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Bubbling analysis near the Dirichlet boundary for approximate harmonic maps from surfaces

JÜRGEN JOST, LEI LIU, AND MIAOMIAO ZHU

For a sequence of maps with a Dirichlet boundary condition from a compact Riemann surface with smooth boundary to a general compact Riemannian manifold, with uniformly bounded energy and with uniformly L^2-bounded tension field, we show that the energy identity and the no neck property hold during a blow-up process near the Dirichlet boundary. We apply these results to the two dimensional harmonic map flow with Dirichlet boundary and prove the energy identity at finite and infinite singular time. Also, the no neck property holds at infinite time.

1. Introduction

Let (M, g) be a compact Riemannian manifold with smooth boundary and (N, h) be a compact Riemannian manifold of dimension n. The energy of

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the mapping u is defined as

$$E(u) = \int_M e(u)dvol_g,$$

where $e(u)$ is the energy density defined by

$$e(u) = \frac{1}{2}|\nabla u|^2 = \mathrm{Trace}_g u^*h,$$

where u^*h is the pull-back of the metric tensor h.

A smooth critical point of the energy E is called a harmonic map.

By Nash's embedding theorem, (N,h) can be isometrically embedded into some Euclidean space $\mathbb{R}^N$. This brings the Euler-Lagrange equation into the form

$$\Delta_g u = A(u)(\nabla u, \nabla u),$$

where A is the second fundamental form of $N \subset \mathbb{R}^N$ and Δ_g is the Laplace-Beltrami operator on M which is defined by

$$\Delta_g := -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\beta}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\right).$$

The tension field $\tau(u)$ of u is defined by

$$\tau(u(x)) = -\Delta_g u(x) + A(u(x))(\nabla u(x), \nabla u(x)). \tag{1.1}$$

Then u is a harmonic map if and only if $\tau(u) = 0$.

In this paper, we shall study the blow-up analysis for a sequence of maps $\{u_n\}$ from a compact Riemann surface M with smooth boundary ∂M to a compact Riemannian manifold N with uniformly L^2-bounded tension fields $\tau(u_n)$, uniformly bounded energy and with Dirichlet boundary

$$u_n(x) = \varphi(x), \quad x \in \partial M. \tag{1.2}$$

In particular, the maps $\{u_n\}$ are not necessarily harmonic, as their tension fields need not vanish, but are only in L^2. Such sequences of maps frequently arise in schemes that have the purpose of proving the existence of harmonic maps, for instance by the heat flow method discussed below, but also in other schemes. Therefore, in this paper we shall systematically study their possible blow-up behavior.

When M is a closed surface, the compactness problem and the blow-up theory (energy identity and no neck property) for a sequence of maps $\{u_n\}$

from M to N with uniformly L^2-bounded tension fields and with uniformly bounded energy have been extensively studied (see e.g. [7, 10, 18, 19, 22, 24, 27–29, 31, 35]). For corresponding results about harmonic map flows, we refer to [22, 28, 29, 33, 34]. For some other related works, see [9, 13, 14, 16, 21].

When M is a compact Riemann surface with smooth boundary, Laurain-Petrides [15] considered a sequence of harmonic maps $\{u_n\}$ from M to the unit ball $B^{n+1} \subset \mathbb{R}^{n+1}$ with free boundary $u_n(\partial M)$ on S^n and with uniformly bounded energy and proved the energy identity. The blow-up theory (including the energy identity and the no neck property) of the more general case of a sequence of maps into a general compact target manifold with free boundary on a general closed supporting submanifold with uniformly L^2 bounded tension fields and with uniformly bounded energy was completed in [12].

Since the interior blow-up case is already well understood, we shall focus on the case where the energy concentration occurs near the Dirichlet boundary and complete the blow-up theory near the Dirichlet boundary for a bubbling sequence. We should point that as a consequence of an old result of Lemaire, it is not possible that a blow-up occurs at the boundary itself, in view of the Dirichlet condition [17]. It is, however, conceivable that there is a sequence of interior points (x_n) converging to a boundary point x_0 such that the maps blow up along that sequence and that in the limit we have a boundary bubble. This, therefore, is the situation investigated in this paper.

Here is our first main result for the local problem:

Theorem 1.1. *Let $u_n \in W^{2,2}(D_1^+(0), N)$ be a sequence of maps with tension fields $\tau(u_n)$ and with Dirichlet boundary data*

$$u_n|_{\partial^0 D_1^+(0)} = \varphi \in C^{2+\alpha}(\partial^0 D_1^+(0))$$

for some $0 < \alpha < 1$, satisfying

(a) $\|u_n\|_{W^{1,2}(D_1^+(0))} + \|\tau(u_n)\|_{L^2(D_1^+(0))} \leq \Lambda,$

(b) *$u_n \to u$ strongly in $W^{1,2}_{loc}(D_1^+(0) \setminus \{0\}, \mathbb{R}^N)$ as $n \to \infty$,*

where $D_1^+(0) := \{(x,y) \in \mathbb{R}^2 | |x|^2 + |y|^2 \leq 1, y \geq 0\}$ and $\partial^0 D_1^+(0) := \{(x,y) \in D_1^+(0) | y = 0\}$.

Then there exist a subsequence of u_n (still denoted by u_n) and a nonnegative integer L such that, for any $i = 1, ..., L$, there exist points x_n^i, positive numbers λ_n^i and a bubble, i.e. a nontrivial harmonic sphere w^i (which we view as a map from $\mathbb{R}^2 \cup \{\infty\}$ to N), such that

(1) $x_n^i \to 0,\ \lambda_n^i \to 0$, *as* $n \to \infty$;

(2) $\frac{dist(x_n^i, \partial^0 D_1^+(0))}{\lambda_n^i} \to \infty$, *as* $n \to \infty$;

(3) $\lim_{n\to\infty}\left(\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \frac{|x_n^i - x_n^j|}{\lambda_n^i + \lambda_n^j}\right) = \infty$ *for any* $i \neq j$;

(4) w^i *is the weak limit of* $u_n(x_n^i + \lambda_n^i x)$ *in* $W_{loc}^{1,2}(\mathbb{R}^2)$;

(5) ***Energy identity:*** *we have*

$$\lim_{n\to\infty} E(u_n, D_1^+(0)) = E(u, D_1^+(0)) + \sum_{i=1}^{L} E(w^i). \tag{1.3}$$

(6) ***No neck property:*** *The image*

$$u(D_1^+(0)) \cup \bigcup_{i=1}^{L} w^i(\mathbb{R}^2) \tag{1.4}$$

is a connected set.

In the free boundary case [12], in general, both harmonic spheres and harmonic disks with free boundary can split off at a boundary energy concentration point. In contrast to the free boundary case, the case that $\frac{dist(x_n^i, \partial^0 D_1^+(0))}{\lambda_n^i}$ is uniformly bounded cannot occur in the Dirichlet boundary case, as we have already explained before the statement of the theorem. Otherwise, one will get a nontrivial harmonic disk with constant boundary data, which is impossible by Lemaire's result [17] (see section 3). Thus, when the energy of the maps concentrates near the Dirichlet boundary, only some harmonic spheres can split off as is described in the above theorem. On the other hand, since the neck domains appearing near the Dirichlet boundary are in general not simply half annuli, we need to apply some finer decomposition of the neck domains (see Section 3) as is done in the free boundary case [12]. This is the main technical achievement of this paper.

Our results complete the blow-up analysis that is needed in the various existence schemes for harmonic maps. In fact, combining Theorem 1.1 and the classical interior blow-up theory of harmonic maps, we have

Theorem 1.2. *Let* $u_n : M \to N$ *be a sequence of* $W^{2,2}$ *maps with Dirichlet boundary* $u_n|_{\partial M} = \varphi(x) \in C^{2+\alpha}(\partial M, N)$ *and with tension fields* $\tau(u_n)$

satisfying

$$E(u_n) + \|\tau(u_n)\|_{L^2(M)} \le \Lambda < \infty.$$

We define the blow-up set

$$\mathcal{S} := \cap_{r>0} \left\{ x \in M \;\middle|\; \liminf_{n\to\infty} \int_{D_r^M(x)} |du_n|^2 dvol \ge \bar{\epsilon}^2 \right\}, \tag{1.5}$$

where $D_r^M(x) = \{y \in M | \; dist(x,y) \le r\}$ *denotes the geodesic ball in* M *and* $\bar{\epsilon} > 0$ *is a constant whose value will be given in* (3.1)*. Then* $\mathcal{S}$ *is a finite set* $\{p_1, ..., p_I\}$*. By taking subsequences,* $\{u_n\}$ *converges weakly in* $W^{2,2}_{loc}(M \setminus \mathcal{S})$ *to some limit map* $u_0 \in W^{2,2}(M,N)$ *with Dirichlet boundary* $u_0|_{\partial M} = \varphi(x)$ *and there are finitely many bubbles: a finite set of nontrivial harmonic spheres* $w_i^l : S^2 \to N$*,* $l = 1, ..., l_i$*,* $i = 1, ..., I$*, such that*

$$\lim_{n\to\infty} E(u_n) = E(u_0) + \sum_{i=1}^{I} \sum_{l=1}^{l_i} E(w_i^l), \tag{1.6}$$

and the image $u_0(M) \cup_{i=1}^{I} \cup_{l=1}^{l_i} (w_i^l(S^2))$ *is a connected set.*

As promised, we shall apply the results in Theorem 1.2 to one of the most important and successful existence schemes, the heat flow for harmonic maps with Dirichlet boundary:

$$\partial_t u(x,t) = \tau(u(x)) \quad (x,t) \in M \times (0,T); \tag{1.7}$$

$$u(\cdot, 0) = u_0(x) \quad x \in M; \tag{1.8}$$

$$u(x,t) = \varphi(x) \in C^{2+\alpha}(\partial M, N), \quad x \in \partial M, \quad \forall\, t \ge 0; \tag{1.9}$$

The existence of a global weak solution of (1.7-1.8) from a closed Riemannian surface with finitely many singularities was first considered by Struwe [33]. Later, Chang [1] considered the harmonic map flow with Dirichlet boundary (1.9) and obtained a global regular solution under some small initial energy assumption. In fact, by combining the results by Struwe [33] and Chang [1], one can get a global weak solution of (1.7-1.8) from a compact Riemann surface with Dirichlet boundary condition (1.9), which is C^2 except at finitely many singularities. For other results for the harmonic map flow with Dirichlet boundary, see [3, 5, 8]. For results of other harmonic map type flows with Dirichlet boundary, we mention [4, 11]. The existence of a global weak solution of the harmonic map flow (1.7-1.8) with free boundary was studied by Ma [25] and the corresponding blow-up theory was further explored in [12].

Let $u : M \times (0,\infty) \to N$ be a global weak solution to (1.7-1.9), which is C^2 away from a finite number of singular points $\{(x_i,t_i)\} \subset M \times (0,\infty)$. In fact, there holds

$$u(x,t) \in C^{2,1,\alpha}_{loc}(M \times (0,\infty) \setminus \{(x_i,t_i)\}) \cap C^\infty((M \setminus \partial M) \times (0,\infty) \setminus \{(x_i,t_i)\}). \tag{1.10}$$

Similarly to the closed surface case (see e.g. [22, 28, 29]) and the free boundary case [12], we shall complete the qualitative picture at the singularities of this flow, where bubbles (nontrivial harmonic spheres) split off.

At infinite time, we have the following

Theorem 1.3. *There exist a harmonic map $u_\infty : M \to N$ with Dirichlet boundary $u_\infty|_{\partial M} = \varphi$, a finite number of harmonic spheres $\{\omega_i\}_{i=1}^m$ and sequences $\{x_n^i\}_{i=1}^m \subset M$, $\{\lambda_n^i\}_{i=1}^m \subset \mathbb{R}_+$ and $\{t_n\} \subset \mathbb{R}_+$ such that*

$$\lim_{t\nearrow\infty} E(u(\cdot,t),M) = E(u_\infty,M) + \sum_{i=1}^m E(\omega_i) \tag{1.11}$$

and

$$\left\| u(\cdot,t_n) - u_\infty(\cdot) - \sum_{i=1}^m \omega_n^i(\cdot) \right\|_{L^\infty(M)} \to 0 \tag{1.12}$$

as $n \to \infty$, where $\omega_n^i(\cdot) = \omega^i\left(\frac{\cdot - x_n^i}{\lambda_n^i}\right) - \omega_i(\infty)$.

For finite time blow-ups, we have

Theorem 1.4. *Let $T_0 < \infty$ and $u \in C^{2,1,\alpha}_{loc}(M \times (0,T_0),N)$ be a solution to (1.7-1.9) with T_0 as its singular time. Then there exist finitely many harmonic spheres $\{\omega_i\}_{i=1}^l$ such that*

$$\lim_{t\nearrow T_0} E(u(\cdot,t),M) = E(u(\cdot,T_0),M) + \sum_{i=1}^l E(\omega_i). \tag{1.13}$$

Remark 1.5. There is the natural and interesting question whether or not the singularity can really occur at the boundary for the problem of (1.7)–(1.9)?

This paper is organized as follows. In Section 2, we recall some well-known results which will be used in this paper. Moreover, we prove some basic lemmas, such as the small energy regularity, removable singularity theorem, Pohozaev's identity in the Dirichlet boundary case. In Section 3, we prove Theorem 1.1 by decomposing the neck domain into several parts including some annulus and some half annulus centered at the boundary, which is similar to the idea in [12]. Combining Theorem 1.1 with the classical interior blow-up theory of harmonic maps, we can then complete the proof of Theorem 1.2. In Section 4, we apply Theorem 1.2 to the harmonic map flow with Dirichlet boundary and prove Theorem 1.3 and Theorem 1.4.

Notation. $D_r(x_0)$ denotes the closed ball in $\mathbb{R}^2$ of radius r and center x_0. Denote

$$\begin{aligned} D_r^+(x_0) &:= \{x=(x^1,x^2)\in D_r(x_0)\mid x^2\geq 0\},\\ D_r^-(x_0) &:= \{x=(x^1,x^2)\in D_r(x_0)\mid x^2\leq 0\},\\ \partial^+D_r(x_0) &:= \{x=(x^1,x^2)\in \partial D_r(x_0)|x^2\geq 0\},\\ \partial^-D_r(x_0) &:= \{x=(x^1,x^2)\in \partial D_r(x_0)|x^2\leq 0\},\\ \partial^0D_r^+(x_0)=\partial^0D_r^-(x_0) &:= \partial D_r^+(x_0)\setminus\partial^+D_r(x_0). \end{aligned}$$

Suppose $a\geq 0$ is a constant, denote

$$\mathbb{R}^2_a := \{(x^1,x^2)|x^2\geq -a\} \ \ \textit{and} \ \ \mathbb{R}^{2+}_a := \{(x^1,x^2)|x^2> -a\}.$$

For simplicity, we denote $D_r=D_r(0)$, $D=D_1(0)$, $D_r^+=D_r^+(0)$, $D^+=D_1^+(0)$, and $\mathbb{R}^2_+=\mathbb{R}^2_a$ when $a=0$.

In this paper, Δ_g denotes the Laplace-Beltrami operator on the Riemannian manifold (M,g) and $\Delta:=\partial_x^2+\partial_y^2$ denotes the usual Laplace operator on $\mathbb{R}^2$.

2. Some basic lemmas

In this section, we will first recall some well known results that are useful for our problem. Then we will prove some basic lemmas for the Dirichlet boundary case, such as small energy regularity, removable singularity and Pohozaev's identity.

First, we recall the interior small energy regularity result (see [7, 18]) which is firstly introduced in [31].

Lemma 2.1. *Let* $u \in W^{2,p}(D,N)$, $1 < p \le 2$ *be a map with tension field* $\tau(u) \in L^p(D)$. *There exist constants* $\epsilon_1 = \epsilon_1(p,N) > 0$ *and* $C = C(p,N) > 0$, *such that if* $\|\nabla u\|_{L^2(D)} \le \epsilon_1$, *then*

$$\|u - u(0)\|_{W^{2,p}(D_{1/2})} \le C(p,N)(\|\nabla u\|_{L^p(D)} + \|\tau(u)\|_{L^p(D)}). \tag{2.1}$$

Moreover, by the Sobolev embedding $W^{2,p}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$, *we have*

$$\begin{aligned}\|u\|_{Osc(D_{1/2})} &= \sup_{x,y\in D_{1/2}} |u(x) - u(y)| \\ &\le C(p,N)(\|\nabla u\|_{L^p(D)} + \|\tau(u)\|_{L^p(D)}).\end{aligned} \tag{2.2}$$

Secondly, we recall a gap theorem for the case of a closed domain.

Lemma 2.2 ([6]). *There exists a constant* $\epsilon_0 = \epsilon_0(M,N) > 0$ *such that if* u *is a smooth harmonic map from a closed Riemann surface* M *to a compact Riemannian manifold* N, *satisfying*

$$\int_M |\nabla u|^2 dvol \le \epsilon_0,$$

then u *is a constant map.*

Thirdly, we state a removable singularity result.

Theorem 2.3. *If* $u : D \setminus \{0\} \to N$ *is a* $W^{2,p}_{loc}(D \setminus \{0\})$ *map for some* $1 < p \le 2$ *with finite energy and satisfies*

$$\tau(u) = g \in L^p(D, TN), \quad \text{in } D \setminus \{0\}, \tag{2.3}$$

then u *can be extended to a map in* $W^{2,p}(D,N)$.

Moreover, if $u : D^+ \setminus \{0\} \to N$ *is a* $W^{2,p}_{loc}(D^+ \setminus \{0\})$ *map for some* $1 < p \le 2$ *with finite energy and with Dirichlet boundary*

$$u|_{\partial^0 D^+} = \varphi \in W^{1,p}(\partial^0 D^+),$$

satisfying

$$\tau(u) = g \in L^p(D^+, TN), \quad \text{in } D^+ \setminus \{0\}, \tag{2.4}$$

then u *can be extended to a map in* $W^{2,p}(D^+,N)$. *Here,* $u \in W^{2,p}_{loc}(D^+ \setminus \{0\})$ *means that* $u \in W^{2,p}(D^+ \setminus D^+_r(0))$ *for any* $0 < r < 1$.

Proof. For the interior case, one can refer to [20]. For the boundary case, one can also use a similar method as in [20] to get the conclusion. Here, we use the regularity theory to prove it.

In fact, on one hand, it is easy to see that u is a weak solution of (2.4) in D^+. On the other hand, it is well known that the equation (2.4) can be written as an elliptic system with an anti-symmetric potential (see [30])

$$\Delta u = \Omega \cdot \nabla u + g$$

with $\Omega \in L^2(D^+, so(N) \otimes \mathbb{R}^2)$ and $g \in L^p(D^+, TN)$ for $1 < p \leq 2$. By taking pure Dirichlet conditions in the boundary regularity Theorem 1.2 in [32] (or see Remark 1.3 in [26]), we know $u \in W^{2,p}(D^+_r, N)$ for some small $r > 0$ and hence $u \in W^{2,p}(D^+, N)$. $\square$

Fourthly, we prove a small energy regularity lemma near the boundary. Here and in the sequel, we shall view φ as the restriction of some $C^{2+\alpha}(M, N)$ map on ∂M and for simplicity, we still denote this map by φ.

Lemma 2.4. *Let $u \in W^{2,p}(D^+, N)$, $1 < p \leq 2$ be a map with tension field $\tau(u) \in L^p(D^+)$ and with Dirichlet boundary $u|_{\partial^0 D^+} = \varphi(x)$, where $\varphi \in C^{2+\alpha}(D)$. There exists $\epsilon_2 = \epsilon_2(p, N) > 0$, such that if $\|\nabla u\|_{L^2(D^+_1)} \leq \epsilon_2$, then*

$$\begin{aligned}(2.5)\qquad & \left\| u - \frac{1}{2}\int_{\partial^0 D^+} \varphi \right\|_{W^{2,p}(D^+_{1/2})} \\ & \leq C(p, N)(\|\nabla u\|_{L^p(D^+)} + \|\nabla \varphi\|_{W^{1,p}(D^+)} + \|\tau(u)\|_{L^p(D^+)}).\end{aligned}$$

Moreover, by the Sobolev embedding $W^{2,p}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$, we have

$$\begin{aligned}(2.6)\qquad \|u\|_{Osc(D^+_{1/2})} &= \sup_{x,y \in D^+_{1/2}} |u(x) - u(y)| \\ & \leq C(p, N)(\|\nabla u\|_{L^p(D^+)} + \|\nabla \varphi\|_{W^{1,p}(D^+)} + \|\tau(u)\|_{L^p(D^+)}).\end{aligned}$$

Proof. Without loss of generality, we assume $\frac{1}{2}\int_{\partial^0 D^+_1} \varphi dx = 0$.

Choosing a cut-off function $\eta \in C^\infty_0(D^+)$ satisfying $0 \leq \eta \leq 1, \eta|_{D^+_{3/4}} \equiv 1, |\nabla \eta| + |\nabla^2 \eta| \leq C$ and computing directly, we get

$$\begin{aligned}(2.7)\qquad |\Delta(\eta\phi)| &= |\eta\Delta\phi + 2\nabla\eta\nabla\phi + \phi\Delta\eta| \\ &\leq C\left(|\phi| + |d\phi| + |d\phi||\eta d\phi| + |\tau|\right) \\ &\leq C|d\phi||d(\eta\phi)| + C\left(|\phi| + |d\phi| + |\tau|\right).\end{aligned}$$

Assume first that $1 < p < 2$, by standard elliptic estimates and Poincare's inequality, we obtain

$$\begin{aligned}\|\eta\phi\|_{W^{2,p}(D^+)} &\leq C\||d\phi||d(\eta\phi)|\|_{L^p(D^+)}\\ &\quad + C(\|\phi\|_{W^{1,p}(D^+)} + \||\varphi|\|_{W^{2,p}(D^+)} + \||\tau|\|_{L^p(D^+)})\\ &\leq C\|d(\eta\phi)\|_{L^{\frac{2p}{2-p}}(D^+)}\|d\phi\|_{L^2(D^+)}\\ &\quad + C(\|d\phi\|_{L^p(D^+)} + \|\nabla\varphi\|_{W^{1,p}(D^+)} + \||\tau|\|_{L^p(D^+)})\\ &\leq C\epsilon_2\|d(\eta\phi)\|_{L^{\frac{2p}{2-p}}(D^+)}\\ &\quad + C(\|d\phi\|_{L^p(D^+)} + \|\nabla\varphi\|_{W^{1,p}(D^+)} + \||\tau|\|_{L^p(D^+)}).\end{aligned}$$

Taking $\epsilon_2 > 0$ sufficiently small, we have

$$\begin{aligned}\|\phi\|_{W^{2,p}(D^+_{3/4})} &\leq \|\eta\phi\|_{W^{2,p}(D^+)}\\ &\leq C(\|d\phi\|_{L^p(D^+)} + \|\nabla\varphi\|_{W^{1,p}(D^+)} + \|\tau\|_{L^p(D^+)}).\end{aligned} \tag{2.8}$$

So, we have proved the lemma in the case $1 < p < 2$.

Next, if $p = 2$, one can first derive the above estimate with $p = \frac{4}{3}$. Such an estimate gives a $L^4(D^+_{3/4})-$ bound for ∇u. Then one can apply the $W^{2,2}-$boundary estimate to the equation and get the conclusion of the lemma with $p = 2$. □

Next, we compute the Pohozaev identity near the Dirichlet boundary.

Lemma 2.5. *For $x_0 \in \partial^0 D^+$, let $u(x) \in W^{2,2}(D^+(x_0))$ be a map with tension field $\tau(u) \in L^2(D^+(x_0))$ and with Dirichlet boundary data $\varphi(x)$ on $\partial^0 D^+(x_0)$. Then, for any $0 < t < 1$, there holds*

$$\begin{aligned}&\int_{\partial^+ D^+_t(x_0)} r\left(\left|\frac{\partial u}{\partial r}\right|^2 - \frac{1}{2}|\nabla u|^2\right)\\ &= \int_{\partial^+(D^+_t(x_0))} \frac{\partial u}{\partial r}\cdot r\varphi_r + \int_{D^+_t(x_0)} r\frac{\partial(u-\varphi)}{\partial r}\tau dx\\ &\quad - \int_{D^+_t(x_0)} \nabla_{e_\alpha} u\cdot\nabla_{e_\alpha}(r\varphi_r)dx\\ &\quad + \int_{D^+_t(x_0)} A(u)(\nabla u,\nabla u)\cdot(r\varphi_r)dx\end{aligned} \tag{2.9}$$

where $(r,\theta) \in (0,1)\times(0,\pi)$ are the polar coordinates at x_0.

Proof. Multiplying $(x-x_0)\nabla(u-\varphi)$ to both sides of the equation

$$\tau = \Delta u + A(u)(\nabla u, \nabla u) \quad a.e.\ x \in D^+(x_0)$$

and integrating by parts, for any $0 < t < 1$, we get

$$\begin{aligned}
(2.10) \qquad & \int_{D_t^+(x_0)} \tau \cdot ((x-x_0)\nabla(u-\varphi))dx \\
&= \int_{D_t^+(x_0)} \Delta u \cdot ((x-x_0)\nabla(u-\varphi))dx \\
&\quad - \int_{D_t^+(x_0)} A(u)(\nabla u, \nabla u) \cdot ((x-x_0)\nabla\varphi)dx \\
&= \int_{\partial(D_t^+(x_0))} \frac{\partial u}{\partial n} \cdot ((x-x_0)\nabla(u-\varphi)) \\
&\quad - \int_{D_t^+(x_0)} \nabla_{e_\alpha} u \cdot \nabla_{e_\alpha}((x-x_0)\nabla(u-\varphi))dx \\
&\quad - \int_{D_t^+(x_0)} A(u)(\nabla u, \nabla u) \cdot ((x-x_0)\nabla\varphi)dx \\
&:= \mathbb{I} + \mathbb{II} + \mathbb{III},
\end{aligned}$$

where $\overrightarrow{n}(x)$ is the outward unite normal vector field for $a.e.\ x \in \partial(D_t^+(x_0))$.

Since $u(x)$ satisfies the Dirichlet boundary condition $u|_{\partial^0 D^+} = \varphi$, we have

$$\begin{aligned}
(2.11) \qquad \mathbb{I} &= \int_{\partial^+(D_t^+(x_0))} \frac{\partial u}{\partial n} \cdot ((x-x_0)\nabla(u-\varphi)) \\
&= \int_{\partial^+(D_t^+(x_0))} r\left|\frac{\partial u}{\partial r}\right|^2 - \int_{\partial^+(D_t^+(x_0))} r\frac{\partial u}{\partial r} \cdot \frac{\partial \varphi}{\partial r}.
\end{aligned}$$

Computing directly and integrating by parts, we get

$$\begin{aligned}
(2.12) \qquad \mathbb{II} &= -\int_{D_t^+(x_0)} |\nabla u|^2 dx - \frac{1}{2}\int_{D_t^+(x_0)} (x-x_0) \cdot \nabla|\nabla u|^2 dx \\
&\quad + \int_{D_t^+(x_0)} \nabla_{e_\alpha} u \cdot \nabla_{e_\alpha}((x-x_0)\nabla\varphi)dx \\
&= -\frac{1}{2}\int_{\partial(D_t^+(x_0))} \langle x-x_0, \overrightarrow{n}\rangle |\nabla u|^2 \\
&\quad + \int_{D_t^+(x_0)} \nabla_{e_\alpha} u \cdot \nabla_{e_\alpha}((x-x_0)\nabla\varphi)dx \\
&= -\int_{\partial^+(D_t^+(x_0))} r\frac{1}{2}|\nabla u|^2 + \int_{D_t^+(x_0)} \nabla_{e_\alpha} u \cdot \nabla_{e_\alpha}(r\varphi_r)dx,
\end{aligned}$$

where the last equality follows from the fact that $\langle x - x_0, \overrightarrow{n}\rangle = 0$ on $\partial^0 D_t^+(x_0)$. Then the conclusion of the lemma immediately follows from (2.10), (2.11) and (2.12). □

Corollary 2.6. *Under the assumptions of Lemma 2.5, we have*

$$\int_{D_{2t}^+(x_0)\setminus D_t^+(x_0)} \left(\left|\frac{\partial u}{\partial r}\right|^2 - \frac{1}{2}|\nabla u|^2 \right) dx \leq Ct,$$

where $C = C(\|\nabla\varphi\|_{C^1}, \|\nabla u\|_{L^2(D^+)}, \|\tau(u)\|_{L^2(D^+)})$ *is a positive constant.*

Proof. From Lemma 2.5, we have

$$\begin{aligned}
&\int_{\partial^+ D_t^+(x_0)} \left(\left|\frac{\partial u}{\partial r}\right|^2 - \frac{1}{2}|\nabla u|^2 \right) \\
&= \frac{1}{t}\bigg(\int_{\partial^+(D_t^+(x_0))} \frac{\partial u}{\partial r} \cdot r\varphi_r + \int_{D_t^+(x_0)} r\frac{\partial(u-\varphi)}{\partial r}\tau dx \\
&\quad - \int_{D_t^+(x_0)} \nabla_{e_\alpha} u \cdot \nabla_{e_\alpha}(r\varphi_r) dx + \int_{D_t^+(x_0)} A(u)(\nabla u, \nabla u)\cdot(r\varphi_r)dx \bigg) \\
&\leq C\bigg(\int_{\partial^+(D_t^+(x_0))} |\nabla u| + \|\nabla(u-\varphi)\|_{L^2(D_t^+(x_0))}\|\tau\|_{L^2(D_t^+(x_0))} \\
&\quad + \frac{1}{t}\int_{D_t^+(x_0)} |\nabla u| dx + \int_{D_t^+(x_0)} |\nabla u|^2 dx \bigg) \\
&\leq C\int_{\partial^+(D_t^+(x_0))} |\nabla u| + C.
\end{aligned}$$

Integrating from t to $2t$, we will get the conclusion of the corollary from Hölder's inequality. □

3. Energy identity and no neck property

In this section, we shall use the idea of [12] to prove Theorem 1.1 and Theorem 1.2. Due to the pointwise constraint of the Dirichlet boundary condition and Theorem 3.2 in [17], a harmonic disk cannot occur in the blow-up process which is different from the free boundary case in [12]. The key point is that we decompose the neck domain into some interior annulus and some half annulus centered at the points on the boundary (see section 5 in [12]).

Proof of Theorem 1.1. By the assumption of Theorem 1.1, we may assume that 0 is the only blow-up point (energy concentration point) of the sequence $\{u_n\}$ in D^+, *i.e.*

$$\liminf_{n\to\infty} E(u_n; D_r^+) \geq \frac{\overline{\epsilon}^2}{4} \text{ for all } r > 0 \tag{3.1}$$

where $\overline{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$. According to the standard argument of blow-up analysis, for any n, there exist sequences $x_n \to 0$ and $r_n \to 0$ such that

$$E(u_n; D_{r_n}^+(x_n)) = \sup_{\substack{x\in D^+, r\leq r_n\\ D_r^+(x)\subset D^+}} E(u_n; D_r^+(x)) = \frac{\overline{\epsilon}^2}{8}. \tag{3.2}$$

Denoting $d_n = dist(x_n, \partial^0 D^+)$, firstly we make the following **Claim 1:** $\limsup_{n\to\infty} \frac{d_n}{r_n} = \infty$.

In fact, if not, then we have $\limsup_{n\to\infty} \frac{d_n}{r_n} < \infty$ and by taking a subsequence, we may assume $\lim_{n\to\infty} \frac{d_n}{r_n} = a \geq 0$. Define

$$v_n(x) := u_n(x_n + r_n x)$$

and

$$B_n := \{x \in \mathbb{R}^2 | x_n + r_n x \in D^+\}.$$

Then we know

$$B_n \to \mathbb{R}_a^2 := \{(x^1, x^2) | x^2 \geq -a\}$$

and for any $x \in \{x^2 = -a\}$ on the boundary, $x_n + r_n x \to 0$ as $n \to \infty$.

It is easy to see that $v_n(x)$ lives in B_n and satisfies

$$\tau(v_n(x)) = \Delta v_n(x) + A(v_n(x))(\nabla v_n(x), \nabla v_n(x)) \; in \; B_n; \tag{3.3}$$

$$v_n(x) = \varphi(x_n + r_n x), \text{ if } x_n + r_n x \in \partial^0 D^+, \tag{3.4}$$

where $\tau(v_n(x)) = r_n^2 \tau(u_n(x))$.

By (3.2), Lemma 2.1 and Lemma 2.4, we get

$$\|v_n\|_{W^{2,2}(D_R(0)\cap B_n)} \leq C(R, N) \tag{3.5}$$

for any $D_R(0) \subset \mathbb{R}^2$. Then there exist a subsequence of v_n (also denoted by v_n) and a harmonic map $v \in W^{2,2}(\mathbb{R}_a^2)$ with constant boundary $v|_{\partial\mathbb{R}_a^2} = \varphi(0)$

such that, for any $R > 0$,

$$\lim_{n\to\infty} \|v_n(x) - v(x)\|_{W^{1,2}(D_R(0)\cap B_n)} = 0.$$

In addition, by 3.2, we have $E(v; D_1(0) \cap \mathbb{R}^2_a) = \frac{\bar{\epsilon}^2}{8}$. However by [17], we know v is a constant map. This is a contradiction. Thus, we proved our **Claim 1**.

Under the condition that $\limsup_{n\to\infty} \frac{d_n}{r_n} = \infty$, we can see that $v_n(x)$ lives in B_n which tends to $\mathbb{R}^2$ as $n \to \infty$. Moreover, for any $x \in \mathbb{R}^2$, when n is sufficiently large, by (3.2), we have

$$E(v_n; D_1(x)) \le \frac{\bar{\epsilon}^2}{8}. \tag{3.6}$$

By Lemma 2.1, we get

$$\|v_n\|_{W^{2,2}(D_R(0))} \le C(R, N).$$

Thus, there exist a subsequence of v_n (we still denote it by v_n) and a harmonic map $v^1(x) \in W^{1,2}(\mathbb{R}^2, N)$ such that, as $n \to \infty$,

$$\begin{aligned} &v_n(x) \rightharpoonup v^1(x) \text{ weakly in } W^{2,2}_{loc}(\mathbb{R}^2), \\ \text{and} \quad &v_n(x) \to v^1(x) \text{ strongly in } W^{1,2}_{loc}(\mathbb{R}^2). \end{aligned} \tag{3.7}$$

Besides, we know $E(v^1; D_1(0)) = \frac{\bar{\epsilon}^2}{8}$. By a standard property of harmonic maps, $v^1(x)$ can be extended to a nontrivial harmonic sphere. We call the above harmonic sphere $v^1(x)$ the first bubble.

We will split the proof of Theorem 1.1 into two parts, energy identity and no neck property. Now, we begin to prove the energy identity.

Energy identity : By the standard induction argument in [7], we just need to prove the theorem in the case where there is only one bubble $v(x)$ which is the strong limit of $u_n(x_n + r_n x)$ in $W^{1,2}_{loc}(\mathbb{R}^2)$. For the case of more than one bubble, i.e. a bubble tree, we just need to distinguish "neck domains" which are almost the same as in the interior blow-up theory. See [2, 21] for details. Then we can estimate the energy concentration on each "neck domain" by using the proof of one bubble case.

Noting that $x_n \to 0$ and the assumption of Theorem 1.1, we have

$$\lim_{\delta\to 0}\lim_{n\to\infty} E(u_n; D^+ \setminus D^+_\delta(x_n)) = E(u; D^+).$$

So, by (3.7), the energy identity is equivalent to

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E(u_n; D^+_\delta(x_n)\setminus D^+_{r_nR}(x_n)) = 0. \tag{3.8}$$

To prove the no neck property, *i.e.* the image of the sets $u(D^+)$ and $v(\mathbb{R}^2\cup\infty)$ are connected in the target manifold, it is enough to show that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} \|u_n\|_{Osc\left(D^+_\delta(x_n)\setminus D^+_{r_nR}(x_n)\right)} = 0. \tag{3.9}$$

Under the “one bubble” assumption, we first make the following:
Claim 2: for any $\epsilon > 0$, there exist $\delta > 0$ and $R > 0$ such that

$$\int_{D^+_{8t}(x_n)\setminus D^+_t(x_n)} |\nabla u_n|^2 dx \le \epsilon^2 \text{ for any } t\in\left(\frac{1}{2}r_nR, 2\delta\right) \tag{3.10}$$

when n is large enough.

In fact, if (3.10) is not true, then there exist a positive constant ϵ_3 and a sequence $t_n\to 0$, such that $\lim_{n\to\infty}\frac{t_n}{r_n}=\infty$ and

$$\int_{D^+_{8t_n}(x_n)\setminus D^+_{t_n}(x_n)} |\nabla u_n|^2 dx \ge \epsilon_3 > 0. \tag{3.11}$$

Passing to a subsequence, we may assume

$$\lim_{n\to\infty}\frac{d_n}{t_n} = b\in[0,\infty].$$

Set

$$w_n(x) := u_n(x_n + t_n x)$$

and

$$B'_n := \{x\in\mathbb{R}^2 | x_n + t_n x\in D^+\}.$$

It is easy to see that $w_n(x)$ lives in B'_n and 0 is also an energy concentration point for w_n. We need to consider the following two cases:

(a) $b<\infty$.

Then B'_n tends to $\mathbb{R}^2_b$ as $n\to\infty$. Here, we also need to consider two cases.

(a-1) w_n has no other energy concentration points except 0.

By Lemma 2.1, Lemma 2.4 and Theorem 2.3, passing to a subsequence, we may assume that w_n converges to a harmonic map $w(x) : \mathbb{R}^2_b \to N$ with constant boundary data $w|_{\partial\mathbb{R}^2_b} = \varphi(0)$ satisfying, for any $R > 0$,

$$\sup_{\lambda>0} \lim_{n\to\infty} \|w_n(x) - w(x)\|_{W^{1,2}\left((D_R(0)\cap B'_n)\setminus D_\lambda(0)\right)} = 0.$$

According to [17], $w(x)$ ia a constant map. However, (3.11) implies

$$\int_{(D_8\setminus D_1)\cap\mathbb{R}^2_b} |\nabla w|^2 dx = \lim_{n\to\infty} \int_{(D_8\setminus D_1)\cap B'_n} |\nabla w_n|^2 dx \geq \epsilon_3 > 0. \tag{3.12}$$

This is a contradiction.

(a-2) w_n has another energy concentration point $p \neq 0$.

Without loss of generality, we may assume p is the only energy concentration point in $D_{r_0}(p)$ for some $r_0 > 0$. By the process of constructing the first bubble, there exist sequences $x'_n \to p$ and $r'_n \to 0$ such that

$$E(w_n; D^+_{r'_n}(x'_n) \cap B'_n) = \sup_{\substack{x\in D^+_{r_0}(p), r\leq r'_n \\ D^+_r(x)\subset D^+_{r_0}(p)}} E(w_n; D^+_r(x) \cap B'_n) = \frac{\bar{\epsilon}^2}{8}. \tag{3.13}$$

By (3.2), we have $r'_n t_n \geq r_n$. Then, by taking a subsequence, we may assume $\lim_{n\to\infty} \frac{d_n}{r'_n t_n} = d \in [0, \infty]$. Furthermore, we know d must be ∞ (the proof is the same as for **Claim 1**). Then similar to the process of constructing the first bubble, there exists a nontrivial harmonic map $v^2(x) : \mathbb{R}^2 \to N$ such that, passing to a subsequence,

$$\lim_{n\to\infty} \|w_n(x'_n + r'_n x) - v^2(x)\|_{W^{1,2}(D_R(0))} = 0$$

for any $R > 0$. This is

$$\lim_{n\to\infty} \|u_n(x_n + t_n x'_n + t_n r'_n x) - v^2(x)\|_{W^{1,2}(D_R(0)} = 0. \tag{3.14}$$

So, $v^2(x)$ is also a bubble for the sequence u_n. This also contradicts the "one bubble" assumption.

(b) $b = \infty$.

In this case, B'_n will tend to $\mathbb{R}^2$ as $n \to \infty$. Again, we need to consider the following two cases.

(b-1) w_n has no other energy concentration points except 0.

According to (3.11), Lemma 2.1 and Theorem 2.3, we know that there exists $v^2(x):\mathbb{R}^2\to N$ which is a nontrivial harmonic map such that, passing to a subsequence,

$$w_n(x)\to v^2(x)\ in\ W^{1,2}_{loc}(\mathbb{R}^2\setminus\{0\}).$$

Then, we get the second bubble $v^2(x)$ which contradicts the "one bubble" assumption.

(b-2) w_n has another energy concentration point $p\neq 0$.

Similar to **Case (a-2)**, there exist sequences $x'_n\to p$ and $r'_n\to 0$ satisfying (3.13) and

$$\lim_{n\to\infty}\frac{d_n}{r'_nt_n}=\infty.$$

Moreover, by the process of constructing the first bubble, there exists a nontrivial harmonic map $v^2(x):\mathbb{R}^2\to N$ such that

$$w_n(x'_n+r'_nx)\to v^2(x)\ strongly\ in\ W^{1,2}_{loc}(\mathbb{R}^2),$$

that is

$$u_n(x_n+t_nx'_n+t_nr'_nx)\to v^2(x)\ strongly\ in\ W^{1,2}_{loc}(\mathbb{R}^2).$$

So, we get the second bubble $v^2(x)$. This also contradicts the "one bubble" assumption. Thus, we proved **Claim 2**.

Suppose $x'_n\in\partial^0D^+$ is the projection of x_n, *i.e.* $d_n=dist(x_n,\partial^0D^+)=|x_n-x'_n|$. Then, we decompose the neck domain $D^+_\delta(x_n)\setminus D^+_{r_nR}(x_n)$ as in [12] as follows

$$\begin{aligned}(3.15)\quad D^+_\delta(x_n)\setminus D^+_{r_nR}(x_n)&=D^+_\delta(x_n)\setminus D^+_{\frac{\delta}{2}}(x'_n)\cup D^+_{\frac{\delta}{2}}(x'_n)\setminus D^+_{2d_n}(x'_n)\\&\quad\cup D^+_{2d_n}(x'_n)\setminus D^+_{d_n}(x_n)\cup D^+_{d_n}(x_n)\setminus D^+_{r_nR}(x_n)\\&:=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4,\end{aligned}$$

when n and R are large.

Noting that $\lim_{n\to\infty}d_n=0$ and $\lim_{n\to\infty}\frac{d_n}{r_n}=\infty$, then we have

$$\Omega_1\subset D^+_\delta(x_n)\setminus D^+_{\frac{\delta}{4}}(x_n),\quad and\quad \Omega_3\subset D^+_{4d_n}(x_n)\setminus D^+_{d_n}(x_n)$$

when n is large enough. Moreover, for any $2d_n\le t\le\frac{1}{2}\delta$, there holds

$$D^+_{2t}(x'_n)\setminus D^+_t(x'_n)\subset D^+_{4t}(x_n)\setminus D^+_{t/2}(x_n).$$

According to assumption (3.10), we have

$$E(u_n;\Omega_1)+E(u_n;\Omega_3)\le\epsilon^2 \tag{3.16}$$

and

$$\int_{D^+_{2t}(x'_n)\setminus D^+_t(x'_n)}|\nabla u_n|^2dx\le\epsilon^2 \text{ for any } t\in\left(2d_n,\frac{1}{2}\delta\right). \tag{3.17}$$

By Lemma 2.1, Lemma 2.4 and the standard scaling argument, we have

$$\begin{aligned} &Osc_{D^+_{2t}(x'_n)\setminus D^+_t(x'_n)}u_n\\ &\le C(\|\nabla u_n\|_{L^2(D^+_{4t}(x'_n)\setminus D^+_{t/2}(x'_n))}+\|\nabla\varphi\|_{L^2(D^+_{4t}(x'_n)\setminus D^+_{t/2}(x'_n))}\\ &\quad+t\|\nabla^2\varphi\|_{L^2(D^+_{4t}(x'_n)\setminus D^+_{t/2}(x'_n))}+t\|\tau(u_n)\|_{L^2(D^+_{4t}(x'_n)\setminus D^+_{t/2}(x'_n))}) \end{aligned} \tag{3.18}$$

for any $t\in(2r_nR,\frac{1}{2}\delta)$.

Since $\Omega_4=D^+_{d_n}(x_n)\setminus D^+_{r_nR}(x_n)=D_{d_n}(x_n)\setminus D_{r_nR}(x_n)$, by the standard blow-up analysis theory of harmonic maps with interior blow-up points, we have

$$\lim_{R\to\infty}\lim_{n\to 0}E(u_n;D_{d_n}(x_n)\setminus D_{r_nR}(x_n))=0. \tag{3.19}$$

and

$$\lim_{R\to\infty}\lim_{n\to 0}Osc(u_n)_{D_{d_n}(x_n)\setminus D_{r_nR}(x_n)}=0. \tag{3.20}$$

See [7, 18, 29] for details.

Thus, we just need to estimate the energy concentration in Ω_2.

Define $\widehat{\Omega_2}:=D_{\frac{\delta}{2}}(x'_n)\setminus D_{2d_n}(x'_n)$, $\mu_n(x):=u_n(x)-\varphi(x),\ x\in\Omega_2$ and

$$\widehat{\mu_n}(x):=\begin{cases}\mu_n(x), & x\in\Omega_2,\\ -\mu_n(x'), & x\in\widehat{\Omega_2}\setminus\Omega_2,\end{cases} \tag{3.21}$$

where $x=(x^1,x^2)$ and $x'=(x^1,-x^2)$. It is easy to see that $\widehat{\mu_n}(x)\in W^{2,2}(\widehat{\Omega_2})$ and satisfies the following equation

$$\Delta\widehat{\mu_n}(x)=\begin{cases}A(u_n(x))(\nabla u_n(x),\nabla u_n(x))+\tau(u_n)(x)-\Delta\varphi(x), & x\in\Omega_2,\\ -A(u_n(x'))(\nabla u_n(x'),\nabla u_n(x'))-\tau(u_n)(x')+\Delta\varphi(x'), & x\in\widehat{\Omega_2}\setminus\Omega_2.\end{cases} \tag{3.22}$$

Set

$$\widehat{\mu_n}^*(r) := \frac{1}{2\pi}\int_0^{2\pi} \widehat{\mu_n}(r,\theta)d\theta,$$

where (r,θ) is the polar coordinates at x'_n. By (3.18) and (3.21), we have

$$\begin{aligned}
(3.23)\qquad & \|\widehat{\mu_n}(x) - \widehat{\mu_n}^*(x)\|_{L^\infty(\widehat{\Omega_2})} \\
&= \sup_{2d_n \le t \le \frac{\delta}{4}} \|\widehat{\mu_n}(x) - \widehat{\mu_n}^*(x)\|_{L^\infty(D_{2t}(x_n)\setminus D_t(x_n))} \\
&\le \sup_{2d_n \le t \le \frac{\delta}{4}} \|\widehat{\mu_n}(x)\|_{Osc(D_{2t}(x_n)\setminus D_t(x_n))} \\
&\le 2 \sup_{2d_n \le t \le \frac{\delta}{4}} \|\mu_n(x)\|_{Osc(D^+_{2t}(x_n)\setminus D^+_t(x_n))} \\
&\le C(N,\Lambda,\|\varphi\|_{C^2})(\epsilon+\delta).
\end{aligned}$$

Without loss of generality, we may assume $\frac{1}{2}\delta = 2^{m_n}(2d_n)$, where m_n is a positive integer which tends to ∞ as $n\to\infty$. Setting $P_i := D^+_{2^{i+1}d_n}(x'_n)\setminus D^+_{2^i d_n}(x'_n)$ and $\widehat{P_i} := D_{2^{i+1}d_n}(x'_n)\setminus D_{2^i d_n}(x'_n)$, then we have

$$\int_{\widehat{P_i}} \nabla\widehat{\mu_n}\nabla(\widehat{\mu_n} - \widehat{\mu_n}^*) = \int_{\partial\widehat{P_i}} (\widehat{\mu_n} - \widehat{\mu_n}^*)\frac{\partial\widehat{\mu_n}}{\partial r} - \int_{\widehat{P_i}} (u_n - u_n^*)\Delta\widehat{\mu_n}.$$

On the one hand, by Jessen's inequality, we get

$$\begin{aligned}
\int_{\widehat{P_i}} \nabla\widehat{\mu_n}\nabla(\widehat{\mu_n} - \widehat{\mu_n}^*) &= \int_{\widehat{P_i}} |\nabla\widehat{\mu_n}|^2 - \int_{\widehat{P_i}} \frac{\partial\widehat{\mu_n}}{\partial r}\frac{\partial\widehat{\mu_n}^*}{\partial r} \\
&\ge \int_{\widehat{P_i}} |\nabla\widehat{\mu_n}|^2 - \left(\int_{\widehat{P_i}} \left|\frac{\partial\widehat{\mu_n}}{\partial r}\right|^2\right)^{\frac{1}{2}} \left(\int_{\widehat{P_i}} \left|\frac{\partial\widehat{\mu_n}^*}{\partial r}\right|^2\right)^{\frac{1}{2}} \\
&\ge \int_{\widehat{P_i}} |\nabla\widehat{\mu_n}|^2 - \int_{\widehat{P_i}} \left|\frac{\partial\widehat{\mu_n}}{\partial r}\right|^2 \\
&= \frac{1}{2}\int_{\widehat{P_i}} |\nabla\widehat{\mu_n}|^2 - \int_{\widehat{P_i}} \left(\left|\frac{\partial\widehat{\mu_n}}{\partial r}\right|^2 - \frac{1}{2}|\nabla\widehat{\mu_n}|^2\right) \\
&= \int_{P_i} |\nabla\mu_n|^2 - 2\int_{P_i} \left(\left|\frac{\partial\mu_n}{\partial r}\right|^2 - \frac{1}{2}|\nabla\mu_n|^2\right).
\end{aligned}$$

By direct computation, we obtain

$$\begin{aligned}
&\int_{P_i} |\nabla \mu_n|^2 - 2\int_{P_i} \left(\left| \frac{\partial \mu_n}{\partial r} \right|^2 - \frac{1}{2}|\nabla \mu_n|^2 \right) \\
&= \int_{P_i} |\nabla u_n|^2 - 2\int_{P_i} \left(\left| \frac{\partial u_n}{\partial r} \right|^2 - \frac{1}{2}|\nabla u_n|^2 \right) \\
&\quad + 4\int_{P_i} \left(\frac{\partial u_n}{\partial r}\frac{\partial \varphi}{\partial r} - \nabla u_n \nabla \varphi \right) + 2\int_{P_i} \left(|\nabla \varphi|^2 - \left| \frac{\partial \varphi}{\partial r} \right|^2 \right) \\
&\geq \int_{P_i} |\nabla u_n|^2 - 2\int_{P_i} \left(\left| \frac{\partial u_n}{\partial r} \right|^2 - \frac{1}{2}|\nabla u_n|^2 \right) - C2^i d_n.
\end{aligned}$$

On the other hand, according to (3.23) and equation (3.22), we have

$$\begin{aligned}
&- \int_{\widehat{P_i}} \Delta \widehat{\mu_n} (\widehat{\mu_n} - \widehat{\mu_n}^*) dx \\
&\leq C(\epsilon + \delta) \int_{P_i} |\nabla u_n|^2 dx + C(\epsilon + \delta) \int_{P_i} (|\tau(u_n)| + |\Delta \varphi|) dx \\
&\leq C(\epsilon + \delta) \int_{P_i} |\nabla u_n|^2 dx + C(\epsilon + \delta)(\|\tau_n\|_{L^2(P_i)} + \|\varphi\|_{C^2}) 2^i d_n.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(3.24) \qquad & (1 - C(\epsilon + \delta)) \int_{P_i} |\nabla u_n|^2 dx \\
&\leq \int_{\partial \widehat{P_i}} \frac{\partial \widehat{\mu_n}}{\partial n} (\widehat{\mu_n} - \widehat{\mu_n}^*) + 2\int_{P_i} \left(\left| \frac{\partial u_n}{\partial r} \right|^2 - \frac{1}{2}|\nabla u_n|^2 \right) dx + C2^i d_n \\
&\leq \int_{\partial \widehat{P_i}} \frac{\partial \widehat{\mu_n}}{\partial n} (\widehat{\mu_n} - \widehat{\mu_n}^*) + C2^i d_n,
\end{aligned}$$

where the last inequality follows from Corollary 2.6.

Summing i from 1 to m_n, we obtain

$$\begin{aligned}
(3.25) \qquad (1 - C(\epsilon + \delta)) \int_{\Omega_2} |\nabla u_n|^2 &\leq \int_{\partial D_{\delta/2}(x_n')} (\widehat{\mu_n} - \widehat{\mu_n}^*) \frac{\partial \widehat{\mu_n}}{\partial r} \\
&\quad - \int_{\partial D_{2d_n}(x_n')} (\widehat{\mu_n} - \widehat{\mu_n}^*) \frac{\partial \widehat{\mu_n}}{\partial r} + C\delta.
\end{aligned}$$

As for the boundary term, using (3.23) and the trace theory, we get

$$
\begin{aligned}
&\int_{\partial D_{\delta/2}(x'_n)} (\widehat{\mu_n} - \widehat{\mu_n}^*)\frac{\partial \widehat{\mu_n}}{\partial r} \\
&\leq C(\epsilon+\delta)\int_{\partial D_{\delta/2}(x'_n)} |\nabla \widehat{\mu_n}| \\
&\leq C(\epsilon+\delta)\int_{\partial^+ D_{\delta/2}(x'_n)} |\nabla \mu_n| \\
&\leq C(\epsilon+\delta)\int_{\partial^+ D_{\delta/2}(x'_n)} (|\nabla u_n| + |\nabla\varphi|) \\
&\leq C(\epsilon+\delta)\big(\|\nabla u_n\|_{L^2(D^+_\delta(x'_n)\setminus D^+_{\frac{1}{4}\delta}(x'_n))} + \delta\|\nabla^2 u_n\|_{L^2(D^+_\delta(x'_n)\setminus D^+_{\frac{1}{4}\delta}(x'_n))} + 1\big) \\
&\leq C(\epsilon+\delta)\big(\|\nabla u_n\|_{L^2(D^+_{\frac{4}{3}\delta}(x_n)\setminus D^+_{\frac{1}{6}\delta}(x_n))} + \|\nabla\varphi\|_{L^2(D^+_{\frac{4}{3}\delta}(x_n)\setminus D^+_{\frac{1}{6}\delta}(x_n))} \\
&\quad + \delta\|\nabla^2\varphi\|_{L^2(D^+_{\frac{4}{3}\delta}(x_n)\setminus D^+_{\frac{1}{6}\delta}(x_n))} + \delta\|\tau_n\|_{L^2(D^+_{\frac{4}{3}\delta}(x_n)\setminus D^+_{\frac{1}{6}\delta}(x_n))} + 1\big) \\
&\leq C(\epsilon+\delta),
\end{aligned}
$$

where the second to last inequality follows from Lemma 2.1 and Lemma 2.4.

Also, there holds

$$\int_{\partial^+ D_{2d_n}(x'_n)} (u_n - u_n^*)\frac{\partial u_n}{\partial r} \leq C(\epsilon+\delta).$$

Combining these results and taking ϵ and δ in (3.25) sufficiently small, we have

$$\int_{\Omega_2} |\nabla u_n|^2 dx \leq C(\delta+\epsilon). \tag{3.26}$$

By (3.16), (3.19) and (3.26), we get (3.8) and we proved the energy identity.

Next, we will show that the base map and the bubbles are connected in the target manifold, *i.e.*, the no neck property in Theorem 1.1.

No neck property. Following the same argument as in the energy identity part, we can decompose the neck domain $D^+_\delta(x_n)\setminus D^+_{r_nR}(x_n) = \Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4$ as in (3.15), when n and R are large. Then, thanks to the no neck results (3.20) (see [18, 29]), we just need to prove that (3.9) holds in $\Omega_1\cup\Omega_2\cup\Omega_3$.

By assumption (3.10), Lemma 2.1 and Lemma 2.4, we have

$$
\begin{aligned}
(3.27)\qquad & \|u_n\|_{Osc(D^+_\delta(x_n)\backslash D^+_{\frac{\delta}{4}}(x'_n))}\\
&\le \|u_n\|_{Osc(D^+_\delta(x_n)\backslash D^+_{\frac{\delta}{5}}(x_n))}\\
&\le C\big(\|\nabla u_n\|_{L^2(D^+_{\frac{4\delta}{3}}(x_n)\backslash D^+_{\frac{\delta}{6}}(x_n))} + \delta\|\tau_n\|_{L^2(D^+_{\frac{4\delta}{3}}(x_n)\backslash D^+_{\frac{\delta}{6}}(x_n))}\\
&\quad + \|\nabla\varphi\|_{L^2(D^+_{\frac{4\delta}{3}}(x_n)\backslash D^+_{\frac{\delta}{6}}(x_n))} + \delta\|\nabla^2\varphi\|_{L^2(D^+_{\frac{4\delta}{3}}(x_n)\backslash D^+_{\frac{\delta}{6}}(x_n))}\big)\\
&\le C(\epsilon+\delta)
\end{aligned}
$$

and

$$
\begin{aligned}
(3.28)\qquad & \|u_n\|_{Osc(D^+_{4d_n}(x'_n)\backslash D^+_{d_n}(x_n))}\\
&\le \|u_n\|_{Osc(D^+_{5d_n}(x_n)\backslash D^+_{d_n}(x_n))}\\
&\le C\big(\|\nabla u_n\|_{L^2(D^+_{6d_n}(x_n)\backslash D^+_{\frac{3d_n}{4}}(x_n))} + d_n\|\tau_n\|_{L^2(D^+_{6d_n}(x_n)\backslash D^+_{\frac{3d_n}{4}}(x_n))}\\
&\quad + \|\nabla\varphi\|_{L^2(D^+_{6d_n}(x_n)\backslash D^+_{\frac{3d_n}{4}}(x_n))} + d_n\|\nabla^2\varphi\|_{L^2(D^+_{6d_n}(x_n)\backslash D^+_{\frac{3d_n}{4}}(x_n))}\big)\\
&\le C(\epsilon+\delta),
\end{aligned}
$$

when n, R are large and δ is small.

Similarly, we may assume $\frac{1}{2}\delta = 2^{m_n}2d_n$. Define $Q(t) := D^+_{2^{t_0+t}2d_n}(x'_n)\setminus D^+_{2^{t_0-t}2d_n}(x'_n)$, $\widehat{Q}(t) := D_{2^{t_0+t}2d_n}(x'_n)\setminus D_{2^{t_0-t}2d_n}(x'_n)$ and

$$f(t) := \int_{Q(t)} |\nabla u_n|^2 dx,$$

where $0\le t_0\le m_n$ and $0\le t\le \min\{t_0, m_n-t_0\}$.

Similar to the proof of (3.24) and (3.25), we get

$$(3.29)\quad (1-C(\epsilon+\delta))\int_{Q(t)}|\nabla u_n|^2dx \le \int_{\partial\widehat{Q}(t)}\frac{\partial\widehat{\mu_n}}{\partial n}(\widehat{\mu_n}-\widehat{\mu_n}^*) + C2^{t_0+t}d_n.$$

As for the boundary, by Hölder's inequality and Poincare's inequality, we have

$$\begin{aligned}
&\int_{\partial D_{2^{t_0+t}2d_n}(x_n')} \frac{\partial \widehat{\mu_n}}{\partial n}(\widehat{\mu_n} - \widehat{\mu_n}^*) \\
&\leq \left(\int_{\partial D_{2^{t_0+t}2d_n}(x_n')} \left| \frac{\partial \widehat{\mu_n}}{\partial r} \right|^2 \right)^{\frac{1}{2}} \left(\int_{\partial^+ D_{2^{t_0+t}2d_n}(x_n')} |\widehat{\mu_n} - \widehat{\mu_n}^*|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{\partial D_{2^{t_0+t}2d_n}(x_n')} \left| \frac{\partial \widehat{\mu_n}}{\partial r} \right|^2 \right)^{\frac{1}{2}} \left(2^{t_0+t} d_n \int_0^{2\pi} \left| \frac{\partial \widehat{\mu_n}}{\partial \theta} \right|^2 \right)^{\frac{1}{2}} \\
&\leq C 2^{t_0+t} d_n \int_{\partial D^+_{2^{t_0+t}2d_n}(x_n')} |\nabla \widehat{\mu_n}|^2 \\
&\leq C 2^{t_0+t} d_n \int_{\partial^+ D^+_{2^{t_0+t}2d_n}(x_n')} |\nabla \mu_n|^2 \\
&\leq C 2^{t_0+t} d_n \int_{\partial^+ D^+_{2^{t_0+t}2d_n}(x_n')} |\nabla u_n|^2 + C(2^{t_0+t} d_n)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{\partial D_{2^{t_0-t}2d_n}(x_n')} \frac{\partial \widehat{\mu_n}}{\partial n}(\widehat{\mu_n} - \widehat{\mu_n}^*) \\
&\leq C 2^{t_0-t} d_n \int_{\partial^+ D^+_{2^{t_0-t}2d_n}(x_n')} |\nabla u_n|^2 + C(2^{t_0-t} d_n)^2.
\end{aligned}$$

Combining these, we obtain

$$\begin{aligned}
(3.30) \qquad &(1 - C(\epsilon + \delta)) \int_{Q(t)} |\nabla u_n|^2 dx \\
&\leq C 2^{t_0+t} 2d_n \int_{\partial^+ (D^+_{2^{t_0+t}2d_n}(x_n'))} |\nabla u_n|^2 \\
&\quad + C 2^{t_0-t} 2d_n \int_{\partial^+ (D^+_{2^{t_0-t}2d_n}(x_n'))} |\nabla u_n|^2 + C 2^{t_0+t} d_n.
\end{aligned}$$

Taking ϵ and δ sufficiently small, then we have

$$\begin{aligned}
\int_{Q(t)} |\nabla u_n|^2 dx &\leq C 2^{t_0+t} 2d_n \int_{\partial^+ (D^+_{2^{t_0+t}2d_n}(x_n'))} |\nabla u_n|^2 \\
&\quad + C 2^{t_0-t} 2d_n \int_{\partial^+ (D^+_{2^{t_0-t}2d_n}(x_n'))} |\nabla u_n|^2 + C 2^{t_0+t} d_n.
\end{aligned}$$

So, we get

$$f(t) \leq \frac{C}{\log 2} f'(t) + C 2^{t_0+t} d_n. \tag{3.31}$$

Therefore,

$$(2^{-\frac{1}{C}t} f(t))' \geq -C 2^{t_0+(1-1/C)t} d_n.$$

Integrating from 2 to L, we arrive at

$$\begin{aligned} f(2) &\leq C 2^{-\frac{1}{C}L} f(L) + C 2^{t_0} d_n \int_2^L 2^{(1-1/C)t} dt \\ &\leq C 2^{-\frac{1}{C}L} f(L) + C 2^{t_0} d_n 2^{(1-1/C)L}. \end{aligned}$$

Let $t_0 = i$ and $L = L_i := \min\{i, m_n - i\}$. Noting that $Q(L_i) \subset D^+_{\delta/2}(x'_n) \setminus D^+_{2d_n}(x'_n) \subset D^+_\delta(x_n) \setminus D^+_{r_n R}(x_n)$, we have

$$\begin{aligned} &\int_{D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n)} |\nabla u_n|^2 dx \\ &\leq C E(u_n, D^+_\delta(x_n) \setminus D^+_{r_n R}(x_n)) 2^{-\frac{1}{C}L_i} + C 2^i d_n 2^{(1-1/C)L_i} \\ &\leq C E(u_n, D^+_\delta(x_n) \setminus D^+_{r_n R}(x_n)) 2^{-\frac{1}{C}L_i} + C 2^i d_n 2^{(1-1/C)(m_n-i)} \\ &\leq C E(u_n, D^+_\delta(x_n) \setminus D^+_{r_n R}(x_n)) 2^{-\frac{1}{C}L_i} + C\delta 2^{(-1/C)(m_n-i)} \\ &\leq C\epsilon 2^{-\frac{1}{C}L_i} + C\delta 2^{(-1/C)(m_n-i)}, \end{aligned}$$

where the last inequality follows from the energy identity (3.8).

By Lemma 2.1 and Lemma 2.4, we obtain

$$\begin{aligned} &Osc_{D^+_{2^{i+1}2d_n}(x'_n) \setminus D^+_{2^{i-1}2d_n}(x'_n)} u_n \\ &\leq C\big(\|\nabla u_n\|_{L^2(D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n))} + \|\nabla \varphi\|_{L^2(D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n))} \\ &\quad + 2^{i+2} d_n \|\nabla^2 \varphi\|_{L^2(D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n))} \\ &\quad + 2^{i+2} d_n \|\tau_n\|_{L^2(D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n))}\big) \\ &\leq C\big(\|\nabla u_n\|_{L^2(D^+_{2^{i+2}2d_n}(x'_n) \setminus D^+_{2^{i-2}2d_n}(x'_n))} + 2^i d_n\big). \end{aligned}$$

Summing over i from 2 to $m_n - 2$, we get

$$\begin{aligned}\|u_n\|_{Osc(D^+_{\delta/4}(x'_n)\backslash D^+_{4d_n}(x'_n))} &\le \sum_{i=2}^{m_n-2} \|u_n\|_{Osc(D^+_{2^{i+1}2d_n}(x'_n)\backslash D^+_{2^{i-1}2d_n}(x'_n))}\\ &\le C\sum_{i=2}^{m_n-2}\left(\epsilon 2^{-\frac{1}{C}L_i} + \delta 2^{(-1/C)(m_n-i)} + 2^i d_n\right)\\ &\le C\sum_{i=2}^{m_n-2} 2^{-\frac{1}{C}i}(\epsilon+\delta) + C\delta \le C(\epsilon+\delta).\end{aligned}$$

Combining this inequality with (3.27), (3.28), we get (3.9). Thus, we have proved that there is no neck during the blow-up process and finished the proof of Theorem 1.1. □

Now, we can prove Theorem 1.2.

Proof of Theorem 1.2. By the blow-up theory of a sequence of maps from a closed Riemann surface with uniformly L^2 bounded tension fields and with uniformly bounded energy developed in [7, 18, 22, 24, 29] and Theorem 1.1, the conclusion of Theorem 1.2 follows from applying the standard blow-up scheme as in [7]. □

4. Application to the harmonic map flow with Dirichlet boundary

With the help of the results in Theorem 1.2, we will study the qualitative behavior near the singularities of the harmonic map flow with Dirichlet boundary and prove Theorem 1.3 and Theorem 1.4 in this section.

Firstly, we have

Lemma 4.1. *Let $u : M \times (0,\infty) \to N$ be a global weak solution to (1.7–1.9), which is C^2−smooth away from a finite number of singular points. Then we have the following estimate*

$$\int_0^\infty \int_M |\partial_t u|^2 dxdt \le E(u_0). \tag{4.1}$$

Moreover, $E(u(\cdot,t))$ is continuous on $[0,\infty)$ and non-increasing.

Proof. The proof is similar to Lemma 3.4 in [33] for the closed case and Lemma 6.1 in [12] for the free boundary case. For any $0 \le t_1 \le t_2 \le \infty$, multiplying the equation (1.7) by $\partial_t u$ and integrating by parts, using the boundary condition that $\partial_t u|_{\partial M} \equiv 0$, we get

$$\begin{aligned}\int_{t_1}^{t_2}\int_M |\partial_t u|^2 dxdt &= \int_{t_1}^{t_2}\int_M -\Delta_g u \cdot \partial_t u dxdt \\ &= \int_{t_1}^{t_2}\int_{\partial M} \frac{\partial u}{\partial \overrightarrow{n}} \cdot \partial_t u - \int_{t_1}^{t_2}\int_M \nabla u \cdot \nabla(\partial_t u) dxdt \\ &= -\int_{t_1}^{t_2}\int_M \frac{1}{2}\partial_t |\nabla u|^2 dxdt = E(u(\cdot, t_1)) - E(u(\cdot, t_2)),\end{aligned}$$

where $\overrightarrow{n}$ is the outward unit normal vector field on ∂M. Then the conclusion of the lemma follows immediately. □

Similarly to the closed surface case (Lemma 2.5 in [22]) and the free boundary case (Lemma 6.2 in [12]), we have

Lemma 4.2. *Let $u \in C^2(M \times (0, T_0), N)$ be a solution to (1.7–1.9). There exists a positive constant R_0 such that, for any $x_0 \in M$, $0 < t \le s < T_0$ and $0 < R \le R_0$, there hold:*

$$E(u(s); B_R^M(x_0)) \le E(u(t); B_{2R}^M(x_0)) + C\frac{s-t}{R^2}E(u_0), \tag{4.2}$$

and

$$\begin{aligned} E(u(t); B_R^M(x_0)) \le E(u(s); B_{2R}^M(x_0)) &+ C\int_t^s\int_M |\partial_t u|^2 dxdt \\ &+ C\frac{s-t}{R^2}E(u_0). \end{aligned} \tag{4.3}$$

Proof. Let $\eta \in C_0^\infty(B_{2R}^M(x_0))$ be a cut-off function such that

$$0 \le \eta \le 1, \quad \eta|_{B_R^M(x_0)} \equiv 1 \quad \text{and} \quad |\nabla \eta| \le \frac{C}{R}.$$

Multiplying (1.7) by $\eta^2 \partial_t u$ and integrating by parts, we have

$$\begin{aligned} &\int_M |\partial_t u|^2 \eta^2 dx + \frac{d}{dt}\left(\frac{1}{2}\int_M |\nabla u|^2 \eta^2 dx\right) \\ &= \int_{\partial M} \frac{\partial u}{\partial \overrightarrow{n}} \cdot \partial_t u \eta^2 - 2\int_M \partial_t u \nabla u \eta \nabla \eta dx \\ &= -2\int_M \partial_t u \nabla u \eta \nabla \eta dx, \end{aligned}$$

where the last equality follows from the boundary condition that $\partial_t u|_{\partial M} \equiv 0$.

Noting that

$$\left|2\int_M \partial_t u \nabla u \eta \nabla \eta dx\right| \leq \frac{1}{2}\int_M |\partial_t u|^2 \eta^2 dx + 2\int_M |\nabla u|^2 |\nabla \eta|^2 dx,$$

we get

$$\begin{aligned} -\frac{3}{2}\int_M |\partial_t u|^2 \eta^2 dx - 2\int_M |\nabla u|^2 |\nabla \eta|^2 dx &\leq \frac{d}{dt}\left(\frac{1}{2}\int_M |\nabla u|^2 \eta^2 dx\right) \\ &\leq 2\int_M |\nabla u|^2 |\nabla \eta|^2 dx. \end{aligned}$$

Integrating this inequality from t to s, we will get the conclusion of the lemma. ☐

By Lemma 4.2 and the standard argument in the closed surface case (Lemma 4.1 in [22]), we obtain the following:

Lemma 4.3. *Let $u \in C^2(M \times (0, T_0), N)$ be a solution to (1.7–1.9) and $x_0 \in M$ be the only singular point at the time T_0. Then there exists a positive number $m > 0$ such that*

$$|\nabla u|^2(x,t)dx \to m\delta_{x_0} + |\nabla u|^2(x, T_0)dx, \tag{4.4}$$

for $t \uparrow T_0$, as Radon measures. Here, δ_{x_0} denotes the $\delta-$mass at x_0.

Now, we begin to prove Theorem 1.3 and Theorem 1.4. Firstly, it is easy to see that Lemma 4.1, Lemma 4.3 and Theorem 1.2 imply Theorem 1.3. In fact,

Proof of Theorem 1.3. By Lemma 4.1, we can find a sequence $t_n \uparrow \infty$ such that

$$\lim_{n\to\infty}\int_M |\partial_t u|^2(\cdot, t_n)dx = 0 \quad \textit{and} \quad E(u(\cdot, t_n)) \leq E(u_0).$$

Take the sequence $u_n = u(\cdot, t_n)$, $\tau(u_n) = \partial_t u(\cdot, t_n)$ in Theorem 1.2. Combining this with Lemma 4.3, the conclusion of Theorem 1.3 follows immediately. ☐

Proof of Theorem 1.4. By Lemma 4.1, Lemma 4.2, Lemma 4.3 and Theorem 1.2, the proof of Theorem 1.4 is almost the same as the proof of Theorem 1.3 in [12] (Page 32). Here, we omit the details. ☐

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MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES
INSELSTRASSE 22, 04103 LEIPZIG, GERMANY
DEPARTMENT OF MATHEMATICS, LEIPZIG UNIVERSITY
04081 LEIPZIG, GERMANY
E-mail address: `jost@mis.mpg.de`

DEPARTMENT OF MATHEMATICS, TSINGHUA UNIVERSITY
HAIDIAN DISTRICT, BEIJING 100084, CHINA
MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES
INSELSTRASSE 22, 04103 LEIPZIG, GERMANY
E-mail address: `leiliu@mis.mpg.de, llei1988@mail.ustc.edu.cn`

SCHOOL OF MATHEMATICAL SCIENCES, SHANGHAI JIAO TONG UNIVERSITY
DONGCHUAN ROAD 800, SHANGHAI 200240, CHINA
E-mail address: `mizhu@sjtu.edu.cn`

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