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Exceptional slopes on manifolds of small complexity

Fionntan Wouter Murray Roukema

It has been observed that most manifolds in the Callahan-Hildebrand-Weeks census of cusped hyperbolic 3-manifolds are obtained by surgery on the minimally twisted 5-chain link. A full classification of the exceptional surgeries on the 5-chain link has recently been completed. In this article, we provide a complete classification of the sets of exceptional slopes and fillings for all cusped hyperbolic surgeries on the minimally twisted 5-chain link, thereby describing the sets of exceptional slopes and fillings for most hyperbolic manifolds of small complexity. The classification produces the description of exceptional fillings for many families of one and two cusped manifolds, and provides supporting evidence for some well-known conjectures. One such family that appears in the classification is an infinite family of 1-cusped hyperbolic manifolds with four Seifert manifold fillings and a toroidal filling.

1. Introduction

The set of exceptional slopes on a boundary component of a hyperbolic manifold has generated a lot of interest in the literature. There are many restrictions on the set of exceptional slopes on a boundary component of a hyperbolic 3-manifold M and its corresponding fillings. For example, no such M has two distinct S^3 fillings [GL1] or more than ten exceptional slopes [LM]. However, it is still not known if there exists a hyperbolic knot exterior in S^3 with a reducible filling, or an M with a pair of exceptional slopes β and β' corresponding to a lens space and toroidal space so that the *distance* (minimal number of intersections) between β and β' is greater than three, or if there is a manifold not equal to the Figure-8 knot exterior with 10 exceptional slopes. Conjecturally, no examples exist, see [GAS], [G1], [Kir, Problem 1.77] respectively.

The distance between two exceptional slopes on a boundary component is at most 8 [LM] (which is realised on both the figure-eight knot exterior and the figure-eight sister manifold) and it is known that only finitely many one cusped 3-manifolds have exceptional slopes at distance more than 5 [Ago2]. It is conjectured that if an orientable 3-manifold has two exceptional slopes at distance greater than five then it is obtained by surgery on the Whitehead link [G3].

Also of interest are manifolds with more than one exceptional reducible filling; it is known that the distance between the fillings is 1 [GL3], and examples are given in [EW], [HM], and [GLi]. Eudave-Muñoz and Wu's examples in [EW] are the only known with more than one boundary component [G2], and Hoffman and Matignon ask in [HM] if reducible pairs must have at least one $L(2,1)$, $L(3,1)$, $L(4,1)$ summand and whether any hyperbolic manifold has three reducible fillings.

In this article, by classifying the sets of exceptional slopes and the corresponding fillings for all manifolds obtained by surgery on the minimally twisted 5-chain link (see the rightmost link in Figure 1), we provide experimental evidence that supports the above conjectures of González-Acuña, Short, and Gordon.

Theorem 1.1. *If M is a cusped hyperbolic manifold obtained by surgery on the minimally twisted 5-chain link and τ is a fixed boundary component of M then:*

1) *If M is the exterior of a knot in S^3 then M does not have a non-prime filling;*
2) *If M has two exceptional slopes on τ at distance greater than 3 apart then they do not correspond to a lens space and a toroidal filling;*
3) *If M has 10 exceptional slopes on τ then M is the figure-8 knot exterior;*

4) If M is a manifold with exceptional slopes on τ at distance greater than 5 then M is obtained by surgery on the Whitehead link.

5) M does not have more than one non-prime filling.

A full analysis of the exceptional fillings of surgeries on the minimally twisted 5-chain link is given to obtain Theorem 1.1. This produces a classification of exceptional filling types for infinitely many 1-cusped and 2-cusped manifolds (see Tables 14–22). These 1-cusped and 2-cusped manifolds are distinct from the examples in [MP] which all have a cyclic filling and at least five exceptional slopes. Among other families, we highlight:

- An infinite family of hyperbolic knots in S^3 with consecutive integral toroidal, small Seifert manifold, toroidal surgeries;
- An infinite family of hyperbolic knots in S^3 with three consecutive integral toroidal surgeries;
- An infinite family of 1-cusped manifolds with a reducible filling and three small Seifert manifold filling at distance one from the reducible filling;
- An infinite family of 1-cusped hyperbolic manifolds with four small Seifert manifold fillings and a toroidal filling;
- An infinite family of 2-cusped manifolds with four fillings on a fixed cusp containing an essential annulus.

These families are not contained in the classification given in [MP]. However, similar examples to the first three families above have already been constructed: see [Eud] for a family of knots with consecutive integral toroidal, small Seifert manifold, toroidal surgeries, and see [Ter] for a family of knots with three consecutive toroidal surgeries. Examples of 1-cusped manifolds with a reducible and small Seifert manifold filling at distance 1 apart are also known, see [Ka2]. I do not know if the families highlighted in this paper agree with the examples of Eudave-Muñoz, Teragaito, or Kang. The specific description of these families and their exceptional fillings can be found in Table 13.

The classification of exceptional fillings in this article does not improve any of the lower bounds on maximal distances to small Seifert manifolds that fibre over the sphere with three exceptional fibres obtained in [MP]. The classification of all exceptional slopes and fillings on manifolds obtained by surgery on the minimally twisted 5-chain link given in this paper enables the enumeration of all manifolds with fixed filling types. The results in this article are used in [AGR] to enumerate all hyperbolic knots obtained by surgery on the 5-chain link with exceptional pairs at maximal known distance. The classification of hyperbolic knots with two lens space surgeries obtained by surgery on the 5-chain link given in [AGR] is consistent with the classification of all 3-manifolds with three cyclic fillings given in [BDH] which uses different methods to reduce the enumeration to a search for examples on the magic manifold.

1.1. The minimally twisted 5-chain link

A notable collection of hyperbolic chain links is described in [MPR]; they are the figure-8 knot, the Whitehead link, the 3-chain link, the 4-chain link with a half twist, and the minimally twisted 5-chain link. These links are shown in Figure 1. We follow [MPR] and denote these chain links by 1CL, 2CL, 3CL, 4CL and 5CL, and their exteriors by M_1, M_2, M_3, M_4, M_5 respectively.

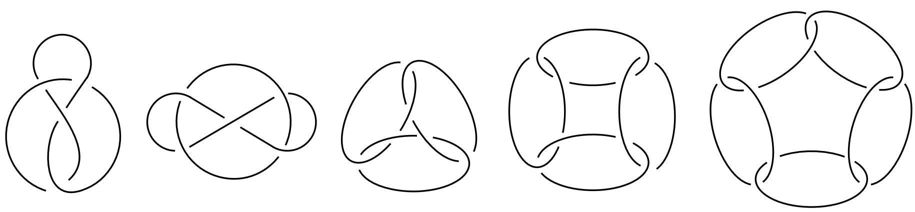

Figure 1: The links 1CL, 2CL, 3CL, 4CL, 5CL in S^3 whose exteriors we denote by M_1, M_2, M_3, M_4, and M_5 respectively.

The significance of this sequence of links comes from the following facts: each M_i is the (or conjecturally the) smallest volume hyperbolic 3-manifold with i cusps, see [Ago1] and [Yos]; more than 80% of the cusped hyperbolic 3-manifolds from the Callahan-Hildebrand-Weeks census [CHW] are surgeries on 5CL (personal communication with Nathan Dunfield). Furthermore, 5CL relates to the program of enumerating exceptional pairs at maximal distance. In particular, all knots realising half-integral toroidal surgeries [GL2], many of the knots realising lens space surgeries [Bak], and all cusped hyperbolic 3-manifolds with distinct reducible and toroidal fillings at maximal distance are obtained by surgery on 5CL [Ka1].

It is easy to see that if ∂M_n is equipped with the usual (meridian, longitude) homology basis then a -1 filling on any boundary component of M_n results in M_{n-1}. As a result, any manifold obtained by surgery on $(n-1)$CL is obtained by surgery on nCL. A classification of the exceptional surgeries on 3CL is given in [MP] together with a complete description of the set of exceptional slopes, and corresponding exceptional fillings, on the boundary components of all hyperbolic manifolds obtained by surgery on 3CL. The statements of Theorem 1.1 are known to hold for any manifold obtained by surgery on 3CL (see the appendix of [MP]).

A classification of exceptional surgeries on 5CL is found in [MPR]. In this article we complete the description of the set of exceptional slopes and corresponding exceptional fillings for manifolds obtained by surgery on 5CL in Theorems 3.1 and 3.3. We then use this classification to verify the statements of Theorem 1.1.

1.2. Article structure

We start with Section 2 where we recall the classification from [MPR]. In order to do so, we begin by recalling and introducing notation and terminology in Sections 2.1–2.3. In Section 2 we also establish some results which turn out to be of great use in the remainder of the paper (see Proposition 2.3 and Lemma 2.7 in Sections 2.4 and 2.5).

Theorems 3.1 and 3.3 are stated and proved in Section 3. These theorems complete the classification of exceptional sets of slopes on cusped hyperbolic manifolds obtained by surgery on 5CL. The proofs are heavily reliant on Proposition 2.3 and Lemma 2.7.

Theorem 1.1 is proved in Section 4. The proof uses Theorems 3.1 and 3.3 to impose restrictions on the filling instructions that can correspond to a counterexample. The exceptional slopes and fillings of many families of manifolds are completely enumerated using Theorems 3.1 and 3.3 in Tables 14–22 found in Section 5. Careful consideration of these tables is needed to complete the proof of Theorem 1.1.

1.3. Acknowledgements and remarks

This is the second version of this article; the initial version of this article omitted many details and was not clear as a result. The feedback from the anonymous referee on the first submission highlighted this. The current presentation has greatly benefited from the first anonymous referee's remarks and from discussions with Marc Lackenby, as well as from the referees' remarks on this version of the article. The results of this article were mainly obtained as a graduate student at the University of Pisa under the supervision of Carlo Petronio and Bruno Martelli. The article has also benefited from discussions with Daniel Matignon, and from email correspondences

with Carlo Petronio, Bruno Martelli, Cameron Gordon, and Nathan Dunfield.

Some of the main results of this revised version have been extended and amended from the original version: Tables 14–22 were not displayed in the original version, and the families of cusped manifolds highlighted in the introduction were not mentioned. Moreover, several typos/omitted examples in Tables 6–12 have been corrected.

2. Background terminology and useful results

2.1. Terminology

We begin with some general terminology, and we introduce the notion of an "exceptional filling instruction" which is used throughout this paper.

Fix an orientable compact 3-manifold X with ∂X consisting of tori:

- A *slope* on a boundary component τ of X is the isotopy class of a non-trivial unoriented simple closed curve on τ;
- A *filling instruction* α for X is a set consisting of either a slope or the empty set for each component of ∂X;
- The *filling* $X(\alpha)$ given by an instruction α is the manifold obtained by attaching one solid torus to ∂X for each (non-empty) slope in α, with the meridian of the solid torus attached to the slope.

We recall that if M is a hyperbolic non-compact finite-volume 3-manifold then $M = \mathrm{int}(X)$ with ∂X consisting of tori, and that $\mathrm{int}(X(\alpha))$ is hyperbolic for all but finitely many α's consisting of one slope and $\varnothing$'s [BH].

- If the interior of X is hyperbolic but the interior of $X(\alpha)$ is not, we say that α is an *exceptional filling instruction* for X and that $X(\alpha)$ is an *exceptional filling* of X;
- We say that an exceptional filling instruction α' on a hyperbolic 3-manifold X with boundary is *properly contained* in α, and write $\alpha' \subset \alpha$ if α' is contained in α and $\alpha' \neq \alpha$ (as sets of slopes).
- We say that an exceptional filling instruction α on a hyperbolic X is *isolated* if $X(\alpha')$ is hyperbolic for all α' properly contained in α; for such an α we call $X(\alpha)$ an *isolated exceptional filling* of X.

A *surgery* on a link L in a manifold M corresponds to a filling of the exterior of L. That is, a surgery on L is a filling of $M\backslash N(L)$ where $N(L)$ is an open regular neighbourhood of L. By a *surgery instruction* for L we mean a filling instruction on the exterior of L.

We now recall some standard notation used in the description of the set of exceptional slopes on a fixed boundary component of a hyperbolic manifold, see for example [G2]. If X is a hyperbolic 3-manifold with boundary consisting of tori and τ is a fixed boundary component of ∂X then the set of exceptional slopes on τ is denoted by $E_\tau(X)$, and the cardinality of $E_\tau(X)$ by $e_\tau(X)$. The subscript τ is dropped whenever the boundary component is clear. To describe $E_\tau(M_5(\alpha))$ we introduce the following definition:

Definition 2.1. Let α be a filling instruction on a manifold X. We say that α *factors through* a manifold Y if there exists some filling instruction $\alpha' \subseteq \alpha$ such that $Y = X(\alpha')$.

We remarked above that a -1 filling on any boundary component of M_n results in M_{n-1}. Therefore, any filling instruction on M_n that contains a -1 slope factors through M_{n-1}. Note that if α is exceptional for X and factors through a hyperbolic Y with $Y = X(\alpha')$, then $\alpha \backslash \alpha'$ is exceptional for Y.

2.2. Notation

Our description of the exceptional fillings of $M_5(\alpha)$ will employ the notation now discussed for Seifert manifolds with orientable base surface. Given integers $p_1, \ldots, p_n, q_1, \ldots, q_n$, with p_i and q_i coprime, and G an orientable surface with $k \geq 0$ boundary components $b_1, \ldots, b_k$, we let Σ denote the surface obtained by removing n open discs from G and we denote by $b_{k+1}, \ldots, b_{k+n}$ the n newly introduced boundary circles. We fix an orientation on $\Sigma \times S^1$ and orient $\{\mu_i, \lambda_i\} = \{b_i \times \{*\}, \{*\} \times S^1\}$ so that μ_i, λ_i is a positive basis of $H_1(b_i \times S^1)$ with $b_i \times S^1$ oriented as $\partial(\Sigma \times S^1)$. We denote by $(G, (p_1, q_1), \ldots, (p_n, q_n))$, the manifold obtained by performing a Dehn filling on each $b_i \times S^1$ along $p_{i-k}\mu_i + q_{i-k}\lambda_i$ for $i > k$. In our case, G will be either the disc D, the annulus A, or the sphere S^2.

Given Seifert manifolds X and Y with orientable base surfaces with boundary as described above, and $B \in \mathrm{GL}(2, \mathbb{Z})$ with $\det(B) = -1$, we define $X \bigcup_B Y$ unambiguously to be the quotient manifold $X \bigcup_f Y$ where $f : T \to U$ for T and U arbitrary boundary components of X and Y respectively, and f acting on homology by B with respect to the bases described above. The case $T, U \subset \partial X$, $T \neq U$ is also allowed and we write the quotient manifold as $X\big/_B$.

The JSJ decomposition and Geometrization theorems tell us that every non-hyperbolic 3-manifold not homeomorphic to the 3-ball either contains an essential sphere, disc, torus, annulus, or is a closed small Seifert space. The closed small Seifert spaces are precisely those manifolds with Heegaard genus 0, Heegaard genus 1 or fibres over the sphere with exactly 3 exceptional fibres. Following [G2] we now assign names to each class of non-hyperbolic manifolds:

- The class of Heegaard genus 0 manifolds (*i.e.* $\{S^3\}$) is denoted by S^H;
- The class of all reducible 3-manifolds is denoted by S;
- The class of manifolds with Heegaard genus 1 (*i.e.* lens spaces) is denoted by T^H;
- The class of manifolds containing an essential torus is denoted by T;
- The class of boundary reducible manifolds is denoted by D;
- The class of manifolds containing an essential annulus is denoted by A;
- The class of Seifert spaces fibering over the sphere with exactly three exceptional fibres is denoted by Z.

We will say that a manifold in a class $\mathcal{C}$ is of *type* $\mathcal{C}$. We remark that the above classes are not mutually exclusive, for example $(D^2 \times S^1)\#(D^2 \times S^1)$ is of type S and of type D, and that $S^1 \times S^2$ is the unique element in $S \cap T^H$.

2.3. Surgery instructions on the chain links

We now explain the convention used to describe surgeries on the chain links. By ordering the components of nCL for $3 \leq n \leq 5$ cyclically as in Figure 2, surgery instructions on nCL can be naturally identified with $(\mathbb{Q} \cup \{\varnothing, \infty\})^n$. By $M_n(x_1, \ldots, x_n)$ we mean the manifold obtained by performing an x_i-surgery on the i^{th} component of nCL.

To establish Theorem 1.1 we will examine all $M_5(\alpha)$. To avoid additional work we introduce the following definition which allows us to identify distinct surgery instructions that correspond to the same surgery.

Definition 2.2. Let α, α' be filling instructions on a 3-manifold X with toroidal boundary components. We will say that α and α' are *equivalent* and write $\alpha \sim \alpha'$ when there exists a homeomorphism $h : X(\alpha) \to X(\alpha')$.

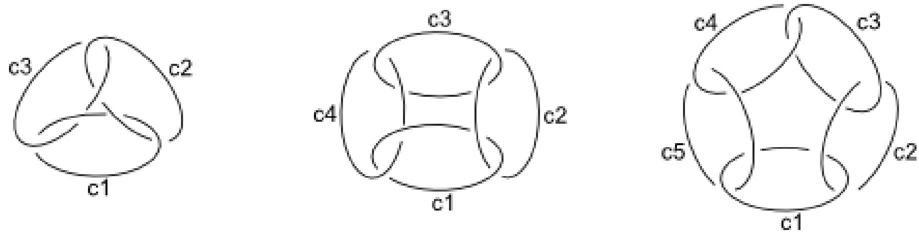

Figure 2: A cyclic ordering of the components of 3CL, 4CL, 5CL.

The appendix of [MP] contains a comprehensive analysis of the set of exceptional slopes on all $M_3(\alpha)$. Therefore, for the purposes of Theorem 1.1 we can omit the investigation of $E(M_5(\alpha))$ when α factors through M_3. As noted in the introduction, a positive twist about a boundary component of M_n with a -1 slope results in M_{n-1} for $n \geq 2$. When we keep track of surgery coefficients we get (2.1)–(2.2) (see [MPR] for precise details).

$$(2.1)\quad M_5\left(\tfrac{p}{q}, -1, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\right) = M_4\left(\tfrac{p+q}{q}, \tfrac{r+s}{s}, \tfrac{u}{v}, \tfrac{x}{y}\right) = M_5\left(\tfrac{p+q}{q}, \tfrac{r}{s}, -1, \tfrac{u-v}{v}, \tfrac{x}{y}\right)$$

$$(2.2)\quad M_4\left(\tfrac{p}{q}, \tfrac{r}{s}, -1, \tfrac{u}{v}\right) = M_3\left(\tfrac{p}{q}, \tfrac{r+s}{s}, \tfrac{u+v}{v}\right).$$

Putting Identities (2.1)–(2.2) we get:

$$(2.3)\quad M_5\left(\tfrac{p}{q}, -1, \tfrac{r}{s}, -1, \tfrac{u}{v}\right) = M_3\left(\tfrac{p+q}{q}, \tfrac{r+2s}{s}, \tfrac{u+v}{v}\right) = M_5\left(\tfrac{p}{q}, -2, -1, \tfrac{r}{s}, \tfrac{u+v}{v}\right).$$

Identities (2.1)–(2.3) will be useful in Section 3.

2.4. The minimally twisted 4-chain link

Most of the exceptional surgeries on 5CL are obtained by surgery on the minimally twisted 4-chain link M4CL shown in Figure 3 (see Proposition 3.5). The proof of Theorem 1.1 will turn unto an investigation of the surgeries on M4CL. We will therefore need to understand the manifolds obtained by surgery on M4CL. Proposition 2.3 shows that all small Seifert spaces as well as many distinct reducible and toroidal manifolds are obtained by surgery on M4CL.

We denote the exterior of M4CL by F. As with nCL, we order the components of M4CL cyclically (see Figure 3) and equip each component of M4CL with the standard choice of meridian and longitude. Surgery instructions on M4CL are naturally identified with $(\mathbb{Q} \cup \{\varnothing, \infty\})^4$; by $F\,(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ we

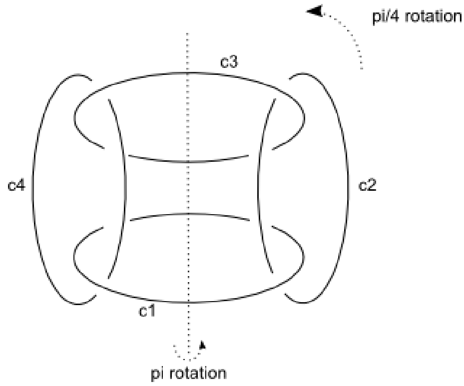

Figure 3: The minimally twisted 4-chain link M4CL.

mean the manifold obtained by performing an α_i-surgery on the i^{th} component of M4CL. It is easy to see from Figure 3 that the symmetry group of M4CL contains the Dihedral group D_4. So, for any $\sigma \in D_4$ we have:

$$F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = F\left(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)}, \alpha_{\sigma(4)}\right). \tag{2.4}$$

It is also useful for us to note that a negative twist about a boundary component of M_5 with a $+1$ slope results in F. When we keep track of surgery coefficients we get:

$$M_5\left(\tfrac{p}{q}, 1, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{x}{y}\right) = F\left(\tfrac{p-q}{q}, \tfrac{r-s}{s}, \tfrac{u}{v}, \tfrac{x}{y}\right) \tag{2.5}$$

Figure 4 highlights an exceptional torus T in F. It is clear that T separates F into two copies of $P \times S^1$, where P is the pair of pants D^2 minus two open discs, which are glued together by identifying a boundary component $\gamma \times S^1$ of one $P \times S^1$ to the other $P \times S^1$ with a horizontal loop $\gamma \times \{*\}$ in the former identified to a fibre $\{*\} \times S^1$ in the latter. Thus F is homeomorphic to $P \times S^1 \bigcup_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} P \times S^1$. We finally remark that F is homeomorphic to the exterior of the open chain link with four components used to describe the exceptional surgeries of 5CL in [MPR] (see Figure 5).

To describe the fillings of the 4-chain link, we employ a flexible notation for Seifert manifolds. We will formally identify an $\varnothing$ slope with $\frac{0}{0}$ and allow all coprime pairs (p_i, q_i) including $p_i, q_i \leq 0$. Moreover, $S^2 \times S^1$, S^3, $\mathbb{RP}^3$ will be written as $L(p, q)$ with $p = 0, 1, 2$ respectively.

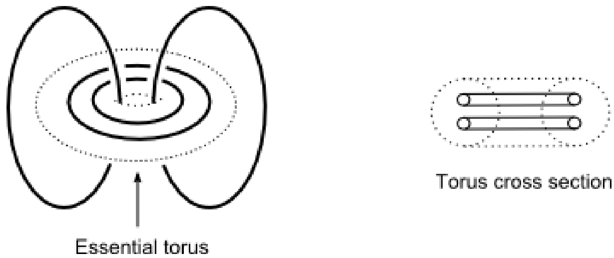

Figure 4: F, the exterior of the minimally twisted 4-chain link.

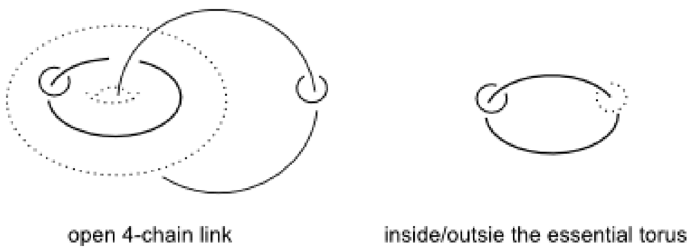

Figure 5: F realised as the exterior of the open 4-chain link.

Proposition 2.3. *If α is a filling instruction on F then up to (2.4) α is equivalent to some $(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ in Tables 1–4 and $F(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ is described in Tables 1–4.*

$\frac{p}{q}$	$\frac{r}{s}$	$\frac{u}{v}$	$\frac{x}{y}$	$F(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$	Type
0	$\varnothing$	$\varnothing$	$\varnothing$	$D^2 \times S^1 \# A \times S^1$	A, D, S
$\frac{1}{n}$	$\varnothing$	$\varnothing$	$\varnothing$	$P \times S^1$	A
$\lvert p\rvert \geq 2$	$\varnothing$	$\varnothing$	$\varnothing$	$\big(A, (p, q)\big) \bigcup_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} P \times S^1$	A, T

Table 1: All manifolds obtained by filling F along a single slope.

Proof. As noted above, the symmetry group of M4CL has a D_4 action on the boundary components of M4CL. Therefore, by (2.4), we may assume that when a filling instruction on F has a single slope it is of the form

$\frac{p}{q}$	$\frac{r}{s}$	$\frac{u}{v}$	$\frac{x}{y}$	$F(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$	Type
0	$\frac{r}{s}$	$\varnothing$	$\varnothing$	$D^2\times S^1\#D^2\times S^1$	D, S
	$\varnothing$	$\lvert u\rvert\neq 1$	$\varnothing$	$L(u,v)\#A\times S^1$	A, S
	$\varnothing$	$\frac{1}{n}$	$\varnothing$	$A\times S^1$	A
$\frac{1}{n}$	$\lvert r\rvert\geq 2$	$\varnothing$	$\varnothing$	$\big(A,(r,s)\big)$	A
	$\frac{1}{k}$	$\varnothing$	$\varnothing$	$A\times S^1$	A
	$\varnothing$	$\lvert v+nu\rvert\geq 2$	$\varnothing$	$\big(A,(v{+}nu,-u)\big)$	A
	$\varnothing$	$\lvert v+nu\rvert=1$	$\varnothing$	$A\times S^1$	A
	$\varnothing$	$-\frac{1}{n}$	$\varnothing$	$S^2\times S^1\#A\times S^1$	A, S
$\lvert p\rvert\geq 2$	$\lvert r\rvert\geq 2$	$\varnothing$	$\varnothing$	$\big(A,(p,q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(A,(r,s)\big)$	A, T
	$\varnothing$	$\lvert u\rvert\geq 2$	$\varnothing$	$\big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}P\times S^1$	A, T

Table 2: All manifolds obtained by filling F along two slopes.

$\frac{p}{q}$	$\frac{r}{s}$	$\frac{u}{v}$	$\frac{x}{y}$	$F(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$	Type
0	$\frac{r}{s}$	$\lvert u\rvert\neq 1$	$\varnothing$	$L(u,v)\#D^2\times S^1$	D, S
		$\frac{1}{n}$	$\varnothing$	$D^2\times S^1$	D
$\frac{1}{n}$	$\frac{1}{k}$	$\frac{u}{v}$	$\varnothing$	$D^2\times S^1$	D
	$\lvert r\rvert\geq 2$	$-\frac{1}{n}$	$\varnothing$	$L(r,s)\#D^2\times S^1$	D, S
		$\lvert v+nu\rvert=1$	$\varnothing$	$D^2\times S^1$	D
		$\lvert v+nu\rvert\geq 2$	$\varnothing$	$\big(D,(r,s),(v{+}nu,-u)\big)$	A
	0	$\lvert v+nu\rvert\neq 1$	$\varnothing$	$L(v{+}nu,-u)\#D^2\times S^1$	D, S
		$\lvert v+nu\rvert=1$	$\varnothing$	$D^2\times S^1$	D
$\lvert p\rvert\geq 2$	$\lvert r\rvert\geq 2$	$\lvert u\rvert\geq 2$	$\varnothing$	$\big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(A,(r,s)\big)$	A, T
	$\frac{1}{n}$	$\lvert u\rvert\geq 2$	$\varnothing$	$\big(D,(p,q),(u,v)\big)$	A
	0	$\lvert u\rvert\geq 2$	$\varnothing$	$D\times S^1\#L(pv{+}qu,pv'{+}qu')$ where $\lvert uv'-vu'\rvert=1$	D, S

Table 3: All manifolds obtained by filling F along three slopes.

$(\frac{p}{q},\varnothing,\varnothing,\varnothing)$, when a filling instruction on F has two slopes it is of the form $(\frac{p}{q},\frac{r}{s},\varnothing,\varnothing)$ or $(\frac{p}{q},\varnothing,\frac{u}{v},\varnothing)$, and when a filling instruction on F has three slopes it is of the form $(\frac{p}{q},\frac{r}{s},\frac{u}{v},\varnothing)$.

The slopes of a filling instruction $\alpha=(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$ can be split according to whether the numerator is 0, ±1, or greater than one in absolute value.

$\frac{p}{q}$	$\frac{r}{s}$	$\frac{u}{v}$	$\frac{x}{y}$	$F(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$	Type
0	$\frac{r}{s}$	$\lvert u\rvert \neq 1$	$\lvert ry+sx\rvert \neq 1$	$L(u,v)\#L(ry{+}sx,\, xs'{+}yr')$ where $\lvert rs'-sr'\rvert = 1$	S
	$\frac{r}{s}$	$\frac{1}{v}$	$\frac{x}{y}$	$L(ry{+}sx,\, xs'{+}yr')$ where $\lvert rs'-sr'\rvert = 1$	T^H/S^H $/S\&T^H$
	$\frac{r}{s}$	$\frac{u}{v}$	$\lvert ry+sx\rvert = 1$	$L(u,v)$	T^H/S^H $/S\&T^H$
$\frac{1}{n}$	$\frac{1}{k}$	$\frac{u}{v}$	$\frac{x}{y}$	$L\big((v{+}nu)(y{+}kx){-}xu,\, (v{+}nu)j{-}ui\big)$ where $\lvert xj-(y+kx)i\rvert = 1$	T^H/S^H $/S\&T^H$
	$\frac{r}{s}$	$v+nu=\epsilon$ $(\epsilon=\pm1)$	$\frac{x}{y}$	$L\big(ry{+}(x{-}\epsilon ru)x,\, rj{+}(s{-}\epsilon ru)i\big)$ where $\lvert xj-yi\rvert = 1$	T^H/S^H $/S\&T^H$
	$\lvert r\rvert \geq 2$	$-\frac{1}{n}$	$\lvert x\rvert \geq 2$	$L(r,s)\#L(x,y)$	S
	$\lvert r\rvert \geq 2$	$\lvert v+nu\rvert \geq 2$	$\lvert x\rvert \geq 2$	$\big(S^2,(v{+}nu,-u),(r,s),(x,y)\big)$	Z
$\lvert p\rvert \geq 2$	$\lvert r\rvert \geq 2$	$\lvert u\rvert \geq 2$	$\lvert x\rvert \geq 2$	$\big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(r,s),(x,y)\big)$	T

Table 4: All manifolds obtained by filling F along four slopes.

Using the symmetry group of M4CL we see that if all slopes are non-empty and $0\in\alpha$ then we may assume $\frac{p}{q}=0$. If $0\notin\alpha$ and $\frac{1}{n}$ is a slope in α we may assume $\frac{p}{q}=\frac{1}{n}$. It is now easy to see that the description of possible slopes in Tables 1–4 is exhaustive.

We have already seen that F is the union of two copies of $P\times S^1$ glued together by the orientation reversing map sending a meridian to a longitude and a longitude to a meridian. Therefore, the Identities (2.6)–(2.11) hold:

$$F = P\times S^1\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} P\times S^1, \tag{2.6}$$

$$F(\tfrac{p}{q},\varnothing,\varnothing,\varnothing) = \big(A,(p,q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} P\times S^1 \tag{2.7}$$

$$F(\tfrac{p}{q},\tfrac{r}{s},\varnothing,\varnothing) = \big(A,(p,q)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(A,(r,s)\big) \tag{2.8}$$

$$F(\tfrac{p}{q},\varnothing,\tfrac{u}{v},\varnothing) = \big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} P\times S^1 \tag{2.9}$$

$$F(\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v},\varnothing) = \big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(A,(r,s)\big) \tag{2.10}$$

$$F(\tfrac{p}{q},\tfrac{r}{s},\tfrac{u}{v},\tfrac{x}{y}) = \big(D,(p,q),(u,v)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D,(r,s),(x,y)\big). \tag{2.11}$$

Keeping the conventions set out in Section 2.2, the description of manifolds given in Tables 1–4 is obtained by repeated use of well-known Identities

(2.12)–(2.22) between graph manifolds (see [FM] for details) to Identities (2.6)–(2.11) until each $F(\frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{x}{y})$ is written as a $L(p,q)$, or a graph manifold with exceptional fibres of the form (a,b) with $|a| \geq 2$ and $|b| \geq 1$.

Seifert manifolds:

$$\begin{aligned} &\big(G, (p_1, q_1), (p_2, q_2), \dots, (p_k, q_k)\big) \\ &= \big(G, (p_1, q_1 - np_1), (p_2, q_2 + np_2), \dots, (p_k, q_k)\big), \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\big(G, (1, q_1), (p_2, q_2), \dots, (p_k, q_k)\big) \\ &= \big(G, (p_2, q_2 + q_1 p_2), \dots, (p_k, q_k)\big), \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\big(G, (p_1, q_1), (p_2, q_2), \dots, (p_k, q_k)\big) \\ &= \big(G, (p_1, q_1 - np_1), (p_2, q_2), \dots, (p_k, q_k)\big) \quad \text{if } \partial G \neq \varnothing. \end{aligned} \tag{2.14}$$

Small Seifert manifolds:

$$\big(S^2, (p, q)\big) = L(q, p), \tag{2.15}$$

$$\big(S^2, (p, q), (r, s)\big) = L(ps + rq, ps' + r'q) \text{ where } rs' - sr' = 1, \tag{2.16}$$

$$\big(S^2, (0, 1), (p, q), (r, s)\big) = L(p, q) \# L(r, s). \tag{2.17}$$

Graph Manifolds:

$$X \bigcup\nolimits_B Y = Y \bigcup\nolimits_{B^{-1}} X, \tag{2.18}$$

$$\big(D, (p, q)\big) \bigcup\nolimits_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} (G, \dots) = \big(G', (ap - bq, cp - dq), \dots\big), \quad \text{where } G' \backslash \text{disc} = G, \tag{2.19}$$

$$\big(G, (p, q), \dots\big) \bigcup\nolimits_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} X = \big(G, (p, q + kp), \dots\big) \bigcup\nolimits_{\left(\begin{smallmatrix} a + kb & b \\ c + kd & d \end{smallmatrix}\right)} X, \tag{2.20}$$

$$\big(D, (0, 1), (p, q)\big) \bigcup\nolimits_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} (G, \dots) = L(p, q) \# \big(G', (b, d), \dots\big), \quad \text{where } G' \backslash \text{disc} = G, \tag{2.21}$$

$$\begin{aligned} &\big(G, (p_1, q_1), \dots, (p_k, q_k)\big) \bigcup\nolimits_{\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)} X \\ &= \big(G, (p_1, -q_1), \dots, (p_k, -q_k)\big) \bigcup\nolimits_{\left(\begin{smallmatrix} -a & c \\ -b & d \end{smallmatrix}\right)} X. \end{aligned} \tag{2.22}$$

□

2.5. Exceptional surgery instructions on 5CL

We now return to 5CL and present the concise description of exceptional surgery instructions given in [MPR]. Given a surgery instruction α on 5CL, the symmetry group of $M_5(\alpha)$ induces a natural action on the boundary components of $M_5(\alpha)$. This action induces an action on the filling instructions on $M_5(\alpha)$. Among the most significant actions arising we mention those coming from the symmetry group of M_5, see (2.23)–(2.25) below, a symmetry of M_4 which may be deduced from the Fenn-Rourke blow-down move on 5CL, see (2.26), and from the amphichirality of the Figure-8 knot $M_5(-1,-2,-2,-2,\varnothing)$, see (2.27) (see [MPR] for full details).

$$(2.23)\qquad (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\longmapsto(\alpha_5,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$$
$$(2.24)\qquad (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\longmapsto(\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1)$$
$$(2.25)\qquad (\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)\longmapsto(\alpha_2^{-1},\alpha_1^{-1},1-\alpha_3,(1-\alpha_4^{-1})^{-1},1-\alpha_5)$$
$$(2.26)\qquad (-1,\alpha_1,\alpha_2,\alpha_3,\alpha_4)\longmapsto(\alpha_1,-1,\alpha_2-1,\alpha_3,\alpha_4+1)$$
$$(2.27)\qquad (-1,-2,-2,-2,\alpha)\longmapsto(-1,-2,-2,-2,-\alpha-6)$$

For a filling instruction α on M_5 we will often simplify notation by omitting empty slopes but leaving the subscripts on non-empty slopes. For example, $((-1)_2,(-1)_4)$ corresponds to the filling instruction $(\varnothing,\mu_2-\lambda_2,\varnothing,\mu_4-\lambda_4,\varnothing)$ with (μ_i,λ_i) the (meridian, longitude) basis of the homology of the i^{th} cusp. Note that for any $\frac{p}{q}\in\mathbb{Q}\cup\{\infty\}$ and $i\neq j$ one has $((\frac{p}{q})_i)=((\frac{p}{q})_j)$ by (2.23), so the fillings $M_5(\frac{p}{q})$ are defined without ambiguity. Our convention will be that a filling instruction $(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y})$ on M_5 with four non-empty slopes and no subscripts represents $(\frac{p}{q},\frac{r}{s},\frac{u}{v},\frac{x}{y},\varnothing)$. We now state the main result of [MPR].

Theorem 2.4. *Every exceptional filling instruction on M_5 is equivalent up to a composition of the symmetries* (2.23)–(2.27) *to a filling instruction containing one of:*

$$1,(-1,-2,-2,-1),(-1,-2,-3,-2,-4),(-1,-2,-2,-3,-5),$$
$$(-1,-3,-2,-2,-3),\left(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2}\right),(-2,-2,-2,-2,-2).$$

Moreover, the following equalities hold:

$$\begin{aligned}
M_5\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},1\right) &= F\left(\tfrac{a-b}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g-h}{h}\right)\\
M_5\left(-1,-2,-2,-1,\tfrac{a}{b}\right) &= \left(A,{\scriptstyle(b,\,-a-b)}\right)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\\
M_5\left(-1,-2,-2,-3,-5\right) &= \left(D,{\scriptstyle(2,1)},{\scriptstyle(2,-1)}\right)\bigcup_{\left(\begin{smallmatrix}-1&2\\1&-1\end{smallmatrix}\right)}\left(D,{\scriptstyle(2,1)},{\scriptstyle(3,1)}\right)\\
M_5\left(-1,-2,-3,-2,-4\right) &= \left(D,{\scriptstyle(2,1)},{\scriptstyle(2,-1)}\right)\bigcup_{\left(\begin{smallmatrix}-1&3\\1&-2\end{smallmatrix}\right)}\left(D,{\scriptstyle(2,1)},{\scriptstyle(3,1)}\right)\\
M_5\left(-1,-3,-2,-2,-3\right) &= \left(D,{\scriptstyle(2,1)},{\scriptstyle(2,-1)}\right)\bigcup_{\left(\begin{smallmatrix}-1&4\\1&-3\end{smallmatrix}\right)}\left(D,{\scriptstyle(2,1)},{\scriptstyle(3,1)}\right)\\
M_5\left(-2,-2,-2,-2,-2\right) &= \left(D,{\scriptstyle(2,1)},{\scriptstyle(2,-1)}\right)\bigcup_{\left(\begin{smallmatrix}-1&5\\1&-4\end{smallmatrix}\right)}\left(D,{\scriptstyle(2,1)},{\scriptstyle(3,1)}\right)\\
M_5\left(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2}\right) &= \left(A,{\scriptstyle(2,-1)}\right)/_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}.
\end{aligned}$$

Remark 2.5. The surgery instructions $(-1,-2,-2,-1,\varnothing)$, $(-1,-2,-2,-3,-5)$, and $(-1,-2,-3,-2,-4)$ factor through M_3, and every non-toroidal exceptional filling of M_5 is obtained by filling $M_5(1)=F$. Furthermore, each isolated exceptional filling of M_5 is a graph manifold and therefore a filling instruction α on M_5 is exceptional if and only if α contains an isolated exceptional surgery instruction.

We now recall the classification of isolated exceptional surgeries on 3CL found in [MP]. It is easy to see that the symmetry group of 3CL is S_3 and so for a filling instruction α on M_3 we can write $M_3(\alpha)$ unambiguously.

Theorem 2.6. [Martelli, Petronio] *Up to the S_3 action on the components of* 3CL*, a surgery instruction on* 3CL *is an isolated exceptional surgery instruction if and only if it is one of*

$$\infty,0,1,2,3,(-1,-1),(4,\tfrac{1}{2}),(\tfrac{3}{2},\tfrac{5}{2}),(5,5,\tfrac{1}{2}),(4,4,\tfrac{2}{3}),(4,\tfrac{3}{2},\tfrac{3}{2}),(4,\tfrac{1}{3},-1),$$
$$(\tfrac{8}{3},\tfrac{3}{2},\tfrac{3}{2}),(\tfrac{5}{2},\tfrac{5}{2},\tfrac{4}{3}),(\tfrac{5}{2},\tfrac{5}{3},\tfrac{5}{3}),(\tfrac{7}{3},\tfrac{7}{3},\tfrac{3}{2}),(-1,-2,-2),(-1,-2,-3),$$
$$(-1,-2,-4),(-1,-2,-5),(-1,-3,-3),(-2,-2,-2).$$

We remark that all non-hyperbolic $M_3(\alpha)$ are described in Theorems 1.1–1.3 in [MP]. To keep the presentation in this article self-contained we describe the exceptional fillings in terms of the manifold F (see Proposition 3.5). To do this the following lemma will be very helpful.

Lemma 2.7. *The action of* $\mathrm{Aut}(M_5)$ *on surgery instructions on* 5CL *is generated by* (2.28)–(2.40). *Moreover, for* $2.30 \le n \le 2.40$ *each* (n) *corresponds to the action of a distinct element of* $\mathrm{Aut}(M_5)/G$ *where* G *is the subgroup generated by the elements* (2.28)–(2.29) *corresponding to the generators of the link symmetry group of* 5CL.

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \longmapsto (\alpha_5,\alpha_1,\alpha_2,\alpha_3,\alpha_4) \tag{2.28}$$

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) \longmapsto (\alpha_5,\alpha_4,\alpha_3,\alpha_2,\alpha_1) \tag{2.29}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{f}{e},\tfrac{j-i}{j},\tfrac{a}{a-b},\tfrac{d-c}{d},\tfrac{h}{g}\right) \tag{2.30}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{b}{b-a},\tfrac{i-j}{i},\tfrac{e-f}{e},\tfrac{d}{d-c},\tfrac{g}{h}\right) \tag{2.31}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{i}{i-j},\tfrac{b-a}{b},\tfrac{f}{e},\tfrac{d}{c},\tfrac{h-g}{h}\right) \tag{2.32}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{j}{j-i},\tfrac{e}{f},\tfrac{b}{b-a},\tfrac{c-d}{c},\tfrac{g-h}{g}\right) \tag{2.33}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{a}{a-b},\tfrac{e}{e-f},\tfrac{i}{i-j},\tfrac{c}{c-d},\tfrac{g}{g-h}\right) \tag{2.34}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h}{g},\tfrac{j}{i},\tfrac{f-e}{f},\tfrac{c}{c-d},\tfrac{b-a}{b}\right) \tag{2.35}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h}{h-g},\tfrac{a}{b},\tfrac{f}{f-e},\tfrac{c-d}{c},\tfrac{i-j}{i}\right) \tag{2.36}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{g}{g-h},\tfrac{f-e}{f},\tfrac{b}{a},\tfrac{d}{c},\tfrac{j-i}{j}\right) \tag{2.37}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{g-h}{g},\tfrac{f}{f-e},\tfrac{i}{j},\tfrac{d}{d-c},\tfrac{a-b}{a}\right) \tag{2.38}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{h-g}{h},\tfrac{b}{a},\tfrac{j}{i},\tfrac{d-c}{d},\tfrac{e}{e-f}\right) \tag{2.39}$$

$$\left(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\tfrac{i}{j}\right) \longmapsto \left(\tfrac{a-b}{a},\tfrac{e-f}{e},\tfrac{h}{h-g},\tfrac{c}{d},\tfrac{j}{j-i}\right). \tag{2.40}$$

Remark 2.8. When a symbol appearing in the argument of one of these maps is $\varnothing$, there is a corresponding symbol $\varnothing$ in the image of the map.

Proof. SnapPy computes the symmetry group of M_5 to be $S_5 \times \mathbb{Z}/2\mathbb{Z}$ and determines its action on slopes (see [CDW]). The maps (2.28)–(2.40) come directly from SnapPy.

Each element of the symmetry group of M_5 acts on the set of boundary components of M_5, and on the set of filling instructions. We will first demonstrate that the action on the set of boundary components of M_5 generated by the maps (2.28)–(2.40) is that of the full S_5, and then we will use this fact to conclude that the action on surgery instructions on 5CL induced by the symmetry group of M_5 is generated by the maps (2.28)–(2.40).

No two of (2.30)–(2.40), considered as permutations of the boundary components, are equal up to the D_5 action from the link symmetry group of 5CL generated by (2.28) and (2.29). Thus, each of (2.30)–(2.40) corresponds to a representative element of a distinct left coset of D_5 in S_5. Since there are $\frac{5!}{10} = 12$ such cosets, and our list of maps (2.30)–(2.40) consists of 11 items, the symmetries of M_5 corresponding to (2.28)–(2.40) generate the full S_5 action.

We recall that SnapPy computes the order of the symmetry group of M_5 to be 240; the generator of the remaining $\mathbb{Z}/2\mathbb{Z}$ corresponds to an involution of M_5 shown in Figure 6 with a trivial action on the set of filling instructions. Thus the equivalence relation on filling instructions induced by the action

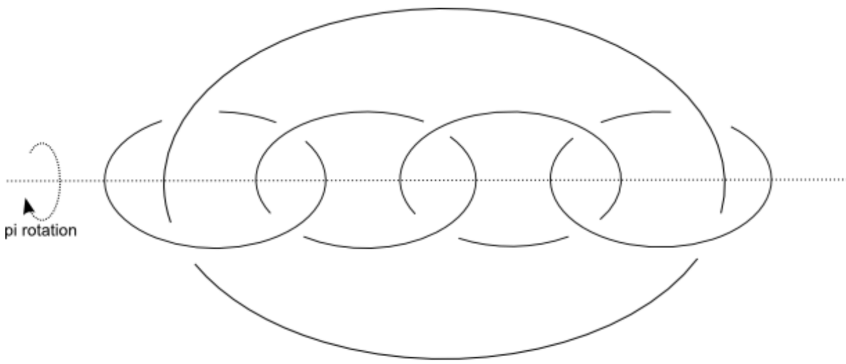

Figure 6: 5CL can be isotoped into the above link diagram. Rotation by π about the highlighted line leaves the link invariant and induces an involution on M_5.

of the symmetry group of M_5 is generated by the maps (2.28)–(2.40). $\square$

Since the action of the maps (2.23) and (2.24) is very easily understandable while that of (2.25)–(2.27) is more involved, we introduce the symbol $[\alpha]$ for the equivalence class of a filling instruction f under (2.23) and (2.24) only, and the symbol $[\![\alpha]\!]$ for the equivalence class of α under (2.23)–(2.27). Note that, $[\alpha] \subseteq [\![\alpha]\!]$ and if $\alpha_1, \alpha_2 \in [\![\alpha]\!]$ then $M_5(\alpha_1) = M_5(\alpha_2)$.

3. Main results

We now precisely describe the classification of exceptional slopes of every hyperbolic surgery on 5CL by describing $E_{\mathcal{T}}(M_5(\alpha))$ for every α not factoring through M_3. We split the result according to whether or not α factors through M_4.

Theorem 3.1. *Let α be a hyperbolic filling instruction on M_5 containing at least one $\varnothing$ and not factoring through M_4, and let τ be a boundary component of $\partial M_5(\alpha)$. Either $e_\tau(M_5(\alpha)) = 3$ and, with respect to the basis induced from M_5, we have $E_\tau(M_5(\alpha)) = \{0, 1, \infty\}$ and*

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(\infty) = F(-\tfrac{a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, -\tfrac{g}{h}) \tag{3.1}$$

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(1) = F(\tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g-h}{h}) \tag{3.2}$$

$$M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(0) = F(\tfrac{b}{b-a}, \tfrac{c-d}{c}, -\tfrac{h}{g}, \tfrac{e-f}{f}) \tag{3.3}$$

or $4 \leqslant e_\tau(M_5(\alpha)) \leqslant 5$ and

- *$M_5(\alpha)$ is homeomorphic to $M_5(f)$ for some f in Tables 6–11 and, with respect to the basis induced from M_5, we have*

$$E_\tau(M_5(f)) = \begin{cases} \{\beta_1, \beta_2, 0, 1, \infty\} & \text{if } f \text{ is found in Table 6;} \\ \{\beta, 0, 1, \infty\} & \text{otherwise,} \end{cases}$$

  where β, β_1, β_2 are found in Tables 6–11.

- *Identities (3.1)–(3.3) hold and $M_5(f)(\beta_i)$, $M_5(f)(\beta)$ are explicitly described in Tables 6–11.*

Remark 3.2. It will be implicitly shown in the proof of Theorem 3.1 that to apply Theorem 3.1 to find the exceptional slopes and fillings on a boundary component τ of a hyperbolic manifold $M_5(\alpha)$ as described above, we:

(a) Check whether $-1, \frac{1}{2}, 2 \in \alpha$. If yes, then α factors through M_4 and Lemma 2.7 can be used to write $M_5(\alpha) = M_5(*, -1, *, *, *)$. Using (2.1), $M_5(\alpha)$ can be used to express $M_5(\alpha)$ as a filling of M_4 and Theorem 3.3 can be applied. Otherwise, proceed to the next step.

(b) Check whether $-2, -\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3 \in \alpha$. If not then $E_\tau(M_5(\alpha)) = \{0, 1, \infty\}$ and the exceptional fillings can be written down using (3.1)–(3.3) and Proposition 2.3. Otherwise, then proceed to the next step.

(c) Use Lemma 2.7 to write $[\![\alpha]\!] = \{[\alpha_1], \ldots, [\alpha_{12}]\}$. Up to the D_5 action on boundary components, either one of the α_i is found in Tables 6–11 and the exceptional slopes and fillings are written down using Theorem 3.1 or no α_i is found in Tables 6–11. In the latter case $E_\tau(M_5(\alpha)) = \{0, 1, \infty\}$ and the exceptional fillings can be written down using (3.1)–(3.3) and Proposition 2.3.

Adopting the convention that a filling instruction $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ on M_4 with three slopes and no subscripts represents $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \varnothing)$, the corresponding result for M_4 is:

Theorem 3.3. *Let α be a hyperbolic filling instruction on M_4 containing at least one $\varnothing$ and not factoring through M_3, and let τ a boundary component of $\partial M_4(\alpha)$. Either $e_\tau(M_4(\alpha)) = 4$ and, with respect to the basis induced from M_4, we have $E_\tau(M_4(\alpha)) = \{0, 1, 2, \infty\}$ and*

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(\infty) = F(\tfrac{b-a}{b}, \tfrac{d}{c-d}, -1, -\tfrac{e}{f}) \tag{3.4}$$

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(2) = F(\tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e-f}{f}, -2) \tag{3.5}$$

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(1) = F(\tfrac{a-2b}{b}, -1, \tfrac{c-d}{d}, \tfrac{e-f}{f}), \tag{3.6}$$

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(0) = F(\tfrac{2d-c}{c-d}, \tfrac{b}{a-2b}, 2, \tfrac{f}{e}), \tag{3.7}$$

or $5 \leqslant e_\tau(M_4(\alpha)) \leqslant 6$ and

- *$M_4(\alpha)$ is homeomorphic to $M_4(f)$ for some f in Table 12, and with respect to the basis induced from M_4, we have*

$$E_\tau(M_4(f)) = \begin{cases} \{\beta_1, \beta_2, 0, 1, 2, \infty\} & \text{if } f = (-2, -2, -2), (2, -\frac{1}{2}, 2); \\ \{\beta, 0, 1, 2, \infty\} & \text{otherwise,} \end{cases}$$

  where β, β_1, β_2 are found in Table 12.

- *Identities (3.4)–(3.7) hold, and the $M_5(f)(\beta_i)$, $M_4(f)(\beta)$ are explicitly described in Table 12.*

Remark 3.4. Theorem 3.3 can be used to find the exceptional slopes on a boundary component of filling $M_4(\alpha)$ and the corresponding fillings in a similar way to Theorem 3.1. In this case, the application is trickier; $M_4(\alpha)$ can be written as a filling on M_5 using (2.1), (3.29) can be used to see if α factors through M_3, and up to (2.28)–(2.40) the filling instruction on M_5 will appear in the list at the end of the proof of Theorem 3.3 and can be located in Table 12 if $e(M_5(\alpha)) > 4$.

3.1. Proofs of main results

In Theorem 2.4 we have a complete description of the exceptional instructions and fillings of M_5. As M_3 is obtained by surgery on 5CL, Theorem 2.4 contains an opaque classification of exceptional surgery instructions contained in Theorem 2.6. Having an explicit classification of the exceptional fillings of M_3 will be important for the proof of Theorem 1.1. Proposition 2.3 can be used to give a complete description of all exceptional fillings of M_3. The description of exceptional fillings in Proposition 3.5 is the same as that given in [MP] up to (2.12)–(2.22). The description of the exceptional fillings in Tables 6–13 comes from Proposition 3.5 not [MP].

To prove Proposition 3.5 we will often use (2.23)–(2.24) in conjunction with previous identities to prove the main results in Section 3. As (2.23)–(2.24) are easy to understand we do not indicate when (2.23)–(2.24) have been used. For example instead of writing

$$\begin{aligned} M_5(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\infty) &\underset{(2.31)}{=} M_5(\tfrac{b}{b-a},1,\tfrac{e-f}{f},\tfrac{d}{d-c},\tfrac{g}{h}) \\ &\underset{(2.24)}{=} M_5(\tfrac{g}{h},\tfrac{d}{d-c},\tfrac{e-f}{f},1,\tfrac{b}{b-a}) \\ &\underset{(2.23)}{=} M_5(\tfrac{b}{b-a},\tfrac{g}{h},\tfrac{d}{d-c},\tfrac{e-f}{f},1) \end{aligned}$$

we will simply write

$$M_5(\tfrac{a}{b},\tfrac{c}{d},\tfrac{e}{f},\tfrac{g}{h},\infty) \underset{(2.31)}{=} M_5(\tfrac{b}{b-a},\tfrac{g}{h},\tfrac{d}{d-c},\tfrac{e-f}{f},1).$$

Proposition 3.5. *The following identities hold:*

$$M_3\,(-1,-1) = P \times S^1 \big/ _{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \tag{3.8}$$

$$M_3\left(-1,-1,\tfrac{a}{b}\right) = \big(A,(b,b-a)\big)\big/ _{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \tag{3.9}$$

$$M_3(-1,-3,-3) = \big(D,(2,1),(2,-1)\big) \bigcup\nolimits_{\left(\begin{smallmatrix}1&2\\1&-1\end{smallmatrix}\right)} \big(D,(2,1),(3,1)\big) \tag{3.10}$$

$$M_3(-2,-2,-2) = \big(D,(2,1),(2,-1)\big) \bigcup\nolimits_{\left(\begin{smallmatrix}-1&3\\1&-2\end{smallmatrix}\right)} \big(D,(2,1),(3,1)\big) \tag{3.11}$$

All other exceptional $M_3(\alpha)$ can be expressed as some filling of F or of $M_3(-1,-1)$ and the correspondence is found in Table 5.

Proof. Firstly, Theorem 2.6 implies that the list of exceptional fillings in Table 5 is complete. Identities (3.8)–(3.11) come directly from [MP]. We prove the remaining equalities in Table 5 using the identities in Lemma 2.7 as well as Identities (2.3), (2.5), (2.26) and (2.27). We indicate which Identity is being employed below the equality sign throughout the proof.

$M_3\big(\frac{p}{q},\frac{r}{s},\infty\big)=F\big(-\frac{1}{2},\frac{q}{q-p},\frac{1}{3},\frac{r}{s}\big)$	$M_3\big(\frac{p}{q},\frac{r}{s},0\big)=F\big(\frac{s}{s-r},2,\frac{q}{3q-p},-3\big)$
$M_3\big(\frac{p}{q},\frac{r}{s},1\big)=F\big(\frac{p-3q}{q},-1,-2,\frac{r-2s}{s}\big)$	$M_3\big(\frac{p}{q},\frac{r}{s},2\big)=F\big(\frac{s}{r},\frac{2q-p}{p-q},-\frac{1}{2},3\big)$
$M_3\big(\frac{p}{q},\frac{r}{s},3\big)=F\big(-2,\frac{p-q}{q},\frac{r-s}{s},-2\big)$	$M_3\big(\frac{p}{q},\frac{3}{2},\frac{5}{2}\big)=F\big(-3,\frac{p-2q}{p-q},2,-2\big)$
$M_3\big(4,\frac{1}{2},\frac{p}{q}\big)=F\big(2,\frac{3}{2},\frac{q}{p-q},-2\big)$	$M_3\big(-1,\frac{1}{3},4\big)=M_3\big(\frac{3}{2},\frac{3}{2},\frac{8}{3}\big)$ $=F\big(-3,\frac{2}{3},2,-2\big)$
$M_3\big(\frac{5}{2},\frac{5}{3},\frac{5}{3}\big)=F\big(2,\frac{3}{2},\frac{2}{3},-2\big)$	$M_3(-1,-2,-2)=F(\frac{1}{3},2,\frac{1}{4},-3)$
$M_3(-1,-2,-3)=F(-\frac{4}{3},-\frac{1}{3},2,-\frac{1}{2})$	$M_3(-1,-2,-4)=F(-\frac{3}{2},-\frac{1}{2},3,-\frac{1}{2})$
$M_3(-1,-2,-5)=F(-2,-2,-2,-3)$	
$M_3\big(\frac{3}{2},\frac{7}{3},\frac{7}{3}\big)=M_3\big(4,4,\frac{2}{3}\big)$ $=M_3\big(-1,-1,\frac{3}{2}\big)$	$M_3\big(5,5,\frac{1}{2}\big)=M_3\big(-1,-1,\frac{1}{2}\big)$
$M_3\big(\frac{5}{2},\frac{5}{2},\frac{4}{3}\big)=M_3\big(-1,-1,\frac{5}{2}\big)$	$M_3\big(4,\frac{3}{2},\frac{3}{2}\big)=M_3\big(-1,-1,4\big)$

Table 5: Homeomorphisms between fillings of M_3 and of F or $M_3(-1,-1)$.

We again highlight the fact that the use of the D_5 action on boundary components is suppressed when used in conjunction with (2.30)–(2.40) throughout the derivations. In all cases below, we pass from a filling of M_3 to a filling of M_5 using the Rolfsen twist. As with our use of (2.30)–(2.40), to save space, we will omit stating when the permutations of boundary components of M_5 and M_3 are used. We already know that M_5 has a D_5 action on boundary components, see (2.23)–(2.24). The link symmetry of 3CL induces an S_3 action on boundary components of M_3 i.e. $M_3(\alpha_1,\alpha_2,\alpha_3)=M_3(\alpha_{\sigma(1)},\alpha_{\sigma(2)},\alpha_{\sigma(3)})$ for all $\sigma\in S_3$. When the link symmetries of 3CL and 5CL are used in conjunction with (2.3) we save a lot of space. For example, instead of writing

$$\begin{aligned}M_3\big(\tfrac{p}{q},\tfrac{r}{s},0\big)&\underset{S_3\text{ symmetry}}{=}M_3\big(\tfrac{r}{s},\tfrac{p}{q},0\big)\underset{(2.3)}{=}M_5(\tfrac{r-s}{s},-2,-1,\tfrac{p-2q}{q},0)\\&\underset{(2.29)}{=}M_5(0,\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s})\\&\underset{(2.28)\circ(2.28)\circ(2.28)\circ(2.28)}{=}M_5(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},0)\end{aligned}$$

we simply write

$$M_3\big(\tfrac{p}{q},\tfrac{r}{s},0\big)\underset{(2.3)}{=}M_5(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},0)$$

In all cases, it is easy to verify that the first equality in each of (3.12)–(3.28) is valid using (2.1) (with (2.28)–(2.29) if necessary) on the right hand side to obtain a filling on M_4 with a -1 surgery coefficient and then using (2.2) (with (2.28)–(2.29) if necessary).

We start with the case where α is an exceptional filling instruction on M_3 found in Theorem 2.6 and $M_3(\alpha)$ is homeomorphic to a filling of F:

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},\infty\big) &\underset{(2.3)}{=} M_5\big(-1,-2,\tfrac{p-q}{q},\tfrac{r}{s},\infty\big) \\ &\underset{(2.31)}{=} M_5\big(\tfrac{1}{2},1,\tfrac{p-2q}{p-q},\tfrac{1}{3},\tfrac{r}{s}\big) \underset{(2.5)}{=} F\big(-\tfrac{1}{2},\tfrac{q}{q-p},\tfrac{1}{3},\tfrac{r}{s}\big) \end{aligned} \tag{3.12}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},0\big) &\underset{(2.3)}{=} M_5\big(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},0\big) \\ &\underset{(2.33)}{=} M_5\big(\tfrac{r-2s}{r-s},2,\tfrac{q}{3q-p},-2,1\big) \underset{(2.5)}{=} F\big(\tfrac{s}{s-r},2,\tfrac{q}{3q-p},-3\big) \end{aligned} \tag{3.13}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},1\big) &\underset{(2.3)}{=} M_5\big(\tfrac{p-2q}{q},-1,-2,\tfrac{r-s}{s},1\big) \\ &\underset{(2.5)}{=} F\big(\tfrac{p-3q}{q},-1,-2,\tfrac{r-2s}{s}\big) \end{aligned} \tag{3.14}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},2\big) &\underset{(2.3)}{=} M_5\big(0,-1,-2,\tfrac{p-q}{q},\tfrac{r}{s}\big) \\ &\underset{(2.35)}{=} M_5\big(\tfrac{s}{r},\tfrac{q}{p-q},1,\tfrac{1}{2},3\big) \underset{(2.5)}{=} F\big(\tfrac{s}{r},\tfrac{2q-p}{p-q},-\tfrac{1}{2},3\big) \end{aligned} \tag{3.15}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{r}{s},3\big) &\underset{(2.3)}{=} M_5\big(-1,\tfrac{p-q}{q},\tfrac{r-s}{s},-1,1\big) \\ &\underset{(2.5)}{=} F\big(-2,-2,\tfrac{p-q}{q},\tfrac{r-s}{s}\big) \end{aligned} \tag{3.16}$$

$$\begin{aligned} M_3\big(\tfrac{p}{q},\tfrac{3}{2},\tfrac{5}{2}\big) &\underset{(2.3)}{=} M_5\big(\tfrac{1}{2},-2,-1,\tfrac{1}{2},\tfrac{p}{q}\big) \\ &\underset{(2.38)}{=} M_5\big(-1,\tfrac{1}{2},\tfrac{p}{q},\tfrac{1}{3},-1\big) \underset{(2.3)}{=} M_5\big(0,\tfrac{1}{2},\tfrac{p-q}{q},-1,\tfrac{1}{3}\big) \\ &\underset{(2.31)}{=} M_5\big(1,-2,\tfrac{p-2q}{p-q},2,-1\big) \underset{(2.5)}{=} F\big(-3,\tfrac{p-2q}{p-q},2,-2\big) \end{aligned} \tag{3.17}$$

$$\begin{aligned} M_3\big(4,\tfrac{1}{2},\tfrac{p}{q}\big) &\underset{(2.3)}{=} M_5\big(2,\tfrac{1}{2},\tfrac{p-q}{q},-2,-1\big) \\ &\underset{(2.33)}{=} M_5\big(\tfrac{1}{2},\tfrac{p-q}{q},-1,-1,\tfrac{3}{2}\big) \underset{(2.3)}{=} M_5\big(-\tfrac{1}{2},-1,\tfrac{p-q}{q},0,\tfrac{3}{2}\big) \\ &\underset{(2.32)}{=} M_5\big(3,\tfrac{3}{2},\tfrac{q}{p-q},-1,1\big) \underset{(2.5)}{=} F\big(2,\tfrac{3}{2},\tfrac{q}{p-q},-2\big) \end{aligned} \tag{3.18}$$

$$M_3\big(-1,\tfrac{1}{3},4\big) \underset{(2.3)}{=} M_5\big(-2,-1,-\tfrac{5}{3},-1,3\big) \underset{(2.37)}{=} M_5\big(\tfrac{1}{2},\tfrac{8}{3},-\tfrac{1}{2},-1,-2\big) \underset{(2.3)}{=} M_3\big(\tfrac{3}{2},\tfrac{3}{2},\tfrac{8}{3}\big) \underset{(2.3)}{=} M_5\big(-1,\tfrac{2}{3},-1,\tfrac{1}{2},\tfrac{1}{2}\big) \underset{(2.32)}{=} M_5\big(2,-1,\tfrac{1}{2},\tfrac{3}{2},-1\big) \underset{(2.3)}{=} M_3\big(4,\tfrac{3}{2},\tfrac{5}{2}\big) \underset{(3.17)}{=} F\big(-3,\tfrac{2}{3},2,-2\big) \tag{3.19}$$

$$M_3\big(\tfrac{5}{2},\tfrac{5}{3},\tfrac{5}{3}\big) \underset{(2.3)}{=} M_5\big(\tfrac{3}{2},-1,-\tfrac{1}{3},-1,\tfrac{2}{3}\big) \underset{(2.31)}{=} M_5\big(-2,-\tfrac{1}{2},4,\tfrac{1}{2},-1\big) \underset{(2.3)}{=} M_3\big(\tfrac{1}{2},4,\tfrac{5}{2}\big) \underset{(3.18)}{=} F\big(2,\tfrac{3}{2},\tfrac{2}{3},-2\big) \tag{3.20}$$

$$M_3\big(-1,-2,-2\big) \underset{(2.3)}{=} M_5\big(-1,-3,-1,-3,-1\big) \underset{(2.26)}{=} M_5\big(-3,-1,-2,-3,0\big) \underset{(2.3)}{=} M_3\big(-1,-2,0\big) \underset{(3.13)}{=} F\big(\tfrac{1}{3},2,\tfrac{1}{4},-3\big) \tag{3.21}$$

$$M_3\big(-1,-2,-3\big) \underset{(2.3)}{=} M_5\big(-1,-2,-2,-2,-5\big) \underset{(2.27)}{=} M_5\big(-1,-2,-2,-2,-1\big) \underset{(2.26)}{=} M_5\big(-2,-1,-3,-2,0\big) \underset{(2.30)}{=} M_5\big(-\tfrac{1}{3},1,\tfrac{2}{3},2,-\tfrac{1}{2}\big) \underset{(2.5)}{=} F\big(-\tfrac{4}{3},-\tfrac{1}{3},2,-\tfrac{1}{2}\big) \tag{3.22}$$

$$M_3\big(-1,-2,-4\big) \underset{(2.3)}{=} M_5\big(-1,-2,-2,-2,-6\big) \underset{(2.27)}{=} M_5\big(-1,-2,-2,-2,0\big) \underset{(2.30)}{=} M_5\big(-\tfrac{1}{2},1,\tfrac{1}{2},3,-\tfrac{1}{2}\big) \underset{(2.5)}{=} F\big(-\tfrac{3}{2},-\tfrac{1}{2},3,-\tfrac{1}{2}\big) \tag{3.23}$$

$$M_3\big(-1,-2,-5\big) \underset{(2.3)}{=} M_5\big(-1,-2,-2,-2,-7\big) \underset{(2.27)}{=} M_5\big(-1,-2,-2,-2,1\big) \underset{(2.5)}{=} F\big(-2,-2,-2,-3\big) \tag{3.24}$$

We now turn to the case where f is a filling instruction of M_3 and $M_3(f)$ is homeomorphic to a filling of $M_3(-1,-1)$:

(3.25)
$$M_3\big(\tfrac{3}{2},\tfrac{7}{3},\tfrac{7}{3}\big) \underset{(2.3)}{=} M_5\big(-2,-1,\tfrac{1}{3},\tfrac{7}{3},\tfrac{1}{2}\big)$$
$$\underset{(2.32)}{=} M_5\big(-1,-\tfrac{4}{3},-1,3,3\big)$$
$$\underset{(2.3)}{=} M_3\big(4,4,\tfrac{2}{3}\big) \underset{(2.3)}{=} M_5\big(2,\tfrac{2}{3},3,-2,-1\big)$$
$$\underset{(2.35)}{=} M_5\big(-\tfrac{1}{2},-1,-2,-2,-1\big) \underset{(2.3)}{=} M_3\big(-1,-1,\tfrac{3}{2}\big)$$

(3.26)
$$M_3\big(5,5,\tfrac{1}{2}\big) \underset{(2.3)}{=} M_5\big(3,\tfrac{1}{2},4,-2,-1\big)$$
$$\underset{(2.35)}{=} M_5\big(-\tfrac{1}{2},-1,-3,-1,-2\big) \underset{(2.3)}{=} M_3\big(-1,-1,\tfrac{1}{2}\big)$$

(3.27)
$$M_3\big(\tfrac{5}{2},\tfrac{5}{2},\tfrac{4}{3}\big) \underset{(2.3)}{=} M_5\big(\tfrac{1}{2},-1,\tfrac{3}{2},\tfrac{1}{3},-1\big)$$
$$\underset{(2.38)}{=} M_5\big(-2,-2,-1,\tfrac{1}{2},-1\big) \underset{(2.3)}{=} M_3\big(-1,-1,\tfrac{5}{2}\big)$$

(3.28)
$$M_3\big(4,\tfrac{3}{2},\tfrac{3}{2}\big) \underset{(2.3)}{=} M_5\big(4,\tfrac{1}{2},-2,-1,-\tfrac{1}{2}\big)$$
$$\underset{(2.35)}{=} M_5\big(-1,-2,3,-1,-3\big) \underset{(2.3)}{=} M_3\big(4,-1,-1\big)$$

This completes the proof. □

Remark 3.6. Identities (2.12)–(2.22) can be used with Proposition 2.3 to show that Proposition 3.5 is consistent with classification of exceptional fillings of the mirror of 3CL given in [MP].

Consideration of the "intersection index" defined in Section 3.3 of [Rou1] together with the classification of exceptional fillings of M_3 given in [MP] demonstrates that $(-1,-3,-2,-2,-3)$, $(-2,-\frac{1}{2},3,3,-\frac{1}{2})$, $(-2,-2,-2,-2,-2)$ do not factor through M_3. This leads to the following useful fact that will be used liberally throughout the proof of Theorems 3.1 and 3.3. By Theorem 2.4 we see that an exceptional filling instruction α on M_5 with no $0,1,\infty$ slopes factors through M_3 if and only if there is a $\gamma \subset \alpha$ with

$\gamma \in [\![((-1)_1, (-2)_2)]\!]$. By Lemma 2.7 and (2.26) we have

$$(3.29)\quad [\![((-1)_1, (-2)_2)]\!] = \{[((\tfrac{1}{2})_1, (\tfrac{2}{3})_3)], [((\tfrac{1}{2})_1, 3_2)], [((\tfrac{1}{2})_1, (\tfrac{3}{2})_2)], [(\tfrac{1}{2})_1, 2_2], [((\tfrac{2}{3})_1, 2_2)], [((\tfrac{1}{2})_1, (\tfrac{1}{3})_3)], [(2_1, (-\tfrac{1}{2})_3)], [((\tfrac{1}{3})_1, 2_2)], [((-1)_1, 3_3)], [((-1)_1, (-2)_2)], [((-1)_1, (\tfrac{3}{2})_3)], [(2_1, (-2)_3)], [((-1)_1, (-\tfrac{1}{2})_2)], [(-1_1, -1_3)], [(-1_1, 2_2)], [(-1_1, (\tfrac{1}{2})_2)], [(-1_1, (\tfrac{1}{2})_3)], [(2_1, 2_2)], [(2_1, 2_3)], [(2_1, (\tfrac{1}{2})_3], [((\tfrac{1}{2})_1, (\tfrac{1}{2})_2)]\}$$

By the remark following Theorem 2.4 there is an obvious strategy to prove Theorem 3.1. Namely, we examine all hyperbolic $M_5(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \varnothing)$ and look for all α_5 so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ contains an isolated exceptional filling instruction. This simplifies matters greatly. If α factors through M_3 then α_5 must be a slope in $[\![(-1)]\!]$ or $[\![(-2)]\!]$, and if α does not factor through M_3 then α contains a slope in $[\![(1)]\!]$ or α is equivalent to one of $(-2, -\frac{1}{2}, 3, 3, -\frac{1}{2})$, $(-1, -3, -2, -2, -3)$, $(-2, -2, -2, -2, -2)$. We now prove Theorems 3.1 and 3.3.

3.1.1. Proof of Theorem 3.1. We start by establishing (3.1)–(3.3). Identity (3.1) is established in [MPR], and (3.2) is exactly the same as (2.5). For (3.3) we have

$$\begin{aligned} M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h}, 0) &\underset{(2.30)}{=} M_5(\tfrac{f}{e}, 1, \tfrac{a}{a-b}, \tfrac{d-c}{d}, \tfrac{h}{g}) \underset{(2.5)}{=} F(\tfrac{f-e}{e}, \tfrac{b}{a-b}, \tfrac{d-c}{d}, \tfrac{h}{g}) \\ &\underset{(2.4)}{=} F(\tfrac{b}{a-b}, \tfrac{d-c}{d}, \tfrac{h}{g}, \tfrac{f-e}{e}) \\ &\underset{(2.18)\ \&\ (2.22)}{=} F(\tfrac{b}{b-a}, \tfrac{c-d}{d}, -\tfrac{h}{g}, \tfrac{e-f}{e}) \\ &\underset{(2.14)}{=} F(\tfrac{b}{b-a}, \tfrac{c-d}{c}, -\tfrac{h}{g}, \tfrac{e-f}{f}). \end{aligned}$$

We let α be a hyperbolic surgery instruction on 5CL containing at least one $\varnothing$ not factoring through M_4. We let τ be a boundary component of $M_5(\alpha)$. Lemma 2.7 allows us to assume that τ comes from the 5th component of 5CL. So, we assume that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \varnothing)$ with $\alpha_i \in \mathbb{Q} \cup \{\varnothing, \infty\}$.

Our assumption that α is hyperbolic and does not factor through M_4 imposes restrictions on the α_i. Theorem 2.4 and the remark that follows tells us that α being hyperbolic means no $\gamma \subseteq \alpha$ contains an isolated exceptional filling instructions on M_5. Identity (2.1) means that if α does not factor through M_4 then no $\alpha_i = -1$. Lemma 2.7 tells us that

$$\llbracket(-1)\rrbracket = [(-1)] \sqcup [(\tfrac{1}{2})] \sqcup [(2)]. \tag{3.30}$$

Thus, no $\alpha_i \in \{-1, \frac{1}{2}, 2\}$ if α does not factor through M_4.

We now examine $E_\tau(M_5(\alpha))$. Theorem 2.4 implies that all slopes in $\llbracket(1)\rrbracket$ are exceptional. By Lemma 2.7

$$\llbracket(1)\rrbracket = [(1)] \sqcup [(\infty)] \sqcup [(0)] \tag{3.31}$$

Therefore, (3.31) implies that no $\alpha_i \in \{0, 1, \infty\}$, that $\{0, 1, \infty\} \subseteq E_\tau(M_5(\alpha))$ and $e(M_5(\alpha)) \geq 3$.

We will now describe all such αs not factoring through M_4 with $e(M_5(\alpha)) > 3$. We define (α, β) to be $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta)$. If β is an exceptional slope of $M_5(\alpha)$ then (α, β) contains an isolated exceptional filling instruction. Lemma 2.7 tells us that

$$\llbracket(-2)\rrbracket = [(-2)] \sqcup [(-\tfrac{1}{2})] \sqcup [(\tfrac{1}{3})] \sqcup [(\tfrac{2}{3})] \sqcup [(\tfrac{3}{2})] \sqcup [(3)]. \tag{3.32}$$

By (3.30) and Theorem 2.4 with (3.32), any such isolated exceptional filling instruction contains at most one slope in $\llbracket(-1)\rrbracket$, and contains two slopes in $\llbracket(-2)\rrbracket$. Thus at least one of the slopes in α belongs to $\llbracket(-2)\rrbracket$. It is a routine consequence of Lemma 2.7 that we may assume without loss of generality that $\alpha_1 = -2$ and that τ remains as the 5th component of ∂M_5.

Finally, the remarks following Theorem 2.4 allow us to conclude that if (α, β) contains an isolated exceptional filling instruction then either β is a slope in $\llbracket(1)\rrbracket$, or β is a slope in $\llbracket(-1)\rrbracket$ and (α, β) factors through M_3, or $M_5(\alpha, \beta)$ is one of

$$m_1 := M_5(-2, -\tfrac{1}{2}, 3, 3, -\tfrac{1}{2}), \quad m_2 := M_5(-1, -3, -2, -2, -3),$$
$$m_3 := M_5(-2, -2, -2, -2, -2).$$

So, we define the following sets of (α, β)s;

- We define l to be the set of exceptional (α, β)s such that $M_5(\alpha, \beta)$ is in $\{m_1, m_2, m_3\}$;
- We define l_{-1} to be the set of exceptional (α, β)s factoring through M_3 with $\beta = -1$;
- We define $l_{\frac{1}{2}}$ to be the set of exceptional (α, β)s factoring through M_3 with $\beta = \frac{1}{2}$;

- We define l_2 to be the set of exceptional (α,β)s factoring through M_3 with $\beta = 2$;

Let p be the projection from the set of filling instructions on M_5 to itself defined by $(\alpha,\beta) \mapsto \alpha$. For α in $p(l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2)$ define B_α to be the set of all β's such that (α,β) is contained in $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$. It is clear that $E\big(M_5(\alpha)\big) = \{0,1,\infty\} \cup B_\alpha$ and that $\{\big(M_5(\alpha), E(M_5(\alpha))\big)\}_\alpha$ is a complete list of all $\big(M_5(\alpha), E(M_5(\alpha))\big)$ pairs with $\alpha = \big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \varnothing\big)$, $M_5(\alpha)$ hyperbolic and $e\big(M_5(\alpha)\big) > 3$.

We now explicitly construct the sets $l, l_{-1}, l_{\frac{1}{2}}, l_2$. Throughout these constructions

$$\alpha_i \notin \{\infty, 0, 1, -1, \tfrac{1}{2}, 2\} \tag{3.33}$$

is used to reduce the number of possible cases (see (3.30) and (3.31)).

In all cases, we are enumerating α not exceptional with (α,β) containing an isolated filling instruction. Theorem 2.4 gives a complete list of isolated filling instructions on M_5 which we use in the construction of l. In the construction of l_{-1}, l_2, $l_{\frac{1}{2}}$ we express $M_5(\alpha,\beta)$ as a filling of M_3 and use the complete list of isolated filling instructions on M_3 from Theorem 2.6.

Construction of the set l: To construct l we look at all (α,β) which contain an isolated exceptional filling instruction with α not exceptional or containing a subfilling in (3.29), $\beta \notin \{0,1,\infty\}$ and, up to (2.28)–(2.29), $\alpha_1 = -2$. This means that $[(\alpha,\beta)]$ is in $[\![-2,-\frac{1}{2},3,3,-\frac{1}{2}]\!] \cup [\![(-1,-3,-2,-2,-3)]\!] \cup [\![(-2,-2,-2,-2,-2)]\!]$. Each of the elements of these sets is examined individually, and the relevant (α,β) are recorded. For example, by Lemma 2.7

$$\begin{aligned}
[\![(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2})]\!] = \{&[(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2})], [(\tfrac{1}{3},\tfrac{3}{2},\tfrac{2}{3},\tfrac{3}{2},\tfrac{1}{3})], [(\tfrac{1}{3},3,\tfrac{2}{3},\tfrac{2}{3},3)],\\
&[(\tfrac{1}{3},3,\tfrac{1}{3},-2,-2)], [(\tfrac{2}{3},3,\tfrac{1}{3},3,\tfrac{2}{3})], [(\tfrac{2}{3},\tfrac{3}{2},\tfrac{1}{3},\tfrac{1}{3},\tfrac{3}{2})],\\
&[(\tfrac{1}{3},-2,-2,\tfrac{1}{3},3)], [(-\tfrac{1}{2},-2,-\tfrac{1}{2},3,3)],\\
&[(\tfrac{3}{2},-2,-\tfrac{1}{2},-2,\tfrac{3}{2})], [(\tfrac{2}{3},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{2}{3},\tfrac{3}{2})],\\
&[(-2,-\tfrac{1}{2},-2,\tfrac{3}{2},\tfrac{3}{2})], [(\tfrac{3}{2},\tfrac{2}{3},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{2}{3})]\}
\end{aligned}$$

Discarding repeats, we examine the elements of

$$\begin{gathered}
\{[(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2})], [(\tfrac{1}{3},\tfrac{3}{2},\tfrac{2}{3},\tfrac{3}{2},\tfrac{1}{3})],\\
[(\tfrac{1}{3},3,\tfrac{2}{3},\tfrac{2}{3},3)], [(\tfrac{1}{3},3,\tfrac{1}{3},-2,-2)], [(\tfrac{3}{2},-2,-\tfrac{1}{2},-2,\tfrac{3}{2})], [(\tfrac{2}{3},-\tfrac{1}{2},-\tfrac{1}{2},\tfrac{2}{3},\tfrac{3}{2})]\}
\end{gathered}$$

We are only interested in having (α,β) with $\alpha_1=-2$. So, we examine the elements of

$$\{[(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2})],[(\tfrac{1}{3},3,\tfrac{1}{3},-2,-2)],[(\tfrac{3}{2},-2,-\tfrac{1}{2},-2,\tfrac{3}{2})]\}$$

Up to the D_5 action on boundary components, we take β on the fifth boundary component i.e. "next to" a -2 slope. So, the relevant (α,β) filling instructions are contained in the following set

$$\begin{gathered}\{[(\alpha_1=-2,-\tfrac{1}{2},3,3,\beta=-\tfrac{1}{2})],[(\tfrac{1}{3},3,\beta=\tfrac{1}{3},\alpha_1=-2,-2)],\\ [(\tfrac{1}{3},3,\tfrac{1}{3},\alpha_1=-2,\beta=-2)],\\ [(\beta=\tfrac{3}{2},\alpha_1=-2,-\tfrac{1}{2},-2,\tfrac{3}{2})],[(\tfrac{3}{2},\alpha_1=-2,\beta=-\tfrac{1}{2},-2,\tfrac{3}{2})]\}.\end{gathered}$$

So, for the isolated filling instruction $(-2,-\frac{1}{2},3,3,-\frac{1}{2})$, the relevant (α,β) fillings instructions are the elements of

$$\begin{gathered}\{(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2}),(-2,-2,\tfrac{1}{3},3,\tfrac{1}{3}),(-2,\tfrac{1}{3},3,\tfrac{1}{3},-2),\\ (-2,-\tfrac{1}{2},-2,\tfrac{3}{2},\tfrac{3}{2}),(-2,\tfrac{3}{2},\tfrac{3}{2},-2,-\tfrac{1}{2})\}.\end{gathered}$$

Arguing using Lemma 2.7 in the same, we see that every (α,β) with $M_5(\alpha,\beta)=m_i$ with $\alpha_1=-2$ and α hyperbolic not factoring through M_4 is contained in the set

$$\begin{aligned}l=\{&\big(-2,\tfrac{1}{4},\tfrac{3}{2},\tfrac{4}{3},\tfrac{1}{2}\big),\big(-2,-\tfrac{1}{2},3,3,-\tfrac{1}{2}\big),\big(-2,\tfrac{1}{3},3,\tfrac{1}{3},-2\big),\\ &\big(-2,-2,\tfrac{1}{3},3,\tfrac{1}{3}\big),\big(-2,-\tfrac{1}{2},-2,\tfrac{3}{2},\tfrac{3}{2}\big),\big(-2,\tfrac{3}{2},\tfrac{3}{2},-2,-\tfrac{1}{2}\big),\\ &\big(-2,-2,-2,-2,-2\big),\big(-2,\tfrac{1}{3},\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{3}\big)\}.\end{aligned}$$

Before constructing the sets $l_{-1},l_{\frac{1}{2}},l_2$ we recall that (α,β) is not in l then (α,β) factors through M_3 if and only if (α,β) contains one of the elements of $[\![((-1)_1,(-2)_2)]\!]$ (see (3.29) and the preceding remarks). We will see that the requirements of α being hyperbolic and not factoring through M_4 (see (3.33)) allow us to completely construct $l_{-1},l_{\frac{1}{2}},l_2$.

Construction of the set l_{-1}: In this case, for $\alpha=(-2,\frac{p}{q},\frac{r}{s},\frac{u}{v})$, we have $M_5(\alpha)(-1)=M_3(\frac{p+q}{q},\frac{r}{s},\frac{u+2v}{v})$ by (2.3). So (α,β) is an exceptional filling of M_5 if and only if $(\frac{p+q}{q},\frac{r}{s},\frac{u+2v}{v})$ contains an isolated exceptional filling instruction on M_3. As noted, (3.33) tells us that if α is hyperbolic and does not factor through M_4 then $\frac{p+q}{q}\notin\{\infty,2,1,0,\frac{3}{2},3\}$, $\frac{r}{s}\notin\{\infty,1,0,-1,\frac{1}{2},2\}$, $\frac{u+2v}{v}\notin\{\infty,3,2,1,\frac{5}{2},4\}$. With these conditions and Theorem 2.6 it is easy to see that β is an exceptional slope on a hyperbolic $M_5(\alpha)$ if and only if one of the following holds:

- $\frac{r}{s} = 3$
- $\frac{u}{v} + 2 = 0$
- $(\frac{p}{q} + 1, \frac{u}{v} + 2)$ belongs to $\{(-1, -1), (\frac{5}{2}, \frac{3}{2}), (4, \frac{1}{2})\}$
- $(\frac{r}{s}, \frac{p}{q} + 1)$ belongs to $\{(\frac{3}{2}, \frac{5}{2}), (4, \frac{1}{2})\}$
- $(\frac{r}{s}, \frac{u}{v} + 2)$ belongs to $\{(\frac{5}{2}, \frac{3}{2}), (4, \frac{1}{2})\}$
- $(\frac{p}{q} + 1, \frac{r}{s}, \frac{u}{v} + 2)$ belongs to

$$\Big\{(5, 5, \tfrac{1}{2}), (\tfrac{1}{2}, 5, 5), (4, 4, \tfrac{2}{3}), (4, \tfrac{3}{2}, \tfrac{3}{2}), (4, \tfrac{1}{3}, -1), (\tfrac{1}{3}, 4, -1), (-1, 4, \tfrac{1}{3}),$$
$$(-1, \tfrac{1}{3}, 4), (\tfrac{8}{3}, \tfrac{3}{2}, \tfrac{3}{2}), (\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{4}{3}), (\tfrac{5}{2}, \tfrac{5}{3}, \tfrac{5}{3}), (\tfrac{5}{3}, \tfrac{5}{2}, \tfrac{5}{3}), (\tfrac{7}{3}, \tfrac{7}{3}, \tfrac{3}{2}), (\tfrac{7}{3}, \tfrac{3}{2}, \tfrac{7}{3}),$$
$$(-1, -2, -2), (-2, -2, -1), (-1, -2, -3), (-3, -2, -1), (-2, -3, -1),$$
$$(-1, -3, -2), (-1, -2, -4), (-4, -2, -1), (-1, -4, -2), (-2, -4, -1),$$
$$(-1, -2, -5), (-5, -2, -1), (-1, -5, -2), (-2, -5, -1), (-1, -3, -3),$$
$$(-3, -3, -1), (-2, -2, -2)\Big\}.$$

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, -1)$ constructed in the above analysis is the set l_{-1}.

Construction of the set l_2: For $(\alpha, \beta) = (-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$ to factor through M_3 we need (α, β) to have a subfilling instruction displayed in (3.29). Comparing all elements of (3.29) with the possible subfillings of $(\alpha, \beta) = (-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$, recalling that $[((\frac{a}{b})_i, (\frac{c}{d})_j)] = \{((\frac{a}{b})_{\sigma(i)}, (\frac{c}{d})_{\sigma(j)}) \mid \sigma \in D_5\}$, and using the restrictions from (3.33), we see that for $(\alpha, \beta) = (-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$ to factor through M_3 we need one of $\frac{p}{q}, \frac{r}{s}$ to be in $\{-2, -\frac{1}{2}\}$ or $\frac{u}{v} \in \{\frac{1}{3}, \frac{2}{3}\}$. We construct all (α, β) for each of these 6 cases individually. The number of subcases is controlled by (3.33).

If $\frac{p}{q} = -2$ then

$$M_5(\alpha)(\beta) \underset{(2.40)}{=} M_5\big(\tfrac{3}{2}, \tfrac{r-s}{r}, \tfrac{v}{v-u}, -2, -1\big) \underset{(2.3)}{=} M_3\big(\tfrac{7}{2}, \tfrac{r-s}{r}, \tfrac{2v-u}{v-u}\big).$$

By (3.33) we have $\frac{r-s}{r} \notin \{\infty, 0, 1, 2, \frac{1}{2}, -1\}$ and $\frac{2v-u}{v-u} \notin \{2, \infty, 1, \frac{3}{2}, 0, 3\}$. From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{r-s}{r} = 3$, $(\frac{r-s}{r}, \frac{2v-u}{v-u}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{r-s}{r}, \frac{2v-u}{v-u}) = (4, \frac{1}{2})$.

If $\frac{p}{q} = -\frac{1}{2}$ then

$$M_5(\alpha)(\beta) \underset{(2.37)}{=} M_5\big(\tfrac{u}{u-v}, \tfrac{s-r}{s}, -\tfrac{1}{2}, -2, -1\big) \underset{(2.3)}{=} M_3\big(\tfrac{3u-2v}{u-v}, \tfrac{s-r}{s}, \tfrac{1}{2}\big).$$

From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{s-r}{s} = 3$, $\frac{3u-2v}{u-v} = 0$, $\frac{s-r}{s} = 4$, $(\frac{s-r}{s}, \frac{3u-2v}{u-v}) = (\frac{5}{2}, \frac{3}{2})$, $(\frac{s-r}{s}, \frac{3u-2v}{u-v}) = (5, 5)$.

If $\frac{r}{s} = -2$ then

$$M_5(\alpha)(\beta) \underset{(2.33)}{=} M_5\big(-1, -2, \tfrac{1}{3}, \tfrac{p-q}{p}, \tfrac{u-v}{u}\big) \underset{(2.3)}{=} M_3\big(\tfrac{4}{3}, \tfrac{p-q}{p}, \tfrac{3u-v}{u}\big).$$

From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{p-q}{p} = 3$, $\frac{3u-v}{u} = 0$, $(\frac{p-q}{p}, \frac{3u-v}{u}) = (4, \frac{1}{2})$, $(\frac{p-q}{p}, \frac{3u-v}{u}) = (\frac{5}{2}, \frac{3}{2})$.

If $\frac{r}{s} = -\frac{1}{2}$ then

$$M_5(\alpha)(\beta) \underset{(2.30)}{=} M_5\big(-2, -1, \tfrac{2}{3}, \tfrac{q-p}{q}, \tfrac{v}{u}\big) \underset{(2.3)}{=} M_3\big(\tfrac{8}{3}, \tfrac{q-p}{q}, \tfrac{v+u}{u}\big).$$

From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{q-p}{q} = 3$, $(\frac{q-p}{q}, \frac{v+u}{u}) = (4, \frac{1}{2})$, $(\frac{q-p}{q}, \frac{v+u}{u}) = (\frac{3}{2}, \frac{5}{2})$.

If $\frac{u}{v} = \frac{1}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.33)}{=} M_5\big(-1, \tfrac{r}{s}, \tfrac{1}{3}, \tfrac{p-q}{p}, -2\big) \underset{(2.3)}{=} M_3\big(\tfrac{r+2s}{s}, \tfrac{1}{3}, \tfrac{2p-q}{p}\big).$$

From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{r+2s}{s} = 0$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (\frac{1}{2}, 4)$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (-1, -1)$, $(\frac{r+2s}{s}, \frac{2p-q}{p}) = (-1, 4)$.

If $\frac{u}{v} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.37)}{=} M_5\big(-2, \tfrac{s-r}{s}, -\tfrac{1}{2}, \tfrac{q}{p}, -1\big) \underset{(2.3)}{=} M_3\big(\tfrac{2s-r}{s}, -\tfrac{1}{2}, \tfrac{2p+q}{p}\big).$$

From Theorem 2.6 and (3.33) we see that $\beta = 2$ is an exceptional slope on $M_5(\alpha)$ if and only if one of the following holds; $\frac{2p+q}{p} = 0$, $(\frac{2s-r}{s}, \frac{2p+q}{p}) = (-1, -1)$, $(\frac{2s-r}{s}, \frac{2p+q}{p}) = (4, \frac{1}{2})$, $(\frac{2s-r}{s}, \frac{2q+p}{p}) = (\frac{5}{2}, \frac{3}{2})$.

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, 2)$ constructed in the above analysis is the set l_2.

Construction of the set $l_{\frac{1}{2}}$: Reasoning as in the case $\beta = 2$, we use the restrictions from (3.33) and the fact that $(\alpha, \beta) = (-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{1}{2})$ factors through M_3 when (α, β) contains a filling instruction shown in (3.29). The result is that if (α, β) factors through M_3 then we need one of $\frac{p}{q}, \frac{r}{s}$ is in $\{\frac{1}{3}, \frac{2}{3}\}$ or $\frac{u}{v} \in \{\frac{3}{2}, 3\}$. We examine each of these 6 cases individually and enumerate all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{1}{2})$ that satisfy (3.33).

If $\frac{p}{q} = \frac{1}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.36)}{=} M_5\big(\tfrac{v}{v-u}, -2, \tfrac{s}{s-r}, -2, -1\big) \underset{(2.3)}{=} M_3\big(\tfrac{3v-2u}{v-u} - 2, \tfrac{2s-r}{s-r}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{3v-2u}{v-u} = 0$, $(\frac{3v-2u}{v-u}, \frac{2s-r}{s-r}) = (-1,-1)$, $(\frac{1}{2}, 4)$, $(\frac{3}{2}, \frac{5}{2})$, $(-1,-2)$, $(-2,-1)$, $(-1,-3)$, $(-3,-1)$, $(-1,-4)$, $(-4,-1)$, $(-1,-5)$, $(-5,-1)$, $(-2,-2)$, $(-3,-3)$.

If $\frac{p}{q} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.34)}{=} M_5\big(\tfrac{2}{3}, \tfrac{r}{r-s}, -1, -2, \tfrac{u}{u-v}\big) \underset{(2.3)}{=} M_3\big(\tfrac{2}{3}, \tfrac{3r-2s}{r-s}, \tfrac{2u-v}{u-v}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{3r-2s}{r-s} = 0$, $(\frac{3r-2s}{r-s}, \frac{2u-v}{u-v}) = (-1,-1)$, $(\frac{3}{2}, \frac{5}{2})$, $(\frac{1}{2}, 4)$.

If $\frac{r}{s} = \frac{1}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.31)}{=} M_5\big(\tfrac{1}{3}, -1, -2, \tfrac{q}{q-p}, \tfrac{u}{v}\big) \underset{(2.3)}{=} M_3\big(\tfrac{7}{3}, \tfrac{2q-p}{q-p}, \tfrac{u}{v}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{u}{v} = 3$, $(\frac{u}{v}, \frac{2q-p}{q-p}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{3}{2}, \frac{7}{3})$, $(4, \frac{1}{2})$.

If $\frac{r}{s} = \frac{2}{3}$ then

$$M_5(\alpha)(\beta) \underset{(2.34)}{=} M_5\big(\tfrac{2}{3}, -2, -1, \tfrac{p}{p-q}, \tfrac{u}{u-v}\big) \underset{(2.3)}{=} M_3\big(\tfrac{5}{3}, \tfrac{3p-2q}{p-q}, \tfrac{u}{u-v}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{3p-2q}{p-q} = 0$, $\frac{u}{u-v} = 3$, $(\frac{3p-2q}{p-q}, \frac{u}{u-v}) = (\frac{3}{2}, \frac{5}{2})$, $(\frac{1}{2}, 4)$, $(\frac{5}{3}, \frac{5}{2})$.

If $\frac{u}{v} = \frac{3}{2}$ then

$$M_5(\alpha)(\beta) \underset{(2.36)}{=} M_5\big(-2, -2, \tfrac{s}{s-r}, \tfrac{p-q}{p}, -1\big) \underset{(2.3)}{=} M_3\big(-1, \tfrac{s}{s-r}, \tfrac{3p-q}{p}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{s}{s-r} = 3$, $\frac{3p-q}{p} = 0, -1$, $(\frac{s}{s-r}, \frac{3p-q}{p}) = (\frac{5}{2}, \frac{3}{2})$, $(4, \frac{1}{2})$, $(-2,-2)$, $(-2,-3)$, $(-3,-2)$, $(-2,-4)$, $(-4,-2)$, $(-2,-5)$, $(-5,-2)$, $(4, \frac{1}{3})$, $(-3,-3)$.

If $\frac{u}{v} = 3$ then

$$M_5(\alpha)(\beta) \underset{(2.32)}{=} M_5\big(-1, 3, \tfrac{s}{r}, \tfrac{q}{p}, -2\big) \underset{(2.3)}{=} M_3\big(5, \tfrac{s}{r}, \tfrac{q+p}{p}\big).$$

Theorem 2.6 and (3.33) tell us that $\beta = \frac{1}{2}$ is an exceptional slope on $M_5\big(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}\big)$ if and only if one of the following holds; $\frac{s}{r} = 3$, $\big(\frac{s}{r}, \frac{q+p}{p}\big) = \big(\frac{3}{2}, \frac{5}{2}\big)$, $(4, \frac{1}{2})$, $(5, \frac{1}{2})$.

Thus, the set of all $(-2, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \frac{1}{2})$ constructed in the above analysis is the set $l_{\frac{1}{2}}$.

This completes the construction of $l \cup l_{-1} \cup l_{\frac{1}{2}} \cup l_2$ and the sets $\{(M_5(\alpha), E(M_5(\alpha))\}$ are now easily computed. The last step is to reduce the size of $\{(M_5(\alpha), E(M_5(\alpha))\}$ using (2.23)–(2.27). Namely, only one $(M_5(\alpha), E(M_5(\alpha))$ is shown for each $[\![\alpha]\!]$. This is done using Lemma 2.7 with the help of an ad hoc Python script [Rou2].

The reduced list of fillings is shown in Tables 6–10. This completes the proof of Theorem 3.1. □

The elementary techniques used to prove Theorem 3.1 can obviously be applied to describe all $E(M_5(\alpha))$ when α factors through M_4 but α does not factor through M_3.

3.1.2. Proof of Theorem 3.3. We first establish (3.4)–(3.7). For (3.4) we have

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \infty) \underset{(2.1)}{=} M_5(\tfrac{a-b}{b}, -1, \tfrac{c-d}{d}, \tfrac{e}{f}, \infty) \underset{(3.1)}{=} F(\tfrac{b-a}{b}, \tfrac{d}{c-d}, -1, -\tfrac{e}{f}).$$

For (3.5) we have

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, 2) \underset{(2.1)}{=} M_5(-1, \tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, 1) \underset{(2.5)}{=} F(-2, \tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e-f}{f}) \underset{(2.4)}{=} F(\tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e-f}{f}, -2). \tag{3.34}$$

For (3.6) we have

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, 1) \underset{(2.1)}{=} M_5(\tfrac{a-b}{b}, -1, \tfrac{c-d}{d}, \tfrac{e}{f}, 1) \underset{(2.5)}{=} F(\tfrac{a-2b}{b}, -1, \tfrac{c-d}{d}, \tfrac{e-f}{f}).$$

For (3.7) we have

$$\begin{aligned} M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, 0) &\underset{(2.1)}{=} M_5(\tfrac{a-b}{b}, -1, \tfrac{c-d}{d}, \tfrac{e}{f}, 0) \\ &\underset{(2.30)}{=} M_5(\tfrac{d}{c-d}, 1, \tfrac{a-b}{a-2b}, 2, \tfrac{f}{e}) \underset{(2.5)}{=} F(\tfrac{2d-c}{c-d}, \tfrac{b}{a-2b}, 2, \tfrac{f}{e}). \end{aligned} \tag{3.35}$$

We now examine the set of exceptional slopes on fillings of M_4 not factoring through M_3. We let α be a hyperbolic surgery instruction on 4CL containing at least one $\varnothing$ not factoring through M_3. We know from (2.1) that $M_5(-1) = M_4$. So we let $\alpha' = (-1, \alpha'_2, \alpha'_3, \alpha'_4, \varnothing)$ be such that $M_4(\alpha) = M_5(\alpha')$. The argument can now proceed exactly as in the proof of Theorem 3.1 to enumerate the $E(M_5(\alpha'))$.

Theorem 3.1 implies that every $\beta' \in \{-1, 0, 1, \infty\}$ is an exceptional slope on $M_5(\alpha')$. So, $e(M_5(\alpha')) \geq 4$ and, with respect to the choice of basis induced from M_5, $\{-1, 0, 1, \infty\} \subseteq E(M_5(\alpha'))$.

As in the proof of Theorem 3.1, (3.29) imposes restrictions on the α'_i. The condition that $M_5(\alpha')$ is hyperbolic with α not factoring through M_3 means that no instruction properly contained in α contains an instruction in $[\![(1)]\!]$ (see Theorem 3.1) or $[\![(-1_1, -2_2)]\!]$ (see (3.29)) or $[\![(-1_1, -1_2)]\!]$ (see Table 1.1.4 from [Rou1]). This imposes the restrictions

$$\alpha'_2 \notin \{-2, -1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, 2, \infty\} \quad \text{and} \quad \alpha'_3, \alpha'_4 \notin \{-1, 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, 3, \infty\}. \tag{3.36}$$

If β' is an exceptional slope on $M_5(\alpha')$, then (α', β') contains an isolated filling instruction. By Theorem 3.1, $\beta' \in \{-1, 0, 1, \infty\}$, or (α', β') factors through M_3, or (α', β') is equivalent to $(-1, -3, -2, -2, -3)$. This implies that β' is in one of $[\![(1)]\!]$, $[\![(-1)]\!]$, $[\![(-2)]\!]$ or, by Lemma 2.7, that (α', β') is one of $(-1, -3, -2, -2, -3)$, $(-1, \frac{1}{3}, \frac{4}{3}, \frac{4}{3}, \frac{1}{3})$, $(-1, -\frac{1}{3}, 4, \frac{2}{3}, 3)$, $(-1, 3, \frac{2}{3}, 4, -\frac{1}{3})$.

If (α', β') factors through M_3, then (3.29) in conjunction with (3.36) tells us that β' is in $\{-2, -\frac{1}{2}, \frac{1}{2}, 2\}$. Every slope in $\{-2, -\frac{1}{2}, \frac{1}{2}, 2\}$ is examined individually as in the proof of Theorem 3.1 to obtain a complete list of all $\big(M_5(\alpha'), E(M_5(\alpha'))\big)$ pairs that have $\alpha' = \big(-1, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \varnothing\big)$, $M_5(\alpha')$ hyperbolic with α' not factoring through M_3 and $e\big(M_5(\alpha')\big) > 4$. The result of the enumeration is that $\beta' = -2$ is an exceptional slope on $M_5(\alpha')$ if and only if α' is one of

$$\begin{gathered}
(-1, -\tfrac{3}{2}, 4, \tfrac{u}{v}), (-1, \tfrac{p}{q}, 4, -\tfrac{1}{2}), (-1, -3, \tfrac{r}{s}, -2), (-1, -\tfrac{3}{2}, 5, 4), \\
(-1, 3, 5, -\tfrac{1}{2}), (-1, -3, 4, -\tfrac{2}{3}), (-1, -\tfrac{5}{3}, 4, -2), (-1, -4, -2, -3), \\
(-1, -\tfrac{1}{3}, \tfrac{5}{2}, \tfrac{2}{3}), (-1, -3, -2, -3), (-1, -4, -2, -2), (-1, -3, -2, -4), \\
(-1, -3, -3, -3), (-1, -4, -3, -2), (-1, -5, -2, -2), (-1, -3, -2, -5), \\
(-1, -3, -4, -3), (-1, -4, -4, -2), (-1, -6, -2, -2), (-1, -3, -2, -6), \\
(-1, -3, -5, -3), (-1, -4, -5, -2), (-1, -7, -2, -2), (-1, -3, -3, -4), \\
(-1, -5, -3, -2)
\end{gathered}$$

$\beta' = -\frac{1}{2}$ is an exceptional slope on $M_5(\alpha')$ if and only if α' is one of

$$(-1, 3, \tfrac{r}{s}, 4), (-1, \tfrac{3}{2}, \tfrac{4}{3}, \tfrac{u}{v}), (-1, \tfrac{p}{q}, \tfrac{4}{3}, \tfrac{5}{2}), (-1, \tfrac{3}{2}, \tfrac{5}{4}, -2), (-1, -3, \tfrac{5}{4}, \tfrac{5}{2}),$$
$$(-1, 3, \tfrac{4}{3}, \tfrac{8}{3}), (-1, \tfrac{5}{3}, \tfrac{4}{3}, 4), (-1, \tfrac{1}{3}, \tfrac{5}{3}, \tfrac{4}{3}), (-1, 3, \tfrac{2}{3}, 5), (-1, 4, \tfrac{2}{3}, 4), (-1, 3, \tfrac{2}{3}, 6),$$
$$(-1, 3, \tfrac{3}{4}, 5), (-1, 4, \tfrac{3}{4}, 4), (-1, 5, \tfrac{2}{3}, 4), (-1, 3, \tfrac{2}{3}, 7), (-1, 3, \tfrac{4}{5}, 5), (-1, 6, \tfrac{2}{3}, 4),$$
$$(-1, 4, \tfrac{4}{5}, 4), (-1, 3, \tfrac{2}{3}, 8), (-1, 3, \tfrac{5}{6}, 5), (-1, 4, \tfrac{5}{6}, 4), (-1, 7, \tfrac{2}{3}, 4), (-1, 3, \tfrac{3}{4}, 6),$$
$$(-1, 5, \tfrac{3}{4}, 4), (-1, 4, \tfrac{2}{3}, 5),$$

$\beta' = \frac{1}{2}$ is an exceptional slope on $M_5(\alpha')$ if and only if α' is one of

$$(-1, \tfrac{1}{3}, \tfrac{r}{s}, \tfrac{4}{3}), (-1, \tfrac{2}{3}, \tfrac{2}{3}, \tfrac{u}{v}), (-1, \tfrac{p}{q}, \tfrac{2}{3}, \tfrac{5}{3}), (-1, \tfrac{2}{3}, \tfrac{5}{4}, \tfrac{2}{3}), (-1, -\tfrac{1}{3}, \tfrac{5}{4}, \tfrac{5}{3}),$$
$$(-1, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{8}{5}), (-1, \tfrac{3}{5}, \tfrac{2}{3}, \tfrac{4}{3}), (-1, 3, \tfrac{1}{3}, 4), (-1, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{5}{4}), (-1, \tfrac{1}{4}, \tfrac{4}{3}, \tfrac{4}{3}),$$
$$(-1, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{6}{5}), (-1, \tfrac{1}{3}, \tfrac{5}{4}, \tfrac{5}{4}), (-1, \tfrac{1}{4}, \tfrac{5}{4}, \tfrac{4}{3}), (-1, \tfrac{1}{5}, \tfrac{4}{3}, \tfrac{4}{3}), (-1, \tfrac{1}{3}, \tfrac{5}{4}, \tfrac{6}{5}),$$
$$(-1, \tfrac{1}{5}, \tfrac{5}{4}, \tfrac{4}{3}), (-1, \tfrac{1}{4}, \tfrac{4}{3}, \tfrac{5}{4}), (-1, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{7}{6}), (-1, \tfrac{1}{3}, \tfrac{6}{5}, \tfrac{5}{4}), (-1, \tfrac{1}{4}, \tfrac{6}{5}, \tfrac{4}{3}),$$
$$(-1, \tfrac{1}{6}, \tfrac{4}{3}, \tfrac{4}{3}), (-1, \tfrac{1}{3}, \tfrac{4}{3}, \tfrac{8}{7}), (-1, \tfrac{1}{3}, \tfrac{7}{6}, \tfrac{5}{4}), (-1, \tfrac{1}{4}, \tfrac{7}{6}, \tfrac{4}{3}), (-1, \tfrac{1}{7}, \tfrac{4}{3}, \tfrac{4}{3}),$$

and $\beta' = 2$ is an exceptional slope on $M_5(\alpha')$ if and only if α' is one of

$$(-1, -\tfrac{1}{3}, \tfrac{r}{s}, \tfrac{2}{3}), (-1, -\tfrac{2}{3}, -2, \tfrac{u}{v}), (-1, \tfrac{p}{q}, -2, \tfrac{1}{3}), (-1, -\tfrac{2}{3}, -3, \tfrac{4}{3}),$$
$$(-1, \tfrac{1}{3}, -3, \tfrac{1}{3}), (-1, -\tfrac{1}{3}, -2, \tfrac{2}{5}), (-1, -\tfrac{3}{5}, -2, \tfrac{2}{3}), (-1, -3, -\tfrac{1}{2}, -2),$$
$$(-1, -\tfrac{1}{3}, 4, \tfrac{3}{4}), (-1, -\tfrac{1}{4}, 4, \tfrac{2}{3}), (-1, -\tfrac{1}{3}, 4, \tfrac{4}{5}), (-1, -\tfrac{1}{3}, 5, \tfrac{3}{4}), (-1, -\tfrac{1}{4}, 5, \tfrac{2}{3}),$$
$$(-1, -\tfrac{1}{5}, 4, \tfrac{2}{3}), (-1, -\tfrac{1}{3}, 4, \tfrac{5}{6}), (-1, -\tfrac{1}{3}, 6, \tfrac{3}{4}), (-1, -\tfrac{1}{4}, 6, \tfrac{2}{3}), (-1, -\tfrac{1}{6}, 4, \tfrac{2}{3}),$$
$$(-1, -\tfrac{1}{3}, 4, \tfrac{6}{7}), (-1, -\tfrac{1}{3}, 7, \tfrac{3}{4}), (-1, -\tfrac{1}{4}, 7, \tfrac{2}{3}), (-1, -\tfrac{1}{7}, 4, \tfrac{2}{3}), (-1, -\tfrac{1}{3}, 5, \tfrac{4}{5}),$$
$$(-1, -\tfrac{1}{5}, 5, \tfrac{2}{3}), (-1, -\tfrac{1}{4}, 4, \tfrac{3}{4}).$$

As with the proof of Theorem 3.1, Identities (2.23)–(2.27) are used to identify equivalent filling instructions.

The final step is to use (2.1) to obtain the filling instructions on M_4 and exceptional slopes shown in Table 11. Namely, the enumerated $M_5(\alpha') = M_5\big(-1, \frac{p}{q}, \frac{r}{s}, \frac{u}{v}, \varnothing\big)$ and $E(M_5(\alpha')) = \{\beta_i\}$ are identified with $M_4(\frac{p}{q}+1, \frac{r}{s}, \frac{u}{v}, \varnothing)$ and $E\big(M_4(\frac{p}{q}+1, \frac{r}{s}, \frac{u}{v}, \varnothing)\big) = \{\beta_i + 1\}$ respectively. These $M_4(\alpha)$, $E(M_4(\alpha))$ are shown in Table 12. □

Remark 3.7. The reduced list in Theorem 3.3 is surprisingly small (see Table 12). This occurs as the fillings listed above with ± 2, $\pm\frac{1}{2}$ are equivalent.

This can be seen by setting $i = 2$ and $j = 1$ below;

$$\begin{aligned} M_5(-1, \tfrac{v-u}{v}, \tfrac{r}{r-s}, \tfrac{q-p}{q}, \tfrac{j}{i}) &\underset{(2.39)}{=} M_5(-1, \tfrac{p}{q}, \tfrac{r}{s}, \tfrac{u}{v}, \tfrac{i}{j}) \\ &\underset{(2.26)}{=} M_5(\tfrac{p+q}{q}, \tfrac{r-s}{s}, -1, \tfrac{u-v}{v}, \tfrac{i+j}{j}) \\ &\underset{(2.30)}{=} M_5(-1, \tfrac{v}{u-v}, \tfrac{2s-r}{s}, \tfrac{p+q}{p}, -\tfrac{i}{j}) \\ &\underset{(2.39)}{=} M_5(-1, -\tfrac{q}{p}, \tfrac{2s-r}{s-r}, \tfrac{u-2v}{u-v}, -\tfrac{j}{i}) \end{aligned}$$

The only filling instructions that appear in more than one of the above lists are $(-1, -3, -\frac{1}{2}, -2)$ and $(-1, -3, -2, -2)$.

4. Families of cusped manifolds and proof of Theorem 1

We finish by showing that Theorem 1.1 will fall out as a consequence of Theorems 3.1 and 3.3. We remarked in Section 1.1 that statements (i)–(iv) from Theorem 1.1 are shown to hold for all hyperbolic $M_3(\alpha)$ in the Appendix of [MP]. Thus, we must show that Theorem 1.1 (i)–(v) hold for all hyperbolic $M_5(\alpha)$ when α does not factor through M_3.

To prove Theorem 1.1 we will need to know the class of every $M_5(\alpha)(\beta)$ for $\beta \in E(M_5(\alpha))$ when α is found in Tables 6–11 and $M_4(\alpha)(\beta)$ for $\beta \in E(M_4(\alpha))$ when α is found in Tables 12. Theorems 3.1 and 3.3 together with Proposition 2.3 make this straightforward. To simplify matters we will say that a slope α_i on the boundary component of a manifold M is of type $\mathcal{C}$ if $M(\alpha_i) \in \mathcal{C}$, and that a set of exceptional slopes $\{\alpha_1, \ldots, \alpha_k\}$ is of type $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ when α_i is of type $\mathcal{C}_i$ for each i. The results are shown in Tables 14–22.

As highlighted in the introduction, Tables 14–22 are of interest in their own right. Among other families, we highlight the family $M_5(-2, k, 3, \frac{k+1}{3k+2})$ of hyperbolic knot exteriors in S^3 with consecutive integral toroidal, type Z, toroidal fillings (see Table 17), the family $M_5(-2, \frac{3}{2}, \frac{3}{2}, \frac{3+14k}{1+5k})$ of hyperbolic knot exteriors in S^3 with three consecutive integral toroidal fillings (see Table 15), the family $M_5(-2, \frac{1}{k}, 3, \frac{k-1}{k})$ with three type Z fillings and a reducible surgery (see Table 17), the family $M_4(-2, \frac{1}{k}, -2)$ with four type Z exceptional fillings and a toroidal filling (see Table 22), and the family $M_5(-2, \frac{1}{k}, 3, \varnothing)$ of 2-cusped manifolds with four annular fillings on the 5^{th} cusp (see Table 17). These families are distinct from any obtained in [MP] because all hyperbolic fillings of M_3 have at least five exceptional slopes and a cyclic filling. As highlighted in the introduction, families with the same filling types as the first three families have already been constructed, see Section 1 for references. The exceptional fillings of these families are written down using Proposition 3.5 and shown in Table 13.

4.1. Proof of Theorem 1.1 (*ii*)–(*iv*)

The maximal distance between a lens space slope and a toroidal slope is known to be either three or four [Lee]. Theorems 3.1 and 3.3 show that the only two $M_5(\alpha)$ with α not factoring through M_3 with two exceptional slopes β, γ at distance greater than 3 are $M_4(-2,-2,-2)$ with $\beta = -2$ and $\gamma = 2$ and $M_4(-2,-\frac{1}{2},-2)$ with $\beta = -1$ and $\gamma = 3$. In all cases the fillings are toroidal (see Table 21). So statement (*ii*) holds.

Theorems 3.1 and 3.3 tell us that if $e(M_5(\alpha)) \geq 6$ then α factors through M_3. So statement (*iii*) holds.

It is well known that the distance between two slopes $\frac{p}{q}$ and $\frac{r}{s}$ is $|ps - rq|$ (see [Sti]). From Theorems 3.1 and 3.3, if the distance between two exceptional slopes on $M_5(\alpha)$ is greater than 4 then α factors through M_3. So statement (*iv*) holds.

4.2. Proof of Theorem 1.1 (*v*)

We now consider the non-prime fillings on M_5. We see directly from Tables 14–22 that if $M_5(f)$ has two exceptional non-prime slopes α, β then $\alpha, \beta \in \{0, 1, \infty\}$ or $M_5(f) = M_4(g)$ for some filling instruction g and $\alpha, \beta \in \{0, 1, 2, \infty\}$. In all cases, the non-prime filling is described as a filling of F by (3.1)–(3.7). From Proposition 2.3, $F(\frac{i}{j}, \frac{k}{l}, \frac{n}{m}, \frac{t}{w})$ is non-prime if and only if one of $\frac{i}{j}, \frac{k}{l}, \frac{n}{m}, \frac{t}{w}$ is zero or one of $\{\frac{i}{j}, \frac{n}{m}\}$, $\{\frac{k}{l}, \frac{t}{w}\}$ equals $\{\frac{1}{\eta}, -\frac{1}{\eta}\}$ for some $\eta \in \mathbb{Z}$.

If $F(\frac{i}{j}, \frac{k}{l}, \frac{n}{m}, \frac{t}{w}) = M_5(f)(\alpha)$ or $F(\frac{i}{j}, \frac{k}{l}, \frac{n}{m}, \frac{t}{w}) = M_4(g)(\beta)$ and one of $\frac{i}{j}, \frac{k}{l}, \frac{n}{m}, \frac{t}{w}$ is the zero slope then, by (3.1)–(3.7), one of the slopes in f is in $0, 1, \infty$ or one of the slopes in g is in $\{0, 1, 2, \infty\}$ which makes $M_5(f)$ and $M_4(g)$ non-hyperbolic by Theorems 3.1 and 3.3.

By (3.4) and Proposition 2.3, if $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \neq 0$ then $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(\infty)$ is reducible if and only if $\frac{b-a}{b} = 1 \Rightarrow \frac{a}{b} = 0$ making $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ non-hyperbolic by Theorem 3.3. In the same way, if $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \neq 0$ then $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(2)$ is non-prime if and only if $\frac{e}{f} = 2$ which makes $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ non-hyperbolic by Theorem 3.3. The remaining pair of possible exceptional slopes on $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ are 0 and 2. If both 0 and 2 are non-prime slopes and $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \neq 0$ then both $\frac{e-f}{f} = \frac{1}{n}$ and $\frac{f}{e} = \frac{1}{m}$ for some integers n, m by (3.5) and (3.7) respectively.

This implies that $\frac{e}{f} \in \{0, 2\}$ which makes $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})$ non-hyperbolic by Theorem 3.3.

The final case to consider is when $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ has a non-prime pair of slopes in $\{0, 1, \infty\}$ and the non-prime fillings $F(\delta)$ and $F(\epsilon)$ have no zero slopes in δ or ϵ. Namely, two of

$$\text{(a)} \quad \left(\tfrac{1}{n}, \tfrac{1}{n}\right) \in \left\{\left(-\tfrac{a}{b}, \tfrac{d}{c}\right), \left(\tfrac{f}{e}, -\tfrac{g}{h}\right)\right\} \quad \text{(b)} \quad \left(\tfrac{1}{m}, \tfrac{1}{m}\right) \in \left\{\left(\tfrac{a-b}{b}, \tfrac{e}{f}\right), \left(\tfrac{c}{d}, \tfrac{g-h}{h}\right)\right\}$$
$$\text{(c)} \quad \left(\tfrac{1}{k}, \tfrac{1}{k}\right) \in \left\{\left(\tfrac{b}{b-a}, -\tfrac{h}{g}\right), \left(\tfrac{c-d}{c}, \tfrac{e-f}{f}\right)\right\}$$

must hold. Each of (a), (b), (c) can hold in two ways. In all twelve ways that two of (a), (b), (c) hold, we find $\{0, 1, \infty\} \cap \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\} \neq \emptyset$ which makes $M_5(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ non-hyperbolic by Theorem 2.4 or that $\{-1, \frac{1}{2}, 2\} \cap \{\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\}$ $\neq \emptyset$ which makes $(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \frac{g}{h})$ factors through M_4 by (3.30).

Finally, M_3 does not have any exceptional reducible pairs (see Table 16 in the Appendix of [MP]). So statement (v) holds.

4.3. Proof of Theorem 1.1 (i)

We first consider the $M_5(\alpha)$ and $M_4(\alpha)$ from Tables 14–22 with a non-prime β not in $\{0, 1, \infty\}$ or $\{0, 1, 2, \infty\}$ respectively. We will show that for such $M_5(\alpha)$, β no $M_5(\alpha)(\gamma) = S^3$ for $\gamma \in \{0, 1, \infty\}$, and for such $M_4(\alpha)$, β no $M_4(\alpha)(\gamma) = S^3$ for $\gamma \in \{0, 1, 2, \infty\}$.

From Tables 14–21 we see that the only $M_5(\alpha)$ or $M_4(\alpha)$ with one boundary component with β non-prime not in $\{0, 1, \infty\}$ or $\{0, 1, 2, \infty\}$ respectively are $M_5(-2, -n, n+3, -2)$ with $n \notin \{-4, -3, -2, -1, 0, 1\}$ and $\beta = -1$ (found in Table 18). It is easy to see from Table 18 that the remaining exceptional slopes are $0, 1, \infty$ none of which give an S^3 filling. In particular, using Table 18, $M_5(-2, n, n-3, -2, 0)$ is of type Z, $M_5(-2, n, n-3, -2, 1)$ is toroidal, and $M_5(-2, n, n-3, -2, \infty)$ is of type T^H. This shows that if β is a non-prime slope on $M_5(\alpha)$ or $M_4(\alpha)$ then β is in $\{0, 1, \infty\}$ or $\{0, 1, 2, \infty\}$ respectively.

Tables 14–21 show no $M_5(\alpha)$ and $M_4(\alpha)$ has an S^H slope β not in $\{0, 1, \infty\}$ or $\{0, 1, 2, \infty\}$ respectively. So, to conclude statement (i) for hyperbolic $M_5(\alpha)$ we need to show that no $M_5(\alpha)$ has non-prime and S^H slopes in $\{0, 1, \infty\}$ and no $M_4(\alpha)$ has a non-prime and S^H slopes in $\{0, 1, 2, \infty\}$.

We already know that an instruction α on M_5 factors through M_3 when α contains an instruction in $[\![((-1)_1, (-2)_2)]\!]$. If $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is an instruction on M_4 that factors through M_3 then the filling instruction $\alpha' = (-1, \alpha_1 - 1, \alpha_2, \alpha_3, \alpha_4 - 1)$ (where $\alpha_i - 1 = \varnothing$ if $\alpha_i = \varnothing$ for $i = 1, 4$) on M_5

factors through M_3. Looking at (3.29) we see that if a filling instruction α on M_4 contains a slope in $\{-1, 3, \frac{3}{2}, \frac{1}{2}\}$ then α factors through M_3.

We have

$$\begin{aligned} M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{\beta_1}{\beta_2}) &\underset{(2.1)}{=} M_5(-1, \tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{\beta_1-\beta_2}{\beta_2}) \\ &\underset{(2.36)}{=} M_5\left(\tfrac{f}{f-e}, -1, \tfrac{d}{d-c}, \tfrac{a-2b}{a-b}, \tfrac{\beta_1-2\beta_2}{\beta_1-\beta_2}\right) \\ &\underset{(2.26)}{=} M_5(-1, \tfrac{f}{f-e}, \tfrac{2d-c}{d-c}, \tfrac{a-2b}{a-b}, \tfrac{\beta_2}{\beta_2-\beta_1}) \\ &\underset{(2.1)}{=} M_4\left(\tfrac{2f-e}{f-e}, \tfrac{2d-c}{d-c}, \tfrac{a-2b}{a-b}, \tfrac{2\beta_2-\beta_1}{\beta_2-\beta_1}\right) \end{aligned}$$

So, when $\beta = \frac{\beta_1}{\beta_2} \in \{0, 1, 2, \infty\}$ is a non-prime slope on $M_4(\alpha)$ with α not factoring through M_3 we only need to consider $\beta \in \{0, 1\}$.

From

$$\begin{aligned} M_4(n+2, \tfrac{c}{d}, -n)(1) &\underset{(3.6)}{=} F(n, -1, \tfrac{c-d}{d}, -n-1) \\ &\underset{\text{Table 4}}{=} \left(S^2, (n,1), (n+2, n+1), (c-d, d)\right) \end{aligned} \tag{4.1}$$

we see that if $M_4(n+2, \frac{c}{d}, -n)(1)$ is non-prime then $n = 0, -2$ or $\frac{c}{d} = 1$ which make $M_4(n+2, \frac{c}{d}, -n)$ non-hyperbolic. So, if $M_4(\alpha)$ is a hyperbolic knot in S^3 with a non-prime slope in $\{0, 1, 2, \infty\}$ then we may assume that the non-prime slope is 0.

In the case when 0 corresponds to a non-prime filling on $M_4(\alpha)$ we have

$$M_4(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f})(0) \underset{(3.7)}{=} F(\tfrac{2d-c}{c-d}, \tfrac{b}{a-2b}, 2, \tfrac{f}{e}).$$

From Proposition 2.3 we see that if $M_4(\frac{a}{b}, \frac{c}{d}, \frac{e}{f})(0)$ is non-prime then one of $2d - c, b, f = 0$ or

$$\frac{b}{a-2b} = \frac{1}{n} = -\frac{f}{e} \tag{4.2}$$

holds. If $2d - c = 0$ then $\frac{c}{d} = 2$ and α factors through M_3 (using (2.2) on the elements of (3.29)). If b or f equal 0 then α is an exceptional filling instruction (see Theorem 3.3). If (4.2) holds then $\frac{a}{b} = 2 + n$ and $\frac{e}{f} = -n$. For $M_4(\alpha) = M_4(2+n, \frac{c}{d}, -n)$ to be hyperbolic we require $n \notin \{-2, -1, 0\}$ and for $M_4(\alpha)$ to not factor through M_3 we require $n \notin \{-3, -1, 1\}$.

When (4.2) holds, the possible S^3 slopes are 0, 1, ∞. We will now investigate each slope individually. Using Theorem 3.3 and the Proposition 2.3

we have

$$M_4(n+2, \tfrac{c}{d}, -n)(\infty) \underset{(3.4)}{=} F(-1-n, \tfrac{d}{c-d}, -1, n) \underset{\text{Table 4}}{=} \big(S^2, (2{+}n, 1{+}n), (n, 1), (d, c{-}d)\big).$$

So $M_4(\alpha) = M_4(n+2, \frac{c}{d}, -n)(\infty)$ is of type Z unless $n \in \{-3, -2, -1, 0, 1\}$ (which means α is exceptional or factors through M_3), or $d \in \{0, \pm 1\}$. If $d = 0$ then α is exceptional. If $d = \pm 1$ then $\frac{c}{d} \in \mathbb{Z}$ so we may assume $d = 1$ with out loss of generality. If $d = 1$ then

$$M_4(\alpha)(\infty) \underset{(3.4)}{=} F(-1-n, \tfrac{1}{c-1}, -1, n) \underset{\text{Table 4}}{=} L(cn^2{+}2cn{+}2, *).$$

This means that if $M_4(\alpha)(\infty) = S^3$ then n divides 1 or 3. The cases $n = -3, \pm 1$ mean α is exceptional or factors through M_3. When $n = 3$ we require $c \in \mathbb{Z}$ to satisfy $c(3)^2 + 2c(3) + 2 = \pm 1$ which is impossible.

From (4.1) we see that $M_4(n+2, \frac{c}{d}, -n)(1)$ is of type Z unless $n \in \{-3, -1, 0, 1\}$ (which make α exceptional or factor through M_3) or $\frac{c}{d} = 1 + \frac{1}{k}$. When $\frac{c}{d} = 1 + \frac{1}{k}$ we find that

$$M_4(n+2, \tfrac{c}{d}, -n)(1) \underset{\text{Table 4}}{=} L\big((k{+}1)n^2{+}2(k{+}1)n{+}2, *\big).$$

So, if $M_4(n+2, \frac{c}{d}, -n)(1) = S^3$ then n divides 1 or 3. The cases $\pm 1, -3$ are excluded, and the case $n = 3$ gives

$$M_4(n+2, \tfrac{c}{d}, -n)(1) = L(15k{+}17, *) \neq S^3$$

for any $k \in \mathbb{Z}$.

We have therefore shown that if $M_4(\alpha)$ is hyperbolic with α not factoring through M_3 with a non-prime slope then $M_4(\alpha)$ does not have an S^3 filling.

The final case to consider is when $M_5(\alpha)(\beta)$ is non-prime and $M_5(\alpha)(\gamma) = S^3$ with $\beta, \gamma \in \{0, 1, \infty\}$ where α is a hyperbolic filling instruction that does not factor through M_4. There are six choices for β and γ but Lemma 2.7 allows us to assume that $\beta = 1$ and $\gamma = 0$. We have

$$M_5(\alpha)(1) = M_5(\tfrac{a}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g}{h})(1) \underset{(2.5)}{=} F(\tfrac{a-b}{b}, \tfrac{c}{d}, \tfrac{e}{f}, \tfrac{g-h}{h})$$

and

$$M_5(\alpha)(\infty) = M_5(\tfrac{n+1}{-n}, \tfrac{n}{-1}, \tfrac{d}{c}, \tfrac{1}{-k})(1) \underset{(3.1)}{=} F(-\tfrac{a}{b}, \tfrac{f}{e}, \tfrac{d}{c}, -\tfrac{g}{h}).$$

From Table 4 we see that if $M_5(\alpha)(1)$ is non-prime then one of $a-b, c, e, g-h=0$ or $\frac{a-b}{b}=\frac{1}{n}=-\frac{e}{f}$ or $\frac{g-h}{h}=\frac{1}{n}=-\frac{c}{d}$. By a composition of (2.28)–(2.29) we only need to consider the cases when one of $a-b, c=0$ or $\frac{a-b}{b}=\frac{1}{n}=-\frac{e}{f}$. If $c=0$ then α is exceptional. If $a-b=0$ then $\frac{a}{b}=1$ and α is exceptional. If $\frac{a-b}{b}=\frac{1}{n}=-\frac{e}{f}$ then $\frac{a}{b}=\frac{n+1}{n}$ and $\frac{e}{f}=-\frac{1}{n}$. Table 4 tells us that in this case

$$M_5(\alpha)(1)=M_5(\tfrac{n+1}{n},\tfrac{c}{d},-\tfrac{1}{n},\tfrac{g}{h})(1)\underset{(2.5)}{=}F(\tfrac{1}{n},\tfrac{c}{d},-\tfrac{1}{n},\tfrac{g-h}{h})=L(c,d)\#L(g-h,h).$$

From Table 4, we see that if

$$M_5(\alpha)(\infty)=M_5(\tfrac{n+1}{n},\tfrac{c}{d},-\tfrac{1}{n},\tfrac{g}{h})(\infty)=F(-\tfrac{n+1}{n},-n,\tfrac{d}{c},-\tfrac{g}{h})=S^3$$

then one of $n+1=\pm1$, $n=\pm1$, $\frac{c}{d}=m\in\mathbb{Z}$ or $\frac{g}{h}=\frac{1}{k}$ must hold. If $n+1=\pm1$ or $n=\pm1$ then α factors through M_4 or is exceptional. If $\frac{c}{d}=m\in\mathbb{Z}$ then, from Table 4,

$$M_5(\alpha)(\infty)=\big(S^2,(n,-1),(g,-h),((1-m)n-m,n+1)\big)=S^3$$

requires one of $n=\pm1$ (already excluded), $g=\pm1$, or $(1-m)n-m=\pm1$. If $(1-m)n-m=\pm1$ then $n=-1$ (which is excluded) or $m\in\{0,2\}$ which make α exceptional or factor through M_4. So, $\frac{g}{h}$ is necessarily of the form $\frac{1}{k}$. From Table 4,

$$\begin{aligned}M_5(\tfrac{n+1}{n},\tfrac{c}{d},-\tfrac{1}{n},\tfrac{1}{k})(\infty)&\underset{(3.1)}{=}F(-\tfrac{n+1}{n},-n,\tfrac{d}{c},-\tfrac{1}{k})\\&=\big(S^2,(d,c),(n+1,-n),(nk+1,n)\big)=S^3\end{aligned}$$

requires one of $\frac{c}{d}=m\in\mathbb{Z}$, $n+1=\pm1$ (which has been excluded), or $nk+1=\pm1$. If $nk+1=\pm1$ then $k=0$ or $n\in\{\pm1,\pm2\}$. The cases $k=0, n=\pm1,-2$ make α exceptional or factor through M_4, and if $n=2$ then $k=-1$ which makes α factor through M_4. So, if 1 is a slope on a one cusped hyperbolic $M_5(\alpha)$ that corresponds to a non-prime filling and $M_5(\alpha)(\infty)=S^3$ then we may assume that $\alpha=(\frac{n+1}{n},m-\frac{1}{n},\frac{1}{k})$ where $k,m\neq\pm1,0,2$ and $n\neq\pm1,0,-2$. From Table 4,

$$\begin{aligned}M_5(\tfrac{n+1}{n},m,-\tfrac{1}{n},\tfrac{1}{k})(\infty)&\underset{(3.1)}{=}F(-\tfrac{n+1}{n},-n,\tfrac{1}{m},-\tfrac{1}{k})\\&=L\big((nm+m-n)(-1-kn)-n(n+1),*\big)=S^3\end{aligned}$$

if and only if

$$m(1+kn+n+kn^2) = \pm 1 + kn^2 - n^2 \Rightarrow m = \frac{(k-1)n^2 \pm 1}{(n+1)(kn+1)} \tag{4.3}$$

because $n \neq 1$. It is straightforward to verify that for $k > 2$ and $n > 1$, or $k > 2$ and $n < -2$, or $k < -1$ and $n > 1$, or $k < -1$ and $n < -2$ that, in all four cases, (4.3) leads to $0 < m < 3$. So, the only valid integer solutions to (4.3) make α exceptional or factor through M_4.

This finishes the argument that no hyperbolic $M_5(\alpha)$ has both a non-prime and S^3 filling.

5. Tables

f	Additional exceptional slopes β_i	Exceptional filling $M_5(f)(\beta_i)$
$(-2, -\frac{1}{2}, 3, 3)$	$\beta_1 = -1$	$\big(S^2, (2,-1), (5,2), (4,1)\big)$
	$\beta_2 = -\frac{1}{2}$	$\big(A, (2,-1)\big)/_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}$
$(-2, \frac{3}{2}, \frac{3}{2}, -2)$	$\beta_1 = -1$	$\big(D, (2,-1), (2,1)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (2,1), (3,-1)\big)$
	$\beta_2 = -\frac{1}{2}$	$\big(A, (2,-1)\big)/_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}$
$(-2, -3, -\frac{1}{2}, -2)$	$\beta_1 = -1$	$\big(S^2, (2,1), (3,-1), (11,-2)\big)$
	$\beta_2 = 2$	$\big(D, (2,1), (3,5)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (3,2), (2,-1)\big)$
$(-2, -\frac{1}{3}, 3, \frac{2}{3})$	$\beta_1 = -1$	$\big(S^2, (2,-1), (7,2), (5,3)\big)$
	$\beta_2 = 2$	$\big(A, (2,3)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
$(-2, -\frac{1}{2}, 3, \frac{2}{3})$	$\beta_1 = -1$	$\big(S^2, (2,-1), (5,2), (5,3)\big)$
	$\beta_2 = 2$	$\big(S^2, (2,1), (3,-1), (11,-2)\big)$
$(-2, -2, -2, -2)$	$\beta_1 = -1$	$\big(S^2, (2,1), (3,-1), (7,-1)\big)$
	$\beta_2 = -2$	$\big(D, (2,1), (2,-1)\big) \bigcup_{\left(\begin{smallmatrix}-1&5\\1&-4\end{smallmatrix}\right)} \big(D, (2,1), (3,1)\big)$

Table 6: All $M_5(f)$ with f not factoring through M_4 and $e_\tau\big(M_5(f)\big) = 5$, $E_\tau\big(M_5(f)\big) = \{\beta_1, \beta_2, 0, 1, \infty\}$.

f	Additional exceptional slopes β	Exceptional filling $M_5(f)$
$(-2,\frac{p}{q},3,\frac{u}{v})$	-1	$F(-2,\frac{p}{q},\frac{u+v}{v},-2)$
$(-2,\frac{p}{q},\frac{r}{s},-2)$	-1	$F(\frac{s}{s-r},2,\frac{q}{2q-p},-3)$
$(-2,\frac{3}{2},\frac{3}{2},\frac{u}{v})$	-1	$F(-3,\frac{u}{u+v},2,-2)$
$(-2,\frac{p}{q},\frac{5}{2},-\frac{1}{2})$	-1	$F(-3,\frac{p-q}{p},2,-2)$
$(-2,-2,\frac{r}{s},-3)$	-1	$\big(A,(s,s-r)\big)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$(-2,-\frac{1}{2},4,\frac{u}{v})$	-1	$F(2,\frac{3}{2},\frac{v}{u+v},-2)$
$(-2,\frac{p}{q},4,-\frac{3}{2})$	-1	$F(2,\frac{3}{2},\frac{q}{p},-2)$

Table 7: All hyperbolic $M_5(f)$ with f not factoring through M_4 and $e_\tau\big(M_5(f)\big)=4$, $E_\tau\big(M_5(f)\big)=\{\beta,0,1,\infty\}$, part 1/5.

f	Additional exceptional slopes β	Exceptional filling $M_5(f)(\beta)$
$(-2,4,5,-\frac{3}{2})$ $(-2,-\frac{1}{2},5,3)$	-1	$\big(A,(2,1)\big)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$(-2,3,4,-\frac{4}{3})$	-1	$\big(A,(2,-1)\big)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$(-2,3,\frac{3}{2},-\frac{1}{2})$	-1	$\big(A,(1,-3)\big)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$
$(-2,-2,4,-\frac{5}{3})$ $(-2,-\frac{2}{3},4,-3)$	-1	$\big(D,(3,-1),(2,1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,3),(2,-1)\big)$
$(-2,\frac{2}{3},\frac{5}{2},-\frac{1}{3})$	-1	$\big(D,(2,1),(2,3)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{4}{3},\frac{3}{2},\frac{1}{3})$	-1	$\big(A,(2,-1)\big)/\begin{pmatrix}0&1\\1&0\end{pmatrix}$

Table 8: All hyperbolic $M_5(f)$ with f not factoring through M_4 and $e_\tau\big(M_5(f)\big)=4$, $E_\tau\big(M_5(f)\big)=\{\beta,0,1,\infty\}$, part 2/5.

f	Additional exceptional slope β	Exceptional filling $M_5(f)(\beta)$
$(-2,-3,-2,-3)$ $(-2,-2,-2,-4)$	-1	$\big(S^2,(2,1),(3,-1),(7,-1)\big)$
$(-2,-4,-2,-3)$ $(-2,-2,-2,-5)$ $(-2,-3,-3,-3)$ $(-2,-2,-3,-4)$	-1	$\big(S^2,(2,1),(4,-3),(5,1)\big)$
$(-2,-2,-4,-4)$ $(-2,-5,-2,-3)$ $(-2,-3,-4,-3)$ $(-2,-2,-2,-6)$	-1	$\big(S^2,(3,-2),(3,1),(4,1)\big)$
$(-2,-2,-2,-7)$ $(-2,-6,-2,-3)$ $(-2,-3,-5,-3)$ $(-2,-2,-5,-4)$	-1	$\big(D,(2,-1),(2,-1)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-1),(3,-1)\big)$
$(-2,-4,-3,-3)$ $(-2,-2,-3,-5)$	-1	$\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}-1&2\\1&-1\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$(-2,-3,-2,-4)$	-1	$\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}-1&3\\1&-2\end{pmatrix}}\big(D,(2,1),(3,1)\big)$
$(-2,\frac{3}{2},\frac{5}{2},-\frac{2}{3})$	-1	$\big(A,(2,-3)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$

Table 9: All hyperbolic $M_5(f)$ with f not factoring through M_4 and $e\big(M_5(f)\big)=4$, $E_\tau\big(M_5(f)\big)=\{\beta,0,1,\infty\}$, part 3/5.

f	Additional exceptional slopes β	Exceptional filling $M_5(f)(\beta)$
$(-2,-2,\frac{1}{4},3)$	$\frac{1}{2}$	$\big(S^2,(3,2),(2,-1),(9,-2)\big)$
$(-2,\frac{2}{5},\frac{3}{4},\frac{3}{2})$	$\frac{1}{2}$	$\big(S^2,(3,2),(2,-1),(3,2)\big)$
$(-2,\frac{1}{5},\frac{4}{3},\frac{3}{2})$	$\frac{1}{2}$	$\big(S^2,(2,1),(4,-3),(5,1)\big)$
$(-2,\frac{1}{4},\frac{2}{3},\frac{5}{3})$	$\frac{1}{2}$	$\big(D,(2,1),(2,3)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{1}{5},\frac{3}{2},\frac{3}{2})$	$\frac{1}{2}$	$\big(S^2,(2,1),(3,-1),(7,-1)\big)$
$(-2,3,\frac{1}{3},4)$	$\frac{1}{2}$	$\big(D,(2,1),(3,4)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{1}{3},\frac{3}{2},\frac{4}{3})$	$\frac{1}{2}$	$\big(A,(1,3)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
$(-2,-2,\frac{1}{5},3)$	$\frac{1}{2}$	$\big(A,(2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
$(-2,\frac{2}{3},\frac{3}{5},\frac{3}{2})$	$\frac{1}{2}$	$\big(D,(2,1),(3,-1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{1}{6},\frac{3}{2},\frac{3}{2})$	$\frac{1}{2}$	$\big(S^2,(2,1),(4,-3),(5,1)\big)$
$(-2,\frac{1}{7},\frac{3}{2},\frac{3}{2})$	$\frac{1}{2}$	$\big(S^2,(3,-2),(3,1),(4,1)\big)$
$(-2,\frac{1}{8},\frac{3}{2},\frac{3}{2})$	$\frac{1}{2}$	$\big(D,(2,-1),(2,-1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-1),(3,-1)\big)$
$(-2,\frac{1}{5},\frac{6}{5},\frac{3}{2})$	$\frac{1}{2}$	$\big(D,(2,-1),(2,-1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,-1),(3,-1)\big)$
$(-2,\frac{3}{8},\frac{3}{4},\frac{3}{2})$	$\frac{1}{2}$	$\big(D,(3,-1),(2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(2,3),(2,-1)\big)$
$(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3})$	$\frac{1}{2}$	$\big(S^2,(3,-1),(2,1),(5,2)\big)$
$(-2,\frac{2}{3},\frac{3}{4},\frac{2}{3})$	$\frac{1}{2}$	$\big(A,(3,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
$(-2,\frac{3}{5},\frac{2}{3},\frac{4}{3})$	$\frac{1}{2}$	$\big(D,(2,1),(3,2)\big)\bigcup_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{1}{4},\frac{3}{2},\frac{4}{3})$	$\frac{1}{2}$	$\big(D,(2,1),(2,-1)\big)\bigcup_{\begin{pmatrix}-1&4\\1&-3\end{pmatrix}}\big(D,(2,1),(3,1)\big)$

Table 10: All hyperbolic $M_5(f)$ with f not factoring through M_4 with $e\big(M_5(f)\big)=4$, $E_\tau\big(M_5(f)\big)=\{\beta,0,1,\infty\}$, part 4/5.

f	Additional exceptional slopes β	Exceptional filling $M_5(f)(\beta)$
$(-2, -\frac{1}{3}, 3, \frac{2}{3})$	2	$\big(A, (2,3)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
$(-2, -\frac{2}{3}, -2, \frac{2}{3})$	2	$\big(D, (3,-1), (2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (2,-1), (2,-1)\big)$
$(-2, -2, -\frac{1}{3}, 3)$	2	$\big(D, (2,1), (2,5)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,2), (2,-1)\big)$
$(-2, -\frac{1}{3}, -2, \frac{2}{5})$	2	$\big(D, (2,1), (3,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,2), (2,-1)\big)$
$(-2, -3, -\frac{1}{2}, -2)$	2	$\big(D, (2,1), (3,5)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,2), (2,-1)\big)$
$(-2, -\frac{1}{2}, -\frac{3}{2}, \frac{1}{3})$	2	$\big(D, (3,-1), (2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,1), (2,-1)\big)$
$(-2, \frac{1}{3}, -3, \frac{1}{3})$	2	$\big(A, (3,2)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}$
$(-2, -\frac{1}{2}, -3, \frac{1}{3})$	2	$\big(D, (2,1), (2,1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,2), (2,-1)\big)$
$(-2, -\frac{1}{2}, -3, \frac{3}{5})$	2	$\big(D, (2,1), (2,-1)\big)/_{\begin{pmatrix}0&1\\1&0\end{pmatrix}}\big(D, (3,2), (2,-1)\big)$

Table 11: All hyperbolic $M_5(f)$ with f not factoring through M_4 with $e\big(M_5(f)\big) = 4$, $E_\tau\big(M_5(f)\big) = \{\beta, 0, 1, \infty\}$, part 5/5.

f	Additional exceptional slopes β_i	Exceptional filling $M_4(f)(\beta_i)$
$(-2,-2,-2)$	$\beta_1=-2$	$\big(D,(2,1),(2,-1)\big)\bigcup_{\left(\begin{smallmatrix}-1&4\\1&-3\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
	$\beta_2=-1$	$\big(A,(1,3)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
$(-2,-\frac{1}{2},-2)$	$\beta_1=-1$	$\big(A,(2,3)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
	$\beta_2=3$	$\big(D,(2,1),(2,3)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(3,2),(2,-1)\big)$
$(-2,\frac{r}{s},-2)$	-1	$\big(A,(s,s-r)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
$(\frac{p}{q},4,-\frac{1}{2})$	-1	$F(2,\frac{3}{2},\frac{q}{p},-2)$
$(4,5,-\frac{1}{2})$		$\big(A,(2,1)\big)/_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
$(-2,4,-\frac{2}{3})$		$\big(D,(3,-1),(2,1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,3),(2,-1)\big)$
$(-2,-5,-3)$	$\beta=-1$	$\big(D,(2,-1),(2,-1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,-1),(3,-1)\big)$
$(-2,-2,-6)$		$\big(D,(2,-1),(2,-1)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(2,-1),(3,-1)\big)$
$(\frac{2}{3},\frac{5}{2},\frac{2}{3})$		$\big(D,(2,1),(2,3)\big)\bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}\big(D,(3,2),(2,-1)\big)$
$(-2,-2,-3)$		$\big(S^2,(2,1),(3,-1),(7,-1)\big)$
$(-2,-3,-3)$,		$\big(S^2,(2,1),(4,-3),(5,1)\big)$
$(-2,-2,-4)$		$\big(S^2,(2,1),(4,-3),(5,1)\big)$
$(-3,-4,-2)$		$\big(S^2,(3,-2),(3,1),(4,1)\big)$
$(-2,-2,-5)$		$\big(S^2,(3,-2),(3,1),(4,1)\big)$
$(-2,-3,-4)$		$\big(D,(2,1),(2,-1)\big)\bigcup_{\left(\begin{smallmatrix}-1&2\\1&-1\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$
$(-3,-2,-3)$		$\big(D,(2,1),(2,-1)\big)\bigcup_{\left(\begin{smallmatrix}-1&3\\1&-2\end{smallmatrix}\right)}\big(D,(2,1),(3,1)\big)$

Table 12: All hyperbolic $M_4(f)$ with f not factoring through M_3 and $e_\tau\big(M_4(f)\big)\geq 5$, together with $E_\tau\big(M_4(f)\big)=\{\beta_1,\beta_2,0,1,2,\infty\}$ if $e_\tau\big(M_4(f)\big)=6$ and $E_\tau\big(M_4(f)\big)=\{\beta,0,1,2,\infty\}$ if $e_\tau\big(M_4(f)\big)=5$.

$k \in \mathbb{Z}\backslash\{\pm 1, 0, 2\}$,	$E(M_5(-2, k, 3, \frac{k+1}{3k+2})) = \{-1, 0, 1, \infty\}$
β	$M_5(-2, k, 3, \frac{k+1}{3k+2})(\beta)$
-1	$\big(D, (2,-1), (k,1)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (2,-1), (4k+3, 3k+2)\big)$
0	$\big(S^2, (k-1, k), (2,1), (8k+5, -3k-2)\big)$
1	$\big(D, (3,1), (3,-1)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (k,1), (2k+1, -3k-2)\big)$
∞	S^3
$k \in \mathbb{Z}$,	$E(M_5(-2, \frac{3}{2}, \frac{3}{2}, \frac{3+14k}{1+5k})) = \{-1, 0, 1, \infty\}$
β	$M_5(-2, \frac{3}{2}, \frac{3}{2}, \frac{3+14k}{1+5k})(\beta)$
-1	$\big(D, (3,-1), (2,1)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (2,-1), (3+14k, 4+19k)\big)$
0	S^3
1	$\big(D, (3,-1), (3,2)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (3,2), (2+9k, 1+5k)\big)$
∞	$\big(D, (2,1), (2,3)\big) \bigcup_{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)} \big(D, (2,3), (3+14k, -1-5k)\big)$
$k \in \mathbb{Z}\backslash\{-1, 0, 1, 2\}$,	$E(M_5(-2, \frac{1}{k}, 3, \frac{k-1}{k})) = \{-1, 0, 1, \infty\}$
β	$M_5(-2, \frac{1}{k}, 3, \frac{k-1}{k})(\beta)$
-1	$\big(S^2, (2,-1), (2k-1, k), (1-2k, 2)\big)$
0	$\big(S^2, (1-k, 1), (1+2k, 1), (2,1)\big)$
1	$L(3,-1) \# L(3,1)$
∞	$\big(S^2, (2,1), (k,1), (2k-3, 1-k)\big)$
$k \in \mathbb{Z}\backslash\{-2, -1, 0, 1, 2\}$,	$E(M_4(-2, \frac{1}{k}, -2)) = \{-1, 0, 1, 2, \infty\}$
β	$M_4(-2, \frac{1}{k}, -2)(\beta)$
-1	$\big(A, (k, k-1)\big) / _{\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)}$
0	$\big(S^2, (2k-1, 1-k), (2,1), (6,1)\big)$
1	$\big(S^2, (4,-1), (4,3), (1-k, k)\big)$
2	$\big(S^2, (3,-1), (3,-1), (1-2k, 2)\big)$
∞	$\big(S^2, (k, 1-k), (2,1), (2,3)\big)$
$k \in \mathbb{Z}\backslash\{-1, 0, 1, 2\}$,	$E(M_5(-2, \frac{1}{k}, 3, \varnothing)) = \{-1, 0, 1, \infty\}$
β	$M_5(-2, \frac{1}{k}, 3, \varnothing)(\beta)$
-1	$\big(D, (2,-1), (1-2k, 2)\big)$
0	$\big(D, (2,1), (1-k, 1)\big)$
1	$\big(D, (3,1), (3,-1)\big)$
∞	$\big(D, (2,1), (k,1)\big)$

Table 13: Exceptional slopes & fillings of five highlighted families

$f=(-2,-\frac{1}{2},3,3)$, $E(M_5(f))=\{-1,-\frac{1}{2},0,1,\infty\}$, types $\{Z,T,S,Z,Z\}$
$f=(-2,\frac{3}{2},\frac{3}{2},-2)$, $E(M_5(f))=\{-1,-\frac{1}{2},0,1,\infty\}$, types $\{T,T,T^H,T,T\}$
$f=(-2,-3,-\frac{1}{2},-2)$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,T,Z,Z,Z\}$
$f=(-2,-\frac{1}{3},3,\frac{2}{3})$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,T,Z,Z,Z\}$
$f=(-2,-\frac{1}{2},3,\frac{2}{3})$, $E(M_5(f))=\{-1,2,0,1,\infty\}$, types $\{Z,Z,Z,Z,Z\}$
$f=(-2,-2,-2,-2)$, $E(M_5(f))=\{-2,-1,0,1,\infty\}$, types $\{T,Z,Z,T,T^H\}$

Table 14: Exceptional sets for $M_5(f)$ for f in Table 6.

$f=(-2,\frac{3}{2},\frac{3}{2},\frac{u}{v})$, $\frac{u}{v}\in\mathbb{Q}\cup\{\varnothing\}\setminus\{-2,-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
types $\begin{cases}\{T,D,T,T\} \text{ if } \frac{u}{v}=\varnothing;\\ \{T,S^H,T,T\} \text{ if } \lvert 5u-14v\rvert=1;\\ \{T,S\&T^H,T,T\} \text{ if } \frac{u}{v}=\frac{14}{5};\\ \{T,T^H,Z,T\} \text{ if } \lvert u-v\rvert=1;\\ \{Z,T^H,T,Z\} \text{ if } \lvert u\rvert=1;\\ \{T,T^H,T,T\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},\frac{5}{2},-\frac{1}{2})$, $\frac{p}{q}\in\mathbb{Q}\cup\{\varnothing\}\setminus\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
types $\begin{cases}\{T\&A,A,T\&A,A\} \text{ if } \frac{p}{q}=\varnothing;\\ \{Z,T^H,T,Z\} \text{ if } \frac{p}{q}=1+\frac{1}{n};\\ \{T,Z,Z,Z\} \text{ if } \lvert p\rvert=1;\\ \{T,Z,T,T^H\} \text{ if } \lvert q\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,-2,\frac{r}{s},-3)$, $\frac{r}{s}\in\mathbb{Q}\cup\{\varnothing\}\setminus\{-1,0,\frac{1}{2},1,2\}$, $E(M_5(f))=\{-1,0,1,\infty\}$
types $\begin{cases}\{T,A,T\&A,A\} \text{ if } \frac{r}{s}=\varnothing;\\ \{T,T^H,T,Z\} \text{ if } \lvert r-s\rvert=1;\\ \{T,Z,Z,Z\} \text{ if } \lvert r\rvert=1;\\ \{T,Z,T,T^H\} \text{ if } \lvert s\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$

Table 15: Exceptional sets for $M_5(f)$ for f in Table 7, part 1/4.

$f=(-2,-\frac{1}{2},4,\frac{u}{v}),\ \frac{u}{v}\in\mathbb{Q}\cup\{\varnothing\}\backslash\{-1,0,\frac{1}{2},1,2\},\ E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{T\&A,A,A,A\} \text{ if } \frac{u}{v}=\varnothing;\\ \{T^H,Z,Z,Z\} \text{ if } \frac{u}{v}=-2;\\ \{Z,S,Z,Z\} \text{ if } \frac{u}{v}=3;\\ \{Z,T^H,Z,S\} \text{ if } \frac{u}{v}=4;\\ \{T,Z,S,Z\} \text{ if } \frac{u}{v}=\frac{3}{2};\\ \{Z,Z,Z,Z\} \text{ if } \frac{u}{v}\in\mathbb{Z}\backslash\{-2,-1,0,1,2,3,4\};\\ \{T,T^H,Z,Z\} \text{ if } \lvert u-3v\rvert=1,\ \frac{u}{v}\neq 4;\\ \{T,Z,T^H,Z\} \text{ if } \lvert 2u-3v\rvert=1;\\ \{T,Z,Z,T^H\} \text{ if } \lvert v-4u\rvert=1;\\ \{T,Z,Z,Z\} \text{ otherwise.}\end{cases}$
$f=(-2,\frac{p}{q},4,-\frac{3}{2}),\ \frac{p}{q}\in\mathbb{Q}\cup\{\varnothing\}\backslash\{-1,0,\frac{1}{2},1,2\},\ E(M_5(f))=\{-1,0,1,\infty\}$
$\text{types}\begin{cases}\{T\&A,A,T\&A,A\} \text{ if } \frac{p}{q}=\varnothing;\\ \{Z,Z,T,T^H\} \text{ if } \lvert q\rvert=1;\\ \{T,T^H,T,Z\} \text{ if } \lvert p-q\rvert=1;\\ \{T,Z,Z,Z\} \text{ if } \lvert p\rvert=1;\\ \{T,Z,T,Z\} \text{ otherwise.}\end{cases}$

Table 16: Exceptional sets for $M_5(f)$ for f in Table 7, part 2/4.

$f=(-2,\frac{p}{q},3,\frac{u}{v})$, $E(M_5(f))=\{-1,0,1,\infty\}$, $\frac{p}{q},\frac{u}{v}\in\mathbb{Q}\cup\{\varnothing\}\setminus\{0,1,-1,\frac{1}{2},2\}$
-1 is of type $\begin{cases} A \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert u+v\rvert=1, \text{ or } \frac{u}{v}=\varnothing \text{ and } \lvert p\rvert=1; \\ T\&A \text{ if } \frac{p}{q}=\frac{u}{v}=\varnothing, \text{ or } \frac{p}{q}=\varnothing \text{ and } \lvert u+v\rvert\neq 1, \\ \qquad \text{or } \frac{u}{v}=\varnothing \text{ and } \lvert p\rvert\neq 1; \\ T^H \text{ if } \lvert p\rvert=\lvert u+v\rvert=1; \\ Z \text{ if } \lvert p\rvert=1 \text{ and } \lvert u+v\rvert\neq 1, \text{ or } \lvert u+v\rvert=1 \text{ and } \lvert p\rvert\neq 1; \\ T \text{ otherwise} \end{cases}$
0 is of type $\begin{cases} A \text{ if } \frac{p}{q}=\varnothing \text{ and } \frac{u}{v}\neq 3,3+\frac{1}{n}, \text{ or } \frac{u}{v}=\varnothing \text{ and } \frac{p}{q}\neq 1+\frac{1}{n}; \\ S\&D \text{ if } \frac{u}{v}=3 \text{ and } \frac{p}{q}=\varnothing; \\ D \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \frac{u}{v}=\varnothing, \text{ or } \frac{u}{v}=3+\frac{1}{n} \text{ and } \frac{p}{q}=\varnothing; \\ S \text{ if } \frac{u}{v}=3 \text{ and } \lvert p-q\rvert\neq 1; \\ S^H \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \lvert(3+2n)u-(7+6n)v\rvert=1; \\ \qquad \text{or } \frac{u}{v}=3+\frac{1}{k} \text{ and } \lvert(3+2k)p-(1+2k)q\rvert=1, \\ S\&T^H \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \frac{u}{v}=\frac{7+6n}{3+2n}, \text{ or } \frac{u}{v}=3+\frac{1}{k} \text{ and } \frac{p}{q}=\frac{1+2k}{3+2k}; \\ T^H \text{ if } \frac{u}{v}=3 \text{ and } \lvert p-q\rvert=1, \\ \qquad \text{or } \frac{p}{q}=1+\frac{1}{n} \text{ and } \lvert(3+2n)u-(7+6n)v\rvert\neq 0,1, \\ \qquad \text{or } \frac{u}{v}=3+\frac{1}{k} \text{ and } \lvert(3+2k)p-(1+2k)q\rvert\neq 0,1; \\ Z \text{ otherwise.} \end{cases}$
1 is of type $\begin{cases} T\&A \text{ if } \frac{p}{q}=\frac{u}{v}=\varnothing, \text{ or } \frac{p}{q}=\varnothing \text{ and } \lvert u-v\rvert\neq 1, \\ \qquad \text{or } \frac{u}{v}=\varnothing \text{ and } \lvert p\rvert\neq 1; \\ A \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert u-v\rvert=1, \text{ or } \frac{u}{v}=\varnothing \text{ and } \lvert p\rvert=1; \\ S \text{ if } \frac{p}{q}=\frac{1}{k}=1-\frac{u}{v}; \\ T^H \text{ if } \frac{p}{q}=\frac{1}{k} \;\&\; \lvert(1-k)v+ku\rvert=1, \text{ or } \frac{u}{v}=1+\frac{1}{k} \;\&\; \lvert kp+q\rvert=1; \\ Z \text{ if } \frac{p}{q}=\frac{1}{k} \;\&\; \lvert(1-k)v+ku\rvert\neq 1, \text{ or } \frac{u}{v}=1+\frac{1}{k} \;\&\; \lvert kp+q\rvert\neq 1; \\ T \text{ otherwise.} \end{cases}$
∞ is of type $\begin{cases} A \text{ if } \frac{p}{q}=\frac{u}{v}=\varnothing, \text{ or } \frac{p}{q}=\varnothing \;\&\; \lvert v-3u\rvert\neq 1, \text{ or } \frac{u}{v}=\varnothing \;\&\; \lvert q\rvert\neq 1; \\ S\&D \text{ if } \frac{u}{v}=\frac{1}{3} \text{ and } \frac{p}{q}=\varnothing; \\ S \text{ if } \frac{u}{v}=\frac{1}{3} \text{ and } \lvert q\rvert\neq 1; \\ S^H \text{ if } \frac{p}{q}=k \text{ and } \lvert(1+2k)v-(1+6k)u\rvert=1, \\ \qquad \text{or } v-3u=\epsilon=\pm 1 \text{ and } \lvert(1+2\epsilon u)q+2p\rvert=1; \\ S\&T^H \text{ if } \frac{p}{q}=k \;\&\; \frac{u}{v}=\frac{2k+1}{6k+1}, \text{ or } v-3u=\epsilon=\pm 1 \;\&\; \frac{p}{q}=-\frac{1+2\epsilon u}{2}; \\ T^H \text{ if } \frac{u}{v}=\frac{1}{3} \;\&\; \lvert q\rvert\neq 1, \text{ or } \frac{p}{q}=k \;\&\; \lvert(1+2k)v-(1+6k)u\rvert\neq 0,1, \\ \qquad \text{or } v-3u=\epsilon=\pm 1 \text{ and } \lvert(1+2\epsilon u)q+2p\rvert\neq 0,1; \\ Z \text{ otherwise.} \end{cases}$

Table 17: Exceptional sets for $M_5(f)$ for f in Table 7, part 3/4.

$f=(-2,\frac{p}{q},\frac{r}{s},-2)$, $E(M_5(f))=\{-1,0,1,\infty\}$, $\frac{p}{q},\frac{r}{s}\in\mathbb{Q}\cup\{\varnothing\}\backslash\{-1,0,\frac{1}{2},1,2\}$
-1 is of type $\begin{cases} T\&A \text{ if } \frac{p}{q}=\frac{r}{s}=\varnothing; \\ A \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert s\rvert=1, \text{ or } \frac{r}{s}=\varnothing \text{ and } \lvert q\rvert=1; \\ S \text{ if } \frac{p}{q}=4-\frac{r}{s}=n \text{ for some } n\in\mathbb{Z}\backslash\{-1,0,1,2\}; \\ T^H \text{ if } \lvert q\rvert=1 \text{ or } \lvert s\rvert=1, \text{ and } 4-\frac{p}{q}+\frac{1}{n}=\frac{r}{s} \text{ for some } n\in\mathbb{Z}; \\ Z \text{ if } \lvert q\rvert=1 \text{ or } \lvert s\rvert=1, \text{ and } 4-\frac{p}{q}+\frac{1}{n}\neq\frac{r}{s} \text{ for any } n\in\mathbb{Z}; \\ T \text{ otherwise.} \end{cases}$
0 is of type $\begin{cases} A \text{ if } \frac{p}{q}=\frac{r}{s}=\varnothing, \text{ or } \frac{p}{q}=\varnothing \;\&\; \lvert r-s\rvert\neq 1, \text{ or } \frac{r}{s}=\varnothing \;\&\; \lvert p-q\rvert\neq 1; \\ D \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert r-s\rvert=1, \text{ or } \frac{r}{s}=\varnothing \text{ and } \lvert p-q\rvert=1; \\ S^H \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \lvert(6-5n)s+(5n-1)r\rvert=1, \\ \quad \text{or } \frac{r}{s}=1+\frac{1}{n} \text{ and } \lvert(5n-1)p+(6-5n)q\rvert=1; \\ S\&T^H \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \frac{r}{s}=\frac{6-5n}{1-5n}, \\ \quad \text{or } \frac{r}{s}=1+\frac{1}{n} \text{ and } \frac{p}{q}=\frac{6-5n}{1-5n}; \\ T^H \text{ if } \frac{p}{q}=1+\frac{1}{n} \text{ and } \lvert(6-5n)s+(5n-1)r\rvert\neq 1,0, \\ \quad \text{or } \frac{r}{s}=1+\frac{1}{n} \text{ and } \lvert(5n-1)p+(6-5n)q\rvert\neq 1,0; \\ Z \text{ otherwise.} \end{cases}$
1 is of type $\begin{cases} T\&A \text{ if } \frac{p}{q}=\frac{r}{s}=\varnothing; \\ A \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert r\rvert=1, \text{ or } \frac{r}{s}=\varnothing \text{ and } \lvert p\rvert=1; \\ T^H \text{ if } \lvert p\rvert=1 \text{ and } \lvert r\rvert=1; \\ Z \text{ if } \lvert p\rvert=1 \text{ and } \lvert r\rvert\neq 1, \text{ or } \lvert r\rvert=1 \text{ and } \lvert p\rvert\neq 1; \\ T \text{ otherwise.} \end{cases}$
∞ is of type $\begin{cases} T\&A \text{ if } \frac{p}{q}=\frac{r}{s}=\varnothing; \\ A \text{ if } \frac{p}{q}=\varnothing \text{ and } \lvert s\rvert=1, \text{ or } \frac{r}{s}=\varnothing \text{ and } \lvert q\rvert=1; \\ T^H \text{ if } \lvert s\rvert=1 \text{ and } \lvert q\rvert=1; \\ Z \text{ if } \lvert s\rvert=1 \text{ and } \lvert q\rvert\neq 1, \text{ or } \lvert q\rvert=1 \text{ and } \lvert s\rvert\neq 1; \\ T \text{ otherwise.} \end{cases}$

Table 18: Exceptional sets for $M_5(f)$ for f in Table 7, part 4/4.

f	$E(M_5(f))$ and types
$(-2,4,5,-\frac{3}{2})$, $(-2,3,4,-\frac{4}{3})$, $(-2,-2,4,-\frac{5}{3})$, $(-2,-2,-3,-5)$, $(-2,-4,-3,-3)$, $(-2,-2,-2,-7)$, $(-2,-6,-2,-3)$, $(-2,-2,-5,-4)$, $(-2,-3,-5,-3)$, $(-2,-3,-2,-4)$,	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, types $\{T,Z,T,T^H\}$
$(-2,-\frac{1}{2},5,3)$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,3,\frac{3}{2},-\frac{1}{2})$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{T,T^H,T,T^H\}$
$(-2,-\frac{2}{3},4,-3)$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{T,Z,T,Z\}$
$(-2,\frac{2}{3},\frac{5}{2},-\frac{1}{3})$, $(-2,\frac{4}{3},\frac{3}{2},\frac{1}{3})$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{T,T^H,T,Z\}$
$(-2,-2,-2,-4)$, $(-2,-3,-2,-3)$, $(-2,-2,-2,-5)$, $(-2,-2,-2,-6)$, $(-2,-5,-2,-3)$, $(-2,-2,-4,-4)$, $(-2,-3,-4,-3)$, $(-2,-3,-3,-3)$, $(-2,-2,-3,-4)$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, types $\{Z,Z,T,T^H\}$
$(-2,-4,-2,-3)$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{Z,Z,T,T^H\}$
$(-2,\frac{3}{2},\frac{5}{2},-\frac{2}{3})$	$E(M_5(\alpha))=\{-1,0,1,\infty\}$, $\{T,T^H,T,T\}$

Table 19: Exceptional sets for $M_5(f)$ for f in Tables 8 and 9.

f	$E(M_5(f))$ and types
$(-2,-\frac{1}{3},3,\frac{2}{3})$, $(-2,-\frac{2}{3},-2,\frac{2}{3})$, $(-2,\frac{1}{3},-3,\frac{1}{3})$, $(-2,-\frac{1}{3},-2,\frac{2}{5})$, $(-2,-3,-\frac{1}{2},-2)$, $(-2,-\frac{1}{2},-\frac{3}{2},\frac{1}{3})$, $(-2,-\frac{1}{2},-3,\frac{1}{3})$, $(-2,-\frac{1}{2},-3,\frac{3}{5})$	$E(M_5(\alpha))=\{2,0,1,\infty\}$ types $\{T,Z,Z,Z\}$
$(-2,-2,-\frac{1}{3},3)$	$E(M_5(f))=\{2,0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,-2,\frac{1}{4},3)$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{Z,S,Z,Z\}$
$(-2,\frac{2}{5},\frac{3}{4},\frac{3}{2})$	$E(M_5(f))=\{\frac{1}{2},0,1,\infty\}$, $\{T^H,T^H,Z,T\}$
$(-2,\frac{1}{5},\frac{4}{3},\frac{3}{2})$, $(-2,\frac{1}{5},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{6},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{7},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{3},\frac{2}{3},\frac{5}{3})$	$E(M_5(\alpha))=\{\frac{1}{2},0,1,\infty\}$, types $\{Z,T^H,Z,T\}$
$(-2,\frac{1}{4},\frac{2}{3},\frac{5}{3})$, $(-2,\frac{1}{4},\frac{3}{2},\frac{4}{3})$, $(-2,\frac{1}{3},\frac{3}{2},\frac{4}{3})$, $(-2,\frac{1}{8},\frac{3}{2},\frac{3}{2})$, $(-2,\frac{1}{5},\frac{6}{5},\frac{3}{2})$, $(-2,\frac{3}{8},\frac{3}{4},\frac{3}{2})$, $(-2,\frac{2}{3},\frac{3}{4},\frac{2}{3})$, $(-2,\frac{2}{3},\frac{3}{5},\frac{3}{2})$	$E(M_5(\alpha))=\{\frac{1}{2},0,1,\infty\}$, types $\{T,T^H,Z,T\}$
$(-2,3,\frac{1}{3},4)$	$E(M_5(\alpha))=\{\frac{1}{2},0,1,\infty\}$, $\{T,T^H,Z,Z\}$
$(-2,-2,\frac{1}{5},3)$	$E(M_5(\alpha))=\{\frac{1}{2},0,1,\infty\}$, $\{T,S,Z,Z\}$
$(-2,\frac{3}{5},\frac{2}{3},\frac{4}{3})$	$E(M_5(\alpha))=\{\frac{1}{2},0,1,\infty\}$, $\{T,T^H,Z,T\}$

Table 20: Exceptional sets for $M_5(f)$ for f in Tables 10 and 11.

f	$E(M_4(f))$ and types
$(-2, -\frac{1}{2}, -2)$	$\{-1, 0, 1, 2, 3, \infty\}$, $\{T, Z, Z, Z, T, Z\}$
$(-2, -2, -2)$	$\{-2, -1, 0, 1, 2, \infty\}$, $\{T, T, Z, Z, T, T^H\}$
$(-3, -2, -3)$, $(-2, 4, -\frac{2}{3})$, $(-2, -3, -4)$, $(4, 5, -\frac{1}{2})$ $(-2, -5, -3)$, $(-2, -2, -6)$,	$E(M_5(f)) = \{-1, 0, 1, 2, \infty\}$ types $\{T, Z, Z, T, T^H\}$
$(-3, -4, -2)$, $(-2, -3, -3)$, $(-2, -2, -4)$, $(-2, -2, -3)$, $(-2, -2, -5)$	$E(M_5(f)) = \{-1, 0, 1, 2, \infty\}$ types $\{Z, Z, Z, T, T^H\}$
$(\frac{2}{3}, \frac{5}{2}, \frac{2}{3})$	$\{-1, 0, 1, 2, \infty\}$, $\{T, Z, Z, Z, Z\}$

Table 21: Exceptional sets for $M_4(f)$ for f in Table 12, part 1/2.

$f = (-2, \frac{r}{s}, -2)$, $\frac{r}{s} \in \mathbb{Q} \cup \{\varnothing\} \backslash \{-2, -1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3\}$, $E(M_4(f)) = \{-1, 0, 1, 2, \infty\}$
$\text{types} \begin{cases} \{T, A, A, T\&A, A\} \text{ if } \frac{r}{s} = \varnothing; \\ \{T, T^H, Z, T, Z\} \text{ if } \frac{r}{s} = 2 + \frac{1}{k}; \\ \{T, Z, T^H, T, Z\} \text{ if } \frac{r}{s} = 1 + \frac{1}{k}; \\ \{T, Z, Z, Z, Z\} \text{ if } \frac{r}{s} = \frac{1}{k}; \\ \{T, Z, Z, T, T^H\} \text{ if } \frac{r}{s} \in \mathbb{Z}; \\ \{T, Z, Z, T, Z\} \text{ otherwise.} \end{cases}$
$f = (\frac{p}{q}, 4, -\frac{1}{2})$, $\frac{p}{q} \in \mathbb{Q} \cup \{\varnothing\} \backslash \{-1, 0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3\}$, $E(M_4(f)) = \{-1, 0, 1, 2, \infty\}$
$\text{types} \begin{cases} \{T\&A, T\&A, A, T\&A, D\} \text{ if } \frac{p}{q} = \varnothing; \\ \{Z, Z, Z, T, T^H\} \text{ if } \frac{p}{q} \in \mathbb{Z}; \\ \{T, T, T^H, T, T^H\} \text{ if } \frac{p}{q} = 2 + \frac{1}{k}; \\ \{T, T, Z, Z, T^H\} \text{ if } \frac{p}{q} = 1 + \frac{1}{k}; \\ \{T, T, Z, T, S^H\} \text{ if } \frac{p}{q} = \frac{n}{6n-1}; \\ \{T, T, Z, T, T^H\} \text{ otherwise.} \end{cases}$

Table 22: Exceptional sets for $M_4(f)$ for f in Table 12, part 2/2.

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School of Mathematics and Statistics, University of Sheffield,
Hicks Building, Hounsfield Road, Sheffield, S3 7RH, UK
E-mail address: `f.roukema@sheffield.ac.uk`

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