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Complete hypersurfaces in Euclidean spaces with finite strong total curvature

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We prove that finite strong total curvature (see definition in Section 2) complete hypersurfaces of $(n+1)$-euclidean space are proper and diffeomorphic to a compact manifold minus finitely many points. With an additional condition, we also prove that the Gauss map of such hypersurfaces extends continuously to the punctures. This is related to results of White [22] and and Müller-Šverák [18]. Further properties of these hypersurfaces are presented, including a gap theorem for the total curvature.

1. Introduction

Let $\phi\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface of the euclidean space $\mathbb{R}^{n+1}$. We assume that $M^n = M$ is orientable and we fix an orientation for M. Let $g\colon M \to S_1^n \subset \mathbb{R}^{n+1}$ be the Gauss map in the given orientation, where S_1^n is

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the unit n-sphere. Recall that the linear operator $A\colon T_pM \to T_pM$, $p \in M$, associated to the second fundamental form, is given by

$$\langle A(X), Y\rangle = -\langle \overline{\nabla}_X N, Y\rangle, \quad X, Y \in T_pM,$$

where $\overline{\nabla}$ is the covariant derivative of the ambient space and N is the unit normal vector in the given orientation. The map $A = -dg$ is self-adjoint and its eigenvalues are the principal curvatures $k_1, k_2, \dots, k_n$.

We say that the total curvature of the immersion is finite if $\int_M |A|^n \, dM < \infty$, where $|A| = \left(\sum_i k_i^2\right)^{1/2}$, i.e., if $|A|$ belongs to the space $L^n(M)$. If $\phi : M^n \to \mathbb{R}^{n+1}$ is a complete minimal hypersurface with finite total curvature then M is (equivalent to) a compact manifold $\overline{M}$ minus finitely many points and the Gauss map extends to the punctures. This was proved by Osserman [19] for $n = 2$ (the equivalence here is conformal and the Gauss map extends to a (anti) holomorphic map $\overline{g}\colon \overline{M}^2 \to S_1^2$; the conformal equivalence had already been proved by Huber [13]). For an arbitrary n, this was proved by Anderson [2] (here the equivalence is a diffeomorphism and the Gauss map extends smoothly).

When ϕ is not necessarily minimal and $n = 2$, the above result, with the additional hypothesis that the Gauss curvature does not change sign at the ends, was shown to be surprisingly true by B. White [22]. The subject was taken up again by Müller-Šverák [18] who answered a question of [22] and obtained further information on the conformal behaviour of the ends.

The results of White [22] and Müller-Šverák [18] start from the fact that, since $\int_{M^2} |A|^2 \, dM \geq 2 \int_{M^2} |K| \, dM$, finite total curvature for $n = 2$ implies, by Huber's theorem, that M is homeormorphic to a compact surface minus finitely many points. For an arbitrary dimension, any generalization of Huber's theorem should require stronger assumptions (see [6] and [7] for a discussion on the theme). Thus, for a generalization of [22] and [18] for $n \geq 3$, a further condition might be necessary to account for the lack of an appropriate generalized Huber theorem.

Here, we assume the hypothesis of *finite strong total curvature*, that is, we assume that $|A|$ belongs to $W_s^{1,q}$, a special Weighted Sobolev space (see Section 2 for precise definitions). We point out that the spaces $W_s^{k,q}(M)$ were used in a seminal work of R. Bartnik [3] for establishing a decay condition on the metric of an n-manifold, $n \geq 3$, in order to prove that the ADM-mass is well-defined. Following the ideas of [3], a lot of related papers also use the norm of $W_s^{k,q}(M)$ to express decay assumptions (see for instance [14], [11], [20]).

We prove the following results.

Theorem 1.1. *Let $\phi\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an orientable, complete hypersurface with finite strong total curvature. Then:*

i) *The immersion ϕ is proper.*
ii) *M is diffeomorphic to a compact manifold $\overline{M}$ minus a finite number of points $q_1, \ldots, q_k$.*

Assume, in addition, that the Gauss-Kronecker curvature $H_n = k_1 k_2 \cdots k_n$ of M does not change sign in punctured neighbourhoods of the q_i's. Then:

iii) *The Gauss map $g\colon M^n \to S_1^n$ extends continuously to the points q_i.*

We point out that the minimal hypersurfaces of $\mathbb{R}^{n+1}$ with finite total curvature have finite strong total curvature (see Example 3).

Theorem 1.2. *Let $\phi\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an orientable complete hypersurface with finite strong total curvature. Assume that the set $\mathcal{N}$ of critical values of the Gauss map g is a finite union of submanifolds of S_1^n with codimension ≥ 3. Then:*

i) *The extended Gauss map $\bar{g}\colon \overline{M} \to S_1^n$ is a homeomorphism.*
ii) *If, in addition, n is even, M has exactly two ends.*

Remark 1.3. The condition on $\mathcal{N}$ can be replaced by a weaker condition on the Hausdorff dimension of $\mathcal{N}$ and the rank of g (See [15], Theorems B and C and Remark 6.7).

It follows from Theorem 1.1 that there is a computable lower bound for the total curvature of the non-planar hypersurfaces of the set C^n defined in the statement below.

Theorem 1.4. (The Gap Theorem) *Let C^n be the set of finite strong total curvature complete orientable hypersurfaces $\phi\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, such that H_n does not change sign in M. Then either $\phi(M^n)$ is a hyperplane, or*

$$\int_M |A|^n \, dM > 2\sqrt{n!}\,(\sqrt{\pi})^{n+1}/\Gamma((n+1)/2),$$

where Γ is the gamma function.

Remark 1.5. For the Gap Theorem it is not enough to requiring that H_n does not change sign at the ends of the hypersurface. This condition should hold on the whole M. Consider the rotation hypersurfaces in $\mathbb{R}^{n+1}$ generated by the smooth curve $x^{n+1} = \varepsilon\, e^{-1/x_1^2}$, $\varepsilon > 0$, $(x_1, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$, around the x_{n+1}-axis. In Example 2, we check that, for all ε, this hypersurface has finite strong total curvature. It is easy to see that H_n does not change sign at the (unique) end of the hypersurface. However, as ε approaches zero, these hypersurfaces approach a hyperplane, and the lower bound for the total curvature of the family is zero.

The paper is organized as follows. In Section 2, we define and present some examples of hypersurfaces with finite strong total curvature. In Section 3, we discuss (Proposition 3.2) the rate of decay at infinity of the second fundamental form of a hypersurface under the hypothesis of finite strong total curvature. In Section 4, we show that each end of such a hypersurface has a unique "tangent plane at infinity" (see the definition before Proposition 4.4) and in Section 5, we prove Theorems 1.1, 1.2 and the Gap Theorem.

2. Definitions and examples

In the rest of this paper, we will be using the following notation for an immersion $\phi\colon M^n \to \mathbb{R}^{n+1}$:

$$\begin{aligned}
\rho &= \text{intrinsic distance in } M \\
d &= \text{distance in } \mathbb{R}^{n+1};\ \ 0 = \text{origin of } \mathbb{R}^{n+1} \\
D_p(R) &= \{x \in M;\ \rho(x,p) < R\} \\
D_p(R,S) &= \{x \in M;\ R < \rho(x,p) < S\} \\
B(R) &= \{x \in \mathbb{R}^{n+1};\ d(x,0) < R\};\ S(R) = \partial B(R) \\
A(R,S) &= \{x \in \mathbb{R}^{n+1};\ R < d(x,0) < S\}.
\end{aligned}$$

We choose a point $p_0 \in M$ and for all $x \in M$, $\rho_0(x)$ will denote the intrinsic distance in M from x to p_0. Now, we set the notation for the norms (see [3, (1.2)]) that will be used in the definition of strong total curvature.

Let $\Omega \subset M$. Given any $q > 0$, we define the *weighted space* $L^q_s(\Omega)$ of all measurable functions of finite norm

$$\|u\|_{L^q_s(\Omega)} = \left(\int_\Omega |u|^q |\rho_0|^{-qs-n}\, dM\right)^{1/q}.$$

We introduce the *weighted Sobolev space* $W_s^{1,q}(\Omega)$ of all measurable functions of finite norm

$$\|u\|_{W_s^{1,q}(\Omega)} = \|u\|_{L_s^q(\Omega)} + \|\nabla u\|_{L_{s-1}^q(\Omega)},$$

where ∇u is the gradient of u in M.

The quantity $\| \, |A| \, \|_{W_{-1}^{1,q}(M)}$ will be called the *strong total curvature* of the immersion by and we say that the immersion has *finite strong total curvature* if

$$|A| \in W_{-1}^{1,q}(M), \quad \text{for } q > n,$$

that is, if

$$\begin{aligned}\| \, |A| \, \|_{W_{-1}^{1,q}(M)} = &\left(\int_M |A|^q |\rho_0|^{q-n} \, dM \right)^{1/q} \\ &+ \left(\int_M |\nabla |A||^q |\rho_0|^{2q-n} \, dM \right)^{1/q} < \infty, \quad \text{for } q > n.\end{aligned}$$

We remark that the function ρ_0 used above to define these norms could be replaced by the distance with respect to any other fixed point $p \in M$. We also remark that the weights used to define the norm $\|.\|_{W_{-1}^{1,q}}$ make it invariant by dilations (see the proof of Proposition 3.2).

Our goal now is to find some interesting examples. We deal with rotational hypersurfaces. We first consider the hypersurfaces obtained by the rotation of a profile curve $(x_1, 0, \ldots, 0, x_{n+1} = f(x_1))$ in $\mathbb{R}^{n+1}$ around the x_1-axis.

A parametrization of M can be given by

$$X(x_1, t_2, \ldots, t_n) = (x_1, f(x_1)\xi), \tag{2.1}$$

where $\xi = \xi(t_2, \ldots, t_n)$ is an orthogonal parametrization of the unit sphere S_1^{n-1}. The basic vector fields associated to X are

$$X_1 = (1, f'(x_1)\xi) \quad \text{and} \quad X_j = (0, f(x_1)u_j), \quad j = 2, \ldots, n,$$

where $\{u_j\}_j$ is a frame of unit vectors tangent to the sphere, and a unit normal field can be chosen to be

$$N = \frac{1}{\sqrt{1 + (f'(x_1))^2}} (f'(x_1), -\xi).$$

In the frame $\{X_1, \ldots, X_n\}$, the coefficients of the metric are given by

$$g_{11} = 1 + f'^2, \qquad g_{ij} = 0, \; i \neq j, \qquad \text{and} \qquad g_{jj} = f^2, \; j = 2, \ldots, n$$

and the volume element of M is then given by

$$dM = (1+f'^2)^{1/2}\, f^{n-1}\, dx_1\, d\mu,$$

where $d\mu$ is the volume element of S_1^{n-1}.

If $h : M \to \mathbb{R}$ is a differentiable function, the gradient of h can be expressed by

$$\nabla h = \sum_{j,k} g^{jk} X_j(h) X_k,$$

where $(g^{jk})_{jk} = (g_{jk})_{jk}^{-1}$.

With our choice for N, the principal curvatures are the following $k_1 = \frac{-f''}{(1+f'^2)^{3/2}}$ along the direction tangent to a copy of the profile curve and $k_2 = \cdots = k_n = \frac{1}{f(1+f'^2)^{1/2}}$ along the directions which are tangent to S_1^{n-1}.

After a translation, if necessary, we can assume that the profile curve touches the x_{n+1}-axis at a point p_0. We choose p_0 to define our distance function, i.e., $\rho_0(p)$ denotes the distance in M from p to p_0. We notice that $\rho_0(p)$ can be estimated by the length of a special curve that links p to p_0 composed by two parts, α and β, suitably chosen. Let $x_1(p)$ denote the x_1-coordinate of p. We choose α to be the geodesic in the $(n-1)$-sphere of radius $f(x_1(p))$, contained in the hyperplane $x_1 = x_1(p)$, that links p to the point $\hat{p} \cong (x_1(p), 0, \ldots, 0, f(x_1(p)))$. β will be the part of the profile curve that joins $\hat{p}$ and p_0. We then have

$$\rho_0(p) \leq \text{length of } \alpha + \text{length of } \beta \leq \pi f(x_1(p)) + \int_0^{x_1(p)} \sqrt{1+f'^2(t)}\, dt,$$

where $(t, f(t))$ is the natural parametrization of β in the x_1x_{n+1}-plane.

Sometimes it is convenient to consider M as the rotation of a curve $x_{n+1} = f(x_1)$ around the x_{n+1}-axis. A suitable parametrization for M is then

$$Y(x_1, t_2, \ldots, t_n) = (x_1\,\xi, f(x_1)), \tag{2.2}$$

where $\xi = \xi(t_2, \ldots, t_n)$ is an orthogonal parametrization of the unit sphere S_1^{n-1}. In this case, the unit normal field and the metric can be given by:

$$N = \frac{1}{\sqrt{1+(f'(x_1))^2}}(\xi f'(x_1), -1).$$

$$g_{11} = 1+f'^2, \qquad g_{ij} = 0,\ i \neq j, \qquad \text{and} \qquad g_{jj} = x_1^2,\ j = 2, \ldots, n.$$

We collect, in the following proposition, some quantities related to the rotational hypersurfaces described above. The result follows from straightforward computation.

Proposition 2.1.

a) *For the hypersurface M in $\mathbb{R}^{n+1}$ obtained by the rotation of the curve $(x_1, 0, \ldots, 0, x_{n+1} = f(x_1))$ around the x_1-axis, with the parametrization given by (2.1), we have*

$$|A|^2 = \frac{n-1}{f^2(1+f'^2)} + \frac{(f'')^2}{(1+f'^2)^3}, \qquad |\nabla|A|| = \left|\frac{\partial|A|}{\partial x_1}\right| \frac{1}{(1+f'^2)^{1/2}}$$

$$\text{and} \quad dM = (1+f'^2)^{1/2}\, f^{n-1}\, dx_1 d\mu.$$

b) *For the hypersurface M in $\mathbb{R}^{n+1}$ obtained by the rotation of the curve $(x_1, 0, \ldots, 0, x_{n+1} = f(x_1))$ around the x_{n+1}-axis, with the parametrization given by (2.2), we have*

$$|A|^2 = \frac{(n-1)f'^2}{x_1^2(1+f'^2)} + \frac{(f'')^2}{(1+f'^2)^3}, \qquad |\nabla|A|| = \left|\frac{\partial|A|}{\partial x_1}\right| \frac{1}{(1+f'^2)^{1/2}}$$

$$\text{and} \quad dM = (1+f'^2)^{1/2}\, x_1^{n-1}\, dx_1 d\mu.$$

Example 1. Here, we prove that the rotational hypersurfaces of $\mathbb{R}^{n+1}$ with vanishing higher order mean curvatures has finite strong total curvature. These hypersurfaces are classified in [16].

Let M be a rotational hypersurface of $\mathbb{R}^{n+1}$ with $H_r = 0$ generated by the rotation of a curve $x_{n+1} = f(x_1)$ around the x_1-axis. In this case, we know from [16] that the function f is even, positive and convex, and satisfies:

$$\begin{aligned}
&f(0) = 1, \quad f \geq 1,\\
&1 + f'^2 = f^v, \text{ where } v = \frac{2(n-r)}{r},\\
&\text{and} \quad ff'' = \frac{v}{2} f^v.
\end{aligned}$$

We can conclude that f is increasing for $x_1 > 0$. Also from [16] (see Lemma 2.1), we know that the behaviour of f can be distinguished into three cases, depending on the value of v. We have:

$$\begin{aligned} f(x_1) &= \mathcal{O}\left(|x_1|^{\frac{2}{2-v}}\right), && \text{if } v<2 \\ f \text{ is defined in a limited interval } &(-L,L), && \text{if } v>2 \\ f(x_1) &= \cosh(x_1), && \text{if } v=2. \end{aligned}$$

<u>Case 1</u>: $v<2$, or equivalently, $n<2r$.

Let M_1 be the restriction of M to the region R_1 where $1<|x_1|<\infty$. It is enough to show that $\| \, |A| \, \|_{W^{1,q}_{-1}(M_1)}<\infty$. By Proposition 2.1 a) we can write

$$\begin{aligned} |A|^2 &= \frac{k}{f^{v+2}}, \text{ with } k=\frac{4(n-1)+v^2}{4}, \\ |\nabla|A|| &= \frac{\tilde{k}(f^v-1)^{1/2}}{f^{v+2}} = \frac{\tilde{k}(1-1/f^v)^{1/2}}{f^{\frac{v+4}{2}}} < \frac{\tilde{k}}{f^{\frac{v+4}{2}}}, \text{ with } \tilde{k}=\frac{\sqrt{k}(v+2)}{2}, \\ \rho_0(p) &\le \pi f(x_1(p)) + \int_0^{x_1(p)} f^{v/2}(t)\, dt \le \pi f(x_1(p)) + f^{v/2}(x_1(p))\, x_1(p) \end{aligned}$$

and

$$dM = f^{\frac{2(n-1)+v}{2}} dx_1 d\mu.$$

We use that $f(x_1)=\mathcal{O}\left(|x_1|^{\frac{2}{2-v}}\right)$ to conclude that

$$|A| \le \text{cte} \,.\, |x_1|^{\frac{v+2}{v-2}}, \qquad |\nabla|A|| \le \text{cte} \,.\, |x_1|^{\frac{v+4}{v-2}}, \qquad \rho_0(p) \le \text{cte} \,.\, |x_1|^{\frac{2}{2-v}}$$

and

$$dM = \tau(x_1) dx_1 d\mu, \text{ where } \tau(x_1) \le \text{cte} \,.\, |x_1|^{\frac{2(n-1)+v}{2-v}}.$$

Then

$$\int_{M_1} |A|^q \rho^{q-n} dM \le \text{cte} \int_{S_1^{n-1}} \int_{R_1} x_1^{-1-\frac{qv}{2-v}} dx_1 d\mu < \infty$$

and

$$\int_{M_1} |\nabla|A||^q \rho^{2q-n} dM \le \text{cte} \int_{S_1^{n-1}} \int_{R_1} x_1^{-1-\frac{qv}{2-v}} dx_1 d\mu < \infty.$$

<u>Case 2</u>: $v>2$, or equivalently, $n>2r$.

In this case, f is defined in a limited interval $(-L, L)$ and tends to infinity when x_1 goes to $\pm L$. Let l, $l \in (0, L)$, be such that $f(l) = 2$ and let $\bar{f}$ be the restriction of f to the interval (l, L). Let M_1 be the hypersurface generated by the rotation of $\bar{f}$ around the x_1-axis. It is clear that if M_1 has finite strong total curvature, the same happens to M. Let $\mathcal{G}(x_{n+1}) = x_1$ be the inverse function of $\bar{f}$. Then, $\mathcal{G}$ is given by (see (2.3) in [16])

$$\mathcal{G}(x_{n+1}) = \int_2^{x_{n+1}} \frac{1}{\sqrt{t^v - 1}}\, dt.$$

Interchanging the role of x_1 and x_{n+1}, we write

$$x_{n+1} = \mathcal{H}(x_1) = \int_2^{x_1} \frac{1}{\sqrt{t^v - 1}}\, dt.$$

and we can see M_1 as the hypersurface obtained by the rotation of $\mathcal{H}(x_1)$, $x_1 \in (2, \infty)$, around the x_{n+1}-axis. We claim that M_1 has finite strong total curvature. We use Proposition 2.1 b) and that $1 + \mathcal{H}'^2 = 1/(1 - 1/x^v)$ is bounded to obtain

$$\begin{aligned} |A| &= \mathrm{cte}\,.\,x_1^{-\frac{v+2}{2}}, \\ |\nabla|A|| &\le \mathrm{cte}\,.\,x_1^{-\frac{v+4}{2}}, \\ \rho_0(p) &\le \pi x_1(p) + \int_2^{x_1(p)} (1 + \mathcal{H}'^2(t))^{1/2}\, dt \le \mathrm{cte}\,.\,x_1 \end{aligned}$$

and

$$dM = \frac{x^{n-1}}{(1 - \frac{1}{x^v})^{\frac{1}{2}}} dx_1 d\mu.$$

Putting things together, we can see that $\|\,|A|\,\|_{W^{1,q}_{-1}(M_1)} < \infty$ and the claim is proved.

Case 3: $v = 2$, or equivalently, $n = 2r$.

This case follows from a straightforward computation.

Example 2. Here, we prove that the hypersurface M obtained by the rotation of the curve $x_{n+1} = f(x_1)$, where $f(x_1) = \varepsilon e^{-1/x_1^2}$, around the x_{n+1}-axis has finite strong total curvature. In order to prove that $\|\,|A|\,\|_{W^{1,q}_{-1}(M)} <$

∞, it is clear that we can make our computation for $x_1 \geq 1$. We have:

$$f(x_1) = \varepsilon e^{-1/x_1^2}, \quad f'(x_1) = \frac{2f}{x_1^3} \quad \text{and} \quad f''(x_1) = \frac{2f}{x_1^6}(2 - 3x_1^2),$$

with $\lim\limits_{x_1 \to \infty} f(x_1) = \varepsilon$ and $\lim\limits_{x_1 \to \infty} f'(x_1) = 0$.

By using Proposition 2.1 b) we may write

$$|A| = \frac{G(x_1)}{x_1^4}, \text{ where } G(x_1) \text{ is a bounded differentiable function,}$$

and, for $x_1 \geq 1$, $|\nabla |A|| \leq \frac{\text{cte}}{x_1^4}$. We also have

$$\rho_0(p) \leq \text{cte} \,.\, x_1 \quad \text{and} \quad dM = (1 + f'^2)^{1/2} x_1^{n-1} dx_1 d\mu.$$

A straightforward computation shows that $\| \, |A| \, \|_{W^{1,q}_{-1}(M)} < \infty$.

Example 3. The minimal hypersurfaces of $\mathbb{R}^{n+1}$ with finite total curvature have finite strong total curvature (see Remark 5.1).

3. The rate of decay of the second fundamental form

Without loss of generality, we assume that $0 \in \phi(M)$ and we choose a point $0 \in M$ such that $0 = \phi(0)$. For $x \in M$, $\rho_0(x)$ will denote the intrinsic distance in M from x to 0. Then, from now on, when we say that the immersion has finite strong total curvature we are implicitly assuming w.l.g. that $0 \in \phi(M)$.

The following lemma will be repeatedly used in this and in the next section.

Lemma 3.1. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded domain with smooth boundary ∂D. Let (W_i) be a sequence of connected n-manifolds and let $\phi\colon W_i \to \mathbb{R}^{n+1}$ be immersions such that $\phi(\partial W_i) \cap D = \emptyset$ and $\phi(W_i) \cap D = M_i$ is connected and nonvoid. Assume that there exists a constant $C > 0$ such that $\sup\limits_{x \in M_i} |A_i(x)|^2 < C$ and that there exists a sequence of points (x_i), $x_i \in M_i$, with a limit point $x_0 \in D$. Then:*

i) *A subsequence of (M_i) converges $C^{1,\lambda}$ on the compact parts (see the definition below) to a union of hypersurfaces $M_\infty \subset D$, where $\lambda < 1$.*

ii) *If, in addition, $\left(\int_{M_i} |A_i|^q \alpha_i \, dM\right)^{1/q} + \left(\int_{M_i} |\nabla |A_i||^q \beta_i \, dM\right)^{1/q} \to 0$, for sequences $(\alpha_i)_i$ and $(\beta_i)_i$ of continuous functions such that*

$\inf_{x\in M_i}\{\alpha_i,\beta_i\} \geq \kappa > 0$. *Then a subsequence of* $|A_i|$ *converges to zero everywhere and* M_∞ *is a union of hyperplanes.*

By $C^{1,\lambda}$ convergence to M_∞ on compact sets we mean that for any $m \in M_\infty$ and each tangent plane T_mM_∞ there exists an euclidean ball B_m around m so that, for i large, the image by ϕ of some connected component of $\phi^{-1}(B_m \bigcap M_i)$ can be graphed over T_mM_∞ by a function g_i^m and the sequence g_i^m converges $C^{1,\lambda}$ to the graph g_∞ of M_∞ over the chosen plane T_mM_∞.

Proof. From the uniform bound of the curvature $|A_i|^2$, we conclude the existence of a number $\delta > 0$ such that for each $p_i \in M_i$ and for each tangent space $T_{p_i}M_i$, M_i can be graphed by a function $f_i^{p_i}$ over a disk $U_\delta(p_i) \subset T_{p_i}M_i$, of radius δ and center p_i in $T_{p_i}M_i$, and that such functions have a uniform C^1 bound (independent of p_i and i). We want to show that we also have a uniform C^2 bound.

Let q be a point in the part of M_i that is a graph over $U_\delta(p_i)$ and let $v \in T_qM_i$. Consider the plane P_q that contains the normal vector $N_i(q)$ and v and take the curve $C_i = P_q \cap M_i$. Parametrize C_i by $c_i(t)$ with $c_i(0) = q$, project it down to $T_{p_i}M_i$ parallely to the normal at p_i . Let $\tilde{c}_i(t)$ be this projection; then, $c_i(t) = \big(\tilde{c}_i(t), f_i^{p_i}(\tilde{c}_i(t))\big)$ and the normal curvature of M_i in q along v is

$$k_v^i(q) = \big(f_i^{p_i}\big)''(0)/\big(1 + \big[\big(f_i^{p_i}\big)'(0)\big]^2\big)^{3/2}, \tag{3.1}$$

where, e.g., $(f_i^p)'(t)$ means the derivative in t of $f_i^{p_i}(\tilde{c}_i(t)) = f_i^{p_i}(t)$. It follows that we have a uniform estimate for second derivatives in any direction v. By a standard procedure (see e.g. [10] p. 280), this implies a uniform C^2-bound on $f_i^{p_i}$. Now, consider the sequence (x_i) with a limit point x_0, and let τ_i be the translation that takes x_i to x_0. The unit normals of $\tau_i(M_i)$ at x_0 have a convergent subsequence, hence a subsequence of the tangent planes $T_{x_0}(\tau_iM_i)$ converges to a plane P containing x_0. For i large, the parts of M_i that were graphs over $U_\delta(x_i)$ are now graphs over $U_{\delta/2}(x_0) \subset P$; we will denote the corresponding functions by $g_i^{x_0}$. By the bounds on the derivatives that we have obtained, the functions $g_i^{x_0}$ and their first and second derivatives are uniformly bounded, say, $|g_i^{x_0}|_{2;U_{\delta/2}(x_0)} < C_1$. By standard arguments using the Mean Value and Arzelá-Ascoli theorems, we conclude that a subsequence of $g_i^{x_0}$ converges $C^{1,\lambda}$ to a function $g_\infty^{x_0}$ (i.e., that the immersion $C^2(U_{\delta/2}(x_0)) \hookrightarrow C^{1,\lambda}(U_{\delta/2}(x_0))$ is compact).

Notice that we have obtained a subsequence of (M_i) with the property that those parts of M_i that are graphs around the points x_i, converge to a hypersurface, again a graph, passing through x_0. We will express this fact by saying that (M_i) *has a subsequence that converges locally at* x_0.

To complete the proof of (i) of Lemma 3.1, we need a covering argument that runs as follows.

Let L be the set of all limit points of sequences of the form (p_i), where $p_i \in M_i$, and let M_∞ be the connected component of L that contains x_0. Let $q_1, q_2, \dots$ be a sequence of points in M_∞ that is dense in M_∞. Let (q_1^i), $q_1^i \in M_i$, be a sequence that converges to q_1. As we did before, we can obtain a subsequence (M_i^1) of (M_i) that converges locally at q_1 (to a hypersurface). From this sequence, we can extract a subsequence (M_i^2) that converges locally at q_1 and q_2. By induction, we can find sequences (M_i^n) that converge locally at $\bigcup_i q_i$, $i = 1, \dots, n$. By using the Cantor diagonal process, we obtain a sequence $M_1^1, M_2^2, \dots$ that converges C^1 to M_∞ and shows that M_∞ is a collection of C^1 hypersurfaces. Clearly M_∞ has no boundary point in the interior of D. Thus M_∞ extends to the boundary of D. Since the local convergence is uniform in compact subsets, it follows that the convergence to M_∞ is uniform in the compact subsets of M_∞. This completes the proof of (i) of Lemma 3.1.

Now we prove (ii) of Lemma 3.1. By (i), a subsequence of M_i converges C^1 to a collection of hypersurfaces, M_∞. As in the proof of (i), given $p \in M_\infty$, we can look upon the part of M_i near p, for large i, as a graph of a function g_i^p over $U_{\delta/2}(p) \subset T_pM_\infty$. The functions g_i^p converge C^1 to the function g^p that defines M_∞ near p.

Let G_i^p be the metric of M_i restricted to $g_i^p(U_{\delta/2}(p))$, G_∞^p be the metric of M_∞ restricted to $g^p(U_{\delta/2}(p))$ and let E be the euclidean metric in T_pM_∞. Notice that since the convergence $M_i \to M_\infty$ is C^1, G_i^p converges to G_∞^p. There exists a constant $\lambda_i > 0$ such that

$$\frac{1}{\lambda_i}E(X,X) \le G_i^p(X,X) \le \lambda_i E(X,X), \text{ for all } X \in T_pM_\infty \simeq \mathbb{R}^n.$$

Then $dM_i = \sqrt{det(G)}dV \ge (\frac{1}{\lambda_i})^{n/2}dV$, where dV is element of volume of $(T_pM_\infty, E) \simeq \mathbb{R}^n$. We obtain

$$\begin{aligned}&\left(\int_{g_i^p(U_{\delta/2}(p))} |A|^q \alpha_i \, dM\right)^{1/q} + \left(\int_{g_i^p(U_{\delta/2}(p))} |\nabla|A||^q \beta_i \, dM\right)^{1/q}\\ &\ge \kappa(\frac{1}{\lambda_i})^{n/2}\left(\int_{U_{\delta/2}(p)} |A|^q \, dV\right)^{1/q} + \kappa(\frac{1}{\lambda_i})^{(n+q)/2}\left(\int_{U_{\delta/2}(p)} |\nabla_{\mathrm{E}}|A||_{\mathrm{E}}^q \, dV\right)^{1/q}.\end{aligned}$$

Since

$$\left(\int_{g_i^p(U_{\delta/2}(p))} |A|^q \alpha_i \, dM\right)^{1/q} + \left(\int_{g_i^p(U_{\delta/2}(p))} |\nabla|A||^q \beta_i \, dM\right)^{1/q} \to 0$$

we conclude that $|A_i| \to 0$ in the usual Sobolev space $W^{1,q}(U_{\delta/2}(p))$. Now, since $q > n$, it follows from the fact that the injection

$$W^{1,q}(U_{\delta/2}(p)) \hookrightarrow C^0(U_{\delta/2}(p), \mathbb{R})$$

is compact (see, for instance, [1], page 168) that a subsequence of $(|A_i|)_i$ (again denoted by $(|A_i|)_i$) converges to zero in $\|.\|_{C^0}$.

Finally, we prove that M_∞ is a collection of hyperplanes by using the fact that $|A_i| \to 0$ everywhere. Since we have not proved that the convergence is C^2, this is not immediate. An argument is as follows. Let $p \in M_\infty$ and again look at the part of M_i near p as a graph of a function g_i^p over $U_{\delta/2}(p) \subset T_pM_\infty$ so that, as before, g_i^p converges C^1 to g^p that defines M_∞ near p. Let $q \in U_{\delta/2}(p)$ and $w \in \mathbb{R}^n$, $|w| = 1$. Set $r(t) = q + tw \subset U_{\delta/2}(p)$, $c_i(t) = (r(t), g_i^p(r(t)))$ and $c(t) = (r(t), g^p(r(t)))$. The fact that $|A_i| \to 0$ is easily seen to imply that $(g_i^p)''(t) \to 0$ in $U_{\delta/2}(p)$ (See (3.1)).

We will prove that M_∞ is a hyperplane over $U_{\delta/2}(p)$; since p is arbitrary, this will yield the result. Since we have a bound for the second derivatives of g_i^p in $U_\delta(p)$, we can use the Dominated Convergence Theorem and the fact that $(g_i^p)'(t) \to (g^p)'(t)$ to obtain

$$\begin{aligned}(g^p)'(t) - (g^p)'(0) &= \lim_{j\to\infty} \{(g_i^p)'(t) - (g_i^p)'(0)\} \\ &= \lim_{j\to\infty} \int_0^t (g_i^p)''(s)\, ds = \int_0^t \lim_{j\to\infty} (g_i^p)''(s)\, ds = 0,\end{aligned}$$

Thus, $c(t)$ is a straight line and, since w is arbitrary, M_∞ is a hyperplane over $U_\delta(p)$, as we asserted. This concludes the proof of Lemma 3.1. □

Remark. For future use, we observe that in the proof that M_∞ is a hyperplane we only use that the convergence is C^1, that we have a bound for the second derivatives of g_i^p and that $|A_i| \to 0$ everywhere.

The proof of the following proposition is inspired by that of [2], Proposition 2.2; for completeness, we present it here. Actually, the crucial point of the proof (Lemma 3.3 below), is also similar to the proof of Proposition 2 in Choi-Schoen [10].

Proposition 3.2. *Let $\phi\colon M^n \to \mathbb{R}^{n+1}$ be a complete immersion with finite strong total curvature. Then, given $\varepsilon > 0$ there exists $R_0 > 0$ such that, for $r > R_0$,*

$$r^2 \sup_{x \in M - D_0(r)} |A|^2(x) < \varepsilon.$$

For the two lemmas below we use the following notation. We denote by $h\colon X^n \to \mathbb{R}^{n+1}$ an immersion into $\mathbb{R}^{n+1}$ of an n-manifold $X^n = X$ with boundary ∂X such that there exists a point $x \in X$ with $D_x(1) \cap \partial X = \emptyset$.

Lemma 3.3. *There exists $\delta > 0$ such that if*

$$\left(\int_{D_x(1)} |A|^q \mu \, dX\right)^{1/q} + \left(\int_{D_x(1)} |\nabla |A||^q \nu \, dX\right)^{1/q} < \delta,$$

for any $h\colon X^n \to \mathbb{R}^{n+1}$ as above and for any pair of continuous functions $\mu, \nu\colon D_x(1) \to \mathbb{R}$ that satisfy $\inf_{D_x(1)} \{\mu, \nu\} > c > 0$, then

$$\sup_{t \in [0,1]} \left[t^2 \sup_{D_x(1-t)} |A_h|^2 \right] \leq 4.$$

Here A_h is the linear map associated to the second fundamental of h.

Proof. Suppose the lemma is false. Then there exist a sequence $h_i\colon X_i \to \mathbb{R}^{n+1}$, a sequence of points $x_i \in X_i$ with $D_{x_i}(1) \cap \partial X_i = \emptyset$ and sequences $(\mu_i)_i, (\nu_i)_i$, with $\inf_{D_x(1)} \{\mu_i, \nu_i\} > c$ such that

$$\left(\left(\int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{x_i}(1)} |\nabla |A_i||^q \nu_i \, dX_i\right)^{1/q}\right) \to 0$$

but

$$\sup_{t \in [0,1]} \left[t^2 \sup_{D_{x_i}(1-t)} |A_i|^2 \right] > 4,$$

for all i, where $A_i = A_{h_i}$.

Choose $t_i \in [0,1]$ so that

$$t_i^2 \sup_{D_{x_i}(1-t_i)} |A_i|^2 = \sup_{t\in[0,1]} \left[t^2 \sup_{D_{x_i}(1-t)} |A_i|^2 \right]$$

and choose $y_i \in \overline{D_{x_i}(1-t_i)}$ so that

$$|A_i|^2(y_i) = \sup_{D_{x_i}(1-t_i)} |A_i|^2.$$

By using that $D_{y_i}(t_i/2) \subset D_{x_i}(1-(t_i/2))$ we obtain

$$\sup_{D_{y_i}(t_i/2)} |A_i|^2 \le \sup_{D_{x_i}(1-(t_i/2))} |A_i|^2 \le \frac{t_i^2}{t_i^2/4} \sup_{D_{x_i}(1-t_i)} |A_i|^2,$$

hence, by the choice of y_i, we have

$$\sup_{D_{y_i}(t_i/2)} |A_i|^2 \le 4|A_i|^2(y_i). \tag{3.2}$$

We now rescale the metric defining $d\tilde{s}_i^2 = |A_i|^2(y_i)ds_i^2$, that is, $d\tilde{s}_i^2$ is the metric on X_i induced by $\tilde{h}_i = d_i \circ h_i$, where d_i is the dilation of $\mathbb{R}^{n+1}$ about $h_i(y_i)$ (by translation, we may assume that $h_i(y_i) = 0$) by the factor $|A_i|(y_i)$. The symbol $\sim$ will indicate quantities measured with respect to the new metric $d\tilde{s}_i^2$.

By assumption, $|A_i|^2(y_i) > 4/t_i^2$. Thus

$$\widetilde{D}_{y_i}(1) = D_{y_i}([|A_i|(y_i)]^{-1}) \subset D_{y_i}(t_i/2) \subset D_{x_i}(1-t_i/2) \subset D_{x_i}(1).$$

It follows that $\widetilde{D}_{y_i}(1) \cap \partial X_i = \emptyset$. Now, we use (3.2) and the fact that

$$|\widetilde{A}_i|(p) = [|A_i|(y_i)]^{-1}|A_i|(p)$$

to obtain

$$\sup_{\widetilde{D}_{y_i}(1)} |\widetilde{A}_i|^2 \le 4.$$

Therefore, the sequence $\tilde{h}_i = \widetilde{D}_{y_i}(1) \to \mathbb{R}^{n+1}$, $\tilde{h}_i(y_i) = 0$, is a sequence of immersions with uniformly bounded second fundamental form.

By using that $\widetilde{D}_{y_i}(1) = D_{y_i}([|A_i|(y_i)]^{-1}) \subset D_{x_i}(1)$ we have

$$\left(\int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{x_i}(1)} |\nabla|A_i||^q \nu_i \, dX_i\right)^{1/q}$$
$$\geq \left(\int_{D_{y_i}([|A_i|(y_i)]^{-1})} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{y_i}([|A_i|(y_i)]^{-1})} |\nabla|A_i||^q \nu_i \, dX_i\right)^{1/q}.$$

Thus, we obtain

$$\left(\left(\int_{\widetilde{D}_{y_i}(1)} |\widetilde{A}_i|^q \mu_i \, |A_i(y_i)|^{q-n} \, d\widetilde{X}_i\right)^{1/q} + \left(\int_{\widetilde{D}_{y_i}(1)} |\widetilde{\nabla}|\widetilde{A}_i||^q \nu_i \, |A_i(y_i)|^{2q-n} \, d\widetilde{X}_i\right)^{1/q}\right)$$
$$\leq \left(\left(\int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{x_i}(1)} |\nabla|A_i||^q \nu_i \, dX_i\right)^{1/q}\right) \to 0.$$

Since $|A_i(y_i)| > \frac{2}{t_i} \geq 2$ we can use Lemma 3.1, with $\alpha_i = \mu_i |A_i(y_i)|^{q-n}$, $\beta_i = \nu_i |A_i(y_i)|^{2q-n}$ and $\kappa = 2c$, to conclude that a subsequence of $|\widetilde{A}_i|$ converges to zero. But $|\widetilde{A}_i|(y_i) = 1$, for all i, hence $|\widetilde{A}_\infty|(y_\infty) = 1$. This is a contradiction, and completes the proof of Lemma 3.3. □

Lemma 3.4. *Given $\varepsilon_1 > 0$, there exists $\delta > 0$, such that if*

$$\left(\int_{D_x(1)} |A|^q \mu \, dX\right)^{1/q} + \left(\int_{D_x(1)} |\nabla|A||^q \nu \, dX\right)^{1/q} < \delta,$$

for any $h\colon X^n \to \mathbb{R}^{n+1}$ as above and for any pair of continuous functions $\mu, \nu\colon D_x(1) \to \mathbb{R}$ that satisfy $\inf_{D_x(1)} \{\mu, \nu\} > c > 0$, then

$$\sup_{D_x(1/2)} |A_h|^2 < \varepsilon_1.$$

Proof. Suppose the lemma is false. Then there exist a sequence $h_i\colon X_i \to \mathbb{R}^{n+1}$, a sequence of points $x_i \in X_i$ with $D_{x_i}(1) \cap \partial X_i = \emptyset$ and sequences

$(\mu_i)_i, (\nu_i)_i$, with $\inf_{D_x(1)} \{\mu_i, \nu_i\} > c$ such that

$$\left(\left(\int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{x_i}(1)} |\nabla |A_i||^q \nu_i \, dX_i\right)^{1/q}\right) \to 0 \tag{3.3}$$

but

$$\sup_{D_{x_i}(1/2)} |A_i|^2 \geq K^2, \tag{3.4}$$

for some constant K.

By Lemma 3.3 (with $t = 1/2$), we have, for i sufficiently large,

$$\sup_{D_{x_i}(1/2)} |A_i|^2 \leq 16.$$

By (3.3) and Lemma 3.1 , a subsequence of $|A_i|$ converges to zero. This is a contradiction to (3.4) and proves Lemma 3.4. □

Proof of Proposition 3.2. We first rescale the immersion ϕ to $\tilde{\phi} = d_{2/r} \circ \phi$, where $d_{2/r}$ is the dilation by the factor $2/r$. Thus the metric induced by $\tilde{x}$ in M is $d\tilde{s}^2 = (4/r^2) ds^2$, where ds^2 is the metric induced by ϕ. We will denote the quantities measured relative to the new metric by the superscript $\sim$. Notice that the second fundamental form $\widetilde{A}$ satisfies $|\widetilde{A}|^2 = \frac{r^2}{4} |A|^2$.

Therefore, Proposition 3.2 will be established once we prove that given $\varepsilon > 0$ there exists R_0 such that, for $r > R_0$,

$$\sup_{M - \widetilde{D}_0(2)} |\widetilde{A}|^2 < \varepsilon/4.$$

Given the above ε, set $\varepsilon_1 < \varepsilon/4$ and let $\delta > 0$ be given by Lemma 3.4. Since M has finite strong total curvature, there exists R_0 such that, for $r > R_0$,

$$\delta > \left(\int_{D_0(r/2,\infty)} |A|^q |\rho_0|^{q-n} \, dM\right)^{1/q} + \left(\int_{D_0(r/2,\infty)} |\nabla |A||^q |\rho_0|^{2q-n} \, dM\right)^{1/q}$$

$$= \left(\int_{\widetilde{D}_0(1,\infty)} |\widetilde{A}|^q |\widetilde{\rho}_0|^{q-n} \, d\widetilde{M}\right)^{1/q} + \left(\int_{\widetilde{D}_0(1,\infty)} |\widetilde{\nabla} |\widetilde{A}||^q |\widetilde{\rho}_0|^{2q-n} \, d\widetilde{M}\right).$$

For $x \in M - \widetilde{D}_0(2)$, we have $\widetilde{D}_x(1) \subset \widetilde{D}_0(1,\infty)$ and then $\inf\limits_{\widetilde{D}_x(1)} \widetilde{\rho}_0 > 1$. Now, Lemma 3.4, with $\mu = |\widetilde{\rho}_0|^{q-n}$ and $\nu = |\widetilde{\rho}_0|^{2q-n}$, and the above inequality imply that

$$\sup_{\widetilde{D}_x(1/2)} |\widetilde{A}|^2 < \varepsilon_1,$$

hence

$$\sup_{M-\widetilde{D}_0(2)} |\widetilde{A}|^2 \leq \varepsilon_1 < \varepsilon/4.$$

This completes the proof of Proposition 3.2. □

4. Uniqueness of the tangent plane at infinity

The proof of our Theorem 1.1 depends on a series of lemmas and a crucial proposition to be presented in a while. In this section, $\phi\colon M^n \to \mathbb{R}^{n+1}$ will always denote a complete hypersurface such that $\phi(M^n)$ passes through the origin 0 of $\mathbb{R}^{n+1}$, with finite strong total curvature.

The following lemma is similar to Lemma 2.4 in Anderson [2].

Lemma 4.1. *Let $\phi\colon M^n \to \mathbb{R}^{n+1}$ be as above and let $r(p) = d(\phi(p), 0)$, where $p \in M$ and d is the distance in $\mathbb{R}^{n+1}$. Then ϕ is proper and the gradient ∇r of r in M satisfies*

$$\lim_{r\to\infty} |\nabla r| = 1.$$

In particular, there exists r_0 such that if $r > r_0$, $\nabla r \neq 0$, i.e., the function r has no critical points outside the ball $B(r_0)$.

Proof. If the immersion is not proper, we can find a ray $\gamma(s)$ issuing from 0 and parametrized by the arc length s such that as s goes to infinity the distance $r(\gamma(s))$ is bounded. Let such a ray be given and set $T = \gamma'(s)$. Let

$$X = (1/2)\overline{\nabla} r^2 = r\overline{\nabla} r,$$

be the position vector field, where $\overline{\nabla} r$ is the gradient of r in $\mathbb{R}^{n+1}$. Then

$$T\langle X, T\rangle = \langle \overline{\nabla}_T X, T\rangle + \langle X, \overline{\nabla}_T T\rangle = 1 + \langle X, \overline{\nabla}_T T\rangle.$$

Since γ is a geodesic in M, the tangent component of $\overline{\nabla}_T T$ vanishes and

$$\overline{\nabla}_T T = \langle \overline{\nabla}_T T, N\rangle N = -\langle \overline{\nabla}_T N, T\rangle N = \langle A(T), T\rangle N.$$

It follows, by Cauchy-Schwarz inequality, that

$$|\langle X, \overline{\nabla}_T T\rangle| \leq |X|\,|A(T)|\,|T| \leq |X|\,|A|,$$

hence

$$T\langle X, T\rangle \geq 1 - |X|\,|A|.$$

By using Proposition 3.2 with $\varepsilon = 1/m^2$, and the facts that $r = |X(s)| \leq s$ and that γ is a minimizing geodesic, we obtain

$$T\langle X, T\rangle(s) \geq 1 - \frac{1}{m}, \tag{4.1}$$

for all $s > R_0$, where R_0 is given by Proposition 3.2. Integration of (4.1) from R_0 to s gives

$$\langle X, T\rangle(s) \geq \left(1 - \frac{1}{m}\right)(s - R_0) + \langle X, T\rangle(R_0). \tag{4.2}$$

Because $r(s) = |X(s)| \geq \langle X, T\rangle(s)$, we see from (3.2) that r goes to infinity with s. This is a contradiction and proves that M is properly immersed.

Now let $\{p_i\}$ be a sequence of points in M such that $\{r(p_i)\} \to \infty$. Let γ_i be a minimizing geodesic from 0 to p_i, and denote again by $\gamma(s)$ the ray which is the limit of $\{\gamma_i\}$. For each γ_i, we apply the above computation, and since

$$\langle X_i, T_i\rangle(s) = \langle r_i \overline{\nabla} r_i, T_i\rangle(s) \leq r_i|\nabla r_i|(s),$$

we have

$$|\nabla r_i|(s) \geq \frac{\langle X_i, T_i\rangle(s)}{s} \geq \left(1 - \frac{1}{m}\right)\left(\frac{s - R_0}{s}\right) + \frac{\langle X_i, T_i\rangle(R_0)}{s},$$

hence, for the ray $\gamma(s)$,

$$|\nabla r|(s) \geq \left(1 - \frac{1}{m}\right)\left(\frac{s - R_0}{s}\right) + \frac{\langle X, T\rangle(R_0)}{s}. \tag{3.3}$$

By taking the limit in (3.3) as $s \to \infty$, we obtain that $\lim_{s\to\infty} |\nabla r| \geq 1 - \frac{1}{m}$. Since m and the sequence $\{p_i\}$ are arbitrary, and $|\nabla r| \leq 1$, we conclude that $\lim_{r\to\infty} |\nabla r| = 1$, and this completes the proof of Lemma 4.1. $\square$

Remark. Related to Lemma 4.1, Bessa, Jorge and Montenegro [5] proved independently that for an immersion $\phi\colon M^n \to \mathbb{R}^N$ (of arbitrary codimension) for which the norm $|\alpha|$ of the second fundamental form α satisfies

$$\lim_{r\to\infty} \sup_{p\in M-D_0(r)} r^2|\alpha|^2 < 1$$

it holds that ϕ is proper and that the distance function $r = d(\phi(p), 0)$, $p \in M$, has no critical point outside a certain ball.

Now, let r_0 be chosen so that r has no critical points in $W = \phi(M) - (B(r_0) \cap \phi(M))$. By Morse Theory, $x^{-1}(W)$ is homeomorphic to $\phi^{-1}[\phi(M) \cap S(r_0)] \times [0, \infty)$. Let V be a connected component of $\phi^{-1}(W)$, to be called an *end* of M. It follows that M has only a finite number of ends. In what follows, we identity V and $\phi(V)$.

Let $r > r_0$ and set

$$\Sigma_r = \frac{1}{r}\,[V \cap S(r)] \subset S(1),$$
$$V_r = \frac{1}{r}\,[V \cap B(r)] \subset B(1).$$

Denote by A_r the second fundamental form of V_r. Then

$$|A_r|^2(x) = r^2|A|^2(rx).$$

Lemma 4.2. *For $r > r_0$, $V \cap B(r)$ is connected.*

Proof. Notice that $V = S \times [0, \infty)$ where S is a connected component of $M \cap S(r_0)$. Assume that $V \cap B(r)$ has two connected components, V_1 and V_2. Since $(V_1 \cup V_2) \cap S(r_0)$ is connected, either $V_1 \cap S(r_0)$ or $V_2 \cap S(r_0)$ is empty. Assume it is $V_2 \cap S(r_0)$.

Let $p \in V_2$. Since all the trajectories of ∇r start from $V_1 \cap S(r_0)$, there exists a trajectory $\varphi(t)$ with $\varphi(0) \in V_1 \cap S(r_0)$ and $\varphi(t_2) = p$. Thus, there exist $t_0, t_1 \in [0, t_2]$, such that a trajectory of ∇r satisfies $|\varphi(t_0)| = |\varphi(t_1)| = r$. We claim that this implies the existence of a critical point of r at some point of $\varphi(t)$.

Indeed, let $f(t) = r(\varphi(t))$. Then $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(t_0) = f(t_1)$. Thus, there exists $\bar{t} \in [t_0, t_1]$ with $f'(\bar{t}) = 0$. But

$$f'(t) = dr\left(\frac{d\varphi}{dt}\right) = dr(\nabla r) = \langle \nabla r, \nabla r\rangle.$$

Therefore,

$$0 = f'(\bar{t}) = |\nabla r(\bar{t})|^2$$

and this proves our claim.

Thus we have reached a contradiction and this proves the lemma. □

Lemma 4.3. *Let $0 < \delta < 1$ be given and fix a ring $A(\delta, 1) \subset B(1)$. Then, given $\varepsilon > 0$, there exists r_1 such that, for all $r > r_1$ and all $x \in V_r \cap A(\delta, 1)$, we have*

$$|A_r|^2(x) < \varepsilon.$$

Proof. By Proposition 3.2, there exists r_0 such that for $r > r_0$

$$r^2 \sup_{x \in M - D_0(r)} |A|^2(x) < \delta^2 \varepsilon. \tag{3.4}$$

Take $r_1 = r_0/\delta$. Then, for $r > r_1$ and $x \in V_r \cap A(\delta, 1)$,

$$r|x| > r\delta > r_0.$$

Thus, by (3.4), for all $x \in V_r \cap A(\delta, 1)$ and $r > r_1$,

$$r^2 |x|^2 \left[\sup_{y \in M - D_0(r|x|)} |A|^2(y) \right] < \delta^2 \varepsilon. \tag{3.5}$$

Now, by using again Proposition 3.2 and (3.5), we obtain that

$$|A_r|^2(x) = r^2 |A|^2(rx) \leq r^2 \sup_{y \in M - D_0(r|x|)} |A|^2(y) < \frac{\delta^2 \varepsilon}{|x|^2} < \varepsilon,$$

for all $x \in V_r \cap A(\delta, 1)$ and $r > r_1$, and this proves Lemma 4.3. □

By Lemma 4.3, we see that $|A_r|^2 \to 0$ uniformly in the ring $A(\delta, 1)$. It follows from this and the fact that V_r is connected that we can apply Lemma 3.1(i) and conclude that a subsequence V_{r_i} of V_r, $r_i \to \infty$, converges C^1 to a union of hypersurfaces π in $A(\delta, 1)$. Again, since $|A_r| \to 0$ uniformly, π is a union of n-planes in $A(\delta, 1)$ (see Remark after the proof of Lemma 3.1). Since δ is arbitrary, a subsequence again denoted by V_{r_i} converges to π in $B(1) - \{0\}$ and the n-planes in π all pass through the origin 0. Thus, each two of them intersect along a linear (n-1)-subspace L and the hypersurfaces $\Sigma_{r_i} \subset S_1^n$, given by the inverse images of the regular values r_i of the distance function r, converge to a family Σ_∞ of equators of S_1^n each two of each intersect along $L \cup S_1^n$. We claim that Σ_∞ contains only one equator. In fact, for r_i large enough, by the basic transversality theorem ([12] Chapter 3, Theorem 2.1), Σ_{r_i} has a self intersection close to $L \cup S_1^n$ and this contradicts the fact that Σ_{r_i} is an embedded hypersurface. It follows that π is a single n-plane passing through 0, possibly with multiplicity $m \geq 1$. Since Σ_∞ covers S_1^{n-1}, which is simply-connected, $m = 1$. Thus V is embedded and π is a single plane that passes through the origin.

The n-plane π spanned by Σ_∞ is called the *tangent plane at infinity of the end V associated to the sequence $\{r_i\}$*. A crucial point in the proof of Theorem 1.1 is to show that this plane does not depend on the sequence $\{r_i\}$. Here we use for the first time the hypothesis on H_n.

Proposition 4.4. *Each end V of M has a unique tangent plane at infinity.*

Proof. Suppose that $\{s_i\}$ and $\{r_i\}$, $s_i, r_i \to \infty$, are sequences of real numbers and that π_1 and π_2 are distinct tangent planes at infinity associated to $\{s_i\}$ and $\{r_i\}$, respectively. We can assume that the sequences satisfy

$$s_1 < r_1 < s_2 < r_2 < \cdots < s_i < r_i < \cdots .$$

Let K be the closure of $B(3/4) - B(1/4)$ and let N_1 be the normal to π_1, obtained as the limit of the normals to

$$K \cap \left\{ \frac{1}{s_i} V \right\} = \frac{1}{s_i}(V \cap s_i K).$$

Similarly, let N_2 be the normal to π_2 obtained as the limit of the normals to $K \cap \{(1/r_i)V\}$.

Now let U_1 and U_2 be neighborhoods in $S^n(1)$ of N_1 and N_2, respectively, such that $U_1 \cap U_2 = \emptyset$. Thus, there exists an index i_0 such that, for $i > i_0$, the normals to $K_i^1 = (s_i K) \cap V$ are in U_1 and the normals to $K_i^2 = (r_i K) \cap V$ are in U_2. If $K_i^1 \cap K_i^2 \neq \emptyset$, for some $i > i_0$, this contradicts the fact that $U_1 \cap U_2 = \emptyset$, and the proposition is proved.

Thus we may assume that, for all $i > i_0$, $K_i^1 \cap K_i^2 = \emptyset$. In this case, we have $(1/4)r_i > (3/4)s_i$; here, and in what follows, we always assume $i > i_0$. Set

$$W_i = V \cap \left(B\left(\frac{1}{4} r_i\right) - B\left(\frac{3}{4} s_i\right) \right).$$

Since H_n does not change sign in V, we have that ([17], Thm. II) $g(\partial W_i) \supset \partial(g(W_i))$. Since

$$g\left(S\left(\frac{1}{4}r_i\right) \cap V\right) \subset U_2,$$
$$g\left(S\left(\frac{3}{4}s_i\right) \cap V\right) \subset U_1,$$

we have $g(\partial W_i) \subset U_1 \cup U_2$. Thus

$$\partial(g(W_i)) \subset g(\partial W_i) \subset U_1 \cup U_2\,. \tag{3.6}$$

We claim that there exists a point $x \in \mathrm{Int}(W_i)$ with $H_n(x) \neq 0$. Suppose that

$$\{x \in \ \mathrm{Int}\ W_i\,; H_n(x) \neq 0\} = \emptyset. \tag{3.7}$$

Since $g(W_i)$ is connected and has nonvoid intersection with U_1 and U_2 which are disjoint, there is a point $x_0 \in$ Int W_i such that $g(x_0) \notin U_1 \cup U_2$. Let rank $A(x_0) = m$. By (3.7), $m < n$. Since the k_i's are continuous, there is a neighborhood V of x_0 such that if $x \in V$,

$$n > \ \mathrm{rank}\ A(x) \geq m,$$

where the left hand inequality follows from (3.7). This implies that either rank A is constant and equal to m in a neighborhood of x_0 or in each neighborhood of x_0 there is a point such that the rank of A at this point is greater than m. In view of (3.7), the latter implies that we can find such a point, to be called y_0, so that about y_0 there is a neighborhood with rank $A = m_0 > m$.

In both cases, we obtain a point and a neighborhood of this point for which rank A is constant. Without loss of generality, we can assume this point to be y_0. Notice that we can assume $g(y_0) \notin U_1 \cup U_2$. By the Lemma of Chern-Lashof ([9], Lemma 2), there passes through y_0 a piece L^p of a p-dimensional plane, $p = n - m_0$, along which g is constant. If L^p intersects ∂W_i, $g(y_0) \in g(\partial W_i) \subset U_1 \cup U_2$, and this contradicts the choice of y_0. If not, a point $\bar{y}_0$ in ∂L^p has again rank $A = m_0$ ([9], Lemma 2), and arbitrarily close to $\bar{y}_0$, we have a point y_1 and a neighborhood of y_1 whose rank is $m_1 > m_0$. Thus, we can repeat the process.

After a finite number of steps, the process will lead either to finding a point with rank $A = n$, what contradicts (3.7), or to finding a piece L of a plane of appropriate dimension with the property that $L \cap \partial W_i \neq \emptyset$. As we have seen above, this is again a contradiction and proves our claim.

Thus, we can assume that there is a point $x \in \operatorname{Int}(W_i)$ with $H_n(x) \neq 0$. Then $g(W_i)$ contains an open set around $g(x)$. We can assume that U_1 and U_2 are small enough so that $g(x) \notin U_1 \cup U_2$. Since $g(W_i)$ is connected and has nonvoid intersection with both U_1 and U_2, the fact that there are interior points in $g(W_i)$ and (3.6) imply that

$$g(W_i) \supset S^n(1) - \{U_1 \cup U_2\}. \tag{3.8}$$

On the other hand, because

$$\left(\Sigma k_i^2\right)^q > C k_1^2 \cdots k_n^2,$$

for a constant $C = C(n)$, we have that

$$|H_n| < \frac{1}{\sqrt{C}} |A|^q.$$

Furthermore, since ϕ has finite strong total curvature,

$$\int_{W_i} |A|^q |\rho_0|^{n-q} \, dM \to 0, \quad i \to \infty.$$

Therefore, since

$$\text{Area } g(W_i) \leq \int_{W_i} |H_n| \, dM < (\frac{1}{\sqrt{C}}) \int_{W_i} |A|^q |\rho_0|^{n-q} \, dM,$$

we have that Area $g(W_i) \to 0$. This a contradiction to (3.8), and completes the proof of Proposition 4.4. □

5. Proofs of Theorems 1.1, 1.2 and 1.4

Proof of Theorem 1.1. (i) has already been proved in Lemma 4.1. To prove (ii), we apply to each end V_i the inversion $I\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$, $I(x) = x/|x|^2$. Then $I(V_i) \subset B(1) - B(0)$ and as $|x| \to \infty$ in V_i, $I(x)$ converges to the origin 0. It follows that each V_i can be compactified with a point q_i. Doing this for each V_i, we obtain a compact manifold $\overline{M}$ such that $\overline{M} - \{q_1, \dots, q_k\}$ is diffeomorphic to M. This prove (ii).

To prove (iii), we use again the above inversion and observe that, by Proposition 4.4, as $|x| \to \infty$ in V_i, the normals at $I(x)$ converge to a unique normal $p_i \in S_1^n$ (namely, to the normal of the unique plane at infinity of V_i). Thus we obtain a continuous extension $\overline{g}\colon \overline{M} \to S_1^n$ of g by setting $\overline{g}(q_i) = p_i$. This proves (iii). □

Remark 5.1. As we mentioned in the introduction, Anderson proved in [2] that a minimal hypersurface $\mathcal{M}$ (in fact, the codimension can be greater than one) with finite total curvature is diffeomorphic to a compact manifold minus finitely many points and that the Gauss map extends smoothly to the punctures. From the proof of Theorem 3.2 in [2], we are able to understand the behaviour of each end of $\mathcal{M}$ and, *a fortiori*, to conclude that $\mathcal{M}$ has finite strong total curvature and that its Gauss-Kronecker curvature does not change sign in each end.

Proof of Theorem 1.2(i). We first observe that $S_1^n - (\mathcal{N})$ is still simply-connected. This comes from the fact that a closed curve C in $S_1^n - (\mathcal{N})$ is homotopic to a simple one and a disk generated by such a curve can, by transversality, be made disjoint of $\mathcal{N}$ by a small perturbation. Thus C is homotopic to a point in $S_1^n - (\mathcal{N})$.

Next, the restriction map

$$\tilde{g}\colon M - \overline{g}^{-1}(\mathcal{N} \cup \{p_i\}) \to S_1^n - (\mathcal{N} \cup \{p_i\})$$

where p_i is defined in the proof of Theorem 1.1, is clearly proper and its Jacobian never vanishes. In this situation, it is known that the map is surjective and a covering map ([23], Corollary 1). Since $S_1^n - (\mathcal{N} \cup \{p_i\})$ is simply-connected, $\tilde{g}$ is a global diffeomorphism.

To complete the proof we must show that if $\overline{g}(n_1) = \overline{g}(n_2) = p$, $n_1, n_2 \in \overline{g}^{-1}(\mathcal{N} \cup \{p_i\})$ then $n_1 = n_2$. Suppose that $n_1 \neq n_2$. Let $W \subset S^n(1)$ be a neighborhood of p. By continuity, there exist disjoint neighborhoods U_1 of n_1 and U_2 of n_2 in $\overline{M}$ such that $\overline{g}(U_1) \subset W$ and $\overline{g}(U_2) \subset W$. Choose $t \in \overline{g}(U_1) \cap \overline{g}(U_2)$, $t \notin \mathcal{N} \cup \{p_i\}$. Then, there exist $r_1 \in U_1$ and $r_2 \in U_2$ such that $\tilde{g}(r_1) = \tilde{g}(r_2) = t$. But this contradicts the fact that $\tilde{g}$ is a diffeomorphism and concludes the proof of (i).

(ii) We will use a result of Barbosa, Fukuoka and Mercuri [4]. By using Hopf's theorem that the Euler characteristic $\chi(\overline{M})$ of $\overline{M}$ is equal to the sum of the indices of a vector field, the following expression is obtained in [4] Theorem 2.3: if n is even,

$$\chi(\overline{M}) = \sum_{i=1}^{k} (1 + I(q_i)) + 2d\sigma.$$

Here $I(q_i)$ is the multiplicity of the end V_i (since $n \geq 3$, $I(q_i) = 1$ in our case), σ is ± 1 depending on the sign of H_n, k is the number of ends and d is the degree of the Gauss map $\overline{g}$. From Theorem 1.2 (i), $\overline{g}$ is a homeomorphism. Thus, $d = 1$ and, since n is even, $\chi(\overline{M}) = 2$. It follows that

$$2 = 2k + 2\sigma.$$

Thus $k = 2$ and $\sigma = -1$, and the result follows. □

Proof of the Gap Theorem. First, we easily compute that

$$|A|^{2n} > (n!)H_n^2.$$

Thus, since H_n is the determinant of the Gauss map $g\colon M^n \to S_1^n$, we obtain

$$\int_M |A|^n \, dM > \sqrt{n!} \int_M |H_n| dM = \sqrt{n!} \text{ area of } g(M) \text{ with multiplicity.}$$

The extended map $\overline{g}\colon \overline{M} \to S_1^n$, which is given by Theorem 1.1, has a well defined degree d, hence

$$\text{area } g(M) = \text{ area } \overline{g}(\overline{M}) = d \text{ area } S_1^n .$$

Now, assume that $\phi(M)$ is not a hyperplane. We claim that $d \neq 0$. To see that, we first show that there exists a point in M where $H_n \neq 0$.

Suppose the contrary holds. Then, since $\phi(M)$ is not a hyperplane, there is a point $x_\ell \in M$ such that rank A at x_ℓ is ℓ, $0 < \ell < n$. Thus, by using the Lemma of Chern-Lashof ([9], Lemma 2) in the same way as we did in Proposition 4.4, we arrive, after a finite number of steps, at one of the two following situations. Either we find a point where $H_n \neq 0$, which is a contradiction, or we find an open set $U_j \subset M$, whose points satisfy rank $A = j \geq \ell$, $j < n$, foliated by $(n-j)$-planes the leaves of which extend to infinity. In the second situation, observe that the Gauss map on each leaf is constant and, since there is only one normal at infinity for each end, the normal map is constant on U_j. Thus U_j is a piece of a hyperplane, and we find again a contradiction, this time to the fact that $n > j \geq \ell > 0$.

Therefore, there exists a point $x_0 \in M$ with $H_n(x_0) \neq 0$. Then, for a neighborhood V of x_0, we have that $H_n(x) \neq 0$, $x \in V$, and that $g(V) \subset S_1^n$ is a neighborhood of $g(x_0)$. By Sard's theorem, the set of critical values of g has measure zero, hence some point of $g(V)$ is a regular value. It follows that the Gauss map g has regular values whose inverse images are not empty. Since H_n does not change sign, this prove our claim.

Furthermore the area σ_n of a unit sphere of $\mathbb{R}^{n+1}$ is given by

$$\sigma_n = \frac{2(\sqrt{\pi})^{n+1}}{\Gamma((n+1)/2)};$$

here Γ is the gamma function, which, in the present case is given by

$$\Gamma((n+1)/2) = ((n-1)/2)!, \text{ if } n \text{ is odd}$$
$$\Gamma((n+1)/2) = \frac{(n-1)(n-3)\cdots 1}{2^{n/2}}\sqrt{\pi}, \text{ if } n \text{ is even.}$$

It follows that, for all non-planar $x \in C^n$,

$$\int_M |A|^n \, dM > 2\sqrt{n!}\,(\sqrt{\pi})^{n+1}/\Gamma((n+1)/2).$$

□

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References

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, Second Edition, Academic Press.

[2] M. Anderson, *The compactification of a minimal submanifold in Euclidean space by the Gauss map*, preprint IHES (1985).

[3] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl.Math. **39** (1986), 661–693.

[4] J. L. Barbosa, R. Fukuoka, and F. Mercuri, *Hypersuperfaces of finite geometric type*, Ann. of Global An. and Geometry **22** (2002), 301–315.

[5] G. P. Bessa, L. Jorge, and J. F. Montenegro, *Complete submanifolds of* $\mathbb{R}^n$ *with finite topology*, Comm. Anal. Geom. **15** (2007), no. 4, 725–732.

[6] G. Carron and M. Herzlich, *The Huber theorem for non-compact conformally flat manifolds*, Comment. Math. Helv. **77** (2002), 192–220.

[7] G. Carron and M. Herzlich, *Erratum to "The Huber theorem for non-compact conformally flat manifolds"*, Comment. Math. Helv. **82** (2007), 451–453.

[8] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet for nonlinear second order equation, III, Functions of the eigenvalues of the hessian*, Acta Math. **155** (1985), 261–301.

[9] S. S. Chern and R. Lashof, *On the total curvature of immersed manifolds*, Am. J. of Math. **78** (1957), 306–318.

[10] H. I. Choi and R. Schoen, *The space of minimal embedding of a surface into a three-dimensional manifold of positive Ricci curvature*, Invent. Math. **81** (1985), 387–394.

[11] X. Dai and L. Ma, *Mass under the Ricci flow*, Comm. Math. Phys. **274** (2007), no. 1, 65–80.

[12] M. W. Hirsch, Differential Topology, Springer-Verlag, New York Heidelberg Berlin (1976).

[13] A. Huber, *On subharmonic functions and differential geometry in the large*, Comment. Math. Helv. **52** (1957), 13–72.

[14] D. A. Lee and P. G. LeFloch, *The positive mass theorem for manifolds with distributional curvature*, Commun. Math. Phys. **339** (2015), 99–120.

[15] C. Matheus and K. Oliveira, *Geometrical versus topological properties of manifolds*, Journal of the Inst. Math. Jussieu **4** (2005), no. 4, 639–651.

[16] J. Hounie and M. L. Leite, *Uniqueness and Nonexistence Theorems for Hypersurfaces with $H_r = 0$.*, Ann. of Gl. An. and Geom. **17** (1999), 397–407.

[17] P. Hartman and L. Nirenberg, *On spherical image maps whose jacobian do not change sign*, Amer. J. of Math. **81** (1959), 901–920.

[18] S. Müller and V. Šverák, *On surfaces of finite total curvature*, J. Diff. Geometry **42** (1995), 229–258.

[19] R. Osserman, *Global properties of minimal surfaces in E^3 and E^n*, Ann. of Math. **80** (1964), 340–364.

[20] T. A. Oliynyk and E. Woolgar, *Rotationally symmetric Ricci flow on asymptotically flat manifolds*, Comm. Anal. Geom. **15** (2007), no. 3, 535–568.

[21] W. Rudin, Real and Complex Analysis, Third Edition, McGraw-Hill Book Company, London, New York (1987).

[22] B. White, *Complete surfaces of finite total curvature*, J. Diff. Geometry **26** (1987), 315–326.

[23] J. Wolf and P. Griffiths, *Complete maps and differentiable coverings*, Michigan Math. J., **10** (1963), 253–255.

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