Communications in
Analysis and Geometry
Volume 27, Number 6, 1325–1354, 2019

A hyperbolic metric and stability conditions on K3 surfaces with $\rho = 1$

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We introduce a hyperbolic metric on the (normalized) space of stability conditions on projective K3 surfaces X with Picard rank $\rho(X) = 1$ and give some applications. We show that all walls are geodesic in the normalized space with respect to the hyperbolic metric. Furthermore we demonstrate how the hyperbolic metric is helpful for us by discussing some topics. As an application of the metric, we give an explicit example of stable complexes in large volume limits.

1. Introduction

In this article we introduce a hyperbolic metric on the (normalized) space of stability conditions on projective K3 surfaces X with Picard rank $\rho(X) = 1$ and give some applications.

The notion of *stability conditions* on arbitrary triangulated categories $\mathcal{D}$ has been introduced in [4]. By virtue of this we can define the "σ-stability" for objects $E \in \mathcal{D}$ with respect to a stability condition σ on $\mathcal{D}$.

Each connected component of the space $\mathrm{Stab}(\mathcal{D})$ of stability conditions on $\mathcal{D}$ is a complex manifold by [4]. The non-emptiness of $\mathrm{Stab}(\mathcal{D})$ is one of the biggest problems. For instance suppose $\mathcal{D}$ is the bounded derived category $D(M)$ of coherent sheaves on a projective manifold M. In the case of $\dim M = 1$, the non-emptiness of $\mathrm{Stab}(D(M))$ was proven in the original article [4]. Furthermore the space $\mathrm{Stab}(D(M))$ was studied in detail by [17], [4] and [15]. In the case of $\dim M = 2$, the non-emptiness was proven by [5] (K3 or abelian surfaces) and [1] (other surfaces). The case of $\dim M = 3$ is discussed in [2]. These are just a handful of many studies.

As we stated before, the space $\mathrm{Stab}(X)$ of stability conditions on the derived category $D(X)$ of a projective K3 surface X is not empty by [5]. This fact is proven by finding a distinguished connected component $\mathrm{Stab}^{\dagger}(X)$. It is conjectured the following:

Conjecture 1.1 ([5, Conjecture 1.2]). *The action of any equivalence* $\Phi \in \mathrm{Aut}(D(X))$ *on* $\mathrm{Stab}(X)$ *preserves the distinguished component* $\mathrm{Stab}^{\dagger}(X)$ *and* $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*

As was proven by [5] and [10], if the conjecture holds then we can determine the group structure of $\mathrm{Aut}(D(X))$ as follows: We have a covering map $\pi\colon \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$ by [5, Theorem 1.1] (see also Theorem 2.4) where $\mathcal{P}_0^+(X)$ is a certain connected subset of $H^*(X,\mathbb{C})$ (see also Section 2.1). By [5] and [10], Conjecture 1.1 implies the following the exact sequence of groups:

$$(1.1)\quad 1 \longrightarrow \pi_1(\mathcal{P}_0^+(X)) \longrightarrow \mathrm{Aut}(D(X)) \xrightarrow{\kappa} O^+_{\mathrm{Hodge}}(H^*(X,\mathbb{Z})) \longrightarrow 1,$$

where $O^+_{\mathrm{Hodge}}(H^*(X,\mathbb{Z}))$ is the group of Hodge isometries of $H^*(X,\mathbb{Z})$ which preserve the orientation of $H^*(X,\mathbb{Z})$. Hence Conjecture 1.1 predicts that the kernel $\mathrm{Ker}(\kappa)$ of the representation κ is given by the fundamental group $\pi_1(\mathcal{P}_0^+(X))$ and that $\mathrm{Aut}(D(X))$ is given by an extension of $\pi_1(\mathcal{P}_0^+(X))$ and $O^+_{\mathrm{Hodge}}(H^*(X,\mathbb{Z}))$.

Recall the right $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action on $\mathrm{Stab}(X)$ where $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is the universal cover of $\mathrm{GL}^+(2,\mathbb{R})$. We define $\mathrm{Stab}^{\mathrm{n}}(X)$ as the quotient of $\mathrm{Stab}^{\dagger}(X)$ by the right $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ action. We call it the *normalized stability manifold.* For a projective K3 surface with $\rho(X) = 1$, we first introduce a hyperbolic metric on $\mathrm{Stab}^{\mathrm{n}}(X)$. We also show that the hyperbolic metric is independent for Fourier-Mukai partners of X:

Theorem 1.2 (=Theorem 3.3). *Assume that* $\rho(X) = 1$.

(1) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is a hyperbolic 2 dimensional real manifold.*

(2) *Let* Y *be a Fourier-Mukai partner of* X *and* $\Phi\colon D(Y) \to D(X)$ *an equivalence which preserves the distinguished component* $\mathrm{Stab}^{\dagger}(X)$. *Then the induced morphism* $\Phi_*^{\mathrm{n}}\colon \mathrm{Stab}^{\mathrm{n}}(Y) \to \mathrm{Stab}^{\mathrm{n}}(X)$ *is an isometry with respect to the hyperbolic metric.*

Next, by using the hyperbolic structure, we observe the simply connectedness of $\mathrm{Stab}^{\dagger}(X)$:

Theorem 1.3 (=Theorem 4.1). *Let* X *be a projective K3 surface with* $\rho(X) = 1$. *The following three conditions are equivalent.*

(1) $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*

(2) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is isomorphic to the upper half plane* $\mathbb{H}$.

(3) *The squares* T_A^2 *of the spherical twists of all spherical locally free sheaves* A *generates a free subgroup* $W(X) \subset \mathrm{Aut}(D(X))$.

We give two remarks. Firstly we could not prove the simply connectedness. However by using the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$, we can deduce the global geometry not only of $\mathrm{Stab}^{\mathrm{n}}(X)$ but also of $\mathrm{Stab}^{\dagger}(X)$ as follows. Since $\mathrm{Stab}^{\dagger}(X)$ is a $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$-bundle over $\mathrm{Stab}^{\mathrm{n}}(X)$, we see $\mathrm{Stab}^{\dagger}(X)$ is simply connected if and only if it is a $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$-bundle over the upper half plane $\mathbb{H}$.

Secondly, if Conjecture 1.1 holds then we see the kernel $\mathrm{Ker}(\kappa)$ is generated by $W(X)$ and the double shift [2]. Since the double shift [2] commutes with any equivalence, the freeness of $W(X)$ implies that $\mathrm{Ker}(\kappa)/\mathbb{Z}[2]$ is free. However in higher Picard rank cases, it is thought that the generators of $\mathrm{Ker}(\kappa)/\mathbb{Z}[2]$ have relations (see also Remark 4.3). Hence the freeness of $W(X)$ is a special phenomenon.

In the third theorem we study chamber structures on $\mathrm{Stab}^{\dagger}(X)$ in terms of the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Before we state the third theorem, let us recall chamber structures.

For a set $\mathcal{S} \subset D(X)$ of objects each of which has bounded mass and an arbitrary compact subset $B \subset \mathrm{Stab}^{\dagger}(X)$, we can define a finite collection of real codimension 1 submanifolds $\{W_\gamma\}_{\gamma \in \Gamma}$ satisfying the following property:

- Let $C \subset B \setminus \bigcup_{\gamma \in \Gamma} W_\gamma$ be an arbitrary connected component. If $E \in \mathcal{S}$ is σ-semistable for some $\sigma \in C$ then E is τ-semistable for all $\tau \in C$.

Each W_γ is said to be a *wall* and each connected component C is said to be a *chamber*. In this paper we call all data of chambers and walls a *chamber structure*. We have to remark that chamber structures on $\mathrm{Stab}^\dagger(X)$ descend to the normalized stability manifold $\mathrm{Stab}^{\mathrm{n}}(X)$. Namely $C/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $\{W_\gamma/\widetilde{\mathrm{GL}}^+(2,\mathbb{R})\}$ also define a chamber structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Our third theorem is the following:

Theorem 1.4 (=Theorem 5.5). *Each wall of any chamber structure on* $\mathrm{Stab}^{\mathrm{n}}(X)$ *is geodesic.*

In generally Fourier-Mukai transformations on X may change chamber structures (This does not mean Fourier-Mukai transformations just permute chambers). By Theorems 3.3 and 5.5, we see that the image of walls by Fourier-Mukai transformations is also geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$. Applying this observation we show the following:

Proposition 1.5 (=Proposition 6.5). *Let X be a projective K3 surface with $\rho(X) = 1$ and Y a Fourier-Mukai partner of X with an equivalence $\Phi\colon D(Y) \to D(X)$.*

If the induced morphism $\Phi_*\colon \mathrm{Stab}(Y) \to \mathrm{Stab}(X)$ *preserves the distinguished component, then Y is isomorphic to the fine moduli space of Gieseker stable torsion-free sheaves.*

We have to mention that a stronger statement was already proven by Orlov in [18]: Any Fourier-Mukai partner of projective K3 surfaces X is isomorphic to a fine moduli space of Gieseker stable sheaves on X. Our proof never needs the global Torelli theorem which was essential for Orlov's proof. Hence our proof gives a new feature of stability condition: The theory of stability conditions substitutes for the global Torelli theorem. Since the strategy of Proposition 6.5 is technical, we will explain it in §6.1.

In [5, §14], it has been expected that the σ-stability in the large volume limit is equivalent to Gieseker twisted stability and has been given an evidence in [5, Proposition 14.2]. However the possibility of stable complexes in the large volume limit is mentioned in [3, Lemma 4.2].

In Corollary 7.3 we prove that a complex $T_A^{-1}(\mathcal{O}_x)$ is stable in the large volume limit where $T_A^{-1}(\mathcal{O}_x)$ is a spherical twist by a spherical locally free sheaf A. The complex $T_A^{-1}(\mathcal{O}_x)$ does not satisfy the assumption in [5, Proposition 14.2] but satisfies the condition in [3, Lemma 4.2].

2. Preliminaries

In this section we prepare basic notations and lemmas. A projective K3 surface with $\mathrm{NS}(X) = \mathbb{Z}L$ is denoted by (X, L). Though almost all notions are defined for general projective K3 surfaces, to simplify the explanations, we focus on K3 surfaces with $\rho(X) = 1$.

2.1. Terminologies

The abelian category of coherent sheaves on X is denoted by $\mathrm{Coh}(X)$. Recall that the numerical Grothendieck group $\mathcal{N}(X)$ is isomorphic to $H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z})$. Put $v(E) = ch(E)\sqrt{td_X}$ for $E \in D(X)$. Then we see $v(E) = r_E \oplus c_E \oplus s_E \in \mathcal{N}(X)$. One can easily check that $r_E = \mathrm{rank}\, E$, c_E is the first Chern class $c_1(E)$ and $s_E = \chi(X, E) - \mathrm{rank}\, E$. Hence for a vector $v = r \oplus c \oplus s \in \mathcal{N}(X)$, the component r is called the *rank* of v.

The Mukai pairing $\langle , \rangle$ on $H^*(X, \mathbb{Z})$ is given by

$$\langle r \oplus c \oplus s, r' \oplus c' \oplus s' \rangle = cc' - rs' - r's.$$

By the Riemann-Roch theorem we see

$$\chi(E, F) = \sum_i (-1)^i \dim \mathrm{Hom}^i_{D(X)}(E, F) = -\langle v(E), v(F) \rangle.$$

An object $A \in D(X)$ is said to be *spherical* if A satisfies

$$\mathrm{Hom}^i_{D(X)}(A, A) = \begin{cases} \mathbb{C} & (i = 0, 2) \\ 0 & (\text{otherwise}). \end{cases}$$

We note that $v(A)^2 = -2$ if A is spherical. By the effort of [19], for a spherical object A we can define the autoequivalence T_A called the *spherical twist* (see also [7, Chapter 8]). By the definition of T_A we have the following distinguished triangle for $E \in D(X)$:

$$\mathrm{Hom}^*_{D(X)}(A, E) \otimes A \xrightarrow{\mathrm{ev}} E \longrightarrow T_A(E), \tag{2.1}$$

where ev is the evaluation map. We note that the vector of $T_A(E)$ can be calculated by $v(T_A(E)) = v(E) + \langle v(E), v(A) \rangle v(A)$.

Let $\Delta(X)$ be the set of (-2)-vectors in $\mathcal{N}(X)$

$$\Delta(X) = \{\delta \in \mathcal{N}(X) \mid \delta^2 = -2\}$$

and let $\Delta^+(X)$ be the set $\{\delta \in \Delta(X) \mid \delta = r \oplus c \oplus s, r>0\}$. Following [5], put

$$\mathcal{P}(X) = \{v \in \mathcal{N}(X) \otimes \mathbb{C} \mid \mathfrak{Re}(v) \text{ and } \mathfrak{Im}(v) \text{ span a positive 2 plane}\}$$

Since $\mathcal{P}(X)$ has two connected components, we define $\mathcal{P}^+(X)$ by the connected component containing $\exp(\sqrt{-1}\omega)$ where ω is an ample class. Then $\mathcal{P}^+(X)$ has a right $\mathrm{GL}^+(2,\mathbb{R})$ action as the change of basis of the planes. Since the action is free, the quotient map $\mathcal{P}^+(X) \to \mathcal{P}^+(X)/\mathrm{GL}^+(2,\mathbb{R})$ gives a principle $\mathrm{GL}^+(2,\mathbb{R})$-bundle with a global section.

Under the assumption $\rho(X) = 1$, $\mathcal{P}^+(X)/\mathrm{GL}^+(2,\mathbb{R})$ is isomorphic to the set $\mathfrak{H}(X) = \{(\beta, \omega) = (xL, yL) \mid x + \sqrt{-1}y \in \mathbb{H}\}$. Clearly $\mathfrak{H}(X)$ is canonically isomorphic to $\mathbb{H}$. Then the global section $\mathfrak{H}(X) \to \mathcal{P}^+(X)$ is given by

$$\mathfrak{H}(X) \ni (x, y) \mapsto \exp(\beta + \sqrt{-1}\omega) \in \mathcal{P}^+(X). \tag{2.2}$$

In particular $\mathcal{P}^+(X)$ is isomorphic to $\mathbb{H} \times \mathrm{GL}^+(2,\mathbb{R})$. We define $\mathcal{P}_0^+(X)$ as

$$\mathcal{P}_0^+(X) = \mathcal{P}^+(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^\perp$$

where $\langle \delta \rangle^\perp$ is the orthogonal complement of δ with respect to the Mukai pairing on $H^*(X, \mathbb{Z})$[1]. Define

$$\mathfrak{H}_0(X) = \{(\beta, \omega) \in \mathfrak{H}(X) \mid \langle \exp(\beta + \sqrt{-1}\omega), \delta \rangle \neq 0 \ (\forall \delta \in \Delta(X))\}.$$

Then we see that $\mathcal{P}_0^+(X)$ is isomorphic to $\mathfrak{H}_0(X) \times \mathrm{GL}^+(2,\mathbb{R})$.

2.2. Stability conditions on K3 surfaces

Let $\mathrm{Stab}(X)$ be the set of numerical locally finite stability conditions on $D(X)$. We put $\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}(X)$ where $\mathcal{A}$ is the heart of a bounded

[1] We remark that the definition of $\mathcal{P}_0^+(X)$ is independent of the assumption $\rho(X) = 1$.

t-structure on $D(X)$ and Z is a central charge. Since the Mukai paring is non-degenerate on $\mathcal{N}(X)$ we have the natural map:

$$\pi\colon \operatorname{Stab}(X) \to \mathcal{N}(X) \otimes \mathbb{C},\ \pi(\sigma) = Z^{\vee}$$

where $Z(E) = \langle Z^{\vee}, v(E)\rangle$.

In $\operatorname{Stab}(X)$, there is a connected component $\operatorname{Stab}^{\dagger}(X)$ which contains the set $U(X)$:

$$U(X) = \Big\{\sigma = (\mathcal{A}, Z) \in \operatorname{Stab}(X) \mid Z^{\vee} \in \mathcal{P}(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^{\perp},$$
$$\mathcal{O}_x \text{ is } \sigma\text{-stable in the same phase for all } x \in X\Big\}.$$

The closure $\bar{U}(X)$ of $U(X)$ in $\operatorname{Stab}(X)$ is the set of stability conditions σ such that $\mathcal{O}_x$ $(\forall x \in X)$ is σ-semistable in the same phase with $Z^{\vee} \in \mathcal{P}(X) \setminus \bigcup_{\delta \in \Delta(X)} \langle \delta \rangle^{\perp}$. The set $\partial U(X) = \bar{U}(X) \setminus U(X)$ is known as the *boundary of* $U(X)$.

Denote by $V(X)$ the set

$$V(X) = \{\sigma = (\mathcal{A}, Z) \in U(X) \mid Z(\mathcal{O}_x) = -1,\ \mathcal{O}_x \text{ is } \sigma\text{-stable with phase } 1\}.$$

One can see $U(X) = V(X) \cdot \widetilde{\mathrm{GL}}^{+}(2, \mathbb{R}) \cong V(X) \times \widetilde{\mathrm{GL}}^{+}(2, \mathbb{R})$ by [5, Proposition 10.3]. Furthermore the set $V(X)$ is parametrized by $(\beta, \omega) \in \mathfrak{H}(X)$ in the following way: An element $\sigma_{(\beta,\omega)} \in V(X)$ is given by the pair $(\mathcal{A}_{(\beta,\omega)}, Z_{(\beta,\omega)}) \in V(X)$ where $\mathcal{A}_{(\beta,\omega)}$ and $Z_{(\beta,\omega)}$ are defined as follows:

$$\mathcal{A}_{(\beta,\omega)} := \left\{ E^{\bullet} \in D(X) \;\middle|\; H^i(E^{\bullet}) \begin{cases} \in \mathcal{T}_{(\beta,\omega)} & (i = 0) \\ \in \mathcal{F}_{(\beta,\omega)} & (i = -1) \\ = 0 & (\text{otherwise}) \end{cases} \right\}$$
$$Z_{(\beta,\omega)}(E) := \langle \exp(\beta + \sqrt{-1}\omega), v(E)\rangle,$$

where

$$\mathcal{T}_{(\beta,\omega)} := \{E \in \operatorname{Coh}(X) \mid E \text{ is a torsion sheaf or } \mu_{\omega}^{-}(E/\text{torsion}) > \beta\omega\} \text{ and}$$
$$\mathcal{F}_{(\beta,\omega)} := \{E \in \operatorname{Coh}(X) \mid E \text{ is torsion-free and } \mu_{\omega}^{+}(E) \leq \beta\omega\}.$$

Here $\mu_{\omega}^{+}(E)$ (resp. $\mu_{\omega}^{-}(E)$) is the maximal slope (resp. minimal slope) of semistable factors of a torsion-free sheaf E with respect to the slope stability.

Since the pair $(\mathcal{T}_{(\beta,\omega)}, \mathcal{F}_{(\beta,\omega)})$ is a torsion pair on $\mathrm{Coh}(X)$, $\mathcal{A}_{(\beta,\omega)}$ is the heart of a bounded t-structure on $D(X)$.

Proposition 2.1 ([5, §6, §7 and Proposition 10.3]). *Assume that (β, ω) satisfies the condition*

$$\langle \exp(\beta + \sqrt{-1}\omega), \delta\rangle \notin \mathbb{R}_{\leq 0}, (\forall \delta \in \Delta^{+}(X)) \tag{2.3}$$

Then the pair $\sigma_{(\beta,\omega)}$ gives a numerical locally finite stability condition on $D(X)$. Furthermore we have

$$V(X) = \{\sigma_{(\beta,\omega)} \in \mathrm{Stab}^{\dagger}(X) \mid (\beta, \omega) \textit{ satisfies the condition } (2.3)\}.$$

Definition 2.2. For a projective K3 surface with $\rho(X) = 1$ we define the subgroup $W(X)$ of $\mathrm{Aut}(D(X))$ as follows

$$W(X) = \langle T_A^2 \mid A = \text{spherical locally free sheaf}\rangle.$$

Then by using $U(X)$ and $W(X)$ we can describe $\mathrm{Stab}^{\dagger}(X)$ in a explicit way:

Proposition 2.3 ([5, Proposition 13.2]). *Let X be a projective K3 with $\rho(X) = 1$. The distinguished connected component $\mathrm{Stab}^{\dagger}(X)$ is given by*

$$\mathrm{Stab}^{\dagger}(X) = \bigcup_{\Phi \in W(X)} \Phi_*(\bar{U}(X)).$$

We remark that the original statement [5, Proposition 13.2] is different from the above, but the above statement is essentially proved in the proposition.

Theorem 2.4 ([5, Proposition 13.2 and Theorem 13.3]). *The natural map $\pi : \mathrm{Stab}^{\dagger}(X) \to \mathcal{N}(X) \otimes \mathbb{C}$ has the image $\mathcal{P}_0^{+}(X)$. Furthermore π is a Galois covering. The covering transformation group is the subgroup generated by equivalences in $\mathrm{Ker}(\kappa)$ which preserve $\mathrm{Stab}^{\dagger}(X)$. Here we recall that κ is the natural representation $\kappa\colon \mathrm{Aut}(D(X)) \to O^{+}_{\mathrm{Hodge}}(H^*(X, \mathbb{Z}))$.*

Corollary 2.5. *For a pair (X, L), the induced map*

$$\pi^{\mathrm{n}}\colon \mathrm{Stab}^{\mathrm{n}}(X) \to \mathfrak{H}_0^{+}(X)$$

is also a Galois covering map.

Proof. We have the following $\mathrm{GL}^+(2,\mathbb{R})$-equivariant diagram:

$$\begin{array}{ccc} \mathrm{Stab}^\dagger(X)/\mathbb{Z}[2] & \xrightarrow{\pi'} & \mathcal{P}_0^+(X) \\ \downarrow & & \downarrow \\ \mathrm{Stab}^{\mathrm{n}}(X) & \xrightarrow{\pi^{\mathrm{n}}} & \mathfrak{H}_0^+(X). \end{array}$$

We note that both vertical maps are $\mathrm{GL}^+(2,\mathbb{R})$-bundles and that π' is also a Galois covering.

By Theorem 2.4 the covering transformation group of π' is a subgroup of $\mathrm{Aut}(D(X))/\mathbb{Z}[2]$. Hence the right $\mathrm{GL}^+(2,\mathbb{R})$-action on $\mathrm{Stab}^\dagger(X)/\mathbb{Z}[2]$ commutes with the covering transformations. Hence π^{n} is also a Galois covering. □

2.3. On the fundamental group of $\mathcal{P}_0^+(X)$

We are interested in the fundamental group $\pi_1(\mathcal{P}_0^+(X))$. Generally speaking, it is difficult to describe the above condition (2.3) explicitly. Because of this difficulty, it becomes difficult to determine the relation between generators of $\pi_1(\mathcal{P}_0^+(X))$. However, under the assumption $\rho(X)=1$ it becomes easier.

Definition 2.6. For $\delta = r \oplus c \oplus s \in \Delta(X)$ there is a unique point $p \in \mathfrak{H}(X)$ such that $\langle \exp(p), \delta \rangle = 0$. We denote the point by $p(\delta)$ and call it a *spherical point*. If δ is the Mukai vector of a spherical object A we denote simply $p(v(A))$ by $p(A)$.

Remark 2.7. Explicitly

$$p(\delta) = \left(\frac{c}{r}, \frac{1}{\sqrt{d}|r|} L \right) \in \mathfrak{H}(X)$$

where $\delta = r \oplus c \oplus s$. This makes sense because $c^2 \geq 0$ so $r \neq 0$, and we have $\Delta(X) = \Delta^+(X) \sqcup -\Delta^+(X)$. Note that $p(\delta) = p(-\delta)$.

Let $\delta = r \oplus c \oplus s \in \Delta^+(X)$. Define the subset $\mathcal{V}(X)$ of $\mathfrak{H}(X)$ as follows.

$$\mathcal{V}(X) = \{(\beta,\omega) \in \mathfrak{H}(X) \mid (\beta,\omega) \text{ satisfies the condition (2.3)}\}.$$

As we remarked in Proposition 2.1, the natural morphism π provides an isomorphism of this set and $V(X)$ as follows: $\pi\colon V(X) \ni \sigma_{(\beta,\omega)} \mapsto Z_{(\beta,\omega)}$ identified with $(\beta,\omega) \in \mathfrak{H}(X)$.

Remark 2.8. Let X be a projective K3 surface (not necessary of Picard rank one). For any $\delta = r \oplus c \oplus s \in \Delta(X)$ with $r \geq 0$, there exists a spherical sheaf A on X such that $v(A) = \delta$ by [13]. In particular if $r > 0$ then we can take A as a locally free sheaf. In addition if we assume $\mathrm{NS}(X) = \mathbb{Z}L$ then we see A is Gieseker-stable by [16, Proposition 3.14]. Since we see $\gcd(r, n) = 1$ where n satisfies $nL = c$, A is μ-stable by [9, Lemma 1.2.14].

Lemma 2.9. *Notations being as above, the set* $\mathfrak{S} = \{p(\delta) \in \mathfrak{H}(X) \mid \delta \in \Delta(X)\}$ *is a discrete set in* $\mathfrak{H}(X)$*. Moreover the set* $\mathcal{V}(X)$ *is open in* $\mathfrak{H}(X)$*.*

Proof. The assertion is a special case of [5, Lemma 11.1]. □

Definition 2.10. Take a base point $p_0 = (0, \omega_0) \in \mathfrak{H}(X)$ such that $\omega_0^2 \gg 2$. We define elements of the fundamental groups $\pi_1(\mathfrak{H}_0(X), p_0)$ and of $\pi_1(\mathrm{GL}^+(2, \mathbb{R}))$ as follows.

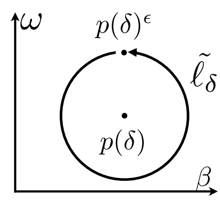

Figure 1: For $p(\delta)$ we define the loop ℓ_δ as the above direction. We also assume that there are no spherical points $p(\delta')$ in the interior of ℓ_δ except for $p(\delta)$ itself.

(1) For a spherical point $p(\delta)$, take a sufficiently small $\epsilon > 0$ such that there is no spherical point in the ball centered at $p(\delta)$ with radius ϵ (with respect to usual metric). The point $p(\delta) + (0, \epsilon L)$ is denoted by $p(\delta)^\epsilon$. We define the loop $\tilde{\ell}_\delta$ as the loop turning round from $p(\delta)^\epsilon$ counterclockwise (see also Figure 1).

The point $p(\delta)^\epsilon$ is in the set $\mathcal{V}(X)$ which is contractible by [5, Lemma 11.1]. Hence there is a unique path γ from p_0 to $p(\delta)^\epsilon$ up to homotopy. To change the base point of ℓ_δ, we define ℓ_δ by the product $\gamma\tilde{\ell}_\delta\gamma^{-1}$ of paths. The homotopy equivalent class of loop ℓ_δ is in $\pi_1(\mathfrak{H}(X), p_0)$.

(2) We define $g \in \pi_1(\mathrm{GL}^+(2, \mathbb{R}))$ by

$$g\colon [0,1] \ni t \mapsto \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \in \mathrm{GL}^+(2, \mathbb{R}).$$

We note that g is a generator of $\pi_1(\mathrm{GL}^+(2, \mathbb{R}))$.

Proposition 2.11. *The fundamental group $\pi_1(\mathcal{P}_0^+(X))$ is isomorphic to*

$$\Big(\mathop{\ast}_{\delta\in\Delta^+(X)}\mathbb{Z}\cdot\ell_\delta\Big)\times\mathbb{Z}\cdot g$$

where $\ast_{\delta\in\Delta^+}\mathbb{Z}\cdot\ell_\delta$ is the free product of infinite cyclic groups $\mathbb{Z}$ generated by ℓ_δ.

Proof. Since $\mathcal{P}_0^+(X)$ is isomorphic to $\mathfrak{H}_0(X)\times\mathrm{GL}^+(2,\mathbb{R})$ we see $\pi_1(\mathcal{P}_0^+(X))\cong\pi_1(\mathfrak{H}_0(X))\times\mathbb{Z}\cdot g$. As we remarked before we have

$$\Delta(X)=\Delta^+(X)\sqcup(-\Delta^+(X)).$$

Hence we see

$$\mathfrak{H}_0(X)=\mathfrak{H}(X)\setminus\bigcup_{\delta\in\Delta(X)}\langle\delta\rangle^\perp=\mathfrak{H}(X)\setminus\bigcup_{\delta\in\Delta^+(X)}\langle\delta\rangle^\perp$$

Since $\mathfrak{H}_0(X)$ is isomorphic to $\mathfrak{H}_0(X)$ it is enough to show that

$$\pi_1(\mathfrak{H}_0(X))=\mathop{\ast}_{\delta\in\Delta^+}\mathbb{Z}\cdot\ell_\delta$$

We choose a base point p of $\mathfrak{H}_0(X)$ so that $p=\sqrt{-1}\omega$ with $\omega^2\gg 2$. Take a loop ℓ whose base point is p. Then there is a compact contractible subset C whose interior C^{in} contains ℓ. Then the set $\{p(\delta)\in C^{in}\mid\delta\in\Delta^+(X)\}$ of spherical points in C^{in} is finite. Since the fundamental group of the complement of n-points in C is the free group of rank n, we see the homotopy equivalence class of ℓ is given by

$$\ell_{\delta_1}^{k_1}\ell_{\delta_2}^{k_2}\cdots\ell_{\delta_m}^{k_m}$$

where each $k_i\in\mathbb{Z}$. In fact if another loop ℓ' is homotopy equivalent to ℓ by $H\colon[0,1]\times[0,1]\to\mathfrak{H}_0(X)$, then there is a contractible compact set C' such that $(C')^{in}$ contains the image of H. Since there are at most finitely many spherical point in $(C')^{in}$, we see the above representation is unique. Thus we have finished the proof. □

To simplify the notations we denote $\ell_{v(A)}$ by ℓ_A. By Remark 2.8, we see

$$\pi_1(\mathfrak{H}_0(X))\cong\langle\ell_A\mid A\text{ is spherical and locally free}\rangle\cong\mathop{\ast}_{A}\mathbb{Z}\ell_A.$$

3. Hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$

Let $\mathrm{Stab}^{\dagger}(X)$ be the connected component of $\mathrm{Stab}(X)$ introduced in §2. In this section we discuss a hyperbolic structure on the normalized stability manifold $\mathrm{Stab}^{\mathrm{n}}(X) = \mathrm{Stab}^{\dagger}(X)/\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$.

To simplify explanations of this section we always use the following notations. Let (X_i, L_i) $(i = 1, 2)$ be projective K3 surfaces with $\mathrm{NS}(X_i) = \mathbb{Z}L_i$ and let $\Phi\colon D(X_2) \to D(X_1)$ be an equivalence between them. The isometry $\mathcal{N}(X_2) \to \mathcal{N}(X_1)$ induced by Φ is denoted by $\Phi^{\mathcal{N}}$.

Remark 3.1. Let X be a projective K3 surface with $\rho = 1$. Then the lattice $\mathcal{N}(X)$ is canonically isomorphic to the abstract lattice

$$N = \left(\mathbb{Z}^{\oplus 3}, \Sigma_d = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2d & 0 \\ -1 & 0 & 0 \end{pmatrix}\right).$$

Thus we can identify $\mathcal{N}(X)$ with N.

Lemma 3.2. *Let $\Phi\colon D(X_2) \to D(X_1)$ be an equivalence between the pair $\{(X_i, L_i)\}_{i=1}^{2}$. Then the equivalence Φ induces an isometry between hyperbolic space $\varphi\colon \mathfrak{H}(X_2) \to \mathfrak{H}(X_1)$.*

Proof. We recall that $\mathfrak{H}(X_i)$ $(i = 1, 2)$ is isomorphic to the upper half plane $\mathbb{H}$. It is well-known that the hyperbolic metric on $\mathbb{H}$ is given by the following construction.

Let $P(N)$ be the set of vectors $v \in N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ such that $v^2 = -1$. Then $P(N)$ has 2 connected components which are isomorphic to $\mathbb{H}$. To fix a component, take $P^{+}(N)$ containing the vector $\sqrt{2}^{-1}(1 \oplus 1 \oplus d+1)$. The orthogonal complement $v^{\perp}$ of the vector v in $N_{\mathbb{R}}$ is a tangent space of $P(N)$ at v and the restriction of Σ_d to $v^{\perp}$ is positive definite. Thus $P(N)$ is a Riemannian manifold. If f is an isometry of N preserving the orientation of positive 2-planes in N, then f induces an isometry of $P^{+}(N)$.

Note that $\mathcal{N}(X_i)$ is canonically isomorphic to the lattice N. Since any equivalence $\Phi\colon D(X_2) \to D(X_1)$ induces an isometry $\Phi^{\mathcal{N}}\colon \mathcal{N}(X_2) \to \mathcal{N}(X_1)$ preserving the orientation by [10], the morphism $\Phi^{\mathcal{N}}$ induces an isometry $\varphi\colon \mathfrak{H}(X_2) \to \mathfrak{H}(X_1)$. □

Theorem 3.3. *Assume that $\rho(X) = 1$.*

(1) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is a hyperbolic 2 dimensional manifold.*

(2) *Let Y be a Fourier-Mukai partner of X and $\Phi\colon D(Y) \to D(X)$ an equivalence. Suppose that Φ preserves the distinguished component* $\mathrm{Stab}^{\dagger}(X)$*. Then the induced morphism* $\Phi_*^{\mathrm{n}}\colon \mathrm{Stab}^{\mathrm{n}}(Y) \to \mathrm{Stab}^{\mathrm{n}}(X)$ *is an isometry with respect to the hyperbolic metric.*

Proof. By Corollary 2.5, we have the normalized covering map

$$\pi^{\mathrm{n}}\colon\ \mathrm{Stab}^{\mathrm{n}}(X) \to \mathfrak{H}_0(X).$$

Since $\mathfrak{H}_0(X)$ is isomorphic to an open subset of $\mathbb{H}$ by Lemma 2.9, we can define the hyperbolic metric on $\mathfrak{H}_0(X)$. Since π^{n} is a covering map, we can also define the hyperbolic metric on $\mathrm{Stab}^{\mathrm{n}}(X)$. Thus $\mathrm{Stab}^{\mathrm{n}}(X)$ is hyperbolic. □

4. Simply connectedness of $\mathrm{Stab}^{\mathrm{n}}(X)$

In this section we always assume $\rho(X) = 1$. Then, as was shown in the previous section, $\mathrm{Stab}^{\mathrm{n}}(X)$ is a hyperbolic manifold. By using the hyperbolic structure, we shall discuss the simply connectedness of $\mathrm{Stab}^{\dagger}(X)$. Namely we show the following:

Theorem 4.1. *The following conditions are equivalent.*

(1) $\mathrm{Stab}^{\dagger}(X)$ *is simply connected.*

(2) $\mathrm{Stab}^{\mathrm{n}}(X)$ *is isomorphic to the upper half plane* $\mathbb{H}$.

(3) *The squares T_A^2 of the spherical twists of all spherical locally-free sheaves A generates a free group $W(X) \subset \mathrm{Aut}(D(X))$.*

Proof. We first show that $\mathrm{Stab}^{\dagger}(X)$ is simply connected if and only if $\mathrm{Stab}^{\mathrm{n}}(X)$ is simply connected. Since the right action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\mathrm{Stab}^{\dagger}(X)$ is free, the natural map $\mathrm{Stab}^{\dagger}(X) \to \mathrm{Stab}^{\mathrm{n}}(X)$ is a $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-bundle. Thus there is an exact sequence of fundamental groups:

$$\pi_1(\widetilde{\mathrm{GL}}^+(2,\mathbb{R})) \longrightarrow \pi_1(\mathrm{Stab}^{\dagger}(X)) \longrightarrow \pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) \longrightarrow 1.$$

Since $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is simply connected we see that $\pi_1(\mathrm{Stab}^{\dagger}(X)) = \{1\}$ if and only if $\pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) = \{1\}$.

Since $\mathrm{Stab}^{\mathrm{n}}(X)$ is a hyperbolic and complex manifold, $\mathrm{Stab}^{\mathrm{n}}(X)$ is isomorphic to $\mathbb{H}$ if and only if $\pi_1(\mathrm{Stab}^{\mathrm{n}}(X)) = \{1\}$ by the Riemann mapping theorem. Thus we have proved that the first condition is equivalent to the second one.

We secondly show that the first condition is equivalent to the third one. Let $\mathrm{Cov}(\pi)$ be the covering transformation group of $\pi\colon \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$. We denote the group $\tilde{W}(X)$ generated by $W(X)$ and by the double shift [2]. Note that $\tilde{W}(X)$ is isomorphic to $W(X) \times \mathbb{Z} \cdot [2]$.

We claim that $\tilde{W}(X)$ is isomorphic to $\mathrm{Cov}(\pi)$. Recall that each spherical sheaf A on X with $\rho(X) = 1$ is μ-stable by Remark 2.8. Hence any $\Phi \in \tilde{W}(X)$ gives a trivial action on $H^*(X,\mathbb{Z})$ and preserves the connected component $\mathrm{Stab}^{\dagger}(X)$. Thus Φ gives a covering transformation by [5, Theorem 13.3]. Thus we have a group homomorphism $\tilde{W}(X) \to \mathrm{Cov}(\pi)$. In particular by Proposition 2.3, we see this morphism is a surjection. Furthermore as is shown in [5, Theorem 13.3], this is injective. Thus we have proved our claim.

Since the covering $\pi\colon \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^+(X)$ is a Galois covering, we have the exact sequence of groups:

$$1 \longrightarrow \pi_1(\mathrm{Stab}^{\dagger}(X)) \longrightarrow \pi_1(\mathcal{P}_0^+(X)) \xrightarrow{\varphi} \mathrm{Cov}(\pi) \longrightarrow 1.$$

As will be shown in Proposition 5.4 we see $\varphi(\ell_A) = T_A^2$ and $\varphi(g) = [2]$. If $\mathrm{Stab}^{\dagger}(X)$ is simply connected then φ is an isomorphism. Hence $W(X)$ is the free group generated by T_A^2. Conversely if $W(X)$ is a free group generated by T_A^2, then φ is an isomorphism. Hence $\mathrm{Stab}^{\dagger}(X)$ is simply connected. □

Remark 4.2. Since the quotient map $\mathrm{Stab}^{\dagger}(X) \to \mathrm{Stab}^{\mathrm{n}}(X)$ is a $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-bundle, we see that $\mathrm{Stab}^{\dagger}(X)$ is simply connected if and only if $\mathrm{Stab}^{\dagger}(X)$ is a $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-bundle over $\mathbb{H}$. Thus we can deduce the global geometry of the stability manifold $\mathrm{Stab}^{\dagger}(X)$.

Remark 4.3. We give some remarks for $W(X)$. Recall that any equivalence $\Phi \in \mathrm{Aut}(D(X))$ induces the Hodge isometry Φ^H of $H^*(X,\mathbb{Z})$ in a canonical way. If Bridgeland's conjecture holds, the group $W(X) \times \mathbb{Z}[2]$ is the kernel $\mathrm{Ker}(\kappa)$ of the natural map

$$\kappa\colon \mathrm{Aut}(D(X)) \to O^+_{\mathrm{Hodge}}(H^*(X,\mathbb{Z}))\colon \Phi \to \Phi^H.$$

Moreover $\mathrm{Ker}(\kappa)$ is given by $\pi_1(\mathcal{P}_0^+(X))$. The freeness of $W(X)$ means that any two orthogonal complements $\langle \delta_1 \rangle^{\perp}$ and $\langle \delta_2 \rangle^{\perp}$ (where δ_1 and $\delta_2 \in \Delta(X)$) do not meet each other in $\mathcal{P}_0^+(X)$.

In more general situations (namely the case of $\rho(X) \geq 2$) there could be some orthogonal complements such that $\langle \delta_1 \rangle^\perp$ and $\langle \delta_2 \rangle^\perp$ meet each other. Hence we expect that the quotient group $\mathrm{Ker}(\kappa)/\mathbb{Z} \cdot [2]$ is not a free group.

5. Walls and the hyperbolic structure

Let X be a projective K3 surface with Picard rank one. We have two goals of this section. The first aim is to show Proposition 5.4 which is necessary for Theorem 4.1. The second aim is to show that any wall is geodesic.

Now we start this section from the following key lemma.

Lemma 5.1. *Any $\sigma \in \partial U(X)$ is in a general position (See also [5, §12]). Namely the point σ lies on only one irreducible component of $\partial U(X)$.*

Before we start the proof, we remark that Maciocia proved a similar assertion in a slightly different situation in [14].

Proof. Suppose that there is an element $\sigma = (\mathcal{A}, Z) \in \partial U(X)$ which is not general. Let W_1 and W_2 be two irreducible components of $\partial U(X)$ such that $\sigma \in W_1 \cap W_2$. By [5, Proposition 9.3] we may assume that all $\tau_1 \in W_1 \setminus \{\sigma\}$ and all $\tau_2 \in W_2 \setminus \{\sigma\}$ are in general positions in a sufficiently small neighborhood of σ. Then, by [5, Theorem 12.1] and the assumption $\rho = 1$, $W_1 \setminus \{\sigma\}$ and $W_2 \setminus \{\sigma\}$ are type $(A_\pm)$-walls. Hence there are two (-2)-vectors $\delta_i \in \Delta^+(X)$ $(i = 1, 2)$ such that for any $\tau_i = (\mathcal{A}_i, Z_i) \in W_i \setminus \{\sigma\}$ the imaginary part $\mathfrak{Im} Z_i(\mathcal{O}_x)\overline{Z_i(\delta_i)} = 0$ where $i \in \{1, 2\}$ and $x \in X$. Since these are closed conditions, the central charge Z of σ also satisfies the following condition:

$$\mathfrak{Im} Z(\mathcal{O}_x)\overline{Z(\delta_1)} = \mathfrak{Im} Z(\mathcal{O}_x)\overline{Z(\delta_2)} = 0. \tag{5.1}$$

Since $\mathrm{NS}(X) = \mathbb{Z}L$, there exists $g \in \mathrm{GL}^+(2, \mathbb{R})$ such that $Z'(E) := g^{-1} \circ Z(E) = \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$ where $(\beta, \omega) \in \mathfrak{H}(X)$.

Now we put $\delta_i = r_i \oplus n_i L \oplus s_i$. Note that $r_i \neq 0$ since $n_i^2 L_i^2 \geq 0$. Since $Z'(\mathcal{O}_x) = -1$ we see $\mathfrak{Im} Z'(\delta_i)$ is zero by the condition (5.1). Thus we see

$$\frac{n_1 L}{r_1} = \frac{n_2 L}{r_2} = \beta.$$

Since $\delta_i^2 = -2$ we see $\gcd(r_i, n_i) = 1$. Hence we have $\delta_1 = \delta_2$. This contradicts $W_1 \neq W_2$. □

By Lemma 5.1 and [5, Theorem 12.1] we see $\partial U(X)$ is a disjoint union of real codimension 1 submanifolds:

$$\partial U(X) = \coprod_{A\text{:spherical locally free}} (W_A^+ \sqcup W_A^-),$$

where W_A^+ (respectively W_A^-) is the set of stability conditions whose type is (A^+) (respectively (A^-)). We remark that there are no type (C_k)-walls by the assumption $\rho = 1$. In the following we give an explicit description of each component $W_A^{\pm}$.

Lemma 5.2. *Let X be a projective K3 surface with* $\mathrm{NS}(X) = \mathbb{Z}L$ *and let A be a spherical locally free sheaf. We put $v(A) = r_A \oplus n_A L \oplus s_A$ and define the set $S(v(A))$ by*

$$S(v(A)) = \left\{ (\beta, \omega) \in \mathfrak{H}(X) \mid \beta = \frac{n_A L}{r_A}, 0 < \omega^2 < \frac{2}{r_A^2} \right\}.$$

Then $W_A^{\pm}$ is isomorphic to $S(v(A)) \times \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. In particular $W_A^{\pm}/\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ is a hyperbolic segment spanning two points in $\mathrm{Stab}^{\mathrm{n}}(X)$ *which is isomorphic to $S(v(A))$.*

Proof. We have to consider two cases: $\sigma \in W_A^+$ or $\sigma \in W_A^-$. Since the proof is similar, we give the proof only for the case $\sigma \in W_A^+$.

Since $\sigma \in W_A^+$, the Jordan-Hölder filtration of $\mathcal{O}_x$ is given by the triangle (2.1)

$$A^{\oplus r_A} \longrightarrow \mathcal{O}_x \longrightarrow T_A(\mathcal{O}_x). \tag{5.2}$$

By applying T_A^{-1} to the triangle (5.2) we have

$$A^{\oplus r_A}[1] \longrightarrow T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x. \tag{5.3}$$

Thus $\mathcal{O}_x$ is $T_{A*}^{-1}(\sigma)$-stable. Hence $T_{A*}^{-1}(\sigma)$ is in $U(X)$.

Now we put $T_{A*}^{-1}(\sigma) = \tau = (\mathcal{A}, Z)$. Since $Z(A[1])/Z(\mathcal{O}_x) \in \mathbb{R}_{>0}$, we see that τ is in the set

$$W' = \left\{ \sigma_{(\beta,\omega)} \in V(X) \mid \beta = \frac{n_A L}{r_A}, \frac{2}{r_A^2} < \omega^2 \right\} \cdot \widetilde{\mathrm{GL}}^+(2, \mathbb{R}).$$

Thus we see $W_A^+ \subset T_{A*}(W')$. To show the inverse inclusion, let $\tau' = (\mathcal{A}', Z')$ be in W'. As we remarked in Remark 2.8, A is μ-stable locally free sheaf.

Then $A[1]$ has no nontrivial subobject in $\mathcal{A}'$ by [8, Theorem 0.2]. Hence $A[1]$ is τ'-stable, in particular, with phase 1. Since $T_A^{-1}(\mathcal{O}_x)$ is given by the extension (5.3) of $\mathcal{O}_x$ and $A^{\oplus r_A}[1]$, the object $T_A^{-1}(\mathcal{O}_x)$ is strictly τ'-semistable. Thus by applying T_A to the triangle (5.3), we obtain the Jordan-Hölder filtration (5.2). Hence we see $W_A^+ = T_{A*}(W')$.

Since the morphism induced on $\mathfrak{H}(X)$ by T_A is given by Lemma 3.2, we see

$$W_A^+ = T_{A*}W' \cong S(v(A)) \times \widetilde{\mathrm{GL}}^+(2,\mathbb{R}). \qquad \square$$

Let us denote $\bar{\mathfrak{H}}(X)$ be the closure of $\mathfrak{H}(X)$. For a spherical locally free sheaf A we define the point $q = p(T_A(\mathcal{O}_x)) \in \bar{\mathfrak{H}}(X)$ by $(\beta,\omega) = (\frac{c_1(A)}{r_A}, 0)$. By the simple calculation we see that

$$\langle \exp(q), v(T_A(\mathcal{O}_x))\rangle = 0.$$

Thus in the sense of Definition 2.6, $p(T_A(\mathcal{O}_x))$ could be regarded as the associated point of the isotropic vector $v(T_A(\mathcal{O}_x))$. In view of this we define the following notion:

Definition 5.3. An associated point $p \in \bar{\mathfrak{H}}(X)$ with a primitive isotropic vector $v \in \mathcal{N}(X)$ is the point which satisfies

$$\langle \exp(p), v\rangle = 0.$$

Clearly if $v = r \oplus nL \oplus s$ then p is given by $\frac{n}{r}$. In particuclar if $v = 0 \oplus 0 \oplus 1$ the associated point is $\infty \in \bar{\mathfrak{H}}(X)$. We denote the point by $p(v)$.

As an application of Lemma 5.2 we complete the proof of Theorem 4.1 by proving:

Proposition 5.4. *Let $\varphi\colon \pi_1(\mathcal{P}_0^+(X)) \to \mathrm{Cov}(\pi)$ be the morphism in the proof of Theorem 4.1. Recall that ℓ_A is the loop which turns around the associated point $p(v(A))$ and g is the generator of $\pi_1(\mathrm{GL}^+(2,\mathbb{R}))$. Then $\varphi(\ell_A) = T_A^2$ and $\varphi(g) = [2]$.*

Proof. We set a base point of $\pi_1(\mathfrak{H}_0(X))$ as $\sqrt{-1}\omega_0$ with $\omega_0^2 \gg 2$. We also define a base point of $\pi_1(\mathcal{P}_0^+(X))$ by $\exp(\sqrt{-1}\omega_0)$. Let $\sigma_0 = \sigma_{(0,\omega_0)} \in V(X)$ be a base point of the covering map $\pi\colon \mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X)$.

Let $\ell_A\colon [0,1] \to \mathfrak{H}_0(X)$ be the loop defined in Definition 2.10 which turns round the point $p(v(A))$ and let $\tilde{\ell}_A$ be the lift of ℓ_A to $\mathrm{Stab}^\dagger(X)$.

The second assertion is almost obvious. In Definition 2.10 we chose g as

$$g\colon [0,1] \to \mathrm{GL}^+(2,\mathbb{R})\colon t \mapsto \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}.$$

Then the induced action of g on $\mathrm{Stab}^\dagger(X)$ is given by the double shift [2]. Hence it is enough to show that $\tilde{\ell}_A(1) = T_{A*}^2 \sigma_0$.

Since there are no spherical points $p(\delta)$ inside the loop ℓ_A except for $p(v(A))$ itself, the intersection $\ell_A([0,1]) \cap \pi(\partial U(X))$ consists of only one point. We may assume the point is given by $\ell_A(1/2)$. Since we have $\tilde{\ell}_A([0,1/2)) \subset U(X)$ we see that $\tilde{\ell}_A(1/2) = \tau$ is in $\partial U(X)$ and that τ is of type (A^+) or (A^-) by Lemma 5.1 and [5, Theorem 12.1].

We finally claim that τ is of type (A^+). To prove the claim we put

$$\tilde{\ell}_A\Big(\frac{1}{2} - \epsilon\Big) = \sigma_\epsilon = (\mathcal{A}_\epsilon, Z_\epsilon) \in \mathrm{Stab}^\dagger(X),$$

for $0 < \epsilon \ll 1$. In fact suppose to the contrary that τ is of type (A^-). By [5, Proposition 9.4] we may assume both A and $T_A^{-1}(\mathcal{O}_x)$ are σ_ϵ-stable for any ϵ. Since we see $\mathfrak{Im} Z_\epsilon(\mathcal{O}_x)/Z_\epsilon(A[2]) < 0$, the distinguished triangle

$$T_A^{-1}(\mathcal{O}_x) \longrightarrow \mathcal{O}_x \longrightarrow A^{\oplus r_A}[2]$$

gives the Harder-Narasimhan filtration of $\mathcal{O}_x$ in σ_ϵ. This contradicts the fact that $\mathcal{O}_x$ is σ_ϵ-stable. Hence $\ell_A(1/2)$ is of type (A^+) and $\tilde{\ell}_A(1/2+\epsilon)$ is in $T_{A*}^2 U(X)$. For $t > 1/2$, since ℓ_A does not meet $\pi(\partial U(X))$, we see $\tilde{\ell}_A(1) = T_{A*}^2(\sigma_0)$. □

Finally we study so called walls in terms of the hyperbolic structure. As we showed in Lemma 5.2 each boundary component of $\partial V(X)$ is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$. More generally we show that any wall is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$.

Let $\mathcal{S}$ be the set of objects which have bounded mass in $\mathrm{Stab}^\dagger(X)$, and B a compact subset of $\mathrm{Stab}^\dagger(X)$. Then by [5, Proposition 9.3] we have a finite set $\{W_\gamma\}_{\gamma\in\Gamma}$ of real codimension 1 submanifolds satisfying the property in the proposition. For the set $\{W_\gamma\}_{\gamma\in\Gamma}$ we put

$$\mathfrak{W}(\mathcal{S}, B) = \Bigg(\bigcup_{\gamma\in\Gamma} W_\gamma\Bigg) \big/ \widetilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Note that $\mathfrak{W}(\mathcal{S}, B)$ is a subset of $\mathrm{Stab}^{\mathrm{n}}(X)$.

Theorem 5.5. *The set* $\mathfrak{W}(\mathcal{S}, B)$ *is geodesic in* $\mathrm{Stab}^{\mathrm{n}}(X)$.

Proof. Following [5, Proposition 9.3] let $\mathcal{T}$ be the set of objects

$$\mathcal{T} = \{A \in D(X) \mid \exists E \in \mathcal{S}, \exists \sigma \in B \text{ such that } m_\sigma(A) \le m_\sigma(E)\}.$$

We denote the set of Mukai vectors in $\mathcal{T}$ by $I = \{v(A) \in \mathcal{N}(X) \mid A \in \mathcal{T}\}$ and let γ be a pair $\gamma = (v_i, v_j) \in I \times I$ such that v_i and v_j are linearly independent. As was shown in [5, Proposition 9.3], each wall component W_γ is of the form

$$W_\gamma = \{\sigma = (\mathcal{A}, Z) \in \mathrm{Stab}^\dagger(X) \mid Z(v_i)/Z(v_j) \in \mathbb{R}_{>0}\}.$$

We denote $W_\gamma / \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ by $\mathfrak{W}_\gamma$. It is enough to prove that $\mathfrak{W}_\gamma$ is geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$.

Let $T^{\mathcal{N}}_{mL}$ be the isomorphism on $\mathcal{N}(X)$ induced by the spherical twist T_{mL} of $mL \in \mathrm{NS}(X)$. Since I is finite set (Recall that $\mathcal{T}$ has bounded mass) we can take a sufficiently large $m \in \mathbb{Z}$ so that the rank of all vectors in $T^{\mathcal{N}}\_mL(I)$ is not 0. For the set $T^{\mathcal{N}}_{mL*}(I)$ we define $\mathfrak{W}^T_\gamma$ by

$$\mathfrak{W}^T_\gamma = \{[\sigma] = [(\mathcal{A}, Z)] \in \mathrm{Stab}^{\mathrm{n}}(X) \mid Z(T^{\mathcal{N}}_{mL}(v_i))/Z(T^{\mathcal{N}}_{mL}(v_j)) \in \mathbb{R}_{>0}\}.$$

We may assume the central charge of $[\sigma] \in \mathfrak{W}^T_\gamma$ is given by

$$Z(E) = \langle \exp(\beta + \sqrt{-1}\omega), v(E) \rangle$$

where $(\beta, \omega) \in \mathfrak{H}(X)$.

We note that $\sigma \in \mathfrak{W}^T_\gamma$ satisfies the following equation

$$\mathfrak{Im} Z(T^H_{mL}(v_i))\overline{Z(T^H_{mL}(v_j))} = 0. \tag{5.4}$$

Then one can easily check that the equation (5.4) defines hyperbolic line in $\mathfrak{H}(X)$. Since a hyperbolic structure is induced from $\mathfrak{H}(X)$ the set $\mathfrak{W}^T_\gamma$ is geodesic also in $\mathrm{Stab}^{\mathrm{n}}(X)$. Since we have $T^{\mathrm{n}}_{mL}\mathfrak{W}^T_\gamma = \mathfrak{W}_\gamma$ the set $\mathfrak{W}_\gamma$ is also geodesic in $\mathrm{Stab}^{\mathrm{n}}(X)$ by Theorem 3.3. □

6. Revisit of Orlov's theorem via hyperbolic structure

In this section we demonstrate applications of the hyperbolic structure on $\mathrm{Stab}^{\mathrm{n}}(X)$. Mainly we prove Orlov's theorem without the global Torelli theorem but assuming the connectedness of $\mathrm{Stab}(X)$ in Proposition 6.5. Hence our application suggests that Bridgeland's theory substitutes for the global Torelli theorem.

6.1. Strategy for Proposition 6.5

Since the proof of Proposition 6.5 is technical, we explain the strategy and the roles of some lemmas which we prepare in §6.2. Proposition 6.5 will be proved in §6.3.

If we have an equivalence $\Phi\colon D(Y) \to D(X)$ preserving the distinguished component then there exists $\Psi \in W(X)$ such that $(\Psi \circ \Phi)_* U(Y) \cap V(X) \neq \emptyset$ by Proposition 2.3. We want to take the large volume limit in the domain $(\Psi \circ \Phi)_* U(Y) \cap V(X)$. Because of the complicatedness of the set $V(X)$, we consider the subset $V(X)_{>2} = \{\sigma_{(\beta,\omega)} \in V(X) \mid \omega^2 > 2\}$ and focus on the domain $D_{>2} = (\Psi \circ \Phi)_* U(Y) \cap V(X)_{>2}$.

To take the large volume limit, we have to know the shape of the domain $D_{>2}$. To know the shape of $D_{>2}$ we have to see where the boundary $(\Psi \circ \Phi)_* \partial U(Y)$ appears in $\mathrm{Stab}^{\dagger}(X)$. As we showed in Lemma 5.2, any connected component of $\partial U(Y)$ is the product of $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ and the hyperbolic segment spanning two associated points. Since any equivalence $D(Y) \to D(X)$ induces an isometry between the normalized spaces $\mathrm{Stab}^{\mathrm{n}}(Y) \to \mathrm{Stab}^{\mathrm{n}}(X)$ by Theorem 3.3, we see that the image $(\Psi \circ \Phi)_* \partial U(Y)$ is also the products of $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$ and hyperbolic segments spanning two associated points (See also Lemma 6.1 below). This is the reason why the hyperbolic metric on $\mathrm{Stab}^{\mathrm{n}}(X)$ is important for us.

Here we have to recall that $\mathrm{Stab}^{\dagger}(X)$ is conjecturally a $\widetilde{\mathrm{GL}}^{+}(2,\mathbb{R})$-bundle over the upper half plane $\mathbb{H}$. Since we don't have the explicit isomorphism $\mathrm{Stab}^{\mathrm{n}}(X) \to \mathbb{H}$ yet, it is impossible to observe the place $(\Psi \circ \Phi)_* \partial U(Y)$ in $\mathrm{Stab}^{\dagger}(X)$. Instead of this observation, we study the numerical information of $(\Psi \circ \Phi)_* \partial U(Y)$, namely the image of $(\Psi \circ \Phi)_* \partial U(Y)$ by the quotient map $\pi_{\mathfrak{H}}\colon \mathrm{Stab}^{\dagger}(X) \to \mathcal{P}_0^{+}(X) \to \mathfrak{H}_0(X)$.

Set $\mathfrak{W} = \pi_{\mathfrak{H}}\big((\Psi \circ \Phi)_* \partial U(Y)\big)$. As we showed in Lemma 5.2, $\mathfrak{W}$ is the disjoint union of hyperbolic segments. As we show in Lemma 6.2 later, there are two types (I) and (II) of components of $\mathfrak{W}$. The type (I) is a hyperbolic segment which does not intersect the domain $\pi_{\mathfrak{H}}(V(X)_{>2})$ and the type (II) is a hyperbolic segment which does intersect $\pi_{\mathfrak{H}}(V(X)_{>2})$. Recall that our basic strategy is to take the limit in the domain $V(X)_{>2}$. If the family of type (II) components is unbounded in $\pi_{\mathfrak{H}}(V(X)_{>2})$, it may be impossible to take the large volume limit. Hence we need the boundedness of type (II) components (Proposition 6.3 and Corollary 6.4).

6.2. Technical lemmas

We prepare some technical lemmas. Throughout this section we use the following notations.

For a K3 surface (X, L) we put $L^2 = 2d$. Suppose that $E \in D(X)$ satisfies $v(E)^2 = 0$ and $A \in D(X)$ is spherical. We put their Mukai vectors respectively

$$v(E) = r_E \oplus n_E L \oplus s_E \text{ and } v(A) = r_A \oplus n_A L \oplus s_A.$$

We denote $(\beta, \omega) \in \bar{\mathfrak{H}}(X)$ by (xL, yL).

The main object is the following set

$$\mathfrak{W}(A, E) = \{(\beta, \omega) \in \bar{\mathfrak{H}}(X) \mid \mathfrak{Im} Z_{(\beta,\omega)}(E)\overline{Z_{(\beta,\omega)}(A)} = 0\}.$$

One can easily check that the condition $\mathfrak{Im} Z_{(\beta,\omega)}(E)\overline{Z_{(\beta,\omega)}(A)} = 0$ is equivalent to

$$N_{A,E}(x, y) = \lambda_E \left(\frac{-1}{r_A} + dr_A y^2 - \frac{d\lambda_A^2}{r_A} \right) - \lambda_A \left(dr_E y^2 - \frac{d\lambda_E^2}{r_E} \right) = 0,$$

where $\lambda_E = n_E - r_E x$ and $\lambda_A = n_A - r_A x$. We also have

$$N_{A,E}(x, y) = d(r_A n_E - r_E n_A)y^2 + d\lambda_E \lambda_A \left(\frac{n_E}{r_E} - \frac{n_A}{r_A} \right) - \frac{\lambda_E}{r_A}. \tag{6.1}$$

Lemma 6.1. *Suppose that $0 < r_E$ and $\frac{n_E}{r_E} \neq \frac{n_A}{r_A}$. Then $\mathfrak{W}(A, E)$ is the half circle passing through the following 4 points:*

$$(x, y) = (\alpha_E, 0), \left(\frac{n_E}{r_E}, 0 \right), \left(\frac{n_A}{r_A}, \frac{1}{\sqrt{d}|r_A|} \right) \quad \textit{and} \quad \left(\alpha_A, \frac{1}{\sqrt{d}|r_A|} \right),$$

where $\alpha_E = \frac{n_A}{r_A} - \frac{1}{dr_A^2(\frac{n_E}{r_E} - \frac{n_A}{r_A})}$ and $\alpha_A = \frac{n_E}{r_E} - \frac{1}{dr_A^2(\frac{n_E}{r_E} - \frac{n_A}{r_A})}$. In particular the set $\mathfrak{W}(A, E)$ is a hyperbolic line passing through the above 4 points.

Proof. We can prove Lemma 6.1 by the simple calculation of (6.1). □

In particular the first two points are associated with $T_A(E)$ and E, respectively. Hence we put them respectively

- $p(T_A(E)) = (\alpha_E, 0)$,
- $p(E) = (\frac{n_E}{r_E}, 0)$,

- $p(A) = (\frac{n_A}{r_A}, \frac{1}{\sqrt{d}|r_A|})$ and
- $q = (\alpha_A, \frac{1}{\sqrt{d}|r_A|})$.

We remark that if $\frac{n_E}{r_E} = \frac{n_A}{r_A}$ then $\mathfrak{W}(A, E)$ is the hyperbolic line $x = \frac{n_E}{r_E}$.

Lemma 6.2. *Suppose that $0 < r_E$ and $0 < \frac{n_E}{r_E} - \frac{n_A}{r_A}$. Then there are two types of the configuration of the above four points on $\mathfrak{W}(A, E)$:*

(I) *If $\frac{1}{\sqrt{d}|r_A|} \le \frac{n_E}{r_E} - \frac{n_A}{r_A}$ then we have $\alpha_E < \frac{n_A}{r_A} \le \alpha_A < \frac{n_E}{r_E}$. See also Figure 2 below.*

(II) *If $0 < \frac{n_E}{r_E} - \frac{n_A}{r_A} < \frac{1}{\sqrt{d}|r_A|}$ then we have $\alpha_E < \alpha_A < \frac{n_A}{r_A} < \frac{n_E}{r_E}$. See also Figure 3 below.*

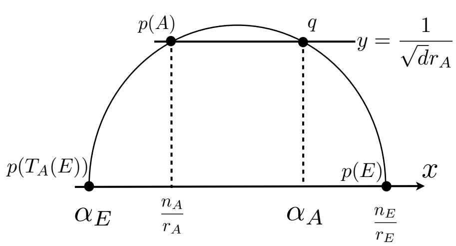

Figure 2: Type(I) in Lemma 6.2.

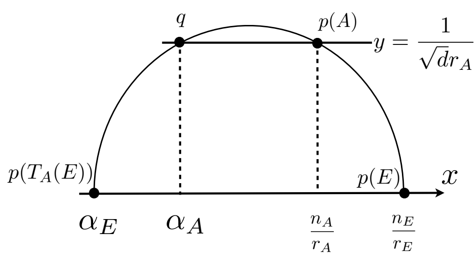

Figure 3: Type(II) in Lemma 6.2.

Proof. Similarly to Lemma 6.1 we could prove the assertion by simple calculations. □

Let $\Phi\colon D(Y) \to D(X)$ be an equivalence preserving the distinguished component. Suppose $E = \Phi(\mathcal{O}_y)$. By Lemma 5.2, $\pi_{\mathfrak{H}}(\Phi_* \partial U(Y))$ is the direct sum of hyperbolic segments $\overline{p(A)p(T_A(E))}$ spanning two points $p(A)$ and $p(T_A(E))$. Clearly the segment $\overline{p(A)p(T_A(E))}$ is a subset of $\mathfrak{W}(A, E)$. Following Lemma 6.2 we have the disjoint sum :

$$\pi_{\mathfrak{H}}(\Phi_* \partial U(Y)) = \coprod_{\text{type(I)}} \overline{p(A')p(T_{A'}(E))} \sqcup \coprod_{\text{type(II)}} \overline{p(A)p(T_A(E))}. \tag{6.2}$$

Since the type (II) segments become obstructions when we take the large volume limit in $V(X)_{>2}$. Hence we have to show the boundedness of type (II) segments. To show this, we give an upper bound of the diameter of the type (II) half circle $\mathfrak{W}(A,E)$ in the following proposition. Clearly from Lemma 6.1 the diameter is given by $\frac{n_E}{r_E}-\alpha_E$.

Proposition 6.3. *Suppose that $r_E>0$ and $0<\frac{n_E}{r_E}-\frac{n_A}{r_A}<\frac{1}{\sqrt{d}|r_A|}$. Then we have*

$$0<\frac{n_E}{r_E}-\alpha_E\le\frac{1}{r_E}+\frac{r_E}{d}.$$

Proof. By the assumption one easily sees $r_A\cdot(r_An_E-r_En_A)>0$. Hence we see

$$\begin{aligned}\frac{n_E}{r_E}-\alpha_E&=\left(\frac{n_E}{r_E}-\frac{n_A}{r_A}\right)+\frac{1}{dr_A^2\left(\frac{n_E}{r_E}-\frac{n_A}{r_A}\right)}\\&=\left|\frac{1}{r_A}\right|\cdot\left(\frac{|r_An_E-r_En_A|}{r_E}+\frac{r_E}{d|r_An_E-r_En_A|}\right)\\&\le\frac{|r_An_E-r_En_A|}{r_E}+\frac{r_E}{d|r_An_E-r_En_A|}.\end{aligned}\tag{6.3}$$

By the assumption we have

$$\frac{|r_An_E-r_En_A|}{r_E}<\frac{r_E}{d|r_An_E-r_En_A|}.$$

Note that the continuous function $f(t)=\frac{1}{t}+\frac{t}{d}$ on $\mathbb{R}_{>0}$ is an increasing function for $\frac{1}{t}<\frac{t}{d}$. Since we have $\frac{r_E}{|r_An_E-r_En_A|}\le r_E$ the following inequality holds:

$$\frac{n_E}{r_E}-\alpha_E\le\frac{1}{r_E}+\frac{r_E}{d}.$$

Thus we have proved the inequality. $\square$

The following corollary is a simple paraphrase of Proposition 6.3. However it is crucial for the proof of the main result of this section.

Corollary 6.4. *Let $\Phi\colon D(Y)\to D(X)$ be an equivalence which preserves the distinguished component. Set $v(\Phi(\mathcal{O}_y))=r\oplus nL_X\oplus s$ and $L_X^2=2d$ and assume $r>0$. Then the image $\pi_{\mathfrak{H}}(\Phi_*\partial U(Y))$ is in the following shaded* *closed region* $R(Y,\Phi)$ *where* $\pi_{\mathfrak{H}}\colon \mathrm{Stab}^{\mathrm{n}}(X) \to \mathfrak{H}_0(X)$:

$$R(Y,\Phi) = \Big\{ (xL_X, yL_X) \in \mathfrak{H}(X) \;\Big|\; \Big(x - \frac{n}{r} + \frac{1}{2}\Big(\frac{1}{d} + \frac{r}{d}\Big)\Big)^2 + y^2 \le \frac{1}{4}\Big(\frac{1}{d} + \frac{r}{d}\Big)^2, \Big(x - \frac{n}{r} - \frac{1}{2}\Big(\frac{1}{d} + \frac{r}{d}\Big)\Big)^2 + y^2 \le \frac{1}{4}\Big(\frac{1}{d} + \frac{r}{d}\Big)^2 \text{ or } y^2 \le \frac{1}{d} \Big\}.$$

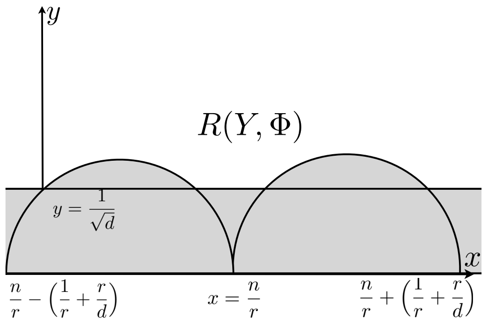

Figure 4: Figure for the region $R(Y,\Phi)$.

Proof. As we explained in (6.2), we see that

$$\pi_{\mathfrak{H}}(\Phi_* \partial U(Y)) = \coprod_{\text{type(I)}} \overline{p(A')p(T_{A'}(\Phi(\mathcal{O}_y)))} \sqcup \coprod_{\text{type(II)}} \overline{p(A)p(T_A(\Phi(\mathcal{O}_y)))}$$

where A and A' are spherical object of $D(X)$. Clearly type (I) hyperbolic segments $\overline{p(A')p(T_{A'}(\Phi(\mathcal{O}_y)))}$ are in the following region:

$$\Big\{ (xL_X, yL_X) \mid y^2 \le \frac{1}{d} \Big\}.$$

By Proposition 6.3, the type (II) hyperbolic segments are in the region

$$\Big\{ (xL_X, yL_X) \in \mathfrak{H}(X) \;\Big|\; \Big(x - \frac{n}{r} + \frac{1}{2}\Big(\frac{1}{d} + \frac{r}{d}\Big)\Big)^2 + y^2 \le \frac{1}{4}\Big(\frac{1}{d} + \frac{r}{d}\Big)^2 \text{ or } \Big(x - \frac{n}{r} - \frac{1}{2}\Big(\frac{1}{d} + \frac{r}{d}\Big)\Big)^2 + y^2 \le \frac{1}{4}\Big(\frac{1}{d} + \frac{r}{d}\Big)^2 \Big\}.$$

This gives the proof. □

6.3. Revisit of Orlov's theorem

We prove the main result of this section.

Proposition 6.5. *Let (X, L_X) be a projective K3 surface with $\rho(X) = 1$ and (Y, L_Y) a Fourier-Mukai partner of (X, L_X). If an equivalence $\Phi : D(Y) \to D(X)$ preserves the distinguished component, then Y is isomorphic to the fine moduli space of Gieseker stable torsion-free sheaves.*

Proof. We first put $L_X^2 = L_Y^2 = 2d$ and $v_0 = v(\Phi(\mathcal{O}_y)) = r \oplus nL_X \oplus s$. If necessary by applying $T_{\mathcal{O}_X}$ and [1], we may assume $r > 0$. We denote the composition of two morphisms $\mathrm{Stab}^\dagger(X) \to \mathcal{P}_0^+(X) \to \mathfrak{H}_0(X)$ by $\pi_{\mathfrak{H}}$. By the assumption we have $\Phi_* U(Y) \subset \mathrm{Stab}^\dagger(X)$.

We can take a stability condition $\tau \in U(Y)$ so that $\pi_{\mathfrak{H}}(\Phi_* \tau) = (\beta_0, \omega_0) = (aL_X, bL_X)$ with

(i) $a < \frac{n}{r} - \left(\frac{1}{r} + \frac{r}{d}\right)$ and

(ii) $2 < \omega_0^2$.

By the second condition (ii) and Lemma 5.2 we see $\pi_{\mathfrak{H}} \circ \Phi_*(\tau)$ does not lie on $\pi_{\mathfrak{H}}(\partial U(X))$. Hence $\Phi_*(\tau)$ is in a chamber of $\mathrm{Stab}^\dagger(X)$ by Proposition 2.3. Hence we see

$$\exists \Psi \in W(X) \times \mathbb{Z}[2] \text{ such that } (\Psi \circ \Phi)_*(\tau) \in U(X).$$

Now we put $\Phi' = \Psi \circ \Phi$ and take $\sigma_0 \in V(X)$ as $\sigma_{(\beta_0, \omega_0)}$. Since $\Phi'_*(\tau)$ and σ_0 belong to the same $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit, σ_0 is in $V(X) \cap \Phi'_*(U(Y))$. We define a family $\mathcal{F}$ of stability conditions as follows:

$$\mathcal{F} = \{\sigma_{(\beta_0, t\omega_0)} \in V(X) \mid 1 < t \in \mathbb{R}\}.$$

Then we see $\pi_{\mathfrak{H}}(\mathcal{F}) \cap R(Y, \Phi') = \emptyset$ by Corollary 6.4. Hence $\mathcal{F}$ does not meet $\Phi'_*(\partial U(Y))$. Since $\sigma_0 \in \Phi'_*(U(Y))$ we see $\mathcal{F} \subset \Phi'_*(U(Y))$ and the object $\Phi'(\mathcal{O}_y)$ is σ-stable for all $\sigma \in \mathcal{F}$. By Bridgeland's large volume limit argument [5, Proposition 14.2] we see that $\Phi'(\mathcal{O}_y)$ is a Gieseker semistable torsion-free sheaf[2]. Moreover by [16, Proposition 3.14] (or the argument of [11, Lemma 4.1]) $\Phi'(\mathcal{O}_y)$ is Gieseker stable. Then the Fourier-Mukai kernel of Φ' is given by a flat family of locally-free sheaves on X (For instance,

[2]Since we are assuming $\rho(X) = 1$, the Gieseker stability is equivalent to the twisted stability.

see[7, Lemma 3.31]). Since $v_0 = v(\Phi'(\mathcal{O}_y))$ is isotropic and there is $u \in \mathcal{N}(X)$ such that $\langle v_0, u\rangle = 1$, there exists the fine moduli space $\mathcal{M}$ of Gieseker stable sheaves (See also [7, Lemma 10.22 and Proposition 10.20]). Hence Y is isomorphic to $\mathcal{M}$. □

Remark 6.6. Clearly the key ingredient of Proposition 6.5 is Corollary 6.4. The role of Corollary 6.4 is to detect the place of the numerical image of walls $\pi_{\mathfrak{H}}(\Phi(\partial U(Y)))$. Without Theorems 3.3 and 5.5, it was difficult to detect the place of $\pi_{\mathfrak{H}}(\Phi(\partial U(Y)))$. By virtue of these theorems, the problem is reduced to the problem with two associated points $p(A)$ and $p(T_A(\Phi(\mathcal{O}_y)))$.

Remark 6.7. We explain the relation between author's work and Huybrechts's question in [8].

In [8, Proposition 4.1], it was proven that any nontrivial Fourier-Mukai partner of a projective K3 surface is given by the a moduli spaces of μ-stable locally free sheaves (See also [8, Proposition 4.1]). We note that this proposition holds for all projective K3 surfaces. If the Picard rank is one, the proof of the proposition is based on the lattice argument. In the proof of [8, Proposition 4.1] Huybrechts asks whether there is a geometric proof.

In the previous work [12, Theorem 5.4], the author gave an answer of Huybrechts's question, that is a geometric proof. However our proof is not completely independent of lattice theories, because it is based on Orlov's theorem which strongly depends on the global Torelli theorem.

As a consequence of Proposition 6.5 and [12, Theorem 5.4], we could give the another proof of [8, Proposition 4.1] which is completely independent of the global Torelli theorem with assuming the connectedness of $\mathrm{Stab}(X)$.

7. Stable complexes in large volume limits

Let A be a spherical sheaf in $D(X)$. At the end of this paper we discuss the stability of the complex $T_A^{-1}(\mathcal{O}_x)$ in the large volume limit[3]. More precisely we shall show that $T_A^{-1}(\mathcal{O}_x)$ is $\sigma_{(\beta,\omega)}$-stable if $\beta\omega < \mu_\omega(A)$ and $\omega^2 > 2$. The possibility of stable complexes in the large volume limit is mentioned in [3, Lemma 4.2 (c)].

[3] We remark that $T_A^{-1}(O_x)$ is a 2-terms complex such that $H^0(T_A^{-1}(\mathcal{O}_x)) = \mathcal{O}_x$ and $H^{-1}(T_A^{-1}(\mathcal{O}_x)) = A^{\oplus r_A}$.

For the vector $v(A) = r_A \oplus n_A L \oplus s_A$ we define the subset $\mathfrak{D}_A \subset \mathfrak{H}(X)$ as follows:

$$\mathfrak{D}_A = \left\{ (xL, yL) \in \mathfrak{H}(X) \middle| \left(x - \frac{n_A}{r_A} \right)^2 + \left(y - \frac{1}{2\sqrt{d}r_A} \right)^2 < \frac{1}{4dr_A^2} \right\}$$

Lemma 7.1. *Notations being as above. In the domain $\mathfrak{D}_A$, there are no spherical point $p(\delta)$ with (-2)-vectors δ. Moreover $\mathfrak{D}_A$ does not intersect $\pi_{\mathfrak{H}} \circ T_{A*}(\partial U(X))$.*

Proof. By the spherical twist T_A, we have the diagram:

$$\begin{array}{ccc} \mathrm{Stab}^{\mathrm{n}}(X) & \xrightarrow{T^{\mathrm{n}}_{A*}} & \mathrm{Stab}^{\mathrm{n}}(X) \\ \downarrow & & \downarrow \\ \mathfrak{H}_0(X) & \xrightarrow{T^{\mathfrak{H}}_A} & \mathfrak{H}_0(X). \end{array}$$

By Lemma 3.2, $T^{\mathfrak{H}}_A$ is given by the liner fractional transformation

$$T^{\mathfrak{H}}_A(x + \sqrt{-1}y) = \frac{1}{dr_A} \cdot \frac{-1}{x + \sqrt{-1}y - \frac{n_A}{r_A}} + \frac{n_A}{r_A}.$$

We remark that $T^{\mathfrak{H}}_A$ is conjugate to the transformation $z \mapsto -1/dr_A z$.

Now we recall that there is no spherical point $p(\delta)$ in the domain

$$\mathfrak{H}(X)_{>2} = \{(\beta, \omega) \in \mathfrak{H}(X) \mid \omega^2 > 2\}.$$

One can easily check that $T^{\mathfrak{H}}_A(\mathfrak{H}(X)_{>2}) = \mathfrak{D}_A$. Moreover it is clear that $\pi_{\mathfrak{H}}(\partial U(X)) \cap \mathfrak{H}(X)_{>2}$. This finishes the proof. □

Define the subset $D^+_A \subset V(X)$ by

$$D^+_A = \left\{ \sigma_{(xL,yL)} \in V(X) \mid x < \frac{n_A}{r_A}, (xL, yL) \in \mathfrak{D}_A \right\}.$$

In the following proposition, we discuss the stability of sheaves $T_A(\mathcal{O}_x)$ in the "small" volume limit D^+_A.

Proposition 7.2. *For any $\sigma \in D^+_A$, $T_A(\mathcal{O}_x)$ is σ-stable. In particular $D^+_A \subset T_{A*}U(X) \cap V(X)$.*

Proof. To simplify the notation we set $A(x) = T_A(\mathcal{O}_x)[-1]$. It is enough to show that $A(x)$ is σ-stable for all $\sigma \in D_A^+$.

One can see $A(x)$ is the kernel of the evaluation map $\mathrm{Hom}(A, \mathcal{O}_x) \otimes A \to \mathcal{O}_x$ and is Gieseker stable. We note that there exists $\sigma \in D_A^+$ such that $A(x)$ is σ-stable by [12, Theorem 4.4 (2)]. In particular we see $D_A^+ \cap T_{A*}(U(X)) \neq \emptyset$. Hence it is enough to show $D_A^+ \cap T_{A*}(\partial U(X)) = \emptyset$. This is obvious by Lemma 7.1. □

We set

$$(D_A^+)^\vee = \left\{ \sigma_{(xL,yL)} \in V(X) \mid (yL)^2 > 2, x > \frac{n_A}{r_A} \right\}.$$

Corollary 7.3. *For any $\sigma \in (D_A^+)^\vee$, $T_A^{-1}(\mathcal{O}_x)$ is σ-stable. In particular $(D_A^+)^\vee \subset T_{A*}^{-1}(U(X))$.*

Proof. Since $D_A^+ \subset T_{A*}(U(X)) \cap U(X)$ by Proposition 7.2, we see

$$T_{A*}^{-1}(D_A^+) \subset U(X) \cap T_{A*}^{-1}(U(X)).$$

Since σ-stability is equivalent to $\sigma \cdot \tilde{g}$-stability for any $\tilde{g} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, it is enough to show that $T_{A*}^{-1}(D_A^+)/\widetilde{\mathrm{GL}}^+(2, \mathbb{R}) = (D_A^+)^\vee/\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. This is obvious from Lemma 3.2. □

Remark 7.4. In the article [3, Lemma 4.2 (c)], the possibility of the stable complexes in large volume limits is referred. Hence Corollary 7.3 gives the proof of this prediction.

Acknowledgement

The author thanks the referee for valuable comments. He was partially supported by Grant-in-Aid for Scientific Research (S), No 22224001.

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Received September 27, 2013
Accepted September 6, 2016