Anisotropic curvature flow of immersed curves

Gwenael Mercier, Matteo Novaga, and Paola Pozzi

We prove short-time existence of φ-regular solutions to the anisotropic and crystalline curvature flow of immersed planar curves.

1. Introduction

In this paper we consider the anisotropic curvature motion for planar immersed curves. More precisely, given an initial curve $u_0 : S^1 \to \mathbb{R}^2$ and a norm φ on $\mathbb{R}^2$, which we call *anisotropy*, we show existence of a family $u(t, x)$ of curves satisfying the evolution equation

$$(1) \qquad\qquad u_t^\perp = \varphi^\circ(\nu)\kappa_\varphi \nu,$$

where $u_t^\perp$ is the normal component of the velocity u_t, ν is the Euclidean normal vector to the support of u, φ^o is the dual norm of φ, and κ_φ denotes the anisotropic curvature (see Section 2). In particular, we prove a short time existence result which holds for any anisotropy φ, under a natural regularity assumption on the initial datum.

After the seminal paper by Brakke [12], the isotropic mean curvature flow has been intensively studied in the past years (see for instance [23, 24, 34, 37] and references therein) and the long-time behavior of the flow as well as its singularities are relatively well-understood.

The first occurrence of anisotropic curvature flow appeared in [4], where the author shows a short time existence result in two dimensions, for smooth and strictly convex anisotropies. A few years later, in [5, 38] it was shown that, at the maximal existence time the evolving curve either looses a self-intersection or shrinks to a point. In particular, if the initial curve is embedded, then it stays embedded, eventually becomes convex and then shrinks to homothetically a point (see [18, 19, 27]).

In the crystalline case, that is, when the anisotropy φ is piecewise linear, equation (1) reduces to a system of ODE's and existence and uniqueness of solutions was proved in [30]. Shortly after, in [39], Stancu proved that a

convex curve remains convex and shrinks homothetically to a point, while its shape approaches the boundary of the unit ball of φ, and in [28] the authors showed that an embedded initial curve becomes eventually convex and then shrinks to a point by the result in [39] (for a classification of the asymptotic behaviour of a polygonal curve when the time approaches the shrinking time see also [1]).

On the other hand few results are available for general anisotropies. In the embedded case, an existence result analogous to the one in [4] was established in [17], by means of an implicit variational scheme introduced in [3] in order to define a global weak solution to the anisotropic mean curvature flow. The consistency of such solution with the solution constructed in [30] was shown in [2]. An important notion in [17] is that of φ-regular flow (see [8, 9] and Definition 7 below), which extends the notion of smooth evolution to general anisotropies. For such flows a uniqueness result was established in [8] (see also [16, 31]).

Let us mention that recently an existence and uniqueness result of global weak solutions to the anisotropic mean curvature flow in the embedded case, which holds in any dimensions and for general anisotropies, has been proved in [13, 14, 33] (see also [29, 32] for a similar result in the crystalline case, in two and three dimensions).

The main purpose of this paper is to extend the existence result in [17] to immersed curves. In order to do this, we regularize both the anisotropy and the initial curve, so that we can apply the result in [4] and obtain a smooth solution for short time. Then, we show that we can pass to the limit in the regularization parameter, to obtain an existence result in the general case. We point out that, since the result in [8] only applies to embedded solutions, at the moment we do not have a general uniqueness result.

The plan of the paper is the following: in Section 2, we introduce the notation which we use in the paper and define the anisotropic curvature flow for immersed curves. In Section 3, we show that a curve with bounded φ-curvature can be approximated by curves with bounded φ_ε-curvature, where φ_ε's are smooth anisotropies converging to φ as $\varepsilon \to 0$. In Section 4.1, we study the evolution of relevant geometric quantities under the flow, and prove that the curvature must blow up at the first singular time, as it happens in the isotropic case. This provides a uniform bound of the existence time of the approximate flows. Finally, in Section 4.2 we pass to the limit in the approximate flows and obtain a solution to the original anisotropic curvature motion in an interval $[0, T)$ where T depends only on the initial data.

2. Notation and preliminary definitions

We consider closed planar curves parametrized by $u : S^1 \times [0, T] \to \mathbb{R}^2$, where S^1 is identified in the obvious way with $[0, 2\pi]$. We denote by s the arc-length parameter of the curve (thus $\partial_s(\cdot) = \partial_x(\cdot)/|u_x|$), by $\tau = u_x/|u_x| = u_s = (\sin\theta, -\cos\theta)$ its unit tangent and $\nu = (\cos\theta, \sin\theta)$ its unit normal. The Euclidean scalar product in $\mathbb{R}^2$ is denoted by $\cdot$. The symbol $\perp$ stands for anti-clockwise rotation by $\pi/2$, therefore $(a, b)^\perp = (-b, a)$. Recall the classical Frenet formulas

$$\tag{2} u_{ss} = \tau_s = \vec{\kappa} = \kappa\nu, \qquad \nu_s = -\kappa\tau.$$

Moreover recall that from the expression for ν_s one infers that for the scalar curvature κ we have

$$\tag{3} \kappa = \theta_s.$$

2.1. Anisotropies

Definition 1. We call *anisotropy* a norm $\varphi : \mathbb{R}^2 \to [0, \infty)$. We say that φ is *smooth* if $\varphi \in C^2(\mathbb{R}^2 \setminus \{0\})$ and φ is *elliptic* if φ^2 is strongly convex, that is, there exists $C > 0$ such that

$$\tag{4} D^2(\varphi^2) \geqslant C\,\mathrm{Id}$$

in the distributional sense.

Definition 2. The set $W_\varphi := \{\varphi \leqslant 1\}$ is called *Wulff shape*. We say that φ is *crystalline* if W_φ is a polygon.

Definition 3. Given an anisotropy φ, we introduce the *polar norm* φ° relative to φ

$$\varphi^\circ(x) = \sup\{\xi \cdot x \,|\, \varphi(\xi) \leqslant 1\}.$$

Remark 1. Note that φ is smooth and elliptic if and only if φ° is smooth and elliptic ([17, § 2]).

The ellipticity condition implies that the Wulff shape is uniformly convex. Moreover, from (4) we infer that

$$(5) \qquad D^2\varphi(\nu)\tau \cdot \tau \geq \widetilde{C}, \qquad \widetilde{C} := \frac{C}{2\max\{\varphi(\tilde{\nu}) \mid \tilde{\nu} \in S^1\}},$$

for unit vectors ν and τ with $\nu \cdot \tau = 0$. Indeed, condition (4) implies that

$$2\varphi(\nu)D^2\varphi(\nu)\xi \cdot \xi + 2(D\varphi(\nu) \cdot \xi)^2 \geq C|\xi|^2$$

for any $\xi \in \mathbb{R}^2$. Given $\tilde{\tau}$, $|\tilde{\tau}| = 1$ with $D\varphi(\nu) \cdot \tilde{\tau} = 0$, then we can write $\tilde{\tau} = \alpha\tau + \beta\nu$ for some $\alpha, \beta \in [-1, 1]$, $\alpha \neq 0$. Then, using the homogeneity properties of φ (namely $D\varphi(\xi) \cdot \xi = \varphi(\xi)$, $D^2\varphi(\xi)\xi = 0$ for $\xi \in \mathbb{R}^2$, $\xi \neq 0$), we get

$$C \leqslant 2\varphi(\nu)D^2\varphi(\nu)\tilde{\tau} \cdot \tilde{\tau} = 2\varphi(\nu)\alpha^2 D^2\varphi(\nu)\tau \cdot \tau \leqslant 2\varphi(\nu)D^2\varphi(\nu)\tau \cdot \tau,$$

from which (5) follows. Inequality (5) will be used in Section 4.1.

2.2. φ-regular curves and the RW_φ-condition

Following [17, § 2] we give the following definition.

Definition 4. Let $R > 0$. We say that a set $E \subset \mathbb{R}^2$ with non-empty interior satisfies the inner (resp. outer) RW_φ-condition if for any $x \in \partial E$, there exists $y \in \mathbb{R}^2$ such that

$$RW_\varphi + y \subseteq \overline{E} \quad (resp.\ RW_\varphi + y \subseteq \overline{E^c}), \quad \text{and} \quad x \in \partial(RW_\varphi + y).$$

We also ask that for every $r < R$ and $x \in \mathbb{R}^2$, the set $(x + rW_\varphi) \cap E^c$ (resp. $(x + rW_\varphi) \cap E$) is connected. We say that a set E satisfies the RW_φ-condition if it satisfies both the inner and the outer RW_φ-condition.

Remark 2. As observed in [17, Remark 1] (and using the connectedness assumption), if E satisfies the RW_φ-condition for some $R > 0$, then ∂E is locally a Lipschitz graph. In particular a pathological set as depicted in [17, Fig. 1] cannot occur. Moreover if in addition φ° and φ are smooth, then ∂E is of class $C^{1,1}$ and $|\kappa_\varphi| \leqslant 1/R$ a.e. on ∂E.

Since we will be working with immersed curves we need a localized version of the RW_φ-condition.

Definition 5. Let u be a closed curve which is locally a Lipschitz graph, that is, for any $x \in S^1$ the image of u coincides with the graph of a function f_x in a neighborhood of $u(x)$. We say that u satisfies locally the inner (reps. outer) RW_φ-condition if for any $x \in S^1$, there exist $y \in \mathbb{R}^2$ and $\rho > 0$ such that $u(x) \in \partial(RW_\varphi + y)$ and

$$(RW_\varphi + y) \cap B_\rho(u(x)) \subseteq S^- \quad (resp. \ (RW_\varphi + y) \cap B_\rho(u(x)) \subseteq S^+),$$

where S^- and S^+ denote respectively the subgraph and the supergraph of the function f_x. We say that u satisfies locally the RW_φ-condition if it locally satisfies both the inner and the outer RW_φ-condition.

Note that the above definition is weaker than Definition 4, in the sense that in the crystalline case information about the curvature gets lost. For instance if $W_\varphi = \{x \in \mathbb{R}^2, \|x\|_{L^\infty} \leqslant 1\}$ is a unit square, then the set $E = \frac{1}{2}W_\varphi$ satisfies locally the RW_φ-condition for any positive R. This "inconsistency" is due to the fact that in the crystalline case the definition of curvature (see Definition 6 below) is no longer a local concept. To retain information about the curvature, the geometrical arguments used in Section 3 will use additional knowledge on the length of the flat sides.

As initial data for the anisotropic curve shortening flow we will consider immersed curves that admit a sufficiently regular anisotropic normal vector field (and hence locally satisfies a RW_φ-condition, by Lemma 1 below). More precisely, we give the following definition.

Definition 6. Let u be a closed curve in $\mathbb{R}^2$ and let $s \in [0, L]$ denote its arc-length parameter. We say that u is φ-*regular* if the image of u is locally a Lipschitz graph, and there exists a Lipschitz vector field $n : [0, L] \to \mathbb{R}^2$, usually called Cahn-Hoffman vector field (see [40]), such that

$$n(s) \in \partial\varphi^\circ(\nu(s)) \text{ a.e. in } [0, L],$$

where $\partial\varphi^\circ$ denotes the sub-differential of the anisotropy φ°.

We also define the (scalar) *anisotropic curvature* of u by

$$\kappa_\varphi := -N_s \cdot \tau,$$

where N is the Cahn-Hoffman vector field that minimizes the energy

$$\mathcal{K}(n) = \int_0^L (n_s \cdot \tau)^2 \varphi^\circ(\nu) ds.$$

Notice that since a φ-regular curve is locally a Lipschitz graph its anisotropic curvature κ_φ is well-defined almost everywhere in s. On the other hand different choices of Cahn-Hoffman vector fields are in general possible. In the following we will always write N when referring to the particular choice of Cahn-Hoffman vector field that realizes the minimum for the functional $\mathcal{K}$. Finally recall that $\kappa_\varphi = -N_s \cdot \tau = -\operatorname{div} N$ and that the Euclidean tangential divergence coincides with the divergence of an extension for N ([10, Lemma 4.5]).

Remark 3. If u is smooth and φ is smooth and elliptic, then

$$\kappa_\varphi = (D^2\varphi^\circ(\nu)\tau \cdot \tau)\kappa$$

and the anisotropic curvature is controlled (from above and below) by the classical curvature κ, thanks to (5).

If u is piecewise linear and φ is crystalline, then κ_φ is no longer a local quantity. Indeed the curvature of an edge $F \subset u(S^1)$ of length l, parallel to an edge of the Wulff shape, is given by the expression

$$\kappa_\varphi^F = \delta_F \frac{l_{W_\varphi}}{l}$$

where l_{W_φ} is the length of the corresponding edge of the Wulff shape, and $\delta_F \in \{0, \pm 1\}$ is a local convexity factor. In particular, if $0 < |\kappa_\varphi| \leqslant C$ then $l \geq l_{W_\varphi}/C$, that is, l must be long at least as the corresponding edge of the rescaled Wulff-shape $\frac{1}{C} W_\varphi$.

Below we will use the following facts.

Lemma 1. *Let u be a φ-regular curve and let*

$$F(s, d) := u(s) + dn(s), \qquad n(s, d) := n(s) \qquad s \in [0, L].$$

Then, the curve $F(\cdot, d)$ is φ-regular with Cahn-Hoffman vector field $n(\cdot, d)$ for all $|d| < \|n_s\|_\infty$. Moreover u satisfies locally the RW_φ-condition with $R = \frac{1}{\|n_s\|_\infty}$.
For the particular choice of $n(s) = N(s)$ we have in addition that $n(s, d) = N(s, d)$ and $\|N_s\|_\infty = \|\kappa_\varphi\|_\infty$.

Proof. First of all notice that n_s is a tangential vector and therefore $n_s = (n_s \cdot \tau)\tau$ (in particular for the Cahn-Hoffman vector field N we infer that $N_s = (N_s \cdot \tau)\tau = -\kappa_\varphi\tau$ and hence $\|N_s\|_\infty = \|\kappa_\varphi\|_\infty$). The fact that n_s is

always parallel to τ is a consequence of $\varphi^2(n) = 1$. Indeed, this implies that $\partial_s(\varphi^2(n(s))) = 0$. Note that thanks to [20, Th. 2.3.10], if $\partial_s n \neq 0$, then $\partial_s(\varphi^2(n(s))) = \partial(\varphi^2)(n(s)) \cdot \partial_s n(s)$, where ∂ denotes the subdifferential. Recalling that

$$\frac{\nu(s)}{\varphi^\circ(\nu(s))} \in \frac{1}{2}\partial(\varphi^2)(n(s)),$$

we then get

$$\nu \cdot \partial_s n = 0,$$

this equality holding also when $\partial_s n = 0$.

The thesis now follows by the argument in [10, Lemmas 3.3 and 3.4]. Indeed, we first prove that F is locally bi-Lipschitz from $[0, L] \times (-1/C, 1/C)$ onto its image in $\mathbb{R}^2$, where $C = \|n_s\|_\infty$. To this aim, we follow [10, Lemma 3.3] and apply the inverse function theorem in a Lipschitz framework (cf. [20, Th. 7.1.1]). We have to show that, if F is differentiable at (s_n, d_n), $(s_n, d_n) \to (s, d)$, and there is a limit of $DF(s_n, d_n)$, then the limit is nonsingular. To see this, we compute

$$DF(s_n, d_n) = (\tau(s_n) + d_n \partial_s n(s_n), n(s_n))$$
$$= ([1 + d_n(n_s(s_n) \cdot \tau(s_n))]\tau(s_n), n(s_n)).$$

Therefore

$$|\det DF(s_n, d_n)| = |1 + d_n(n_s(s_n) \cdot \tau(s_n))||n(s_n) \cdot \tau^\perp(s_n)|$$
$$= \varphi^\circ(\nu(s_n))|1 + d_n(n_s(s_n) \cdot \tau(s_n))|.$$

This shows that as long as $|d|$ is smaller than $1/C$, the determinant cannot degenerate. Moreover, since $n(s_n) \to n(s)$, we have that convex combinations of the above limits are still matrices with full rank; hence the implicit function theorem applies. It follows that F is locally bi-Lipschitz and $F(\cdot, d)$ is φ-regular for every $|d| < 1/C$.

Notice also that $\varphi(F(s, d) - u(s)) = d$. To conclude that u satisfies locally the RW_φ-condition with $R = 1/C$, it is enough to observe that $\varphi(F(s, t) - u(s')) \geq d$ for s' in a neighborhood of s. This is proven [10, Lemma 3.4] under the assumption that u is an embedding, but it also applies to our case as we can replace u with a curve $\hat{u}$ such that $\hat{u}$ is a φ-regular embedding that coincides with u in a neighborhood of s, with $\|\hat{n}_s\|_\infty \leqslant C$.

Finally observe that if $n(s) = N(s)$ then $N(\cdot, d)$ is by construction the Cahn-Hoffman vector field that minimizes $\mathcal{K}$ as in Definition 6. (In principle this must be only verified on flat segments of u where the curvature is not

zero: here $N(\cdot)$ linearly interpolates between the values of N at the end of the segment (as discussed for instance [6, Ex. 7.1]) and the same holds for $N(\cdot, d)$ by definition.) □

Remark 4. Notice that, letting $(s(x), d(x))$ be a local inverse of F in a neighborhood U of $u(\bar{s})$, and letting $\widetilde{n}(x) = n(s(x))$, for $x \in U$ there holds

$$(6) \qquad \widetilde{n}(x) \in \partial\varphi^\circ(\nabla d(x)), \qquad \operatorname{div}\widetilde{n}(x) = \operatorname{div} n(s(x)) + O(d(x)).$$

Notice that d is a local φ-distance function and s is a local projection function on the support of the curve u in the neighborhood U, thus $\nabla d(x) = \frac{\nu(s(x))}{\varphi^\circ(\nu(s(x)))}$ and $\varphi^\circ(\nabla d(x)) = 1$.

In the case of smooth and elliptic anisotropies a stronger statement holds.

Lemma 2. *Let u be a closed curve which is locally a Lipschitz graph. Let φ (and φ°) be smooth and elliptic. The curve u is φ-regular if and only if it locally satisfies the RW_φ-condition for some $R > 0$. Moreover we have $R = \frac{1}{\|N_s\|_\infty}$.*

Proof. One part of the statement follows from Lemma 1. On the other hand, suppose now that u satisfies locally the RW_φ-condition. Then (thanks to the regularity and ellipticity of φ), the curve is $\mathcal{C}^{1,1}$ which means that the Euclidian normal vector ν is Lipschitz with respect to the arc-length s. Then, since φ° is regular, the vector field

$$N := D\varphi^\circ(\nu)$$

is also Lipschitz with respect to s and is therefore the Cahn-Hoffman vector field. □

2.3. φ-regular flows

Definition 7. We say that a family of curves $u : S^1 \times [0, T] \to \mathbb{R}^2$ is a φ-regular flow if

- u is Lipschitz continuous,

- $u(\cdot, t)$ is φ-regular for every $t \in [0, T]$, and there exists a family of Cahn-Hoffman vector fields $n(\cdot, t)$ for $u(\cdot, t)$ such that $\|n_s(\cdot, t)\|_\infty \leq C$ for every $t \in [0, T]$,

• for almost every (x, t) there holds

$$(7) \qquad\qquad u_t \cdot \nu = -(\operatorname{div} n) \, n \cdot \nu \,.$$

It is easy to check that the Wulff shape shrinks self-similarly under (7) (see [27, 39]).

Remark 5. Given a φ-regular flow u we can define the functions $F(s, d, t)$, $n(s, d, t)$, as in Lemma 1. We let $(s(x, t), d(x, t))$ be the inverse of F in a (space-time) neighborhood V of $u(\bar{s}, \bar{t})$, and we let $\widetilde{n}(x, t) = n(s(x, t), t) \in \partial \varphi^\circ(\nabla d)$. By (6) for $(x, t) \in V$ we get

$$\operatorname{div} \widetilde{n}(x, t) = \operatorname{div} n(s(x, t), t) + O(d(x, t)) \,,$$

moreover, using the equality $x = u(s(x, t), t) + d(x, t) n(s(x, t), t)$ and assuming that the curve $u(S^1, t)$ moves in direction n, one can show

$$d_t(x, t) = -u_t(s(x, t), t) \cdot \frac{\nu(s(x, t), t)}{\varphi^\circ(\nu(s(x, t), t))} \,.$$

As a consequence, in analogy to [17, Definition 2]), (7) is equivalent to

$$(8) \qquad\qquad d_t = \operatorname{div} \widetilde{n} + O(d) \qquad a.e. \, in \, V.$$

When u is smooth and the anisotropy φ is smooth and elliptic, the classical formulation of the anisotropic curvature flow (see [4]) is given by the equation

$$(9) \qquad\qquad u_t = \varphi^\circ(\nu) \kappa_\varphi \nu.$$

Notice that a solution u of (9) also satisfies (7). By setting

$$(10) \qquad\qquad \phi(\theta) := \varphi^\circ(\nu) = \varphi^\circ(\cos \theta, \sin \theta),$$

a straightforward calculation gives

$$(11) \qquad\qquad \phi(\theta) + \phi''(\theta) = D^2 \varphi^\circ(\nu) \tau \cdot \tau,$$

so that we can rewrite the flow (9) as

$$(12) \qquad\qquad u_t = \phi(\theta)(\phi(\theta) + \phi''(\theta)) \, \kappa \nu = \psi(\theta) \kappa \nu,$$

where κ is the Euclidean curvature and

$$(13) \qquad \psi(\theta) := \phi(\theta)(\phi(\theta) + \phi''(\theta)).$$

Note that by (5), the ellipticity of φ implies that $\psi \geq \tilde{C} > 0$.

3. Approximation of φ-regular curves

In [7, Lemma 3] (see also [17, Remark 3]) it is shown that given an anisotropy φ, it is possible to construct a sequence of anisotropies φ_ε, such that φ_ε and $\varphi_\varepsilon^\circ$ are elliptic and belong to $C^\infty(\mathbb{R}^2 \setminus \{0\})$, and such that φ_ε converges uniformly to φ on compact subsets of $\mathbb{R}^2$ as $\varepsilon \to 0$.

In this section we approximate a φ-regular curve u by a family u_ε of smooth φ_ε-regular curves, where $\varphi_\varepsilon \geqslant \varphi$ is a family of $C^\infty(\mathbb{R}^2 \setminus \{0\})$ elliptic anisotropies approximating φ.

Lemma 3. *Let u be a φ-regular curve with $|\kappa_\varphi| \leqslant C$. Then, for a given sequence $\varphi_\varepsilon \to \varphi$ with $\varphi_\varepsilon \in C^\infty(\mathbb{R}^2 \setminus \{0\})$, elliptic and $\varphi \leqslant \varphi_\varepsilon$ (which is equivalent to $W_{\varphi_\varepsilon} \subset W_\varphi$), there exists a sequence of curves u_ε of class C^∞, such that $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$ and $|\kappa_{\varphi_\varepsilon}| \leqslant C'$ uniformly in ε, where $C' > C$ is arbitrarily close to C.*

Proof. The main idea is to localize [17, Lemma 1], where the result is proved for an embedded curve u which is the boundary of a set E. The argument employed in [17, Lemma 1] is to take the union of all the Wulff shapes of type $x + RW_{\varphi_\varepsilon}$, $R = 1/C$, contained in the set E (think for instance of the situation where E is a "thick" $\sqcup$-shaped set and φ is the L^∞-norm). The boundary of this new set $\tilde{E}_\varepsilon$ essentially regularizes the curve u ("from inside") attaching arcs of Wulff shapes RW_{φ_ε} where ∂E has corners. Repeating the same operation on the complementary $\tilde{E}_\varepsilon^c$ one smoothes out also the remaining corners. We shall adapt this method to the case of a general immersed curve, by exploiting the fact that the curve is locally a Lipschitz graph. Roughly speaking, the idea is to consider portions $u([x - \delta, x + \delta])$ of the curve u, extend them with straight lines in order to obtain a global graph, and then perform the construction mentioned above.

Step 1: covering $u(S^1)$ with graphs. Note that the curve u is locally a Lipschitz graph. We cover $u(S^1)$ by $u([x_i - \alpha_i, x_i + \beta_i])$ such that

(i) $u(S^1) = \cup_{i=1}^N u(I_i)$ where $I_i = (x_i - \alpha_i, x_i + \beta_i)$,

(ii) $u|_{[x_i - \alpha_i, x_i + \beta_i]}$ is a graph and u is differentiable at $x_i - \alpha_i$ and $x_i + \beta_i$,

(iii) The intersection of the images of two consecutive graphs $u(I_i) \cap u(I_{i+1})$ contains a neighborhood of a point $y_i = u(w_i)$ such that u and the Cahn-Hoffman vector field N are differentiable at w_i and $\partial_s N(w_i) \neq 0$. In addition, if y_i belongs to (the closure of) a straight line, we require that this line is entirely included in $u(I_i) \cap u(I_{i+1})$.

This is possible since N is Lipschitz and therefore differentiable almost everywhere and since every region of zero anisotropic curvature can be included in a graph. Indeed, in such regions, the Cahn-Hoffman vector field N is constant. Therefore, since we have $\frac{\nu(x)}{\varphi^\circ(\nu(x))} \in \partial\varphi(N(x))$ (where $\nu(x)$ denotes the locally oriented Euclidean normal to u at x) and since at a point N_{const} the subdifferential $\partial\varphi(N_{const})$ comprises at most a segment of finite length (recall that the duality map maps a unit ball into its dual ball), the normal ν is forced to remain in a non-flat cone and the region to remain a graph.

Step 2: smoothing construction for graphs. We prolongate each $u(I_i)$ by attaching two half lines with slope $\partial_x u(x_i - \alpha_i)$ and $\partial_x u(x_i + \beta_i)$. We then obtain the graph Γ_f of a Lipschitz function f with $u(I_i) \subset \Gamma_f$. From our assumptions on u and Lemma 1, it follows that Γ_f satisfies the RW_φ-condition (indeed, if this were not the case we get a contradiction by using [11, Lemma 8.2]). Therefore also the $R'W_\varphi$-condition is satisfied for any $0 < R' < R$. In the next step, we will choose a suitable $R' < R$ needed to glue together our local constructions. We now apply the result in [17, Lemma 1] to the subgraph S_f of f, with R' instead of R, and obtain a regularized set

$$S_{x_i,\varepsilon} := \bigcup \left\{ (p + R'W_{\varphi_\varepsilon}) \mid (p + R'W_{\varphi_\varepsilon}) \subset S_f \right\}.$$

Moreover by construction, $S_{x_i,\varepsilon}$ satisfies the inner $R'W_{\varphi_\varepsilon}$-condition and the outer $R'W_\varphi$-condition.

Recalling that $\varphi_\varepsilon \geq \varphi$, we have that the set $\partial S_{x_i,\varepsilon} \setminus \Gamma_f$ is a union of arcs of Wulff shape $R'W_{\varphi_\varepsilon}$. We now define the curve u_ε^- by replacing $u(I_i) \setminus \partial S_{x_i,\varepsilon}$ by these arcs of Wulff shape. We show next that the construction is compatible in some subregion of $u(I_i) \cap u(I_{i+1})$ so that the approximation curves u_ε^- of two adjacent local graphs can be well connected/glued across this subregion.

Step 3: connecting the approximate graphs. Let us now prove the main claim of Lemma 3, that is the constructions of $S_{x_i,\varepsilon}$ and $S_{x_{i+1},\varepsilon}$ are compatible: they can be connected in a canonical way. To do so, we will take advantage of the point y_i.

First, notice that the points $y \in u(S^1)$ where u and N are differentiable and $\partial_s N \neq 0$ are of two kinds. Either y belongs to a flat edge of $u(S^1)$
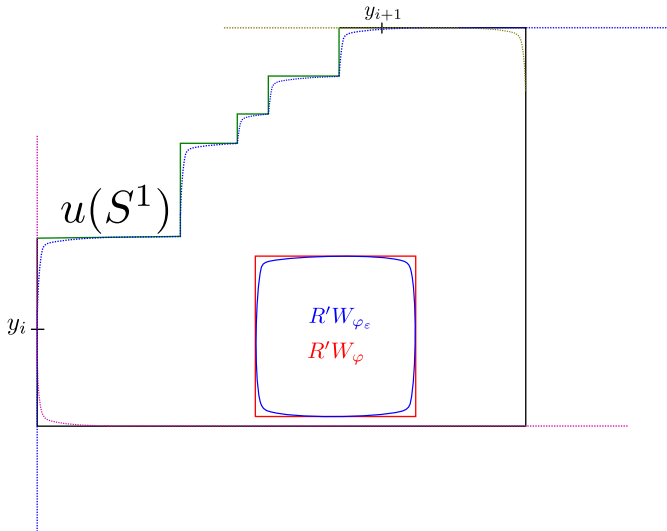
Figure 1: Typical situation when y_i belongs to a flat edge of $u(S^1)$.

around which u is locally convex, which is parallel to an edge of the Wulff shape RW_φ and whose length is bigger than the corresponding edge of RW_φ (recall for instance Remark 3), or u is not flat around y and the Wulff shape $p + RW_\varphi \subset S_f$ which touches Γ_f at y from inside is not flat either around y.

We now prove that for ε small enough, $S_{x_i,\varepsilon}$ and $S_{x_{i+1},\varepsilon}$ coincide around y_i, and therefore can be connected. We first deal with the case y_i belongs to a flat edge ℓ of $u(S^1)$. Thanks to our assumptions $u(I_i) \cap u(I_{i+1})$ contains the whole ℓ. The smoothing procedure leaves a part of this line unchanged (because of the reduction of R into R') when building $S_{x_i,\varepsilon}$ as well as $S_{x_{i+1},\varepsilon}$, and the two curves can therefore be connected (see Figure 1).

Now, let us assume that y_i belongs to a non-flat part of $u(S^1)$ (see Figure 2). Then, the curve u at y_i has a contact with a Wulff shape $p_i + RW_\varphi$, which cannot be flat around y_i either. (In fact $u(S^1)$ and $\partial(p_i + RW_\varphi)$ might even coincide in a neighborhood of y_i; however the Wulff shape $\partial(p_i + RW_\varphi)$ is locally strictly convex at y_i.) As a result, the contact between $u(S^1)$ and the Wulff shape $p_i + R'W_\varphi$ takes place at y_i only. Let $\eta_i > 0$ be such that $(B_{\eta_i}(y_i) \cap u(I_i)) \subset u(I_i) \cap u(I_{i+1})$. Since the approximate Wulff shape $R'W_{\varphi_\varepsilon}$ Hausdorff converges to $R'W_\varphi$, it must touch $u(S^1)$ inside the ball of radius η_i, provided ε is small enough (depending on η_i, but these are in finite
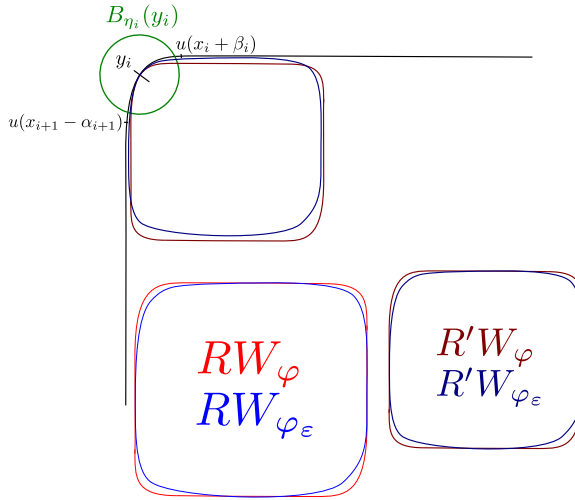
Figure 2: Typical situation when y_i belongs to a non-flat part of $u(S^1)$.

number anyway). Hence, every graph which coincides with $u(S^1)$ around y_i is smoothed similarly on $B_{\eta_i}(y_i)$. We can therefore connect $S_{x_i,\varepsilon}$ and $S_{x_{i+1},\varepsilon}$.

Eventually, we can connect all these pieces of graphs and obtain a curve u_ε^- which is locally a Lipschitz graph, and satisfies locally the inner $R'W_{\varphi_\varepsilon}$-condition and the outer $R'W_\varphi$-condition, where outer (resp. inner) means in the direction of the local orientation given by the vector ν (resp. $-\nu$). Notice also that, by construction, the Hausdorff distance between $u(I_i)$ and $u_\varepsilon^-(I_i)$ is bounded above by the Hausdorff distance between $R'\partial W_\varphi$ and $R'\partial W_{\varphi_\varepsilon}$, hence in particular it goes to zero as ε goes to zero.

On the other hand the "added arcs" $u(I_i) \setminus S_{x_i,\varepsilon}$ which constitute parts of u_ε^-, locally satisfies the $R'W_{\varphi_\varepsilon}$ condition, which is equivalent to say that these arcs are φ_ε-regular with $|\kappa_{\varphi_\varepsilon}| \leqslant C'$, $C' = 1/R'$ (see Lemma 2). This means that the added arcs are of class $C^{1,1}$ (and must admit a good local parametrization) and by construction concave (with respect to the local orientation). In particular their length is controlled above by the length of the replaced piece of curve and below by the distance of the endpoints of the arcs, and the length of u_ε^- tends to the length of u as $\varepsilon \to 0$.

Step 4: building an actual curve. We have explained so far how to construct geometrically the approximation u_ε^-. However we want some control of its parametrization too, since we claim uniform convergence of the approximate

curves u_ε towards u. This can be done by exploiting Dini's theorem on each local graph $u(\bar{I}_i)$ and using the fact that we have a finite number of them.

We proceed similarly with u_ε^- instead of u and by smoothing on the other side of the curve to build our approximation u_ε, i.e we perform the analogous construction on the curve u_ε^- (which has the same orientation of u) by considering the envelope of the Wulff shapes $R'W_{\varphi_\varepsilon}$ that are locally *above u_ε^-*.

Finally, by a convolution argument we can assume that the curves u_ε are of class C^∞ and satisfy $|\kappa_{\varphi_\varepsilon}| \leqslant C'$, with $C' > C$ arbitrarily close to C. $\qquad\square$

Remark 6. Note that by construction the lengths of the curves $u_\varepsilon(S^1)$ are uniformly bounded from above and below.

4. Existence of φ-regular flows

4.1. Smooth and elliptic anisotropies

Goal of this section is to show that if we have a bound of type $|\kappa_{\varphi_\varepsilon}(0)| \leqslant C'$ for initial smooth curves $u_\varepsilon(0)$, then we can find a time interval (independent of ε) where the anisotropic curve shortening flows (9) associated to the anisotropies φ_ε and initial data $u_\varepsilon(0)$ exist.

In this section, for simplicity of notation we drop the index ε and write φ instead of φ_ε. In other words we assume $\varphi \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ to be an elliptic anisotropy.

First of all we collect some important properties of the anisotropic curvature flow (9) (or equivalently (12)). For analogous results in the isotropic case see for instance [15].

In what follows it is important to keep in mind that (5) holds, hence $\psi \geq \tilde{C}$ (recall (13)).

Theorem 1. *[4, Theorem 3.1] There exists a smooth solution to* (9) *on* $[0, T)$.

We start by deriving the evolution laws of relevant geometric quantities.

Lemma 4. *The following holds*

$$\partial_t \partial_s(\cdot) = \partial_s \partial_t(\cdot) + \psi(\theta)\kappa^2 \partial_s(\cdot)$$
$$\tau_t = (\psi(\theta)\kappa)_s \nu$$
$$\nu_t = -(\psi(\theta)\kappa)_s \tau$$
(14)
$$\kappa_t = (\psi(\theta)\kappa)_{ss} + \psi(\theta)\kappa^3$$
$$\theta_t = (\psi(\theta)\kappa)_s.$$

Proof. Let $f : S^1 \to \mathbb{R}^2$, $f = f(x,t)$. Then, we compute (note that the derivatives in x and t can commute)

$$\partial_t \partial_s f = \partial_t \left(\frac{\partial_x f}{|u_x|} \right) = -\frac{\langle u_{tx}, u_x \rangle}{|u_x|^3} \partial_x f + \frac{\partial_t \partial_x f}{|u_x|}$$
$$= -\langle \partial_s(\psi(\theta)\kappa\nu), \tau \rangle \partial_s f + \frac{\partial_x}{|u_x|} \partial_t f = \psi(\theta)\kappa^2 \partial_s f + \partial_s \partial_t f.$$

Applying this formula to the other quantities, we get

$$\tau_t = \partial_t(\partial_s u) = \partial_s(\partial_t u) + \psi(\theta)\kappa^2 \partial_s u = \partial_s(\psi(\theta)\kappa\nu) + \psi(\theta)\kappa^2 \tau = (\psi(\theta)\kappa)_s \nu.$$

We prove the third formula similarly.
Writing $\kappa = \langle \tau_s, \nu \rangle$, we get

$$\kappa_t = \langle \partial_t \tau_s, \nu \rangle + \langle \tau_s, \partial_t \nu \rangle.$$

Since τ_s is proportional to ν and ν_t is proportional to τ, the second term vanishes. We obtain

$$\kappa_t = \langle \partial_s((\psi(\theta)\kappa)_s \nu) + \psi(\theta)\kappa^2 \tau_s, \nu \rangle = (\psi(\theta)\kappa)_{ss} + \psi(\theta)\kappa^3.$$

Finally, recalling that $\nu = (\cos\theta, \sin\theta)$, one has $\nu_t = -\theta_t(\sin\theta, -\cos\theta)$, which implies the last formula. $\square$

We now show that we can find a time interval where the anisotropic curvature does not blow up, provided we know that its values at the initial time are bounded by a constant C'. This result holds for *any* smooth elliptic anisotropy φ whose associated anisotropic curvature κ_φ is bounded by C' on the initial curve.

Proposition 1. *Suppose $\|\kappa_\varphi(0)\|_{L^\infty(S^1)} \leqslant C'$. Then the anisotropic curvature κ_φ remains bounded on any time interval $[0,T] \subset [0, \frac{1}{2C'^2})$ (provided the*

flow exists on this time interval). Precisely we have

$$\|\kappa_\varphi(t)\|_{L^\infty(S^1)} \leqslant \frac{C'}{\sqrt{1 - 2T\|\kappa_\varphi(0)\|_{L^\infty(S^1)}^2}}.$$

Proof. First of all recall that due to (11) we have $\kappa_\varphi = \kappa(\phi + \phi'')$. For simplicity we denote by h the quantity $\phi + \phi''$. By Lemma 4 we infer that

$$
\begin{aligned}
\partial_t \kappa_\varphi = \partial_t(\kappa h) &= h[(\partial_{ss}(\psi(\theta)\kappa) + \kappa^3\psi] + \kappa h' \partial_s(\psi(\theta)\kappa) \\
&= h[\partial_s(\kappa^2\psi' + \kappa_s\psi) + \kappa^3\psi] + \kappa h'(\kappa^2\psi' + \kappa_s\psi) \\
&= h(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \kappa^3\psi) + \kappa^3 h'\psi' + \kappa\kappa_s h'\psi
\end{aligned}
$$

and

$$\partial_{ss}\kappa_\varphi = \partial_{ss}(\kappa h) = \partial_s(\kappa_s h + \kappa^2 h') = \kappa_{ss}h + 3\kappa_s\kappa h' + \kappa^3 h''.$$

Noting that

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\varphi^2}{2} = \kappa_\varphi\partial_t\kappa_\varphi - \psi\kappa_\varphi\partial_{ss}\kappa_\varphi - \psi(\partial_s\kappa_\varphi)^2$$

we get

$$
\begin{aligned}
(\partial_t - \psi\partial_{ss})\frac{\kappa_\varphi^2}{2} &= -\psi\kappa_\varphi\left[\kappa_{ss}h + 3\kappa_s\kappa h' + \kappa^3 h''\right] - \psi(\partial_s\kappa_\varphi)^2 \\
&\quad + \kappa_\varphi\left[h(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \kappa^3\psi) + \kappa^3 h'\psi' + \kappa\kappa_s h'\psi\right] \\
&= \kappa_s\kappa_\varphi\kappa(3h\psi' - 2h'\psi) + \kappa_\varphi\kappa^3(h\psi + h\psi'' + h'\psi' - \psi h'') \\
&\quad - \psi(\partial_s\kappa_\varphi)^2.
\end{aligned}
$$

Now, note that since

$$\psi' = (h\phi)' = h'\phi + h\phi',$$

$$\psi'' = h''\phi + 2h'\phi' + h\phi'',$$

we obtain

$$3h\psi' - 2h'\psi = 3h^2\phi' + hh'\phi$$

and

$$
\begin{aligned}
h\psi + h\psi'' + h'\psi' - \psi h'' &= h^2\phi + hh''\phi + 2hh'\phi' + h^2\phi'' \\
&\quad + (h')^2\phi + hh'\phi' - hh''\phi \\
&= h^3 + 3hh'\phi' + (h')^2\phi.
\end{aligned}
$$

As a result,

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\varphi^2}{2} = \kappa_s\kappa_\varphi\kappa(3h^2\phi' + hh'\phi)$$
$$+ \kappa_\varphi\kappa^3(h^3 + 3hh'\phi' + (h')^2\phi) - \psi(\partial_s\kappa_\varphi)^2.$$

Since

$$\partial_s\frac{\kappa_\varphi^2}{2} = \kappa_\varphi(\kappa_s h + \kappa^2 h'),$$

we can write

$$\kappa_s\kappa_\varphi\kappa(3h^2\phi' + hh'\phi) + \kappa^3\kappa_\varphi(3hh'\phi' + (h')^2\phi) = (3\kappa h\phi' + h'\kappa\phi)\partial_s\frac{\kappa_\varphi^2}{2}$$

which yields

$$(\partial_t - \psi\partial_{ss})\frac{\kappa_\varphi^2}{2} \leqslant (3\kappa h\phi' + h'\kappa\phi)\partial_s\frac{\kappa_\varphi^2}{2} + \kappa_\varphi^4.$$

At a maximal point for κ_φ^2, the quantity $\partial_s\kappa_\varphi^2$ vanishes and $\partial_{ss}\kappa_\varphi^2$ is nonpositive. As a result, letting $g := \max_{S^1} \kappa_\varphi^2$ and by [37, Lemma 2.1.3], we have $\frac{d}{dt}g \leqslant 2g^2$, which implies

$$(15) \qquad\qquad g(t) \leqslant \frac{g(0)}{1 - 2tg(0)}$$

as long as $1 - 2tg(0) > 0$. Since by assumption $g(0) \leqslant (C')^2$, the anisotropic curvature κ_φ cannot blow up on a time interval $[0, T] \subset [0, \frac{1}{2C'^2})$. $\square$

The rest of this section is devoted to showing that if the maximal time of existence of the flow is finite, then both the isotropic and anisotropic curvatures have to blow-up.

For the evolution of the derivatives of the curvature we have

Lemma 5. *For $j \in \mathbb{N}$, $j \geq 1$ we have*

$$(16) \ \ \partial_t(\partial_s^j\kappa) = \psi(\theta)(\partial_s^j\kappa)_{ss} + (j+3)\psi'(\theta)\kappa(\partial_s^j\kappa)_s + P_j(\psi, \psi', \psi'', \kappa, \kappa_s)\partial_s^j\kappa$$
$$+ Q_j(\psi, \psi', \ldots, \psi^{(j+2)}, \kappa, \ldots, \partial_s^{j-1}\kappa)$$

where $P_j(\cdot)$ and $Q_j(\cdot)$ are polynomials in the given variables and $\psi^{(m)} = \partial_\theta^m\psi$.

Proof. The proof is by induction on j and relies on Lemma 4 and the fact that $\psi(\theta)_s = \psi'(\theta)\kappa$. For $j = 1$ we have that

$$
\begin{aligned}
(17) \qquad \partial_t \partial_s \kappa &= \partial_s \partial_t \kappa + \psi(\theta)\kappa^2 \partial_s \kappa \\
&= \partial_s((\psi(\theta)\kappa)_{ss} + \psi(\theta)\kappa^3) + \psi(\theta)\kappa^2 \partial_s \kappa \\
&= \psi(\theta)\kappa_{sss} + 4\psi'(\theta)\kappa\kappa_{ss} + \psi^{(3)}(\theta)\kappa^4 + \psi'(\theta)\kappa^4 \\
&\quad + (6\psi''(\theta)\kappa^2 + 4\psi(\theta)\kappa^2 + 3\psi'(\theta)\kappa_s)\kappa_s.
\end{aligned}
$$

The induction step follows with similar arguments. $\qquad\square$

Lemma 6. *Let $w := \log|u_x|$. There holds*

$$
(18) \qquad\qquad\qquad w_t = -\psi(\theta)k^2.
$$

In particular $\|u_x(t)\|_\infty \leqslant \|u_x(0)\|_\infty$.

Proof. A direct computation gives

$$
w_t = \tau \cdot \partial_s u_t = \tau \cdot \psi(\theta)\kappa\nu_s = -\psi(\theta)k^2.
$$

The second statement follows from $\psi \geq 0$. $\qquad\square$

Note that if we have a bound on the curvature, then from $-w_t \leqslant C(\|\kappa\|_\infty, \|\psi\|_\infty)$ we also infer that $|u_x(t)| \geq (\inf_{S^1}|u_x(0)|)e^{-C(\|\kappa\|_\infty, \|\psi\|_\infty)t}$.

Lemma 7. *Assume that (9) has a smooth solution on $[0, \bar{t}]$, with $\bar{t} > 0$. Then*

$$
\max_{S^1 \times [0,\bar{t}]} |\partial_s^j \kappa| \leqslant C_j, \qquad (j \in \mathbb{N}),
$$

where C_j depends on $\bar{t}$, $\tilde{C}$ (as in (5)), $\|\psi^{(l)}\|_\infty$ for $l = 0, \ldots, j+2$, C_l for $l \leqslant j-1$, $\|\partial_s^j \kappa(0)\|_\infty$, and $\max_{S^1 \times [0,\bar{t}]} |\kappa|$.

Proof. The proof goes by induction on j. Let $v = \partial_s^j \kappa$. From Lemma 5 we know that

$$
\begin{aligned}
v_t &= \psi(\theta)v_{ss} + (j+3)\psi'(\theta)\kappa v_s + P_j(\psi, \psi', \psi'', \kappa, \kappa_s)v \\
&\quad + Q_j(\psi, \psi', \ldots, \psi^{(j+2)}, \kappa, \ldots, \partial_s^{j-1}\kappa)
\end{aligned}
$$

(where recall that $\partial_s = \frac{1}{|u_x|}\partial_x$ and $v_{ss} = \frac{1}{|u_x|^2}v_{xx} - \frac{v_x}{|u_x|^2}\tau \cdot \frac{u_{xx}}{|u_x|^2}$). Together with $\psi(\theta) \geq \tilde{C} > 0$ (the anisotropy is elliptic) we obtain a parabolic quasi-linear equation for which we can apply arguments given in [35, Thm. 9.5].

More precisely let us look at the case where $j \geq 2$. Without loss of generality we may assume that there exists a point in $S^1 \times (0, \bar{t}]$ where v attains a positive maximum (if not argue with $-v$). The map v satisfies an equation of type

$$0 = -v_t + \psi(\theta)v_{ss} + a(s, v, v_s)$$

where, in view of the induction hypothesis, we have that $a(s, v, 0) \leq c(|v| + 1) \leq \alpha|v| + \frac{\beta}{|v|}$ with positive constants α and β depending on $\|\psi^{(l)}\|_\infty$ for $l = 0, \ldots, j+2$, C_l for $l \leq j-1$, and $\max_{S^1 \times [0, \bar{t}]} |\kappa|$. Set $\lambda = -\alpha - 1$. Suppose $P = (x, t) \in S^1 \times (0, \bar{t}]$ is a point in which $m := e^{\lambda t}v$ attains a positive maximum. Then $m_t = \lambda e^{\lambda t}v + e^{\lambda t}v_t$ and at P we have $m_x = 0$, $m_t \geq 0$ (thus $m_s = v_s = 0$), $m_{xx} \leq 0$ (thus $m_{ss} \leq 0$, $v_{ss} \leq 0$). At P (where $v > 0$) we have

$$0 = -v_t + \psi(\theta)v_{ss} + a(s, v, v_s)$$
$$\leq -v_t + \alpha|v| + \frac{\beta}{|v|} = (\lambda + \alpha)v + \frac{\beta}{v} = -v + \frac{\beta}{v}.$$

Thus $v(P) \leq \sqrt{\beta}$ and we infer that

$$\sup_{S^1 \times [0, \bar{t}]} v \leq e^{(\alpha+1)\bar{t}} \left(\sqrt{\beta} + \sup_{S^1} v^+(0) \right).$$

Arguing in the same way with $-v$ instead of v, we get a bound also on v^- and therefore on $|v|$.

For the case $j = 1$, the conclusion of Lemma 5 does not let us apply the same maximum principle (see the quadratic term appearing in (17)). To cope with this difficulty, we will study the quantity $v := \psi(\theta)^{\frac{3}{2}}\kappa$. To this aim, let us compute using Lemma 4 (for simplicity we drop the dependence on θ in the formulas for ψ and its derivatives)

$$\partial_t(\psi^{3/2}(\theta)\kappa) = \frac{3}{2}(\kappa^2\psi' + \kappa_s\psi)\psi'\psi^{1/2}\kappa$$
$$+ \psi^{3/2}(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \psi\kappa^3)$$

whereas

(19)
$$\partial_s(\psi^{3/2}\kappa) = \frac{3}{2}\kappa^2\psi'\psi^{1/2} + \psi^{3/2}\kappa_s.$$

Then, we have

$$\partial_{ss}(\psi^{3/2}\kappa) = 3\kappa\kappa_s\psi'\psi^{1/2} + \frac{3}{2}\kappa^3\psi''\psi^{1/2} + \frac{3}{4}\kappa^3(\psi')^2\psi^{-1/2}$$
$$+ \frac{3}{2}\kappa\psi'\psi^{1/2}\kappa_s + \psi^{3/2}\kappa_{ss}$$
$$= \frac{9}{2}\kappa\kappa_s\psi'\psi^{1/2} + \frac{3}{2}\kappa^3\psi''\psi^{1/2} + \frac{3}{4}\kappa^3(\psi')^2\psi^{-1/2} + \psi^{3/2}\kappa_{ss}$$

and

$$\partial_{sss}(\psi^{3/2}\kappa) = \frac{9}{2}\kappa_s^2\psi'\psi^{1/2} + \frac{9}{2}\kappa\kappa_{ss}\psi'\psi^{1/2} + \frac{9}{2}\kappa^2\kappa_s\psi''\psi^{1/2} + \frac{9}{4}\kappa^2\kappa_s(\psi')^2\psi^{-1/2}$$
$$+ \frac{9}{2}\kappa^2\kappa_s\psi''\psi^{1/2} + \frac{3}{2}\kappa^4\psi'''\psi^{1/2} + \frac{3}{4}\kappa^4\psi''\psi'\psi^{-1/2}$$
$$+ \frac{9}{4}\kappa^2\kappa_s(\psi')^2\psi^{-1/2} + \frac{3}{2}\kappa^4\psi'\psi''\psi^{-1/2} - \frac{3}{8}\kappa^4(\psi')^3\psi^{-3/2}$$
$$+ \frac{3}{2}\kappa\psi'\psi^{1/2}\kappa_{ss} + \psi^{3/2}k_{sss}$$
$$= \frac{9}{2}\kappa_s^2\psi'\psi^{1/2} + 6\kappa\kappa_{ss}\psi'\psi^{1/2} + 9\kappa^2\kappa_s\psi''\psi^{1/2} + \frac{9}{2}\kappa^2\kappa_s(\psi')^2\psi^{-1/2}$$
$$+ \frac{3}{2}\kappa^4\psi'''\psi^{1/2} + \frac{9}{4}\kappa^4\psi''\psi'\psi^{-1/2} - \frac{3}{8}\kappa^4(\psi')^3\psi^{-3/2} + \psi^{3/2}k_{sss}.$$

Recall Lemma 4 and the calculations performed in Lemma 5, we also compute

$$\partial_t(\partial_s(\psi^{3/2}\kappa)) = 3\kappa\kappa_t\psi'\psi^{1/2} + \frac{3}{2}\kappa^2(\psi'\kappa^2 + \kappa_s\psi)\psi''\psi^{1/2}$$
$$+ \frac{3}{4}\kappa^2(\psi')^2(\psi'\kappa^2 + \kappa_s\psi)\psi^{-1/2}$$
$$+ \frac{3}{2}(\psi'\kappa^2 + \kappa_s\psi)\psi'\psi^{1/2}\kappa_s + \psi^{3/2}\partial_t\kappa_s$$
$$= 3\kappa\psi'\psi^{1/2}(3\kappa\kappa_s\psi' + \kappa^3\psi'' + \kappa_{ss}\psi + \psi\kappa^3)$$
$$+ \frac{3}{2}\kappa^2(\psi'\kappa^2 + \kappa_s\psi)\psi''\psi^{1/2} + \frac{3}{4}\kappa^2(\psi')^2(\psi'\kappa^2 + \kappa_s\psi)\psi^{-1/2}$$
$$+ \frac{3}{2}(\psi'\kappa^2 + \kappa_s\psi)\psi'\psi^{1/2}\kappa_s + \psi^{5/2}\kappa_{sss} + 4\psi^{\frac{3}{2}}\psi'\kappa\kappa_{ss}$$
$$+ \psi^{\frac{3}{2}}\psi^{(3)}\kappa^4 + \psi^{\frac{3}{2}}\psi'\kappa^4 + \psi^{\frac{3}{2}}(6\psi''\kappa^2 + 4\psi\kappa^2 + 3\psi'\kappa_s)\kappa_s$$
$$= 9\kappa^2\kappa_s(\psi')^2\psi^{1/2} + 3\kappa^4\psi'\psi''\psi^{1/2} + 3\kappa\kappa_{ss}\psi'\psi^{3/2} + 3\kappa^4\psi'\psi^{3/2}$$
$$+ \frac{3}{2}\kappa^2(\psi'\kappa^2 + \kappa_s\psi)\psi''\psi^{1/2} + \frac{3}{4}\kappa^2(\psi')^2(\psi'\kappa^2 + \kappa_s\psi)\psi^{-1/2}$$
$$+ \frac{3}{2}(\psi'\kappa^2 + \kappa_s\psi)\psi'\psi^{1/2}\kappa_s + \psi^{5/2}\kappa_{sss} + 4\psi^{3/2}\psi'\kappa\kappa_{ss}$$
$$+ \psi^{3/2}\psi^{(3)}\kappa^4 + \psi^{3/2}\psi'\kappa^4 + \psi^{3/2}(6\psi''\kappa^2 + 4\psi\kappa^2 + 3\psi'\kappa_s)\kappa_s.$$

As a result, we get

$$\partial_t(\partial_s(\psi^{3/2}\kappa)) = \psi\partial_{ss}(\partial_s(\psi^{3/2}\kappa)) + \kappa\psi'(\partial_s(\psi^{3/2}\kappa))_s$$
$$+ \partial_s(\psi^{3/2}\kappa) \cdot P(\kappa, \psi, \psi^{-1}, \psi', \psi'')$$
$$+ Q(\kappa, \psi^{1/2}, \psi^{-1/2}, \psi^{3/2}, \psi, \psi', \psi'', \psi^{(3)}),$$

where P and Q are polynomials in the given variables.

Finally, as before, $v = \partial_s(\psi^{3/2}\kappa)$ satisfies an equation of the type

$$v_t = \psi(\theta)v_{ss} + a(s, v, v_s),$$

where $a(s, v, 0) \leqslant c(|v| + 1)$ and we can conclude again that v is bounded. This, thanks to (19) and the result for $j = 0$ (κ is bounded), implies that κ_s is bounded and concludes the proof. $\qquad\square$

Proposition 2. *Let T be the maximal time of existence of* (9) *and assume that $T < \infty$. Then*

$$(20) \qquad\qquad \limsup_{t \to T} \|\kappa\|_\infty = +\infty.$$

Proof. Assume that $|\kappa|$ is uniformly bounded for all $t \in [0, T)$. Then the previous lemmas imply a uniform bound on $|u_x|$, $|u_x|^{-1}$ and $|\partial_s^j\kappa|$. Using (9) we can write

$$u(x, t_2) - u(x, t_1) = \int_{t_1}^{t_2} \varphi^\circ(\nu(x))(D^2\varphi^\circ(\nu(x))\tau(x) \cdot \tau(x))\kappa(x)\nu(x)dt.$$

The bounds on κ and on the anisotropy map imply that $u(x, t_2)$ has a limit when $t_2 \to T$. It remains to show that the convergence of $u(\cdot, t)$ is in fact in $\mathcal{C}^\infty$. This is achieved by showing that we can get uniform bounds (in time) for all derivatives of the map u in the original parametrization.

First of all note that for a function $h : S^1 \to \mathbb{R}$ we have that

$$(21) \qquad \partial_x^m h - |u_x|^m\partial_s^m h = P_m(|u_x|, \dots, \partial_x^{m-1}|u_x|, h, \dots, \partial_s^{m-1}h),$$

where P_m is a polynomial in the given variables. Therefore if we can show that uniform bounds hold for the derivatives of the length element $|u_x|$ then using Lemma 7 we obtain bounds for the derivatives $|\partial_x^m\kappa|$ and $|\partial_x^m u_x|$ on $(0, T)$.

It remains to show that the derivatives of the length element $z = |u_x|$ stay bounded. Differentiating the PDE for $z = |u_x|$, namely

$$z_t = -\psi(\theta)\kappa^2 z,$$

we can write

$$(22) \qquad (\partial_x^m z)_t = -\psi(\theta)\kappa^2 \partial_x^m z + \sum_{i+j=m,j\leqslant m-1} c(i,j,m)\partial_x^i(\psi(\theta)\kappa^2)\partial_x^j z$$

for some coefficients $c(i,j,m)$. Here we proceed by induction. Assuming that $|\partial_x^j z|$ is bounded up to order $m-1$, then equation (21) applied to $h = \psi(\theta)\kappa^2$ and Lemma 7 give boundedness of the terms appearing in (22), so that we infer

$$(\partial_x^m z)_t \leqslant -\psi(\theta)\kappa^2 \partial_x^m z + c.$$

A Gronwall argument yields then boundedness of $\|\partial_x^m z\|_\infty$ on $(0,T)$.

Having achieved $\mathcal{C}^\infty$ convergence, we can now extend u past T, which gives a contradiction. The claim follows. $\qquad\square$

Note that since $\kappa_\varphi = D^2\varphi^\circ(\nu)\tau \cdot \tau\kappa$ the previous proposition implies that also the anisotropic curvature blows-up if the maximal time of existence of the flow (9) is finite.

Similarly to the isotropic case we get a lower bound for the curvature as follows.

Lemma 8. *Let T be the maximal time of existence of* (9) *and suppose $T < \infty$. Then*

$$(23) \qquad \liminf_{t\to T} \sqrt{T-t}\|\kappa\|_{L^\infty} \geq \frac{1}{\sqrt{2\alpha}}$$

where $\alpha = \max_{S^1} |\psi + \psi''|$.

Proof. Let $w := \kappa^2$. Then from (14) we infer that

$$w_t = \psi(\theta)w_{ss} + 2(\psi(\theta) + \psi''(\theta))w^2 + 3\psi'(\theta)w_s\sqrt{w}\,\text{sign}(\kappa) - 2\psi(\theta)(k_s)^2$$
$$\leqslant \psi(\theta)w_{ss} + 2(\psi(\theta) + \psi''(\theta))w^2 + 3\psi'(\theta)w_s\sqrt{w}\,\text{sign}(\kappa).$$

Let $M(t) := \max_{S^1} w \geq 0$. Then using [37, Lemma 2.1.3] we infer

$$\frac{d}{dt}M(t) \leqslant 2|\psi + \psi''|M^2(t) \leqslant 2\alpha M^2(t) \leqslant 2\alpha(M(t) + \delta)^2,$$

where $\alpha = \max_{S^1} |\psi + \psi''|$ and $\delta > 0$. Integrating on $[t, s] \subset [0, T)$ we obtain

$$-\frac{1}{M(s) + \delta} + \frac{1}{M(t) + \delta} \leqslant 2\alpha(s - t).$$

Letting $s \to T$ along a sequence of times so that $M(s) \to \infty$ by the Proposition 2, and choosing δ arbitrary small we get the claim. $\square$

4.2. General anisotropies

We now show short time existence of φ-regular flows, starting from a φ-regular initial curve.

Theorem 2. *Let u_0 be a closed φ-regular curve. Then there exist $T > 0$ and a φ-regular flow u on $[0, T]$, with $u(0) = u_0$.*

Proof. Let $\varphi_\varepsilon \geqslant \varphi$ be a family of smooth and elliptic anisotropies converging to φ as $\varepsilon \to 0$. Let $u_{\varepsilon,0}$ be the approximations of u_0 described in Lemma 3. Recalling Remark 6, up to a reparametrization, we can assume that $|(u_{\varepsilon,0})_x| \leq C$, so that the curves $u_{\varepsilon,0}$ are equi-Lipschitz in ε.

Denote by u_ε the solutions to (7) with initial data $u_{\varepsilon,0}$, that is, the functions u_ε solve the equation

$$(24) \qquad (u_\varepsilon)_t = \kappa_{\varphi_\varepsilon} N_\varepsilon,$$

where $N_\varepsilon = D\varphi_\varepsilon^\circ(\nu_\varepsilon)$ is the Cahn-Hoffman vector field.

In view of Lemma 3 and Proposition 1 the curves u_ε have curvature $\kappa_{\varphi_\varepsilon}$ bounded by a constant Λ which does not depend on ε and t, as long as t does not reach a certain T which depends only on u_0. As a consequence, by Proposition 2 we can assume that the solutions u_ε are all defined in the same time interval $[0, T]$. Our goal is to pass to the limit in u_ε as $\varepsilon \to 0$.

Thanks to Lemma 6 the maps u_ε are equi-Lipschitz in space. Moreover, by Proposition 1 and (24) the solutions u_ε are also equi-Lipschitz in time. Using Arzelà-Ascoli Theorem we can ensure that u_ε converge, up to a subsequence, to some Lipschitz function u such that $u(0) = u_0$. We claim that u is a φ-regular flow.

Fix $(\bar{x}, \bar{t}) \in S^1 \times (0, T)$. By Remark 5, in a neighborhood V of $(u(\bar{x}, \bar{t}), \bar{t})$ with $u(\bar{x}, \bar{t}) = \lim_\varepsilon u_\varepsilon(\bar{x}, \bar{t})$ equation (24) can be rewritten as

$$(25) \qquad \widetilde{N}_\varepsilon = D\varphi_\varepsilon^\circ(\nabla d_\varepsilon), \qquad (d_\varepsilon)_t = \operatorname{div}\widetilde{N}_\varepsilon + O(d_\varepsilon) \qquad \text{a.e. in } V.$$

where the functions d_ε, $\widetilde{N}_\varepsilon$ are defined as in Remarks 4 and 5. In particular, d_ε is the φ_ε-distance function from the support of u_ε, restricted to a neighborhood of $(u(\bar{x},\bar{t}),\bar{t})$ and $\widetilde{N}_\varepsilon$ is a suitable extension of the Cahn-Hoffman field N_ε.

From the convergence of u_ε to u we immediately get the uniform convergence of d_ε to some function d in V. Notice that d is the φ-distance function to the support of u, restricted to a neighborhood of $(u(\bar{x},\bar{t}),\bar{t})$. We now show that the fields $\widetilde{N}_\varepsilon$ also converge to some field $\widetilde{n}$ which is the extension of a Cahn-Hoffman vector field n for u. By the fact that the curvatures $\kappa_{\varphi_\varepsilon}$ are uniformly bounded and the control on the lengths from above and below (Remark 6, Lemma 6 and comments below), it follows for every fixed time t that $N_\varepsilon(\cdot,t)$ are equi-Lipschitz with respect to arc-length and (up to a subsequence) converge uniformly to a Lipschitz field $n(s,t)$. Let $\widetilde{n}$ be the extension in V (cf. Remark 5) satisfying $n(s(x),t) = \widetilde{n}(s(x),t) = \widetilde{n}(x,t)$. Since $\widetilde{N}_\varepsilon$ are uniformly bounded in V, they converge, up to a subsequence in the weak* topology of $L^\infty(V)$, to the vector field $\widetilde{n} \in L^\infty(V)$. By $\varphi_\varepsilon \to \varphi$ we infer that $\varphi(\widetilde{n}) \le 1$. Again using the uniform boundedness of the anisotropic curvatures $\kappa_{\varphi_\varepsilon}$ and (6) we infer that $(\operatorname{div}\widetilde{N}_\varepsilon)$ weak*- converges to $(\operatorname{div}\widetilde{n})$ in $L^\infty(V)$. We can now pass to the limit in (25) and obtain that d satisfies

$$(26) \qquad\qquad d_t = \operatorname{div}\widetilde{N} + O(d) \qquad \text{a.e. in } V.$$

We now show that $\widetilde{N}$ satisfies the inclusion

$$(27) \qquad\qquad \widetilde{N} \in \partial\varphi^o(\nabla d) \qquad \text{a.e. in } V.$$

Indeed, letting $\psi \in C_c^1(V)$ and recalling that $\widetilde{N}_\varepsilon \cdot \nabla d_\varepsilon = \varphi_\varepsilon^o(\nabla d_\varepsilon) = 1$, we have

$$\int_V \psi\,dxdt = \int_V \psi\,\widetilde{N}_\varepsilon \cdot \nabla d_\varepsilon\,dxdt = -\int_V d_\varepsilon\big(\widetilde{N}_\varepsilon \cdot \nabla\psi + \psi\operatorname{div}\widetilde{N}_\varepsilon\big)dxdt.$$

Passing to the limit in the right-hand side, we get

$$\int_V \psi\,dxdt = -\int_V d\big(\widetilde{N}\cdot\nabla\psi + \psi\operatorname{div}\widetilde{N}\big)dxdt = \int_V \psi\,\widetilde{N}\cdot\nabla d\,dxdt,$$

which is equivalent to (27).

It then follows that u is a φ-regular flow on $[0,T]$, which proves the thesis. $\qquad\square$

The uniqueness of φ-flows for general anisotropies and initial data is still an open problem. However, it has been proved in [8] that the evolution is

unique if the initial curve is embedded. Moreover, in the purely crystalline case, that is when W_φ is a polygon and when the initial curve is piecewise linear and φ-regular, the problem reduces to a family ODE's and the solution is therefore unique (see [30]).

References

[1] B. Andrews, *Singularities in crystalline curvature flows*, Asian J. Math. **6** (2002), no. 1, 101–121.

[2] F. J. Almgren and J. E. Taylor, *Flat flow is motion by crystalline curvature for curves with crystalline energies*, J. Diff. Geom. **42** (1995), 1–22.

[3] F. J. Almgren, J. E. Taylor, and L.-H. Wang, *Curvature-driven flows: a variational approach*, SIAM J. Control Optim. **31** (1993), no. 2, 387–438.

[4] S. Angenent, *Parabolic equations for curves on surfaces part I. Curves with p-integrable curvature*, Annals of Mathematics **132** (1990), no. 3, 451–483.

[5] S. Angenent, *Parabolic equations for curves on surfaces, Part II. Intersections, blow-up and generalized solutions*, Annals of Mathematics **133** (1991), 171–215.

[6] G. Bellettini, *Anisotropic and crystalline mean curvature flow*, A Sampler of Riemann-Finsler Geometry, 49–82, Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, Cambridge, (2004).

[7] G. Bellettini, V. Caselles, A. Chambolle, and M. Novaga. *Crystalline mean curvature flow of convex sets*, Arch. Rat. Mech. Anal. **179** (2006), no. 1, 109–152.

[8] G. Bellettini and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in* $\mathbb{R}^N$, Math. Mod. Meth. Appl. Sc. **10** (2000), 1–10.

[9] G. Bellettini, M. Novaga, and M. Paolini, *Facet-breaking for three dimensional crystals evolving by mean curvature*, Interfaces Free Bound. **1** (1999), 39–55.

[10] G. Bellettini, M. Novaga, and M. Paolini, *On a crystalline variational problem, part I: first variation and global L^∞ regularity*, Arch. Rational Mech. Anal. **157** (2001), no. 3, 165–191.

[11] G. Bellettini, M. Novaga, and M. Paolini, *Characterization of facet breaking for nonsmooth mean curvature flow in the convex case*, Interfaces Free Bound. **3** (2001), no. 4, 415–446.

[12] K. A. Brakke, The Motion of A Surface by Its Mean Curvature, Volume 20 of Mathematical Notes, Princeton University Press, Princeton, N.J., (1978).

[13] A. Chambolle, M. Morini, M. Novaga and M. Ponsiglione, *Existence and uniqueness for anisotropic and crystalline mean curvature flows*, J. Amer. Math. Soc. **32** (2019), no. 3, 779–824.

[14] A. Chambolle, M. Morini, and M. Ponsiglione, *Existence and uniqueness for a crystalline mean curvature flow*, Comm. Pure Appl. Math. **70** (2017), no. 6, 1084–1114.

[15] A. Cesaroni, M. Novaga, and E. Valdinoci, *Curve shortening flow in heterogeneous media*, Interfaces Free Bound. **13** (2011), no. 4, 485–505.

[16] A. Chambolle and M. Novaga, *Implicit time discretization of the mean curvature flow with a discontinuous forcing term*, Interfaces Free Bound. **10** (2008), 283–300.

[17] A. Chambolle and M. Novaga, *Existence and uniqueness for planar anisotropic and crystalline curvature flow*, Advanced Studies in Pure Mathematics **67** (2015), 87–113.

[18] K.-S. Chou and X.-P. Zhu, The Curve Shortening Problem, Chapman & Hall/CRC, Boca Raton, FL, (2001).

[19] K.-S. Chou and X.-P. Zhu, *A convexity theorem for a class of anisotropic flows of plane curves*, Indiana Univ. Math. J. **48** (1999), no. 1, 139–154.

[20] F. H. Clarke, Optimization and Nonsmooth Analysis, Volume 5 of Classics in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, (1990).

[21] M. G. Crandall, H. Ishii, and P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Am. Math. Soc. **27** (1992), 1–67.

[22] Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), no. 3, 749-786.

[23] K. Ecker and G. Huisken, *Mean curvature evolution of entire graphs*, Ann. of Math. (2) **130** (1989), no. 3, 453–471.

[24] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[25] L. C. Evans and J. Spruck, *Motion of level sets by mean curvature. II*, Trans. Amer. Math. Soc. **330** (1992), 321–332.

[26] M. Gage and R. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), 69–95.

[27] M. E. Gage and Y. Li, *Evolving plane curves by curvature in relative geometries. II*, Duke Math. J. **75** (1994), no. 1, 79–98.

[28] M.-H. Giga and Y. Giga, *Crystalline and level set flow - Convergence of a crystalline algorithm for a general curvature flow in the plane*, GAKUTO International Series Mathematical Sciences and Applications **13** (2000), 64–79.

[29] M.-H. Giga and Y. Giga, *Generalized motion by nonlocal curvature in the plane*, Arch. Ration. Mech. Anal. **159** (2001), no. 4, 295–333.

[30] Y. Giga and M. E. Gurtin, *A comparison theorem for crystalline evolutions in the plane*, Quarterly of Applied Mathematics **LIV** (1996), 727–737.

[31] Y. Giga, T. Ohtsuka, and R. Schätzle, *On a uniform approximation of motion by anisotropic curvature by the Allen-Cahn equations*, Interfaces Free Bound. **8** (2006), no. 3, 317–348.

[32] Y. Giga and N. Pozár, *A level set crystalline mean curvature flow of surfaces*, Adv. Differential Equations **21** (2016), no. 7-8, 631–698.

[33] Y. Giga and N. Pozár, *Approximation of general facets by regular facets with respect to anisotropic total variation energies and its application to crystalline mean curvature flow*, Comm. Pure Appl. Math. **71** (2018), no. 7, 1461–1491.

[34] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[35] G. M. Lieberman, Second Order Parabolic Differential Equations, World scientific, (2005).

[36] S. Luckhaus and T. Sturzenecker, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differ. Equ. **3** (1995), 253–271.

[37] C. Mantegazza, Lectures on Mean Curvature Flow, Progress in Mathematics **290**, Birkhäuser/Springer Basel, Basel, (2011).

[38] J. A. Oaks, *Singularities and self-intersections of curves evolving on surfaces*, Indiana Univ. Math. J. **43** (1994), 959–981.

[39] A. Stancu, *Uniqueness of self-similar solutions for a crystalline flow*, Indiana Univ. Math. J. **45** (1996), no. 4, 1157–1174.

[40] J. E. Taylor, *Crystalline variational problems*, Bull. Amer. Math. Soc. (N.S.) **84** (1978), 568–588.

RICAM, Johann Radon Institute
Altenberger Strasse 69, 4040 Linz, Austria
E-mail address: `gwenael.mercier@ricam.oeaw.ac.at`

Dipartimento di Matematica, Università di Pisa
Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
E-mail address: `matteo.novaga@unipi.it`

Universität Duisburg-Essen, Fakultät für Mathematik
Thea-Leymann-Strasse 9, 45127 Essen Germany
E-mail address: `paola.pozzi@uni-due.de`