{33} - \sigma_{23}^2 = 1/2$, we obtain that $\sigma_{22} - \sigma_{33} = 0$, which contradicts, using (4.1), the equation of its derivative with respect to X . Hence M has not constant mean curvature, and we get (2c). \square

As a consequence of Corollary 1 we classify locally the homogeneous hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$.

Corollary 2. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface . If M is locally homogeneous, then $\Phi(M)$ is congruent to either an open subset of $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1]$, or an open subset of M_t , $t \in (-1, 1)$.*

Proof.

□

Let N be a unit normal vector field. We fix a point $p_0 \in M$. Then, as M is locally homogeneous, for any $p \in M$ there exist open sets $p_0 \in U_0$, $p \in U$ and an isometry F of $\mathbb{S}^2 \times \mathbb{S}^2$ such that $(F \circ \Phi)(U_0) = \Phi(U)$ and $F(\Phi(p_0)) = \Phi(p)$. Then $N_p = \pm dF_{p_0}(N_{p_0})$ and $dF_{p_0} \circ P = \pm P \circ dF_{p_0}$. Hence $C(p) = \pm C(p_0)$ and as M is connected, C is constant. Also, it is clear that F keeps the second fundamental form, and so the mean curvature and the scalar curvature of M are also constant. Hence Corollary 2 follows from Corollary 1.

5. Isoparametric hypersurfaces

In this section we classify the isoparametric hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$. If M is an isoparametric hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$, then there exists an isoparametric function $F : \mathbb{S}^2 \times \mathbb{S}^2 \rightarrow \mathbb{R}$ such that $M = F^{-1}(t_0)$ for some regular value t_0 . Then it is well-known that $M_t := F^{-1}(t)$ are also hypersurfaces for $t \in (t_0 - \delta, t_0 + \delta)$, parallel to M and with constant mean curvature. We start classifying hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ satisfying this property.

Theorem 2. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface. If the parallel hypersurfaces $\Phi_t : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$, $t \in (-\epsilon, \epsilon)$, $\Phi_0 = \Phi$, have constant mean curvature, then $\Phi(M)$ is congruent either to an open subset of $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1]$, or to an open subset of M_t , $t \in (-1, 1)$.*

Proof. We are going to consider the open subset of M defined by $O = \{p \in M \mid C^2(p) < 1\}$. If O is empty, then M has constant mean curvature and $C^2 = 1$, and hence Theorem 1 says that $\Phi(M)$ is congruent to an open subset of $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1]$.

Now we suppose that O is not empty. We write $\Phi = (\phi, \psi) : O \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$, with $\phi, \psi : O \rightarrow \mathbb{S}^2$. If $N = (N_1, N_2)$ is a unit normal vector field to Φ , then

the parallel hypersurfaces $\Phi_t : O \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ are given by

$$\Phi_t = (\exp_\phi tN_1, \exp_\psi tN_2), \quad t \in (-\epsilon, \epsilon), \quad \Phi_0 = \Phi,$$

where \exp denotes the exponential map in \mathbb{S}^2 . As $|N_1|^2 = (1+C)/2$ and $|N_2|^2 = (1-C)/2$, then $\Phi_t = (\phi_t, \psi_t)$ is defined by

$$\begin{aligned} \phi_t &= \cos(C^+t)\phi + (1/C^+)\sin(C^+t)N_1, \\ \psi_t &= \cos(C^-t)\psi + (1/C^-)\sin(C^-t)N_2, \end{aligned}$$

where $C^+ = \sqrt{1+C}/\sqrt{2}$ and $C^- = \sqrt{1-C}/\sqrt{2}$.

Now, it is straightforward to check that $N^t = (N_1^t, N_2^t)$ defined by

$$\begin{aligned} N_1^t &= \cos(C^+t)N_1 - C^+ \sin(C^+t)\phi, \\ N_2^t &= \cos(C^-t)N_2 - C^- \sin(C^-t)\psi, \end{aligned}$$

is a unit normal vector field to the hypersurface Φ_t . Under these conditions it is easy to see that

$$C_t = C, \quad J_1N^t = J_1N, \quad J_2N^t = J_2N, \quad \forall t \in (-\epsilon, \epsilon).$$

Hence, in $(\Phi_t)_*(TM|_O)$ we can consider the orthonormal reference $\{E_i^t, i = 1, 2, 3\}$ where

$$E_1^t = \frac{X_t}{\sqrt{1-C^2}}, \quad E_2^t = \frac{J_1N^t + J_2N^t}{\sqrt{2(1+C)}}, \quad E_3^t = \frac{J_1N^t - J_2N^t}{\sqrt{2(1-C)}}.$$

Taking into account the above relations, if we denote $E_i^0 = E_i$, it is clear that $E_2^t = E_2, E_3^t = E_3, \forall t \in (-\epsilon, \epsilon)$.

Now, let $\{e_1, e_2, e_3\}$ be a local orthonormal reference on O such that $\Phi_*(e_i) = E_i, i = 1, 2, 3$. We denote $\sigma_{ij} = \langle \Phi_*(Ae_i), \Phi_*(e_j) \rangle$ the second fundamental form of Φ associated to the normal field N . Then, from a simply but long computation, we obtain that

$$\begin{aligned} (\Phi_t)_*(e_i) &= (\delta_{1i} - t\sigma_{1i})E_1^t + \left(\delta_{2i} \cos(C^+t) - \sigma_{2i} \frac{\sin(C^+t)}{C^+} \right) E_2^t \\ &\quad + \left(\delta_{3i} \cos(C^-t) - \sigma_{3i} \frac{\sin(C^-t)}{C^-} \right) E_3^t \end{aligned}$$

and

$$\begin{aligned}
 (\Phi_t)_*(A^t e_i) &= \sigma_{1i} E_1^t + (\sigma_{2i} \cos(C^+t) + \delta_{2i} C^+ \sin(C^+t)) E_2^t \\
 &\quad + (\sigma_{3i} \cos(C^-t) + \delta_{3i} C^- \sin(C^-t)) E_3^t,
 \end{aligned}$$

where A^t is the shape operator of Φ_t associated to the normal field N^t .

If we denote $(\Phi_t)_*(e_i) = \sum_{j=1}^3 Q_{ij} E_j^t$, it is clear from the above expressions that $(\Phi_t)_*(A^t e_i) = -\sum_{j=1}^3 Q'_{ij} E_j^t$, where $'$ stands for derivative with respect to t . Hence the induced metric g_t on O by the immersion Φ_t and the second fundamental form σ^t of Φ_t are given by

$$g_t = QQ^T, \quad \sigma^t = -Q(Q^T)',$$

where Q is the matrix $Q = (Q_{ij})$ and $(\cdot)^T$ stands for the transpose. The mean curvatures of the immersions Φ_t are given by

$$3H(t) = \text{tr}(g_t^{-1}\sigma^t) = -\text{tr}((Q^T)^{-1}Q^{-1}Q(Q^T)') = -\frac{(\det Q)'}{\det Q},$$

where tr stands for the trace and \det stands for the determinant. Hence $(\det Q)' = -3H(t) \det Q$ and as $Q(0) = Id$, an inductive argument says that

$$\left(\frac{d^k \det Q}{dt^k}\right)(0), \quad k \geq 0,$$

are constants functions on O .

From the definition, the determinant of Q is given by

$$\begin{aligned}
 \det Q &= (1 - t\sigma_{11}) \cos(C^+t) \cos(C^-t) + (H_{23} - tK) \frac{\sin(C^+t) \sin(C^-t)}{C^+C^-} \\
 &\quad + (-\sigma_{22} + tH_{12}) \frac{\sin(C^+t) \cos(C^-t)}{C^+} \\
 &\quad + (-\sigma_{33} + tH_{13}) \frac{\cos(C^+t) \sin(C^-t)}{C^-},
 \end{aligned}$$

where $H_{ij} = \sigma_{ii}\sigma_{jj} - \sigma_{ij}^2$ and $K = \det A$ is the Gauss-Kronecker curvature of M .

Now, computing the Taylor serie of the function $\det Q$ around $t = 0$ and from a very long computation we get that

$$\begin{aligned} \det Q &= 1 - 3Ht + \frac{\rho - 3}{2}t^2 + \frac{9H - 6K - (1 + C)\sigma_{22} - (1 - C)\sigma_{33}}{3!}t^3 \\ &+ \frac{2(3 - \rho) - C^2 - 4((1 - C)H_{12} + (1 + C)H_{13})}{4!}t^4 \\ &+ \frac{-5(2 - C^2)3H + 20K + 4(1 + C - C^2)\sigma_{22} + 4(1 - C - C^2)\sigma_{33}}{5!}t^5 \\ &+ \frac{4(\rho - 3) + (5 - \rho)C^2 + 4((1 - C)(4 + C)H_{12} + (1 + C)(4 - C)H_{13})}{6!}t^6 \\ &+ \dots \end{aligned}$$

As all the coefficients of the above serie are constant functions, we get that not only the mean curvature but also the scalar curvature ρ of M is constant.

Now we work on the open subset $V = \{p \in O \mid C(p) \neq 0\}$. Taking into account that the coefficients corresponding to t^4 and t^6 are constant and that ρ is also constant, we obtain on V that

$$H_{12} = \frac{(1 + C)R_2(C)}{8C(1 - C^2)}, \quad H_{13} = \frac{(1 - C)R_3(C)}{8C(1 - C^2)},$$

where R_2 and R_3 are non-trivial polynomials of degree 3 in C with constant coefficients. Computing the term of the Taylor serie corresponding to t^8 , we have that

$$\begin{aligned} -8(\rho - 3) + 4(\rho - 4)C^2 + C^4 + 8(-6 + 4C + 3C^2 - C^3)H_{12} \\ + 8(-6 - 4C + 3C^2 + C^3)H_{13} = \lambda, \end{aligned}$$

for certain constant λ . Using the above expressions of H_{12} and H_{13} in this equation, we finally prove that C satisfies a non trivial polynomial of degree 7 with constant coefficients. This means that C is constant on each connected component of the open set V .

Hence the function C on the connected hypersurface M takes only a discrete number of values. This means that the function C is constant and the result follows from Corollay 1. □

Corollary 3. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an isoparametric hypersurface. Then M is congruent either to $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1]$, or to M_t , $t \in (-1, 1)$.*

6. Hypersurfaces with constant principal curvatures

In this section we are going to study orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with constant principal curvatures.

6.1. Hypersurfaces with one or two constant principal curvatures

In the following result we classify locally the orientable hypersurfaces of $\mathbb{S}^2 \times \mathbb{S}^2$ with at most two constant principal curvatures.

Theorem 3. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface of $\mathbb{S}^2 \times \mathbb{S}^2$. If M has at most two constant principal curvatures, then $\Phi(M)$ is either an open subset of $\mathbb{S}^1 \times \mathbb{S}^2$ (M is totally geodesic), or $\Phi(M)$ is an open subset of $\mathbb{S}^1(r) \times \mathbb{S}^2$, $r \in (0, 1)$ (if M has two constant principal curvatures).*

Remark 3. When the hypersurface has only one principal curvature, not necessarily constant, i.e., when the hypersurface is umbilical, it is easy to conclude that the mean curvature is constant and hence the hypersurface is totally geodesic.

Proof. If M has only one constant principal curvature, then M is an umbilical hypersurface with constant mean curvature. Then in [TU], Proposition 1, it was proved that M is congruent to an open subset of $\mathbb{S}^1 \times \mathbb{S}^2$.

Now, we suppose that M has two constant principal curvatures. Let λ_1, λ_2 be the corresponding different principal curvatures having λ_1 multiplicity one and λ_2 multiplicity two.

Under these conditions, let E_1 be a unit vector field on M such that $AE_1 = \lambda_1 E_1$. Then, we have that the second fundamental form and its covariant derivative are given by

$$(6.1) \quad \begin{aligned} \sigma(V, W) &= \lambda_2 \langle V, W \rangle + (\lambda_1 - \lambda_2) \langle V, E_1 \rangle \langle W, E_1 \rangle, \\ (\nabla \sigma)(Z, V, W) &= (\lambda_1 - \lambda_2) (\langle W, E_1 \rangle \langle V, \nabla_Z E_1 \rangle + \langle V, E_1 \rangle \langle W, \nabla_Z E_1 \rangle), \end{aligned}$$

for any vector fields V, W, Z on M .

Using (6.1), Codazzi equation and the fact that M has constant mean curvature, we obtain

$$\begin{aligned}
 0 &= \sum_{i=1}^3 (\nabla\sigma)(Z, e_i, e_i) \\
 &= \sum_{i=1}^3 (\nabla\sigma)(e_i, Z, e_i) + \frac{1}{2} \sum_{i=1}^3 \{ \langle X, Z \rangle \langle Pe_i, e_i \rangle - \langle X, e_i \rangle \langle PZ, e_i \rangle \} \\
 &= (\lambda_1 - \lambda_2) (\langle E_1, Z \rangle \operatorname{div} E_1 + \langle \nabla_{E_1} E_1, Z \rangle) - \frac{1}{2} (C \langle X, Z \rangle + \langle PX, Z \rangle) \\
 &= (\lambda_1 - \lambda_2) (\langle E_1, Z \rangle \operatorname{div} E_1 + \langle \nabla_{E_1} E_1, Z \rangle),
 \end{aligned}$$

for any vector field Z on M , being $\{e_1, e_2, e_3\}$ an orthonormal reference on M and div the divergence operator.

As $|E_1|^2 = 1$, from the last equation we get that the vector field E_1 is a geodesic vector field with zero divergence, i.e.,

$$(6.2) \quad \operatorname{div} E_1 = 0 \quad \text{and} \quad \nabla_{E_1} E_1 = 0.$$

Using (6.2) in the second equation of (6.1) we obtain

$$(\nabla\sigma)(E_1, V, W) = 0, \quad (\nabla\sigma)(V, E_1, W) = (\lambda_1 - \lambda_2) \langle \nabla_V E_1, W \rangle,$$

and hence, the Codazzi equation says that

$$(6.3) \quad \nabla_V E_1 = \frac{\langle X, V \rangle P^T E_1 - \langle X, E_1 \rangle P^T V}{2(\lambda_1 - \lambda_2)},$$

for any vector field V tangent to M , where P^T denotes the tangential component of P , i.e., $P^T Z = PZ - \langle X, Z \rangle N$.

Now, from (6.2) and (6.3) and making a direct computation, we obtain that the Ricci curvature of E_1 is given by

$$\operatorname{Ric}(E_1) = -\frac{\langle X, E_1 \rangle^2}{2(\lambda_1 - \lambda_2)^2}.$$

But, on the other hand, the Gauss equation says that

$$\operatorname{Ric}(E_1) = \frac{1}{2} + 2\lambda_1\lambda_2 + \frac{\langle X, E_1 \rangle^2}{2} - \frac{C \langle PE_1, E_1 \rangle}{2},$$

and hence finally we get that

$$(6.4) \quad 1 + 4\lambda_1\lambda_2 = C \langle PE_1, E_1 \rangle - \left(1 + \frac{1}{(\lambda_1 - \lambda_2)^2} \right) \langle X, E_1 \rangle^2.$$

Now we proceed as follows. From Lemma 1,(1) and (6.3) it is clear that

$$\begin{aligned} E_1(C) &= -2\lambda_1\langle X, E_1\rangle, & E_1(\langle PE_1, E_1\rangle) &= 2\lambda_1\langle X, E_1\rangle, \\ E_1(\langle X, E_1\rangle) &= \lambda_1(C - \langle PE_1, E_1\rangle). \end{aligned}$$

Hence, taking derivatives in (6.4) with respect to E_1 and taking into account the above expressions, we get that

$$0 = \lambda_1\langle X, E_1\rangle(\langle PE_1, E_1\rangle - C).$$

If $\lambda_1 = 0$, then from (6.4) we get

$$1 \leq 1 + \left(1 + \frac{1}{\lambda_2^2}\right)\langle X, E_1\rangle^2 = C\langle PE_1, E_1\rangle \leq 1,$$

because $-1 \leq C, \langle PE_1, E_1\rangle \leq 1$. So the equality holds in the above equation and in particular $C = 1$. This fact contradicts the classification given at the end of section 3.1. Hence $\lambda_1 \neq 0$ and we finally get

$$0 = \langle X, E_1\rangle(\langle PE_1, E_1\rangle - C).$$

Taking derivatives again this equation with respect to E_1 and using again the above expressions we obtain $4\langle X, E_1\rangle^2 = (\langle PE_1, E_1\rangle - C)^2$, and hence the above equation implies that

$$\langle X, E_1\rangle = 0, \quad \langle PE_1, E_1\rangle = C.$$

Now, (6.4) becomes in $C^2 = 1 + 4\lambda_1\lambda_2$. This means that C is a constant function and the result follows from Corollary 1. □

6.2. Hypersurfaces with three constant principal curvatures

From now on, let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface with three different constant principal curvatures $\{\lambda_1, \lambda_2, \lambda_3\}$. Then there exists a trivialization of M by unit vector fields $\{E_1, E_2, E_3\}$ with $AE_i = \lambda_i E_i$, $i = 1, 2, 3$. These principal curvatures are the roots of the polynomial

$$(6.5) \quad \lambda^3 - 3H\lambda^2 + \frac{\rho - 2}{2}\lambda - K = 0.$$

For simplicity, we will denote

$$\begin{aligned} \sigma_{ij} &= \sigma(E_i, E_j), \quad \nabla\sigma_{ijk} = (\nabla\sigma)(E_i, E_j, E_k), \quad P_{ij} = \langle PE_i, E_j \rangle, \\ b_i &= \langle X, E_i \rangle, \quad \Gamma_{ij}^k = \langle \nabla_{E_i} E_j, E_k \rangle. \end{aligned}$$

Lemma 2. *The above functions satisfy the following properties*

- 1) $\Gamma_{ij}^k + \Gamma_{ik}^j = 0, \quad 1 \leq i, j, k \leq 3.$
- 2) $\nabla\sigma_{ijk} + (\lambda_k - \lambda_j)\Gamma_{ij}^k = 0, \quad 1 \leq i, j, k \leq 3.$
- 3) $(\lambda_k - \lambda_j)\Gamma_{ij}^k - (\lambda_k - \lambda_i)\Gamma_{ji}^k = \frac{1}{2}(b_j P_{ik} - b_i P_{jk}), \quad 1 \leq i, j, k \leq 3$
- 4) $\Gamma_{ii}^j = \frac{b_i P_{ij} - b_j P_{ii}}{2(\lambda_i - \lambda_j)}, \quad 1 \leq i, j \leq 3, \quad i \neq j.$
- 5) $(\lambda_i - \lambda_j)\Gamma_{ii}^j = -(\lambda_k - \lambda_j)\Gamma_{kk}^j, \quad 1 \leq i, j \leq 3, \quad i \neq j \neq k.$

Proof. (1) is trivial. Taking derivatives in $\sigma_{jk} = \lambda_j \delta_{jk}$ with respect to E_i we get (2). From Codazzi equation and (2) we prove (3). Taking $k = i$ in (3) and using (1) we get (4). Finally from (2.3) and (4) it follows (5). □

Lemma 3. *The functions $\Lambda_i = b_i^2 - CP_{ii}, i = 1, 2, 3,$ satisfy the following equation*

$$B\Lambda = B_0 + D,$$

where

$$B = \begin{pmatrix} \lambda_2 & -\lambda_1 & 2(\lambda_1 - \lambda_2) \\ \lambda_3 & 2(\lambda_1 - \lambda_3) & -\lambda_1 \\ 2(\lambda_2 - \lambda_3) & \lambda_3 & -\lambda_2 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} (\lambda_1 - \lambda_2)(1 + 2\lambda_1\lambda_2) \\ (\lambda_1 - \lambda_3)(1 + 2\lambda_1\lambda_3) \\ (\lambda_2 - \lambda_3)(1 + 2\lambda_2\lambda_3) \end{pmatrix}, \quad D = \begin{pmatrix} D_{12} \\ D_{13} \\ D_{23} \end{pmatrix}$$

and $D_{ij}, i \neq j$ are defined by

$$D_{ij} = 4(\lambda_i - \lambda_j)((\Gamma_{ii}^j)^2 + (\Gamma_{jj}^i)^2) + 4(\lambda_k - \lambda_j)(\Gamma_{ij}^k)^2 - 4(\lambda_k - \lambda_i)(\Gamma_{ji}^k)^2,$$

with $k \neq i \neq j$.

Remark 4. We observe that Lemma 3 provides that the functions Λ_i satisfy a compatible linear system because

$$\det B = -6(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \neq 0.$$

Proof. For $i, j \in \{1, 2, 3\}$, $i \neq j$, let K_{ij} be the sectional curvature of the plane spanned by $\{E_i, E_j\}$. Then from the definition of the sectional curvature we have that

$$K_{ij} = E_i(\Gamma_{jj}^i) + E_j(\Gamma_{ii}^j) - (\Gamma_{ii}^j)^2 - (\Gamma_{jj}^i)^2 - \Gamma_{ii}^k \Gamma_{jj}^k - \Gamma_{ij}^k(\Gamma_{jk}^i + \Gamma_{kj}^i) + \Gamma_{ji}^k \Gamma_{kj}^i,$$

where $k \neq i$ and $k \neq j$.

Now, using Lemma 2 and Lemma 1, (1) and from a straightforward computation it is easy to conclude that

$$K_{ij} = -2((\Gamma_{ii}^j)^2 + (\Gamma_{jj}^i)^2) - \frac{1}{2}(P_{ii}P_{jj} - P_{ij}^2) + \frac{1}{2(\lambda_j - \lambda_i)}(\lambda_i \Lambda_j - \lambda_j \Lambda_i) + \frac{2}{(\lambda_j - \lambda_i)}((\lambda_k - \lambda_j)(\Gamma_{ij}^k)^2 - (\lambda_k - \lambda_i)(\Gamma_{ji}^k)^2), \quad k \neq i \neq j.$$

Finally using the Gauss equation, $2K_{ij} = 1 + P_{ii}P_{jj} - P_{ij}^2 + 2\lambda_i \lambda_j$, and (2.3) in the above equation we obtain that

$$\lambda_j \Lambda_i - \lambda_i \Lambda_j + 2(\lambda_i - \lambda_j) \Lambda_k = (\lambda_i - \lambda_j)(1 + 2\lambda_i \lambda_j) + D_{ij}, \quad i \neq j \neq k,$$

and the Lemma is proved. □

Lemma 4. *If there exists a point p on M with $C^2(p) = 1$, then*

$$\Lambda_i(p) = \frac{2}{3}(\lambda_i^2 - 3H\lambda_i + \rho - \frac{1}{2}) + \frac{12|\nabla\sigma|^2(p)}{(\det B)^2} \{2(\rho - 2 - 6H^2)\lambda_i^2 + 3(3K + 12H^3 - \frac{5}{2}H(\rho - 2))\lambda_i + (\rho - 2)(\rho - 2 - 3H^2) - 18HK\}, \quad i \in \{1, 2, 3\},$$

and

$$1 - 2\rho = 18\{(\rho - 2)(\rho - 2 - 3H^2/2) - 27HK\} \frac{|\nabla\sigma|^2(p)}{(\det B)^2}.$$

Proof. Suppose now that p is a point with $C^2(p) = 1$. Then $X_p = 0$ and so $b_i(p) = 0$, $1 \leq i \leq 3$. Using Lemma 2 we have that

$$D_{ij}(p) = \frac{4(\lambda_j - \lambda_i)(\lambda_k - \lambda_j)}{\lambda_k - \lambda_i} (\Gamma_{ij}^k)^2(p), \quad 1 \leq i, j, k \leq 3, \quad i < j, \quad k \neq i, \quad k \neq j.$$

On the other hand, using again Lemma 2 it is easy to check that

$$|\nabla\sigma|^2(p) = 6(\lambda_k - \lambda_j)^2(\Gamma_{ij}^k)^2(p), \quad i \neq j \neq k.$$

From the last two equations it follows that

$$D(p) = \frac{4|\nabla\sigma|^2(p)}{\det B} \begin{pmatrix} (\lambda_1 - \lambda_2)^2 \\ -(\lambda_1 - \lambda_3)^2 \\ (\lambda_2 - \lambda_3)^2 \end{pmatrix}$$

Now, solving the system of Lemma 3 and using (6.5) we get the first part. The last assertion in the Lemma follows from the fact that $\sum_{i=1}^3 \Lambda_i = 1$, which easily follows from Lemma 2. \square

Theorem 4. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable hypersurface with three constant principal curvatures.*

- 1) *If the scalar curvature ρ satisfies $2\rho \neq 1$ and the Gauss-Kronecker curvature $K = 0$, then $C^2 < 1$.*
- 2) *If p_0 is a critical point of the function C , then either $C^2(p_0) = 1$ or $C(p_0) = 0$. Moreover, in the second case, the Gauss-Kronecker curvature $K = 0$ and the scalar curvature $\rho = 1$.*

Proof. First we are going to prove part (1).

Suppose that there exists a point $p_0 \in M$ such that $C^2(p_0) = 1$ and without loss of generality we can take $C(p_0) = 1$. Then we are going to get a contradiction.

As $K = 0$, one of the principal curvatures is zero, for instance $\lambda_1 = 0$. Let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ be the integral curve of E_1 with $\gamma(0) = p_0$. Then

$$C'(t) = \langle (\nabla C)(t), E_1(t) \rangle = -2\langle AX(t), E_1(t) \rangle = 0,$$

and hence $C(t) = 1$ for all $t \in (-\epsilon, \epsilon)$. Now, as the Gauss-Kronecker curvature $K = 0$, from Lemma 4 it follows that

$$(1 - 2\rho)(\det B)^2 = 18(\rho - 2)(\rho - 2 - 3H^2/2)|\nabla\sigma|^2(t).$$

As $1 - 2\rho \neq 0$, we have that $|\nabla\sigma|^2(t) = 6(\lambda_3 - \lambda_2)^2(\Gamma_{12}^3)^2(t)$ is a non-null constant function and that $\rho \neq 2$. Hence, from Lemma 4, the functions

$\Lambda_i(t) = -P_{ii}(t)$ are also constant. So

$$0 = P'_{22}(t) = 2\Gamma_{12}^3(t)P_{23}(t),$$

which implies that $P_{23}(t) = 0$. Derivating again

$$0 = P'_{23}(t) = \Gamma_{12}^3(t)(P_{33}(t) - P_{22}(t)),$$

and so $P_{22}(t) = P_{33}(t)$. Using all the information about the P_{ij} in (2.3) we obtain that

$$P_{11}(t) = -P_{22}(t) = -P_{33}(t) = 1.$$

Using the first part of Lemma 4, and as $\Lambda_2(t) = \Lambda_3(t)$ and $3H = \lambda_2 + \lambda_3$ we get that

$$18H(\rho - 2) \frac{|\nabla\sigma|^2(t)}{(\det B)^2} = 0,$$

which implies that $H = 0$ because $\rho \neq 2$.

Now, as $\Lambda_1(t) = -1$, $H = 0$ and $\lambda_1 = 0$, first part of Lemma 4 says that

$$-(\rho + 1) = 18(\rho - 2)^2 \frac{|\nabla\sigma|^2(t)}{(\det B)^2},$$

that joint with the information given in the second part of Lemma 4

$$1 - 2\rho = 18(\rho - 2)^2 \frac{|\nabla\sigma|^2(t)}{(\det B)^2}$$

imply that $\rho = 2$, which is a contradiction. This proves (1).

Now, we are going to prove part (2).

Let p_0 be a point of M such that $(\nabla C)(p_0) = 0$ and $C^2(p_0) < 1$. We are going to prove that $C(p_0) = 0$.

In this case, $AX(p_0) = 0$ and as $|X|^2 = 1 - C^2(p_0) > 0$, one of the three principal curvatures is zero. For instance $\lambda_1 = 0$ and so

$$X(p_0) = \sqrt{1 - C^2(p_0)}E_1(p_0).$$

Now, let $\gamma : (-\epsilon, \epsilon) \rightarrow M$ be the integral curve of E_1 with $\gamma(0) = p_0$. As

$$C'(t) = \langle (\nabla C)(t), E_1(t) \rangle = -2\langle AX(t), E_1(t) \rangle = 0,$$

because $AE_1 = 0$, then C is constant along γ and $(\nabla C)(t) = \alpha(t)E_2(t) + \beta(t)E_3(t)$. Hence, derivating with respect to t we have that

$$\begin{aligned} (\nabla^2 C)(E_1, E_2)(t) &= \alpha'(t) - \Gamma_{12}^3(t)\beta(t), \\ (\nabla^2 C)(E_1, E_3)(t) &= \beta'(t) + \Gamma_{12}^3(t)\alpha(t). \end{aligned}$$

On the other hand, from Lemma 1 we obtain that

$$\begin{aligned} (\nabla^2 C)(E_1, E_2) &= -2\lambda_2\Gamma_{22}^1\langle X, E_2 \rangle + 2\lambda_3\Gamma_{21}^3\langle X, E_3 \rangle, \\ (\nabla^2 C)(E_1, E_3) &= 2\lambda_2\Gamma_{31}^2\langle X, E_2 \rangle - 2\lambda_3\Gamma_{33}^1\langle X, E_3 \rangle. \end{aligned}$$

But, by definition $\alpha(t) = -2\lambda_2\langle X, E_2 \rangle(t)$ and $\beta(t) = -2\lambda_3\langle X, E_3 \rangle(t)$, and so the above information about the Hessian of C says that α and β satisfy the following ODE system

$$\begin{aligned} \alpha'(t) &= \Gamma_{22}^1(t)\alpha(t) + (\Gamma_{12}^3(t) - \Gamma_{21}^3(t))\beta(t) \\ \beta'(t) &= -(\Gamma_{12}^3(t) + \Gamma_{31}^2(t))\alpha(t) + \Gamma_{33}^1(t)\beta(t). \end{aligned}$$

As $(\nabla C)(0) = 0$, i.e. $\alpha(0) = \beta(0) = 0$, it follows that $\alpha = \beta = 0$ is the only solution to the above system with that initial conditions. Hence, $(\nabla C)(t) = 0, \forall t$, and so

$$X(t) = \sqrt{1 - C^2(p_0)}E_1(t).$$

This relation between X and E_1 joint with (2.3) and Lemma 2 imply that on the curve γ we have the following relations

$$\begin{aligned} b_1(t) &= \sqrt{1 - C^2(p_0)}, \quad b_2(t) = b_3(t) = 0, \\ \Lambda_1(t) &= 1, \quad \Lambda_2(t) + \Lambda_3(t) = 0, \quad P_{12}(t) = P_{13}(t) = 0, \\ \Gamma_{11}^2(t) &= \Gamma_{11}^3(t) = \Gamma_{22}^3(t) = \Gamma_{33}^2(t) = 0, \quad \lambda_2\Gamma_{32}^1(t) = \lambda_3\Gamma_{23}^1(t). \end{aligned}$$

Now, using the above relations and Lemma 2, the linear system of Lemma 3 becomes in

$$\begin{aligned} (6.6) \quad & 2\lambda_2(\Gamma_{22}^1)^2(t) + 2(\lambda_2 - \lambda_3)(\Gamma_{12}^3)^2(t) + 2\lambda_3(\Gamma_{21}^3)^2(t) = -\lambda_2(1 + \Lambda_2(t)), \\ & -\frac{2\lambda_2^2}{\lambda_3}(\Gamma_{22}^1)^2(t) + 2(\lambda_2 - \lambda_3)(\Gamma_{12}^3)^2(t) - \frac{2\lambda_3^2}{\lambda_2}(\Gamma_{21}^3)^2(t) = \lambda_3(1 - \Lambda_2(t)), \\ & \frac{-4\lambda_3}{\lambda_2}(\Gamma_{21}^3)^2(t) = 1 - 2\lambda_2\lambda_3 + \frac{\lambda_3 + \lambda_2}{\lambda_2 - \lambda_3}\Lambda_2(t). \end{aligned}$$

This means that the functions $(\Gamma_{22}^1)^2(t), (\Gamma_{12}^3)^2(t)$ and $(\Gamma_{21}^3)^2(t)$ satisfy a linear system whose determinant is $16(\lambda_3 - \lambda_2)(\lambda_3 + \lambda_2)$.

In what follows we are going to consider two cases:

First case: $H = \lambda_2 + \lambda_3 = 0$. In this case, (6.6) becomes in

$$(6.7) \quad \begin{aligned} \Lambda_2(t) &= 0, \\ 4(\Gamma_{21}^3)^2(t) &= 1 + 2\lambda_2^2, \\ 2(\Gamma_{22}^1)^2(t) + 4(\Gamma_{12}^3)^2(t) &= -\frac{1}{2} + \lambda_2^2. \end{aligned}$$

Firstly we prove that $C(p_0) = 0$. If not, as $\Lambda_2(t) = b_2^2(t) - C(p_0)P_{22}(t) = -C(p_0)P_{22}(t)$, it follows from (6.7) that $P_{22}(t) = 0$ and so $P_{23}(t)^2 = 1$. Hence

$$0 = P'_{22}(t) = 2\Gamma_{12}^3(t)P_{23}(t),$$

which means that $\Gamma_{12}^3(t) = 0$. But, from Lemma 2, $\Gamma_{22}^1(t) = 0$ and so (6.7) says that $\lambda_2^2 = 1/2$ and $(\Gamma_{21}^3)^2(t) = 1/2$. Finally, from Lemma 2 it follows that $\lambda_2\Gamma_{21}^3(t) = -b_1P_{23}(t)/2$, which implies using the above information that $b_1^2 = 1$. This is a contradiction because $b_1^2 = 1 - C^2(p_0)$ and we are assuming that $C(p_0) \neq 0$.

Secondly we prove that the scalar curvature $\rho = 1$. From Lemma 2 we have that

$$(6.8) \quad P_{22}(t) = -2\lambda_2\Gamma_{22}^1(t), \quad P_{23}(t) = 4\lambda_2\Gamma_{12}^3(t) - 2\lambda_2\Gamma_{21}^3(t).$$

Now using (6.7), (6.8) and the fact that $P_{22}^2(t) + P_{23}^2(t) = 1$, we obtain that $\Gamma_{12}^3(t)$ satisfies the following non-trivial second order equation

$$8\lambda_2^2(\Gamma_{12}^3)^2(t) - 16\lambda_2^2\Gamma_{21}^3\Gamma_{12}^3(t) + 4\lambda_2^4 - 1 = 0.$$

As from (6.7) the function $\Gamma_{21}^3(t)$ is constant, the coefficients of the above polynomial are constant, and so $\Gamma_{12}^3(t)$ is also constant. Now (6.7) says that $\Gamma_{22}^1(t)$ is also a constant function and from (6.8) we get that $P_{22}(t)$ and $P_{23}(t)$ are constant functions too. Hence

$$0 = P'_{22}(t) = 2\Gamma_{12}^3P_{23}, \quad 0 = P'_{23}(t) = \Gamma_{12}^3P_{33} + \Gamma_{13}^2P_{22} = -2\Gamma_{12}^3P_{22}.$$

So $\Gamma_{12}^3(t) = 0$ and from the above equation it follows that $\lambda_2^2 = 1/2$, i.e., the scalar curvature $\rho = 1$.

Second case: $H \neq 0$. In this case the system (6.6) is compatible and it is easy to check that

$$(6.9) \quad \frac{2\lambda_2}{\lambda_3}(\Gamma_{22}^1)^2(t) = -\frac{1 + 2\lambda_2\lambda_3}{2} + \frac{\lambda_3^2 + \lambda_2^2 - 6\lambda_3\lambda_2}{2(\lambda_3^2 - \lambda_2^2)}\Lambda_2(t).$$

Firstly we prove that $C(p_0) = 0$. If not, from Lemma 2, it follows that

$$\Gamma_{22}^1(t) = -\frac{b_1C(P_0)P_{22}(t)}{2\lambda_2C(P_0)} = \frac{b_1\Lambda_2(t)}{2\lambda_2C_0}.$$

Putting this information in (6.9), it follows that $\Lambda_2(t)$ satisfies a non-trivial polynomial of degree two with constant coefficients. This means that $\Lambda_2(t)$ is a constant function and hence $P_{22}(t)$ and $P_{23}(t)$ are also constant. Using that $P'_{22}(t) = P'_{23}(t) = 0$, like in the above minimal case, we get that $\Gamma_{12}^3(t) = 0$. But then, the solution of $\Gamma_{12}^3(t)$ which provides (6.6) implies that

$$(6.10) \quad \Lambda_2(t) = \frac{(\lambda_3 + \lambda_2)(1 + 2\lambda_3\lambda_2)}{\lambda_3 - \lambda_2}.$$

From (6.9) and (6.10) it follows that

$$(\Gamma_{22}^1)^2(t) = -\frac{\lambda_3^2(1 + 2\lambda_3\lambda_2)}{(\lambda_3 - \lambda_2)^2} = -\frac{\lambda_3^2}{\lambda_3^2 - \lambda_2^2}\Lambda_2(t).$$

But we know that

$$(\Gamma_{22}^1)^2(t) = \frac{(1 - C^2(p_0))\Lambda_2^2}{4\lambda_2^2C^2(p_0)},$$

which implies that

$$\Lambda_2 \left(\frac{(1 - C^2(p_0))\Lambda_2}{4\lambda_2^2C^2(p_0)} + \frac{\lambda_3^2}{\lambda_3^2 - \lambda_2^2} \right) = 0.$$

Hence we have two possibilities. If $\Lambda_2 = 0$, then $P_{22} = 0$ and from (6.10) it follows that $\lambda_2\lambda_3 = -1/2$ and $\Gamma_{22}^1 = 0$. Now (6.6) says that $\Gamma_{12}^3 = 0$ and $(\Gamma_{21}^3)^2 = \lambda_2^2$. Finally, from Lemma 2, (3) we get that $1 - C^2(p_0) = 1$ which is a contradiction.

On the other hand, if $\frac{(1 - C^2(p_0))\Lambda_2}{4\lambda_2^2C^2(p_0)} + \frac{\lambda_3^2}{\lambda_3^2 - \lambda_2^2} = 0$, then (6.10) implies that

$$C^2(p_0) = \frac{(\lambda_3 + \lambda_2)^2(1 + 2\lambda_3\lambda_2)}{(\lambda_3 + \lambda_2)^2 + 2\lambda_3\lambda_2(\lambda_3^2 + \lambda_2^2)} > 1,$$

which is also a contradiction. Hence we have proved that $C(p_0) = 0$.

Secondly we prove that the scalar curvature $\rho = 1$. In this case, as

$$\Lambda_2(t) = b_2^2(t) - C(p_0)P_{22}(t) = 0,$$

(6.6) becomes in the following equations

$$\begin{aligned} (\Gamma_{21}^3)^2(t) &= \frac{\lambda_2(2\lambda_2\lambda_3 - 1)}{4\lambda_3}, \\ -2\lambda_2(\Gamma_{22}^1)^2(t) + 2(\lambda_3 - \lambda_2)(\Gamma_{12}^3)^2(t) &= \lambda_2(\lambda_2\lambda_3 + 1/2), \\ \frac{2\lambda_2^2}{\lambda_3}(\Gamma_{22}^1)^2(t) + 2(\lambda_3 - \lambda_2)(\Gamma_{12}^3)^2(t) &= -\lambda_3(\lambda_2\lambda_3 + 1/2), \end{aligned}$$

which implies that

$$(\Gamma_{22}^1)^2(t) = \frac{-\lambda_3(\lambda_2\lambda_3 + 1/2)}{2\lambda_2}, \quad (\Gamma_{12}^3)^2(t) = -\frac{(\lambda_2\lambda_3 + 1/2)}{2}.$$

As $(\Gamma_{22}^1)^2(t)$ and $(\Gamma_{12}^3)^2(t)$ are non-negative functions, the above equations say that $\lambda_2\lambda_3 + 1/2 = 0$ and so the scalar curvature of M is 1. \square

Corollary 4. *Let $\Phi : M \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be an orientable compact hypersurface with three different constant principal curvatures. If the scalar curvature ρ satisfies $2\rho \neq 1$ and the Gauss-Kronecker curvature $K = 0$, then $\Phi(M)$ is congruent to M_t for some $t \in (-1, 1)$.*

Proof. From Theorem 4, (1), the function C satisfies $C^2 < 1$. As M is compact, from Theorem 4, (2), the maximum and the minimum of C is zero. So $C \equiv 0$ and the result follows from Corollary 1. \square

References

- [BD] J. Berndt and J. C. Díaz-Ramos, *Real hypersurfaces with constant principal curvatures in the complex hyperbolic plane*, Proc. Amer. Math. Soc. **135** (2007), 3349–3357.
- [C] E. Cartan, *Sur les familles remarquables d’hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45** (1939), 335–367.
- [CU] I. Castro and F. Urbano, *Lagrangian surfaces of $\mathbb{S}^2 \times \mathbb{S}^2$* , Comm. Anal. and Geom. **1** (2007), 903–906.
- [K] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.

- [MP] W. H. Meeks III and J. Pérez, *Constant mean curvature surfaces in metric Lie groups*, Geometric Analysis **570** (2012), 25–110. Contemporary Mathematics, edited by J. Galvez and J. Pérez.
- [M] H. Münzner, *Isoparametrische hyperflächen in sphären I*, Math. Ann. **251** (1980), 57–71.
- [T] R. Takagi, *Real hypersurfaces in a complex projective space with constant principal curvatures*, J. Math. Soc Japan **27** (1975), 43–53.
- [TU] F. Torralbo and F. Urbano, *Surfaces with parallel mean curvature vector in $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{H}^2 \times \mathbb{H}^2$* , Trans. Amer. Math. Soc. **1** (2012), 903–906.

DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE GRANADA, 18071 GRANADA, SPAIN
E-mail address: `furbano@ugr.es`

RECEIVED JUNE 24, 2016

ACCEPTED JUNE 4, 2017