

Asymptotically hyperbolic normalized Ricci flow and rotational symmetry

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We consider the normalized Ricci flow evolving from an initial metric which is conformally compactifiable and asymptotically hyperbolic. We show that there is a unique evolving metric which remains in this class, and that the flow exists up to the time where the norm of the Riemann tensor diverges. Restricting to initial metrics which belong to this class and are rotationally symmetric, we prove that if the sectional curvature in planes tangent to the orbits of symmetry is initially nonpositive, the flow starting from such an initial metric exists for all time. Moreover, if the sectional curvature in planes tangent to these orbits is initially negative, the flow converges at an exponential rate to standard hyperbolic space. This restriction on sectional curvature automatically rules out initial data admitting a minimal hypersphere.

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1. Introduction

Conformal compactification techniques have proved to be very helpful in general relativity, where they were pioneered by Roger Penrose [18]. Subsequently they were also found to facilitate the study of Riemannian manifolds with asymptotic ends, especially asymptotically hyperbolic manifolds, on which the sectional curvatures approach -1 at infinity. The most important case is that of complete *Poincaré-Einstein* manifolds. A Poincaré-Einstein manifold is one whose metric is conformally compactifiable and Einstein. These manifolds were studied in a seminal paper of Fefferman and Graham [7] and play an important role in the AdS/CFT correspondence of string-theoretic physics. This line of enquiry has led to important advances in conformal geometry, yielding hitherto unknown invariants for the conformal metric carried on the boundary-at-infinity associated to the conformal compactification.

There are many unanswered and tantalizing questions about Poincaré-Einstein (PE) manifolds and the related but more general class of *asymptotically Poincaré-Einstein* (APE) manifolds [3]. For example, can one construct a non-constant curvature PE manifold whose conformal infinity is a connected, compact hyperbolic space? Of particular interest for physics, can one construct an APE metric with this conformal infinity which obeys the so-called *static Einstein equations* with non-constant lapse function? The existence of such metrics is not even confirmed, and an analytic construction would likely be difficult.

It is thus conceivable that the detailed study of these metrics will proceed mostly by numerical techniques.¹ Figueras and Wiseman [8] have pursued the numerical construction of Poincaré-Einstein metrics using a relaxation method based on the Ricci flow (see [21] for an overview), suitably normalized to account for sectional curvatures which asymptote to -1 . These numerical results have already had significant impact in the physics of the Randall-Sundrum model. Yet the rigorous analytical analysis of the underlying flow technique is not well-developed. While the relevant physics may be too complicated to study effectively without numerics, the successful use of the flow technique in this numerical approach and the important physics conclusions which follow from it provide compelling reasons for mathematicians to study the long-time existence and convergence behaviour of the flow in those settings which are simple enough to address analytically.

¹A similar situation arises for Calabi-Yau 3-folds; see [6] for a discussion.

The flow in question is the *normalized Ricci flow* (NRF) on an n -manifold (\mathcal{M}, g) , given by the evolution equation

$$(1.1) \quad \frac{\partial g}{\partial t} = -2(\operatorname{Ric}(g) + (n-1)g) =: -2E(g).$$

The analysis of this flow was initiated in [2], wherein the short-time existence in the asymptotically hyperbolic category was proved. Qing, Shi, and Wu [19] independently studied this flow, and in addition to the short-time existence also obtained long-time existence if the initial curvature was sufficiently pinched.

This raises the question of long-time existence for “large data”; i.e., when the curvature pinching is removed. Obviously this is a very difficult question in general, but ought to be tractable under conditions of high symmetry at least. Apart from its mathematical interest, there is already much to be explained in the numerical work discussed above, which rigorous convergence results could address. In the present paper we study long-time existence (and convergence) of this flow for the simplest nontrivial case of a complete metric evolving according to equation (1.1), namely that which starts from initial data that are rotationally symmetric and asymptotically hyperbolic.

Our approach is to study the long-time existence of the flow (1.1) for rotationally symmetric metrics that asymptotically hyperbolic in a weak sense. These metrics have curvatures approaching -1 near infinity but require no additional smoothness of a conformal compactification.

We begin by considering the general short-time existence of solutions of the normalized Ricci flow within the asymptotically hyperbolic category, without any symmetry assumptions. It has been known since the work of WX Shi [20] that for given smooth initial data on a complete manifold there is a unique Ricci flow evolution of this data for an interval of time $0 \leq t < T$ (for some $T \in (0, \infty]$) such that the curvature is bounded (by a T -dependent bound), a result that also applies to normalized Ricci flow. In Section 2 we collect a number of facts that imply, if the initial data are conformally compact and asymptotically hyperbolic, that there is a unique asymptotically hyperbolic evolution $g(t)$ under normalized Ricci flow on this time interval. We show that the asymptotically hyperbolic evolution can be continued to $t = T$ and beyond unless the norm of the Riemann curvature diverges as $t \nearrow T$. Let K be the type-(0, 4) algebraic curvature tensor

$$(1.2) \quad K_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}.$$

We prove

Theorem 1.1 (AH existence and unique continuation). *Suppose that g_0 is smoothly conformally compact and asymptotically hyperbolic. Then the normalized Ricci flow has a unique solution which remains conformally compact and asymptotically hyperbolic in $[0, T_M)$. If $T_M < \infty$, then*

$$(1.3) \quad \limsup_{t \nearrow T_M} \sup_{p \in \mathcal{M}} |\text{Rm} + K|_g = \infty.$$

The important point here is that conformally compact asymptotically hyperbolic metrics have a certain asymptotic expansion near infinity. One must show first that initial data which have such an expansion will always yield a unique flow with bounded curvature, at least for a finite interval of time, and next that the form of the expansion is preserved for as long as the flow exists. The existence and uniqueness of a bounded curvature flow follows from earlier work [2, 4, 20], and indeed the ingredients for the unique continuation are already in the literature. We thus take this opportunity to explicitly state the unique continuation result.

Beginning in Section 3, we specialize to rotational symmetry. We consider a related, gauge-fixed flow for rotationally symmetric metrics in which the metric is expressed at all times in area-radius coordinates. This flow reduces to a single parabolic equation for a function f . This function describes the norm of the radial coordinate vector field for an evolving, rotationally symmetric metric. This metric can be pulled back to a normalized Ricci flow, which for the given initial data is the unique Ricci flow of Shi. Certain combinations of f and its first radial derivative can be shown to be uniformly bounded along the gauge-fixed flow and these combinations pull back to the sectional curvatures of the evolving Ricci flow metric. Let λ denote the sectional curvature in 2-planes tangential to the orbits of rotational symmetry. We prove the following result.

Theorem 1.2 (Long-time existence). *Let $g(t)$ be the unique normalized Ricci flow of bounded curvature on $[0, T_{\max}) \times \mathbb{R}^n$ developing from an initial metric $g(0) = g_0$ such that g_0 is complete asymptotically hyperbolic, and rotationally symmetric. Let T_{\max} be the maximal time of existence. Then*

- (a) $g(t)$ is complete, rotationally symmetric and asymptotically hyperbolic for all $t \in [0, T_{\max})$ and
- (b) if the initial sectional curvature in 2-planes tangential to the orbits of rotational symmetry is $\lambda(g_0) \leq 0$, then
 - (i) $T_{\max} = \infty$,
 - (ii) $\lambda(g(t)) \leq 0$ for all $t > 0$, and

- (iii) if $\lambda(g_0) < 0$, there are constants $C, c > 0$ depending only on n and g_0 such that $|\text{Rm} + K| < Ce^{-ct}$.

Remark 1.3. Assumption (b) above implies that the initial data contain no closed minimal hypersurfaces.

Proof of Remark. Consider first a minimal hypersphere that is an orbit of the rotational symmetry group. But in rotational symmetry the Gauss-Codazzi equations for the orbits give that

$$(1.4) \quad \frac{1}{r^2} = \lambda + \frac{H^2}{(n-1)^2},$$

where r is the intrinsic radius of curvature of an orbit and H is its mean curvature. When $\lambda \leq 0$, then

$$(1.5) \quad \frac{1}{r^2} \leq \frac{H^2}{(n-1)^2},$$

so $H > 0$, and then no orbit can be a minimal hypersphere.

The general result now follows from the standard barrier argument. A closed minimal hypersurface Σ has a point p which maximizes the distance from the origin of rotational symmetry. At p , Σ is tangent to a distance sphere from the origin (and thus an orbit of rotational symmetry) bounding a closed ball B . By the above paragraph, this distance sphere is mean convex (indeed, by symmetry, convex). But by the maximum principle for minimal surfaces, any minimal hypersurface tangent to a mean convex surface at p cannot lie entirely within B , contradicting that p maximizes distance from the origin. \square

This leaves open the possibility that for initial data containing no minimal hypersphere but having λ somewhere positive, a neckpinch could form during the flow. This intriguing possibility cannot happen for rotationally symmetric, asymptotically flat Ricci flow [17], and would be worthy of further study (possibly numerically).

By Theorem 1.2, provided that the sectional curvatures in planes tangent to the orbits of symmetry are initially $\lambda < 0$, the norm of the Riemann tensor is not only uniformly bounded but all sectional curvatures decay to -1 exponentially along the flow. We are then able to show that the flow converges in the infinite-time limit to hyperbolic space.

Theorem 1.4 (Convergence). *Let $g(t)$, g_0 be as in Theorem 1.2. Choose any increasing, unbounded sequence t_n . Let $O \in \mathcal{M}$ denote the origin of rotational symmetry. Then $(\mathcal{M}, g(t_n), O)$ converges in the pointed Cheeger-Gromov sense to standard hyperbolic space.*

This paper is organized as follows. Section 2 contains background on conformally compactifiable and asymptotically hyperbolic manifolds. We also present certain standard Ricci flow results, which we adapt to the normalized Ricci flow 1.1 for use in the proof of Theorem 1.1 given at the end of that section. In Section 3, we specialize to rotational symmetry. We outline our strategy, which involves introducing a gauge-fixed, rotationally symmetric flow which pulls back to normalized Ricci flow but which has the virtue of reducing to a single parabolic flow equation, a strategy also used in earlier work on Ricci flow in asymptotically flat manifolds [10, 17]. We discuss related work of Ma and Xu [15] in this section. We also list certain basic results that we will need later. Section 4 contains a parabolic maximum principle adapted to the singular PDEs which arise, based on the strong maximum principle of Hopf. We apply this principle to prove that no neckpinches can form under our rotationally symmetric flow if no minimal hypersphere is present in the initial data. In Section 5, we find bounds on the sectional curvature λ in 2-planes tangent to the orbits of the rotational symmetry. In Section 6, we find that the difference between λ and the sectional curvature κ in planes containing the radial direction decays exponentially to zero, provided $\lambda < 0$. Combining these results with the continuation principle of Theorem 1.1, the proofs of Theorems 1.2 and 1.4 follow easily. These are given in Section 7.

1.1. Conventions

Our conventions generally follow those of the book [5]. However, we lower the index of the Riemann tensor $\text{Riem} = R_{kli}{}^j$ into the fourth position, so $g(\cdot, \text{Riem}) = g_{jp} R_{kli}{}^p = R_{klij}$. We sometimes denote this object (i.e., the Riemann tensor with four lower indices) by Rm .

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2. Flow of conformally compact asymptotically hyperbolic metrics

2.1. Definitions

A manifold (\mathcal{M}, g) with an asymptotic end is said to be (*smoothly*) *conformally compactifiable* if there is a (smooth) manifold-with-boundary (M, \tilde{g}) , an embedding $i : \mathcal{M} \hookrightarrow M$ that maps \mathcal{M} onto the interior of M , and a function $\Omega : M \rightarrow [0, \infty)$ such that $g = i^*(\Omega^{-2}\tilde{g})$ where Ω vanishes to first order precisely on ∂M . Then (\mathcal{M}, g) is *conformally compact asymptotically hyperbolic* if there is a smooth boundary defining function Ω , such that $|d\Omega|_{\Omega^2 g}^2 = 1$ on ∂M and $\Omega^2 g$ extends to a smooth metric on ∂M . For any representative metric $g_{(0)}$ of the induced conformal class on the boundary, we may solve for a new *special defining function* x so that $|dx|_{x^2 g}^2 \equiv 1$ on a collar of the boundary. In this neighborhood, the metric may be decomposed as

$$(2.1) \quad g = \frac{dx^2 + g_x}{x^2},$$

with g_x a smooth family of metrics on ∂M and $g_x|_{x=0} = g_{(0)}$.

Let K denote the +1 constant curvature 4-tensor

$$(2.2) \quad K_{ijkl} = g_{il}g_{jk} - g_{ik}g_{jl}.$$

A computation shows that asymptotically hyperbolic metrics are complete and of bounded curvature; moreover

$$(2.3) \quad |\text{Rm} + K|_g = O(x).$$

2.2. Normalized Ricci flow: Unique continuation

We consider the NRF given by (1.1) in two related settings, that of complete metrics of bounded curvature, and that of conformally compact asymptotically hyperbolic metrics.

The following result allows us to apply certain standard Ricci flow results to the normalized Ricci flow.

Lemma 2.1. *Define*

$$(2.4) \quad \begin{aligned} u(t) &:= \frac{e^{2(n-1)t} - 1}{2(n-1)}, \\ \hat{g} \circ u &:= e^{2(n-1)t}g = (1 + 2(n-1)u)g. \end{aligned}$$

Then $g(t)$ is a normalized Ricci flow for $t \in [0, T]$ with initial metric $g(0) = g_0$ if and only if $\hat{g}(u)$ is a Ricci flow for $u \in [0, U]$ with initial metric $\hat{g} = g_0$, where $U = \frac{e^{2(n-1)T} - 1}{2(n-1)}$.

Proof. Compute $\frac{\partial}{\partial t} \hat{g} \circ u = \frac{\partial}{\partial t} (e^{2(n-1)t}g)$ and use (1.1). □

We recall the fundamental existence result for the Ricci flow on non-compact manifolds:

Theorem 2.2 (WX Shi [20]). *Let $(M, g_{ij}(x))$ be an n -dimensional complete noncompact Riemannian manifold with Riemann curvature Rm satisfying $|R_{ijkl}|^2 \leq k_0$ on M for some $k_0 \in \mathbb{R}$. Then there exists a constant $T(n, k_0) > 0$ such that the evolution equation*

$$(2.5) \quad \begin{aligned} \partial_t g &= -2\text{Rc}, \\ g(0) &= g_0, \end{aligned}$$

has a smooth solution $g_{ij}(t, x) > 0$ for a short time $0 \leq t \leq T(n, k_0)$. Furthermore, for any integer $m \geq 0$, there exist constants $c_m = c_m(n, m, k_0)$ such that

$$(2.6) \quad \sup_{x \in \mathcal{M}} |\nabla^m R_{ijkl}(t, x)|^2 \leq \frac{C_m}{t^m}, \quad 0 \leq t \leq T(n, k_0).$$

Regarding uniqueness, we have the following:

Theorem 2.3 (Chen and Zhu [4]). *Let $(M^n, g_{ij}(x))$ be a complete non-compact Riemannian manifold of dimension n with bounded curvature. Let $g_{ij}(t, x)$ and $\bar{g}_{ij}(t, x)$ be two solutions to the Ricci flow on $[0, T] \times M$ with the same $g_{ij}(x)$ as initial data and with bounded curvatures. Then $g_{ij}(t, x) = \bar{g}_{ij}(t, x)$ for all $(t, x) \in [0, T] \times M^n$.*

As a consequence, we immediately obtain the following

Corollary 2.4. *Theorems 2.2 and 2.3 apply with the flow (2.5) replaced by (1.1), and the estimates (2.6) become*

$$(2.7) \quad \sup_{x \in M} |\nabla_g^m R_{ijkl}(t, x)|^2 = \sup_{x \in M} |e^{-2(n-1)t} \nabla_{\hat{g}}^m \hat{R}_{ijkl}(t, x)|^2 \leq \frac{C_m e^{-2(n-1)t}}{t^m}.$$

Proof. Simply substitute $g = e^{-2(n-1)t} \hat{g}$ in (1.1). The decay estimate (2.7) follows from an elementary Riemann curvature scaling argument, which yields that $\text{Rm}(g) = e^{-2(n-1)t} \text{Rm}(\hat{g})$. \square

As an AH metric g_0 is complete and of bounded curvature, the results above imply that there is a solution to the NRF starting at g_0 for a short time. We call this *Shi's flow*. Here is the basic long-time existence criterion for this flow [see [5], page 118].

Theorem 2.5. *If g_0 is a smooth metric on a noncompact complete manifold with bounded curvature, the normalized Ricci flow has a unique solution $g(t)$ with $g(0) = g_0$ on a maximum time interval $0 \leq t < T_M \leq \infty$. If $T_M < \infty$, then*

$$(2.8) \quad \limsup_{t \rightarrow T_M} \sup_p |\text{Rm} + K|_g = \infty.$$

Theorem 1.1 concerns the NRF starting at a conformally compact asymptotically hyperbolic metric. The results above alone do not imply that the short time solution to the NRF remains within this class. However, by earlier work of the first author, the unique solution to the NRF remains conformally compact AH for a short time. One could worry that $g(t)$ ceases to be conformally AH at time earlier than the maximal time of existence as a Shi flow. Theorem 1.1 guarantees that this does not happen. We now prove that theorem.

Proof of Theorem 1.1. In the statement of the theorem, the time T_M represents the maximal interval of existence of the NRF as a Shi flow. Let $T_1 > 0$ denote the maximal interval of existence within the class of smoothly conformally compact asymptotically hyperbolic metrics. Suppose by way of contradiction that $T_1 < T_M$.

On $[0, T_1]$, $g(t)$ is a complete metric with uniformly bounded curvature, and so by the work of Chen-Zhu [4] (see in particular Theorem 2.1 and observe that the estimates allow us to continue a solution to $[0, T_1]$), there exists a solution to the harmonic map heat flow coupled with the Ricci flow on $[0, T_1]$ that produces a solution $\tilde{g}(t)$ to the normalized Ricci DeTurck flow

on $[0, T_1]$ with uniform control of curvature and all covariant derivatives. But now since $\tilde{g}(0) = g_0$ is conformally compact asymptotically hyperbolic the regularity results in [2, Section 5] give that $\tilde{g}(t)$ is smoothly conformally compact AH on $[0, T_1]$. Pullback by the DeTurck diffeomorphism now extends the conformally compact AH solution of the NRF on $[0, T_1]$, contradicting the maximality of T_1 . So $T_1 = T_M$.

The conformally compact flow now coincides with the Shi flow, and so the previous extension theorem applies. \square

Note that this theorem says nothing about the regularity of conformal compactification of a *limiting* Poincaré-Einstein metric. Should the flow converge to a limit, nothing here requires the limit metric to be smoothly conformally compact.

3. Rotational symmetry

3.1. Introduction

There are a number of approaches to rotationally symmetric Ricci flow, see for example [1, 12, 15, 17]. A time-dependent rotationally symmetric metric can be parametrized as

$$(3.1) \quad g = \phi^2(t, \rho)d\rho^2 + \psi^2(t, \rho)g(\mathbb{S}^{n-1}, \text{can}).$$

The Ricci flow of this metric leads to a system of two equations for ϕ and ψ which, as usual, is not strictly parabolic. One can use the DeTurck trick, but for long-time existence this is inconvenient since it leads to a parabolic system with two unknown functions. Instead, other methods are available which effectively reduce the problem to a single parabolic equation for a single function.

One such method is to fix a coordinate gauge, and modify the Ricci flow by a diffeomorphism term to preserve the gauge during the flow. The two obvious choices are area-radius coordinates (in which $\psi(t, \rho) = \rho$ for all $t \geq 0$) and normal coordinates about the origin of rotational symmetry (in which $\phi(t, \rho) = 1$ for all $t \geq 0$). The former leads to a single parabolic equation for ψ , while the latter leads to a considerably more complicated coupled system which we will not consider further. However, a variant on the normal coordinate strategy leads to a common approach in the literature, in which the coordinate vector field $\frac{\partial}{\partial \rho}$ is replaced by $\frac{\partial}{\partial s} := \frac{1}{\varphi} \frac{\partial}{\partial \rho}$. Then $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ form a non-commuting set of vector fields on $[0, T) \times M$ (the partial

derivative notation notwithstanding). This approach decouples ϕ from the ψ equation, leaving a single parabolic equation for ψ ; the solution for ϕ can be found after the ψ equation is solved.

While both the area-radius gauge approach and the non-commuting vector fields approach lead to a single parabolic differential equation, in each case the resulting equation has a singular point at $\rho = 0$, as one expects when employing polar coordinates. However, this may be more than just a coordinate artifact since, if sectional curvatures were to become unbounded, symmetry would suggest that this will occur at the origin. This issue does not arise, or at least is not central, in treatments of short-time existence or in attempts to find exact, rotationally symmetric solutions, but for us it will consume the bulk of the effort in what follows.

In the sequel, we have chosen the area-radius approach. This approach at first glance seems to have a disadvantage. Area-radius coordinates break down at minimal hyperspheres, so this approach is valid only when there are none. We will, however, show that if the initial data contain no minimal hyperspheres then none form during the evolution. Hence, this is a restriction on the initial data, and as noted by Remark 1.3 it is a consequence of a restriction that we already impose on the sectional curvature.

Were we to employ the alternative approach of non-commuting vector fields, it would be unlikely that we could remove this restriction and thereby allow minimal hyperspheres in the initial data,² even though the issue of coordinate breakdown at minimal hyperspheres would no longer arise. To see this, consider the following gedankenexperiment, posed in the $n = 3$ case for simplicity. Consider $\mathbb{R} \times \mathbb{S}^2$ with an asymptotically hyperbolic (at both ends) rotationally symmetric metric; a time-symmetric slice of a Kottler (i.e., AdS-Schwarzschild) metric will do. This contains a topologically essential minimal sphere, a neck. Under normalized Ricci flow, the neck may at first expand, but a standard argument from the analysis of the Ricci flow of closed 3-manifolds with nontrivial second homotopy can be adapted to this setting and shows that the neck eventually will contract and will form a neckpinch in finite time (see, e.g., [16, pp 420–429, especially Lemma 18.11]). Allow the contraction to proceed almost to the singular time, but stop the flow before singularity formation. The neck will now have area of order ϵ^2 for a small positive ϵ .

²Perhaps we could *relax* the restriction without removing it entirely. However, we are able to study the $\lambda(0) \leq 0$ (as opposed to $\lambda(0) < 0$) case with our current methods, and cannot find evidence for convergence for data with $\lambda(0)$ not strictly negative. This may be (weak) evidence that $\lambda < 0$ may be necessary for convergence.

Now consider a 2-sphere of radius much larger than $1/\epsilon^2$, lying in one of the two asymptotic regions. Perform a surgery that removes the asymptotic region beyond that sphere, and smoothly attach in its place a sufficiently large (and thus nearly flat) 3-sphere with a disk removed. This will require some smoothing near the surgery region. The result is \mathbb{R}^3 with a very small minimal surface, and the geometry for quite some distance around that minimal surface is insensitive to the surgery.

Now restart the normalized Ricci flow. Pseudo-locality considerations, if they can be applied here, would suggest that the evolution near the neck will be nearly unaffected by the surgery, and since a singularity would have formed immediately had the flow not been stopped, the same will happen now.

It therefore seems plausible that the restrictions inherent in the area-radius approach may not be as onerous as they may at first seem — perhaps greater generality is possible, but plausibly some form of restriction will always be necessary if the flow is to exist for all $t > 0$ and converge. The current form of these restrictions accommodates our approach in which the analysis becomes relatively straightforward (though not entirely so, since the singular point at the origin must be carefully dealt with).

We take this opportunity to mention related work by Ma and Xu [15]. These authors study the behavior of the un-normalized Ricci flow within a class of rotationally symmetric metrics of asymptotically hyperbolic type, using the non-commuting vector fields approach. Nonetheless, they require *all* sectional curvatures to be negative, whereas we have no need to restrict the sectional curvature in radial 2-planes. It is thus implicit in their assumptions, as in ours, that there are no initial minimal hyperspheres. Their flow can be transformed to a flow of our type, but then the asymptotic sectional curvatures would vary in time (according to a simple scaling law). In this transformed picture, their work amounts to the study of the same flow equation and initial conditions, but with a different boundary (rather, asymptotic) condition. They find long-time existence and convergence. Their arguments appear to require a maximum principle for singular differential equations, but the precise nature of this principle is not made explicit in [15] (*cf* our Propositions 4.1 and Corollary 4.2, which perhaps can be used to close this apparent gap).

3.2. Rotational symmetry in area-radius coordinates

Our strategy here is similar to that of [17, Section 4.1], but the smoothly conformally compactifiable setting means that differences arise in the treatment of the asymptotics.

By Theorem 1.1 and using that Ricci flow (including normalized Ricci flow) preserves isometries, if we start from a rotationally symmetric metric $g_0 = g(0)$, then we may write the solution to (1.1) in the form of (3.1) on some time interval $t \in [0, T_M)$ (possibly $T_M = \infty$). A computation shows that the normalized Ricci flow (1.1) for the ansatz of (3.1) devolves to

$$(3.2) \quad \begin{aligned} \frac{\partial \phi}{\partial t} &= \frac{(n-1)}{\psi \phi} \frac{\partial^2 \psi}{\partial \rho^2} - \frac{(n-1)}{\psi \phi^2} \frac{\partial \phi}{\partial \rho} \frac{\partial \psi}{\partial \rho} - (n-1)\phi, \\ \frac{\partial \psi}{\partial t} &= \frac{1}{\phi^2} \frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\phi^3} \frac{\partial \psi}{\partial \rho} \frac{\partial \phi}{\partial \rho} + \frac{(n-2)}{\phi^2 \psi} \left(\frac{\partial \psi}{\partial \rho} \right)^2 - \frac{n-2}{\psi} - (n-1)\psi. \end{aligned}$$

As an initial condition, we may choose ρ to be the distance from the centre of symmetry with respect to the g_0 metric, so $\phi(0, \rho) \equiv 1$. Since the flow is smooth, ϕ remains bounded and even as a function of ρ near $\rho = 0$, and $\phi(t, 0) = 1$ (all of these statements hold at least for a short time). Similarly, $\psi(t, \rho) \sim \rho$ remains an odd function of ρ near $\rho = 0$. Since g_0 is conformally compact AH, then $\psi(0, \rho) \sim e^\rho$, as $\rho \rightarrow \infty$. By Theorem 1.1, $g(t)$ remains conformally compact AH. This ensures that $\psi(t, \rho) \sim e^\rho$ as $\rho \rightarrow \infty$ at least for a short time as well. From these bounds, it follows that distance with respect to $g(t)$ is comparable to distance with respect to g_0 so that $\Omega = e^{-\rho}$ is a boundary defining function for the conformal infinity.

In view of the discussion of the last subsection, we prefer the ansatz

$$(3.3) \quad \hat{g}(t) = f^2(t, r)dr^2 + r^2g(\mathbb{S}^{n-1}, \text{can})$$

for the evolving metric on (\mathbb{R}^n, g) . We now derive the equation that f must satisfy. By the change-of-variables formula, using the notation $\phi_t(r) := \phi(t, \rho)$, $\psi_t(r) := \psi(t, \rho)$, we have

$$(3.4) \quad f(t, r) = \frac{\phi_t \circ \psi_t^{-1}(r)}{(\partial_\rho \psi_t) \circ \psi_t^{-1}(r)}.$$

This change of variables arises from a diffeomorphism of \mathcal{M} . The next lemma discusses this, after which a subsequent lemma extends the diffeomorphism to M such that the asymptotic structure is preserved. ³

Lemma 3.1. *Let $g(t) = \phi_t^2(\rho)d\rho^2 + \psi_t^2(\rho)g(S^{n-1}, \text{can})$ be the smooth, rotationally symmetric, conformally compactifiable and asymptotically hyperbolic flow of metrics obeying (1.1) on \mathbb{R}^n developing from an initial metric $g_0 = g(0)$ that has no minimal hypersphere. Let $F_t : \mathcal{M} \rightarrow \mathcal{M}$ be the map with components $(r, \theta^A) = F_t(\rho, \theta^A)$ such that $\hat{g} = (F_t^{-1})^* g(t) = f^2(t, r)dr^2 + r^2g(S^{n-1}, \text{can})$. Then F_t is a smooth diffeomorphism and \hat{g} is smooth.*

Proof. Since we are working in polar coordinates, strictly speaking, the map F_t defined above is defined on \mathcal{M} punctured at the pole. Since F_t is the identity in the angular components, we must check that the radial map $\psi_t : (0, \infty)_\rho \rightarrow (0, \infty)_r$ is invertible and has odd parity in r as $r \rightarrow 0$. For convenience we extend ψ_t by continuity to zero; i.e., we regard $\psi_t : [0, \infty)_\rho \rightarrow [0, \infty)_r$, however we only use that ψ_t and its inverse are smooth on $(0, \infty)$, and we use their asymptotic behaviour.

Now it is elementary that the mean curvature of constant- ρ hypersurfaces

$$(3.5) \quad H = \frac{(n-1)\partial_\rho\psi_t}{\phi_t\psi_t} = \frac{n-1}{rf}$$

tends to infinity at a fixed point of rotational symmetry (i.e., the origin) and tends to $n - 1$ on an asymptotically hyperbolic end. Furthermore, if g_0 has no minimal hypersphere, then either the mean curvature $H(0, \rho) = \frac{(n-1)\partial_\rho\psi_0}{\psi_0}$ of the constant- ρ hypersurfaces is greater than $n - 1$ or it achieves a minimum $H_{\min} > 0$ at some finite ρ ; in either case, it is bounded away from zero. Then so are both $H(t, \cdot)$ and $\frac{\partial_\rho\psi_t}{\psi_t}$ for some interval $t \in [0, T)$. Likewise, taking a smaller T if necessary, we have $0 < C \leq \phi_t$ (using that $\phi_0 = 1$). Since $H(t, \rho) := \frac{(n-1)\partial_\rho\psi_t}{\phi_t\psi_t} > 0$ we may conclude that $\frac{\partial_\rho\psi_t}{\psi_t} > 0$ for $t \in [0, T)$. Then ψ_t is monotonic in ρ . Hence ψ_t is injective.

Surjectivity follows because $\psi_t(0) = 0$ ($SO(n)$ fixes an origin in \mathbb{R}^n), $\psi_t(\rho) \rightarrow \infty$ as $\rho \rightarrow \infty$ (there are $SO(n)$ orbits of arbitrarily large area in a conformally compactifiable, rotationally symmetric manifold), and $\psi_t(\cdot)$ is continuous. Thus ψ_t is invertible, and hence so is F_t ,

Smoothness of $\psi_t^{-1}(r)$ on the open interval $(0, \infty)$ follows from the same simple inverse function theorem argument by differentiating the composition

³Equation (3.4) corresponds to [17, equation (4.13)], except that the latter has a minor error in its denominator: ∂_r should be ∂_ρ .

$\psi_t^{-1}(\psi_t(\rho)) = \rho$ that will be used to prove Lemma 3.4 below. Moreover, since ψ_t is an odd function of ρ as $\rho \rightarrow 0$, one may check that $\psi_t^{-1}(r)$ will be odd as $r \rightarrow 0$.

The explicit expression for $f(t, r)$ given in equation (3.4) and the parity conditions for ϕ_t and ψ_t and its inverse now show that f is an even function of r . It is easy to check that the leading coefficient is 1.

Finally, although the diffeomorphism F_t is not defined at the pole, the resulting pullback metric will still be smooth. Since r is just the area-radius coordinate for $g(t)$, the condition that $g(t)$ is smooth at $r = 0$ is precisely the condition that there is a coordinate system in which the metric at $r = 0$ becomes the flat metric in polar coordinates $\hat{g} = dr^2 + r^2g(S^{n-1}, \text{can})$ plus corrections given by even powers of r . Thus since f has leading coefficient 1 and is even in r , $\hat{g}(t)$ is smooth across $r = 0$. □

Thus the flow (3.3) arises from (3.1) by pullback along the inverse of the time-dependent diffeomorphism $(r, \theta^A) = F_t(\rho, \theta^A) = (\psi_t(\rho), \theta^A)$. Let the generator of the inverse diffeomorphism flow be denoted by $X_t \equiv X(t, \cdot)$. This is not a DeTurck type vector field defined by the difference between the connection of the flowing metric and a fixed background connection. Since F_t is constant in time in the angular variables, X_t will have vanishing angular components and

$$(3.6) \quad X_t = - \left(\frac{\partial \psi_t}{\partial t} \circ \psi_t^{-1}(r) \right) \frac{\partial}{\partial r}.$$

From this and equations (3.2) it follows that

$$(3.7) \quad X_t = \left[\frac{1}{f^3(r)} \frac{\partial f}{\partial r} + \frac{(n-2)}{r} \left(1 - \frac{1}{f^2(r)} \right) + (n-1)r \right] \frac{\partial}{\partial r}.$$

While individual terms in X_t are singular at $r = 0$, from the proof of Lemma 3.1, we have that $f \sim 1 + c(t)r^2 + \dots$ as $r \rightarrow 0$, and so X_t is not singular. Rather, it vanishes at the origin, as it must. Moreover, if the flow (3.1) solves (1.1), then the metric $\hat{g} = (F_t^{-1})^*g$ of (3.3) solves

$$(3.8) \quad \frac{\partial \hat{g}}{\partial t} = -2(\text{Rc}(\hat{g}) + (n-1)\hat{g}) + \mathcal{L}_{X_t}\hat{g}.$$

The angular components are constant in time, thus preserving ansatz (3.3) (providing another derivation of (3.7)). The evolution of the rr -component

of the metric is given by

$$(3.9) \quad \frac{\partial f}{\partial t} = \frac{1}{f^2} \frac{\partial^2 f}{\partial r^2} - \frac{2}{f^3} \left(\frac{\partial f}{\partial r} \right)^2 + \left(\frac{n-2}{r} - \frac{1}{rf^2} + (n-1)r \right) \frac{\partial f}{\partial r} - \frac{(n-2)}{r^2 f} (f^2 - 1),$$

We may write (3.9) as

$$(3.10) \quad \begin{aligned} \frac{\partial f}{\partial t} &= \Delta f - \frac{1}{f} |df|^2 + Y(f) - \frac{(n-2)}{r^2 f} (f^2 - 1) \\ Y &:= \left(\frac{n-2}{r} - \frac{n}{rf^2} + (n-1)r \right) \frac{\partial}{\partial r}. \end{aligned}$$

where, for any sufficiently differentiable function u of r alone (and possibly t), Δ denotes the scalar Laplacian $\Delta := \nabla^i \nabla_i$ for the ansatz (3.3), and so obeys

$$(3.11) \quad \Delta u(r) = \frac{1}{r^{n-1} f} \frac{\partial}{\partial r} \left(\frac{r^{n-1}}{f} \frac{\partial u}{\partial r} \right) = \frac{1}{f^2} \frac{\partial^2 u}{\partial r^2} + \frac{(n-1)}{rf^2} \frac{\partial u}{\partial r} - \frac{1}{f^3} \frac{\partial f}{\partial r} \frac{\partial u}{\partial r}.$$

We also employ the notation $|\omega| \equiv |\omega|_g := \sqrt{g^{-1}(\omega, \omega)}$ for ω a one-form, so that for the radial one-form df we have $|\omega|^2 = \left(\frac{1}{f} \frac{\partial f}{\partial r} \right)^2$.

Along the flow (3.3), define two functions

$$(3.12) \quad \kappa := \frac{1}{rf^3} \frac{\partial f}{\partial r},$$

$$(3.13) \quad \lambda := \frac{1}{r^2} \left(1 - \frac{1}{f^2} \right).$$

These are the sectional curvatures of \hat{g} , with κ being the sectional curvature in 2-planes containing $\frac{\partial}{\partial r}$ and λ being the sectional curvature in 2-planes orthogonal to $\frac{\partial}{\partial r}$. They pull back to the sectional curvatures along the normalized Ricci flow (3.1). By asymptotic hyperbolicity the latter tend to -1 as $\rho \rightarrow \infty$. Thus so do κ and λ as $r \rightarrow \infty$. Observe that then $f \sim 1/r$ for large r . Also, κ and λ are related to each other by the Bianchi identity, which in this case can be read off from (3.12) and (3.13):

$$(3.14) \quad r \frac{\partial \lambda}{\partial r} = 2(\kappa - \lambda).$$

We summarize:

Lemma 3.2. *Given a rotationally symmetric, smoothly conformally compactifiable, asymptotically hyperbolic metric g_0 with no closed minimal hypersurfaces, there is a $T > 0$ such that the flow (3.1) exists for $0 \leq t < T$, and is the pullback along F_t^{-1} of the flow (3.3). For each $t \in [0, T)$, the family of diffeomorphisms F_t is smooth in t and preserves $\text{sec} \rightarrow -1$, $f \sim 1/r$, as $r \rightarrow \infty$. For each $t \in [0, T)$, the function f defined by (3.4) is smooth in t and r and obeys equation (3.9).*

3.3. Extended diffeomorphism

As stated above F_t and its inverse are only diffeomorphisms of \mathcal{M} . However we will show that this diffeomorphism extends to a diffeomorphism of M . We prove this for thoroughness, but is not used in the sequel.

We first state a condition that allows us to check the preservation of conformal compactness.

Lemma 3.3. *Consider a rotationally symmetric metric $g(t)$, $t \in [0, T)$, of the form (3.1), with $g(0) = g_0$ and with ρ the g_0 -distance from the centre of symmetry. Then, for any $t \in [0, T)$, $g(t)$ is conformally compact if and only if any number of applications of the operator $e^\rho \partial_\rho$ to both $\phi_t^2(\rho)$ and $e^{-2\rho} \psi_t^2(\rho)$ remain bounded as $\rho \rightarrow \infty$ with t fixed.*

Proof. Recall that $\Omega = e^{-\rho}$ is a boundary defining function for M the compact manifold with boundary that contains \mathcal{M} . The condition that g is conformally compact means that $\Omega^2 g$ extends to a smooth metric on M . Thus, replacing the ρ coordinate by Ω and multiplying by Ω^2 , we obtain

$$(3.15) \quad \Omega^2 g(t) = \phi_t^2(-\log \Omega) d\Omega^2 + \Omega^2 \psi_t^2(-\log \Omega) g(\mathbb{S}^{n-1}, \text{can}).$$

Then $\phi_t^2(-\log \Omega)$ and $\Omega^2 \psi_t^2(-\log \Omega)$ are smooth on M ; i.e., any number of Ω -derivatives of these functions remains bounded as $\Omega \rightarrow 0$. Converting to ρ we find $\partial_\Omega = -e^\rho \partial_\rho$, so that if g is conformally compact, then for any $k \in \mathbb{N}$, $(-e^\rho \partial_\rho)^k(\phi_t^2(\rho))$ and $(-e^\rho \partial_\rho)^k(e^{-2\rho} \psi_t^2(\rho))$ remain bounded as $\rho \rightarrow \infty$. Conversely, if this condition holds for any k as $\rho \rightarrow \infty$, then we see that any Ω -derivative of $\phi_t^2(-\log \Omega)$ and $\Omega^2 \psi_t^2(-\log \Omega)$ remains bounded as $\Omega \rightarrow 0$. But then each Ω -derivative of the compactified metric component satisfies a Lipschitz condition which allows us to extend each derivative to the manifold with boundary. So g is conformally compact. □

Lemma 3.4. *The diffeomorphism $F_t : \mathcal{M} \rightarrow \mathcal{M} : F_t(\rho, \theta^A) \mapsto (r, \theta^A)$ defined above extends to a diffeomorphism $F_t : M \rightarrow M$ of manifolds with boundary. Consequently $\hat{g} = (F_t^{-1})^* g$ remains conformally compact.*

Proof. The diffeomorphism F_t is the identity in the angular directions and so we need only consider the radial direction. In the interior, the diffeomorphism $(0, \infty)_\rho$ to $(0, \infty)_r$ is given by $r = \psi_t(\rho)$. Rescaling, we set $\Omega = e^{-\rho}$ as before and let $s = \frac{1}{r}$. We will be interested in the composition $s = s(\Omega)$, initially defined for $\Omega > 0$ (and obviously $r > 0$) by

$$(3.16) \quad s = \frac{1}{r} = \frac{1}{\psi_t(\rho)} = \frac{1}{\psi_t(-\log \Omega)},$$

and its inverse $\Omega = \Omega(s)$, defined for $s > 0$ by

$$(3.17) \quad \Omega = e^{-\psi_t^{-1}(1/s)}.$$

To show that these diffeomorphisms extend to $\Omega = 0$ and $s = 0$, we convert the requisite Ω and s derivatives to ρ and then argue that the control on ϕ_t and ψ_t coming from conformal compactness via Lemma 3.3 gives a bound on the Ω and s derivatives down to zero. Then these derivatives are Lipschitz and extend to the boundary, proving that F_t is a diffeomorphism of manifolds with boundary.

Let \mathcal{E} denote the algebra of functions such that any number of $e^\rho \partial_\rho$ derivatives remain bounded as $\rho \rightarrow \infty$. Obviously, $e^{-\rho} \in \mathcal{E}$ and, by Lemma 3.3, $e^{-\rho} \psi(\rho) \in \mathcal{E}$. Further, $\partial_\rho \psi_t - \psi_t = e^\rho \partial_\rho (e^{-\rho} \psi_t) \in \mathcal{E}$, so that $e^{-\rho} \partial_\rho \psi_t \in \mathcal{E}$ as well. Consider the function $s = \frac{1}{\psi(-\log \Omega)}$. We find

$$(3.18) \quad \frac{ds}{d\Omega} = \frac{1}{\Omega} \frac{\partial_\rho \psi_t(-\log \Omega)}{\psi_t^2(-\log \Omega)} = \frac{\partial_\rho \psi_t(\rho)}{e^{-\rho} \psi_t^2(\rho)} = \frac{e^{-\rho} \partial_\rho \psi_t(\rho)}{e^{-2\rho} \psi_t^2(\rho)} \in \mathcal{E}$$

and so $\frac{ds}{d\Omega}$ is bounded as $\rho \rightarrow \infty$ (equivalently $\Omega \rightarrow 0$). Noting that $\frac{\partial}{\partial \Omega} = -e^\rho \frac{\partial}{\partial \rho} : \mathcal{E} \rightarrow \mathcal{E}$, we conclude that the forward map extends to the boundary.

We now consider the inverse map. We may differentiate the equation $\psi_t^{-1}(\psi_t(\rho)) = \rho$, to obtain $(\psi_t^{-1})'(r) = \frac{1}{\partial_\rho \psi_t(\rho)}$.⁴ Further, $\frac{\partial}{\partial s} = -\frac{\psi_t^2(\rho)}{\partial_\rho \psi_t(\rho)} \frac{\partial}{\partial \rho}$.

⁴Because of the absence of closed minimal hyperspheres, cf. Remark 1.3, this derivative exists everywhere (excluding the origin, since r is a polar coordinate), but here we only need it to exist on a collar neighbourhood of infinity.

Differentiating (3.17), we obtain

$$(3.19) \quad \begin{aligned} \frac{d\Omega}{ds} &= \frac{1}{s^2} e^{-\psi_t^{-1}(1/s)} (\psi_t^{-1})'(1/s) = \frac{e^{-\rho} \psi_t^2(\rho)}{\partial_\rho \psi_t(\rho)} \\ &= (e^{-\rho} \psi_t(\rho)) \left(\frac{e^{-\rho} \psi_t(\rho)}{e^{-\rho} \partial_\rho \psi_t(\rho)} \right), \end{aligned}$$

so that $\frac{d\Omega}{ds} \in \mathcal{E}$, which in turn shows that the s -derivative of Ω remains bounded as $\rho \rightarrow \infty$ ($s \rightarrow 0$). This same argument shows that

$$\frac{\partial}{\partial s} = -\frac{e^{-2\rho} \psi_t^2(\rho)}{e^{-\rho} \partial_\rho \psi_t(\rho)} e^\rho \frac{\partial}{\partial \rho},$$

so that in fact $\frac{d}{ds} : \mathcal{E} \rightarrow \mathcal{E}$. So F_t extends to a diffeomorphism of manifolds with boundary.

Finally, since s and Ω pull back to each other under the rescaled diffeomorphism, a short computation shows that \hat{g} is conformally compact. \square

4. Uniform C^0 Control of f

4.1. Maximum principle

As discussed above, the main issue we have to deal with is the possibility that the metric and curvature quantities that we wish to study become unbounded at the singular point at the origin of symmetry. We therefore would like a version of the usual maximum principle which is adapted to this situation.

We denote $\Omega_T = (0, T) \times (0, \infty)$, $\bar{\Omega}_T := [0, T] \times [0, \infty)$, and $\hat{\Omega}_T := (0, T] \times (0, \infty)$. Then

Proposition 4.1. *Let $u \in C^{0,0}(\bar{\Omega}_T) \cap C^{1,2}(\Omega_T)$ satisfy*

$$(4.1) \quad 0 \leq L(u) := a(t, r) \frac{\partial^2 u}{\partial r^2} + b(t, r) \frac{\partial u}{\partial r} + c(t, r) u - \frac{\partial u}{\partial t}$$

with $0 < C_1 \leq a(t, r) \leq C_2$, on Ω_T , with $a, b \in C^{(0,0)}(\Omega_T)$ and $c \in C^{0,0}(\hat{\Omega}_T)$. Furthermore, assume that $u(t, 0) = u_0$, that $u(t, r) \rightarrow u_\infty \in \mathbb{R}$ as $r \rightarrow \infty$, and that $\max\{u_0, u_\infty\} \geq 0$. Either

- 1) $\sup_{\bar{\Omega}_T} u = \max\{u_0, u_\infty\}$, or
- 2) $\sup_{\bar{\Omega}_T} u$ is a maximum realized at $(0, r^*)$ for some $r^* \in (0, \infty)$, or

- 3) $\sup_{\bar{\Omega}_T} u$ is a maximum realized at some point $(t^*, r^*) \in (0, T] \times (0, \infty)$, and $c(t^*, r^*) \geq 0$.

We have extended the continuity condition on c to include $t = T$ even though (4.1) holds only in Ω_T since otherwise the third alternative in the theorem could fail to hold as stated. This condition will always hold in our applications.

Proof. If neither possibility 1 nor possibility 2 above applies, then u achieves a positive maximum at some point $(t^*, r^*) \in (0, T] \times (0, \infty)$. Then $v(t, r) = u(t, r) - u(t^*, r^*)$ has maximum $v(t^*, r^*) = 0$ in $(0, T] \times (0, \infty)$ and obeys $L(v) \geq -c(t, r)u(t^*, r^*)$. Consider a small rectangle $R = [t^* - \epsilon, t^*] \times [r^* - \epsilon, r^* + \epsilon] \subset (0, T] \times (0, \infty)$ about (t^*, r^*) , such that $u(t, r) > 0$ in R . If $v(t, r)$ is not identically zero in R , it is necessary that $c(t, r) > 0$ somewhere in R so as not to contradict the strong maximum principle [14, Theorem 2.7], and by repeating the argument as we take $\epsilon \rightarrow 0$, we obtain a sequence of points (t_i, r_i) converging to (t^*, r^*) such that $c(t_i, r_i) > 0$. Thus $c(t^*, r^*) \geq 0$. Alternatively, if $v(t, r) \equiv 0$ in R , then $u(t, r) \equiv u(t^*, r^*)$ and therefore $u(t, r) > 0$ in R , and also $L(v) = L(0) = 0$ in R . But since $L(v) \geq -c(t, r)u(t^*, r^*)$, then $c(t, r) \geq 0$ throughout R , and so again $c(t^*, r^*) \geq 0$. \square

Though the inequality (4.1) applies when right-hand side is linear in u , we will apply the proposition when the right side of the equation is only quasi-linear. Thus the coefficients a , b , and c may arise from compositions of continuous functions; e.g., $b(t, r) = \beta(t, r; p, q)$ with $p = u(t, r)$ and $q = \frac{\partial u}{\partial r}(t, r)$. Alternatively, in the proof one instead could appeal to the strong maximum principle for quasi-linear equations [13, Theorem 2]. We conclude this subsection with the corresponding parabolic minimum principle.

Corollary 4.2. *Let $u \in C^{0,0}(\bar{\Omega}_T) \cap C^{1,2}(\Omega_T)$ satisfy*

$$(4.2) \quad 0 \geq L(u) := a(t, r) \frac{\partial^2 u}{\partial r^2} + b(t, r) \frac{\partial u}{\partial r} + c(t, r)u - \frac{\partial u}{\partial t}$$

with $0 < C_1 \leq a(t, r) \leq C_2$, on Ω_T , with $a, b \in C^{(0,0)}(\Omega_T)$ and $c \in C^{0,0}(\hat{\Omega}_T)$. Furthermore, assume that $u(t, 0) = u_0$, that $u(t, r) \rightarrow u_\infty \in \mathbb{R}$ as $r \rightarrow \infty$, and that $\min\{u_0, u_\infty\} \leq 0$. Either

- 1) $\inf_{\bar{\Omega}_T} u = \min\{u_0, u_\infty\}$, or
- 2) $\inf_{\bar{\Omega}_T} u$ is a minimum realized at $(0, r^*)$ for some $r^* \in (0, \infty)$, or
- 3) $\inf_{\bar{\Omega}_T} u$ is a minimum realized at some point $(t^*, r^*) \in (0, T] \times (0, \infty)$, and $c(t^*, r^*) \geq 0$.

Proof. Replace u by $-u$ in the proof of Proposition 4.1. Note that the sign condition on $c(t^*, r^*)$ in the third alternative above is *not* reversed. \square

4.2. No neckpinches form

Letting $w = f^2$, we obtain from (3.9) the evolution equation

$$(4.3) \quad \frac{\partial w}{\partial t} = \Delta w - \frac{1}{w} |dw|^2 + Y(w) - \frac{2(n-2)}{r^2} (w-1).$$

By expanding terms and dropping certain terms of definite sign, we obtain

$$(4.4) \quad \frac{\partial w}{\partial t} \leq \frac{1}{w} \frac{\partial^2 w}{\partial r^2} + \left(\frac{n-2}{r} - \frac{n}{rw} + (n-1)r \right) \frac{\partial w}{\partial r} - \frac{2(n-2)}{r^2} (w-1).$$

There is a singular point at $r = 0$ but the divisions by w cause no problem since, by long-time existence, w does not become zero for $t \leq T$ (though it tends to zero as $r \rightarrow \infty$). We may now apply the maximum principle.

Lemma 4.3. *Suppose that the flow (3.1) obeys (1.1) for $t \in [0, T_{\max})$ where $T_{\max} \in (0, \infty]$ is the maximal time of existence. Furthermore suppose that the initial metric $g_0 = g(0, \cdot)$ does not admit a minimal hypersphere.⁵ Then f is uniformly bounded for $t \in [0, T_{\max}]$, no minimal hypersphere forms, and the coordinate transformation (3.4) exists (and is invertible) for $t \in [0, T_{\max})$.*

Proof. By (3.4) and the condition that no minimal hypersphere is present initially, we then have that $\sup_r f(0, r) \leq C$ for some $C \geq 1$ (since by smoothness at $t = 0$, we have $f(0, 0) = 1$). Now let $T_0 > 0$ be the first time at which a minimal hypersphere forms along the flow (3.1), and let $T_1 := \min\{T_0, T_{\max}\}$. Consider expression (4.4) on $(0, T_1] \times (0, \infty)$, so as to avoid the singularity at $r = 0$. This expression has the same form as (4.1) with $u = w - 1$, $a = \frac{1}{w}$, $b = \frac{n-2}{r} - \frac{1}{rw} + (n-1)r$, and $c = -\frac{2(n-2)}{r^2}$. As well, $w = 1$ at $r = 0$ and $w \rightarrow 0$ as $r \rightarrow \infty$, so Proposition 4.1 applies and asserts that $w(t, r)$ is bounded above by C^2 . Thus f is uniformly bounded on $(0, T_1] \times (0, \infty)$. But by (3.5), a compact minimal hypersurface cannot form unless f diverges, and so $T_1 = T_{\max}$. \square

⁵We remind the reader that the absence of initial minimal hyperspheres follows from our assumption $\lambda(0) \leq 0$, as explained in Remark 1.3. As well, from an elementary argument presented as part of the proof of Remark 1.3, we see that the absence of minimal hyperspheres at any time t precludes the existence of any closed minimal hypersurface at time t .

We note in passing that this does not forbid compact minimal hypersurfaces from forming if some are already present initially, since then the assumption $\sup_{x \in \mathcal{M}} f(0, x) = C$ does not hold and our coordinate system fails to cover the initial manifold.

Corollary 4.4 (Preservation of nonpositive λ). *If $\sup \lambda(0, r) \leq 0$ for $r \in [0, \infty)$, then $\sup_r \lambda(t, r) \leq 0$ for $(t, r) \in [0, T_{\max}] \times [0, \infty)$.*

Proof. We first observe that if $\sup \lambda(0, r) \leq 0$, then $\sup f(0, r) \leq 1$ by equation (3.13), and then in fact we have $\max f(0, r) = 1$ since $f(0, 0) = 1$ by smoothness. But by Lemma 4.3, if $\max f(0, r) = 1$, then $\max f(t, r) = 1$ for all $(t, r) \in [0, T_{\max}] \times [0, \infty)$, and then it follows from equation (3.13) that $\sup \lambda(t, r) \leq 0$ all along the flow. \square

In the next section we will prove decay estimates for λ .

5. Evolution of λ

5.1. Evolution equation for λ

From (4.3), we can write

$$(5.1) \quad \frac{\partial}{\partial t} \left(1 - \frac{1}{f^2} \right) = \Delta \left(1 - \frac{1}{f^2} \right) + f^2 \left| d \left(1 - \frac{1}{f^2} \right) \right|^2 + Y \left(1 - \frac{1}{f^2} \right) + \frac{2(n-2)}{r^2} \left(1 - \frac{1}{f^2} \right)^2 - \frac{2(n-2)}{r^2} \left(1 - \frac{1}{f^2} \right).$$

We also need several identities, the first of which is

$$(5.2) \quad \begin{aligned} \Delta r &= \frac{1}{r^{n-1} f} \frac{\partial}{\partial r} \left(\frac{r^{n-1}}{f} \right) = \frac{n-1}{r f^2} - \frac{1}{f^3} \frac{\partial f}{\partial r} \\ &= -(n-1) \frac{1}{r} \left(1 - \frac{1}{f^2} \right) + \frac{n-1}{r} - r \kappa \\ &= -nr \lambda - r(\kappa - \lambda) + \frac{n-1}{r} \\ &= -nr \lambda - \frac{1}{2} r^2 \frac{\partial \lambda}{\partial r} + \frac{n-1}{r}. \end{aligned}$$

By straightforward manipulations, some of which use (3.13), we then further obtain the identities

$$(5.3) \quad \frac{1}{r^2}d\left(1 - \frac{1}{f^2}\right) = d\lambda + 2\lambda\frac{dr}{r},$$

$$(5.4) \quad \frac{1}{r^2}f^2\left|d\left(1 - \frac{1}{f^2}\right)\right|^2 = r^2f^2|d\lambda|^2 + 4r\lambda f^2 dr \cdot d\lambda + 4\lambda^2,$$

$$(5.5) \quad \frac{1}{r^2}\Delta\left(1 - \frac{1}{f^2}\right) = \Delta\lambda + (4 - r^2f^2\lambda)\frac{dr}{r} \cdot d\lambda - 2(n+1)\lambda^2 + \frac{2n}{r^2}\lambda$$

$$(5.6) \quad \frac{1}{r^2}Y\left(1 - \frac{1}{f^2}\right) = \left(nr\lambda - \frac{2}{r} + (n-1)r\right)f^2 dr \cdot d\lambda + 2n\lambda^2 + 2\left(n-1 - \frac{2}{r^2}\right)\lambda,$$

where $dr \cdot d\lambda := g^{-1}(dr, d\lambda) \equiv \frac{1}{f^2}\frac{\partial\lambda}{\partial r}$. Multiplying (5.1) by $1/r^2$ and using the above equations, we obtain the desired evolution equation

$$(5.7) \quad \frac{\partial\lambda}{\partial t} = \Delta\lambda + Z \cdot d\lambda + r^2f^2|d\lambda|^2 + 2(n-1)\lambda(\lambda+1),$$

where we define

$$(5.8) \quad Z := \left[\frac{(n+1)}{r} - \frac{(n-1)}{rf^2} + (n-1)r\right]f^2 dr = \left[\frac{2}{r} + (n-1)r(\lambda+1)\right]f^2 dr.$$

5.2. Decay of λ

It is convenient to define $\Lambda(t, r)$ by

$$(5.9) \quad \lambda(t, r) =: -1 + \Lambda(t, r)e^{-2(n-1)\delta^2 t},$$

where δ will be assigned values $\delta = 0$ and $\delta = 1$ when needed below. Then

$$(5.10) \quad \frac{\partial\Lambda}{\partial t} = \Delta\Lambda + Z \cdot d\Lambda + e^{-2(n-1)\delta^2 t}r^2f^2|d\Lambda|^2 + 2(n-1)(\lambda + \delta^2)\Lambda.$$

The idea now is to show that Λ is bounded uniformly (and, even better, converging) in t . A lower bound seems obvious from inspection of (5.10). However, the norm of Z diverges at $r = 0$ so when (5.10) is expressed in any coordinate system that covers the origin, the result is a singular PDE. This

was also the case with Y in (4.3), and there the strategy was to work on the domain $[0, T] \times [\epsilon, 1/\epsilon]$, $T < T_{\max}$, and then take $\epsilon \searrow 0$, using that $w \rightarrow 1$ as $r \searrow 0$. We will follow the same strategy here, but this alone is insufficient because we have no uniform control of Λ at the origin. Instead, we borrow a trick from [17] and study the functions

$$(5.11) \quad \Lambda_\alpha := \frac{r^\alpha}{(r^\alpha + \alpha)} \Lambda, \quad \alpha \in (0, 1),$$

subsequently taking $\alpha \searrow 0$. The factor $\frac{r^\alpha}{r^\alpha + \alpha}$ is chosen because we will have $\Lambda_\alpha(t, 0) = 0$, while maintaining $\Lambda_\alpha(t, r) \sim \Lambda \rightarrow 0$ as $r \rightarrow \infty$, and we will have $\lim_\alpha \Lambda_\alpha = \Lambda$ (non-uniformly).

We again need some identities. First,

$$(5.12) \quad \Lambda = \left(1 + \frac{\alpha}{r^\alpha}\right) \Lambda_\alpha,$$

so

$$(5.13) \quad d\Lambda = \left(1 + \frac{\alpha}{r^\alpha}\right) d\Lambda_\alpha - \frac{\alpha^2}{r^{\alpha+1}} \Lambda_\alpha dr.$$

Omitting the exponential factor for brevity, this allows us to write out the term in (5.10) that is quadratic in $d\Lambda$ as

$$(5.14) \quad \begin{aligned} r^2 f^2 |d\Lambda|^2 &= r^2 f^2 \left(1 + \frac{\alpha}{r^\alpha}\right)^2 |d\Lambda_\alpha|^2 \\ &\quad - 2\alpha^2 r^{1-\alpha} f^2 \Lambda_\alpha \left(1 + \frac{\alpha}{r^\alpha}\right) dr \cdot d\Lambda_\alpha + \frac{\alpha^4}{r^{2\alpha}} \Lambda_\alpha^2 \\ &= r^2 f^2 \left(1 + \frac{\alpha}{r^\alpha}\right)^2 |d\Lambda_\alpha|^2 \\ &\quad - 2\alpha^2 r^{1-\alpha} f^2 \Lambda_\alpha \left(1 + \frac{\alpha}{r^\alpha}\right) dr \cdot d\Lambda_\alpha \\ &\quad + \frac{\alpha^4 e^{2(n-1)\delta^2 t}}{r^\alpha (r^\alpha + \alpha)} (\lambda + 1) \Lambda_\alpha, \end{aligned}$$

using $f^2 |dr|^2 = 1$. In the last line, we have rewritten the quadratic combination Λ_α^2 in terms of $(\lambda + 1)\Lambda_\alpha$. This is for convenience in what follows (though the Λ_α^2 form also plays a role below), where by imposing sign conditions on λ we will convert the differential equation for Λ_α into an inequality with linear nonderivative terms (*cf.* the nonderivative term in equation (5.10)).

The $Z \cdot d\Lambda$ term is

$$\begin{aligned}
 (5.15) \quad Z \cdot d\Lambda &= \left(1 + \frac{\alpha}{r^\alpha}\right) Z \cdot d\Lambda_\alpha - \frac{\alpha^2}{r^{\alpha+1}} \Lambda_\alpha \left(\frac{2}{r} + (n-1)r(\lambda+1)\right) \\
 &= \left(1 + \frac{\alpha}{r^\alpha}\right) Z \cdot d\Lambda_\alpha - \frac{2\alpha^2}{r^{\alpha+2}} \Lambda_\alpha - \frac{\alpha^2(n-1)}{r^\alpha} (\lambda+1) \Lambda_\alpha.
 \end{aligned}$$

Next we have to work out the relationship between $\Delta\Lambda$ and $\Delta\Lambda_\alpha$. Taking the divergence of (5.13), we have

$$\begin{aligned}
 (5.16) \quad \Delta\Lambda &= \left(1 + \frac{\alpha}{r^\alpha}\right) \Delta\Lambda_\alpha - \frac{2\alpha^2}{r^{\alpha+1}f^2} \frac{\partial\Lambda_\alpha}{\partial r} \\
 &\quad + \frac{\alpha^2(\alpha+1)}{r^{\alpha+2}f^2} \Lambda_\alpha - \frac{\alpha^2}{r^{\alpha+1}} \Lambda_\alpha \Delta r \\
 &= \left(1 + \frac{\alpha}{r^\alpha}\right) \Delta\Lambda_\alpha - \frac{2\alpha^2}{r^{\alpha+1}f^2} \frac{\partial\Lambda_\alpha}{\partial r} + \frac{\alpha^2(\alpha+1)}{r^{\alpha+2}} \Lambda_\alpha \\
 &\quad - \frac{\alpha^2(\alpha+1)}{r^\alpha} \Lambda_\alpha \lambda + \frac{\alpha^2}{r^{\alpha+1}} \Lambda_\alpha \left[nr\lambda + \frac{1}{2}r^2 \frac{\partial\lambda}{\partial r} - \frac{n-1}{r} \right],
 \end{aligned}$$

where we replaced factors of $\frac{1}{r^2 f^2}$ using $\frac{1}{r^2 f^2} = \frac{1}{r^2} - \lambda$ and we used (5.2) to replace Δr by

$$(5.17) \quad \Delta r = -nr\lambda - \frac{1}{2}r^2 \frac{\partial\lambda}{\partial r} + \frac{n-1}{r}.$$

Next, we need to replace $\frac{\partial\lambda}{\partial r}$ in favour of $\frac{\partial\Lambda_\alpha}{\partial r}$, using (5.9) and (5.13). After considerable manipulation, the result can be written as

$$\begin{aligned}
 (5.18) \quad \Delta\Lambda &= \left(1 + \frac{\alpha}{r^\alpha}\right) \Delta\Lambda_\alpha + \frac{\alpha^2}{r^\alpha} \left[\frac{1}{2}r(\lambda+1) - \frac{2}{rf^2} \right] \frac{\partial\Lambda_\alpha}{\partial r} \\
 &\quad - \frac{\alpha^2}{r^\alpha} \left[\frac{n-\alpha-2}{r^2} + \frac{\alpha^2}{2(r^\alpha+\alpha)} \right] \Lambda_\alpha \\
 &\quad + \frac{\alpha^2}{r^\alpha} \left[n-\alpha-1 - \frac{\alpha^2}{2(r^\alpha+\alpha)} \right] \lambda \Lambda_\alpha.
 \end{aligned}$$

Plugging all this into (5.10) and organizing terms, we get the evolution equation for Λ_α :

$$\begin{aligned}
 (5.19) \quad \frac{\partial \Lambda_\alpha}{\partial t} &= \Delta \Lambda_\alpha + Z \cdot d\Lambda_\alpha + e^{-2(n-1)\delta^2 t} r^2 \left(1 + \frac{\alpha}{r^\alpha}\right) \left(\frac{\partial \Lambda_\alpha}{\partial r}\right)^2 \\
 &\quad - \frac{\alpha^2}{(r^\alpha + \alpha)} \left[\frac{3}{2}r(\lambda + 1) + \frac{2}{rf^2}\right] \frac{\partial \Lambda_\alpha}{\partial r} \\
 &\quad + 2(n-1)(\delta^2 + \lambda) \Lambda_\alpha \\
 &\quad - \frac{\alpha^2}{(r^\alpha + \alpha)} \left[n - 1 - \frac{\alpha^2}{2(r^\alpha + \alpha)} + \frac{(n - \alpha)}{r^2}\right] \Lambda_\alpha \\
 &\quad + \frac{\alpha^3}{(r^\alpha + \alpha)} \left[\frac{\alpha}{2(r^\alpha + \alpha)} - 1\right] \lambda \Lambda_\alpha.
 \end{aligned}$$

Comparing to (4.2), we have

$$\begin{aligned}
 (5.20) \quad c(t, r) &= 2(n-1)(\delta^2 + \lambda) \\
 &\quad - \frac{\alpha^2}{(r^\alpha + \alpha)} \left[n - 1 - \frac{\alpha^2}{2(r^\alpha + \alpha)} + \frac{(n - \alpha)}{r^2}\right] \\
 &\quad + \frac{\alpha^3}{(r^\alpha + \alpha)} \left[\frac{\alpha}{2(r^\alpha + \alpha)} - 1\right] \lambda.
 \end{aligned}$$

Proposition 5.1. $\lambda(t, r) \geq -1 + e^{-2(n-1)t} \inf_{r \in [0, \infty)} (\lambda(0, r) + 1)$ for all $(t, r) \in [0, T_{\max}) \times [0, \infty)$.

Proof. Choose some $T < T_{\max}$. If $\Lambda_\alpha \geq 0$ for all $(t, r) \in [0, T] \times [0, \infty)$ then $\lambda + 1 \geq 0$ and then $\inf_{r \in [0, \infty)} (\lambda(0, r) + 1) = \lim_{r \rightarrow \infty} (\lambda(0, r) + 1) = 0$, so the proposition is true.

Otherwise, Λ_α has a negative minimum on $[0, T] \times [0, \infty)$. In this case, consider equations (5.19, 5.20) but set $\delta = 1$ so that

$$(5.21) \quad \lambda = -1 + e^{-2(n-1)t} \left(1 + \frac{\alpha}{r^\alpha}\right) \Lambda_\alpha.$$

Now replace the λ factors in (5.20) using (5.21) and of course set $\delta = 1$. This yields

$$\begin{aligned}
 (5.22) \quad c(t, r) &= -\frac{\alpha^2}{(r^\alpha + \alpha)} \left[n - 1 - \alpha + \frac{(n - \alpha)}{r^2}\right] \\
 &\quad + e^{-2(n-1)t} \left[2(n-1) + \frac{\alpha(2(n-1) - \alpha^2)}{r^\alpha} + \frac{\alpha^4}{2r^\alpha(r^\alpha + \alpha)}\right] \Lambda_\alpha.
 \end{aligned}$$

Take $\alpha > 0$ small enough; indeed, $\alpha \in (0, 1)$ will do. Now, if Λ_α achieves a negative minimum at (t^*, r^*) in $(0, T] \times (0, \infty)$, then from the right-hand side of (5.22) we obtain

$$(5.23) \quad c(t^*, r^*) < 0.$$

But then Corollary 4.2 would imply that $t^* = 0$.

Thus in either case we may write

$$\Lambda_\alpha(t, r) \geq \inf_{r \in [0, \infty)} \Lambda_\alpha(0, r) = \inf_{r \in [0, \infty)} \frac{r^\alpha}{r^\alpha + \alpha} \Lambda(0, r) \geq \inf_{r \in [0, \infty)} \Lambda(0, r),$$

where the last inequality holds because the infimum is not positive. This yields $\Lambda_\alpha(t, r) \geq \inf_{r \in [0, \infty)} (\lambda(0, r) + 1)$ and since this bound is independent of α , we may take $\alpha \searrow 0$, yielding $\Lambda(t, r) \geq \inf_{r \in [0, \infty)} (\lambda(0, r) + 1)$. Then the proposition follows from (5.21) and from the fact that the bound does not depend on T so we can take $T \nearrow T_{\max}$. \square

On the other hand, by selecting other values of δ we can bound λ from above. For example, setting $\delta = 0$ in (5.19, 5.20) yields

$$(5.24) \quad c(t, r) = 2(n - 1)\lambda - \frac{\alpha^2}{(r^\alpha + \alpha)} \left[n - 1 - \frac{\alpha^2}{2(r^\alpha + \alpha)} + \frac{(n - \alpha)}{r^2} \right] + \frac{\alpha^3}{(r^\alpha + \alpha)} \left[\frac{\alpha}{2(r^\alpha + \alpha)} - 1 \right] \lambda,$$

with

$$(5.25) \quad \lambda = -1 + \left(1 + \frac{\alpha}{r^\alpha} \right) \Lambda_\alpha.$$

Then we obtain

Proposition 5.2. *Assume that $\sup_{r \in [0, \infty)} \lambda(0, r) \leq -1$. Then $\lambda(t, r) \leq -1$ for all $(t, r) \in [0, T_{\max}) \times [0, \infty)$.*

Proof. Assume to the contrary that $\sup_r \lambda(0, r) \leq -1$ but $\lambda(t, r) > -1$ for some (t, r) with $t > 0$. Then there is some $T < T_{\max}$ such that $\lambda < 0$ for all $t \in [0, T]$ but λ is somewhere greater than -1 . Since $\lambda + 1$ and Λ_α have the same sign, then $\Lambda_\alpha > 0$ somewhere. But $\Lambda_\alpha(0, r) \leq 0$, $\Lambda_\alpha(t, 0) = 0$, and $\Lambda_\alpha(t, r) \rightarrow 0$ as $r \rightarrow \infty$, so by Proposition 4.1 we must conclude that $c(t^*, r^*) > 0$ at some $(t^*, r^*) \in (0, T] \times (0, \infty)$.

Nonetheless, $\lambda < 0$ on $[0, T] \times [0, \infty)$ and so, for $\alpha > 0$ chosen to be small enough, then from (5.24) we see that $c(t, r)$ is strictly negative in $(0, T] \times (0, \infty)$, a contradiction. Thus we conclude that $\lambda(t, r) \leq -1$ for all $(t, r) \in [0, T] \times (0, \infty)$. \square

We may in fact weaken our assumptions to allow $\lambda(0, r) < 0$. We have seen that the bound from below is exponential and decays to -1 as $e^{-2(n-1)t}$. The decay from above is also exponential, but potentially slower.

Proposition 5.3. *Assume that $\lambda(0, r) < 0$ for all $r \in [0, \infty)$. Then there is a constant $\beta > 0$ depending only on the initial metric $g(0, \cdot) = g_0$ and n such that $\lambda(t, r) < -1 + e^{-\beta t}$ for all $(t, r) \in [0, T_{\max}] \times [0, \infty)$.*

Proof. Since Proposition 5.1 deals with the bound from below, we restrict attention to bounding $\lambda(t, r)$ from above. Thus assume that $\lambda(0, r) > -1$ for some $r \in [0, \infty)$. Since $\lambda(0, r) < 0$ for all $r \in [0, \infty)$ and tends to -1 as $r \rightarrow \infty$, then $\max_r \lambda(0, r)$ exists and lies in the interval $(-1, 0)$.

Choose any $\delta \in (0, 1)$ such that $\max_r \lambda(0, r) < -\delta^2$. Let $T_\delta \in (0, \infty]$ be the first time such that $\delta^2 + \lambda(T_\delta, r) = 0$ for some $r \in [0, \infty)$; if there is no such time then $T_\delta := T_{\max}$ (possibly infinite). But by (5.20) and for $0 < \alpha < \epsilon$ with ϵ small enough, we have

$$(5.26) \quad c(t, r) < 2(n - 1) (\delta^2 + \lambda) \leq 0$$

for $t \in [0, T_\delta]$. Thus, by Proposition 4.1, for each $\alpha \in (0, \epsilon)$, the maximum of Λ_α on $[0, T_\delta] \times [0, \infty)$ occurs at $t = 0$. Then for $0 \leq t \leq T_\delta$ and each $\alpha \in (0, \epsilon)$ we have $\Lambda_\alpha(t, r) \leq \sup_{r \in [0, \infty)} \Lambda_\alpha(0, r) = \sup_{r \in [0, \infty)} \frac{r^\alpha}{\alpha + r^\alpha} \Lambda(0, r) \leq \sup_{r \in [0, \infty)} \Lambda(0, r) = \sup_{r \in [0, \infty)} (1 + \lambda(0, r)) < 1 - \delta^2$. Since this last bound is independent of α , we may take $\alpha \rightarrow 0$ (first for $r \neq 0$, then extending to $r = 0$ by continuity), yielding

$$(5.27) \quad \lambda(t, r) \leq -1 + (1 - \delta^2) e^{-2(n-1)\delta^2 t}$$

on $[0, T_\delta] \times [0, \infty)$, and thus $T_\delta = T_{\max}$. \square

6. Evolution of κ

6.1. Evolution equation for κ

Lemma 6.1. *The evolution equation for the sectional curvature κ is*

$$(6.1) \quad \frac{\partial \kappa}{\partial t} = \Delta \kappa + [n - 1 + \kappa + (n - 2)\lambda] \left(r \frac{\partial \kappa}{\partial r} + 2\kappa \right) + \frac{2(n - 2)}{r^2 f^2} (\lambda - \kappa).$$

Proof. Starting from (4.3), we obtain

$$(6.2) \quad \begin{aligned} \frac{\partial}{\partial t} \left(-\frac{1}{f^2} \right) &= \Delta \left(-\frac{1}{f^2} \right) + \left(\frac{\partial}{\partial r} \left(-\frac{1}{f^2} \right) \right)^2 \\ &\quad + \left[\frac{(n - 2)}{r} - \frac{n}{r f^2} + (n - 1)r \right] \frac{\partial}{\partial r} \left(-\frac{1}{f^2} \right) \\ &\quad + \frac{2(n - 2)}{r^2} \left[\left(-\frac{1}{f^2} \right) + \left(-\frac{1}{f^2} \right)^2 \right]. \end{aligned}$$

Differentiating both sides of (6.2) with respect to r , multiplying by $\frac{1}{2r}$, and using that $\frac{1}{2r} \frac{\partial}{\partial r} \left(-\frac{1}{f^2} \right) = \frac{1}{r f^3} \frac{\partial f}{\partial r} = \kappa$, then we get that

$$(6.3) \quad \begin{aligned} \frac{\partial \kappa}{\partial t} &= \frac{1}{2r} \frac{\partial}{\partial r} \left(\Delta \left(-\frac{1}{f^2} \right) \right) + 4r\kappa \frac{\partial \kappa}{\partial r} + 4\kappa^2 \\ &\quad + \left[\frac{(n - 2)}{r} - \frac{n}{r f^2} + (n - 1)r \right] \frac{\partial \kappa}{\partial r} + 2[n - 1 + n\kappa] \kappa \\ &\quad + \frac{2(n - 2)}{r^4 f^2} \left(1 - \frac{1}{f^2} \right) + \frac{2(n - 2)}{r^2} \left(1 - \frac{2}{f^2} \right) \kappa. \end{aligned}$$

On the other hand, we have that

$$(6.4) \quad \frac{1}{2r} \frac{\partial}{\partial r} \left(\Delta \left(-\frac{1}{f^2} \right) \right) = \frac{1}{2r} \frac{\partial}{\partial r} \left[\frac{1}{r^{n-1} f} \frac{\partial}{\partial r} \left(\frac{r^{n-1}}{f} \frac{\partial}{\partial r} \left(-\frac{1}{f^2} \right) \right) \right],$$

whose right-hand side can be re-expressed to yield

$$(6.5) \quad \frac{1}{2r} \frac{\partial}{\partial r} \left(\Delta \left(-\frac{1}{f^2} \right) \right) = \Delta \kappa + \frac{2}{r f^2} \frac{\partial \kappa}{\partial r} - 3r\kappa \frac{\partial \kappa}{\partial r} - 2(n + 1)\kappa^2.$$

Substituting this into (6.3) yields

$$(6.6) \quad \frac{\partial \kappa}{\partial r} = \Delta \kappa + r \kappa \frac{\partial \kappa}{\partial r} + \left[\frac{(n-2)}{r} \left(1 - \frac{1}{f^2} \right) + (n-1)r \right] \frac{\partial \kappa}{\partial r} + 2 \left[\kappa + n - 1 + \frac{(n-2)}{r^2} \left(1 - \frac{2}{f^2} \right) \right] \kappa + \frac{2(n-2)}{r^4 f^2} \left(1 - \frac{1}{f^2} \right).$$

Simplifying and using (3.13) (the definition of λ), we obtain (6.1). □

6.2. The difference of sectional curvatures

Lemma 6.2. *The difference of sectional curvatures obeys the evolution equation*

$$(6.7) \quad \frac{\partial}{\partial t}(\kappa - \lambda) = \Delta(\kappa - \lambda) + [n - 1 + \kappa + (n - 2)\lambda] r \frac{\partial}{\partial r}(\kappa - \lambda) + 2 \left[2(n - 1)\lambda - \frac{n}{r^2} + n - 1 \right] (\kappa - \lambda).$$

Proof. Subtract (5.7) from (6.1) and write the resulting equation in terms of the difference $\kappa - \lambda$. The Bianchi identity (3.14) is then used to convert some of the $r \frac{\partial \lambda}{\partial r}$ factors into $2(\kappa - \lambda)$. After some manipulation, (6.7) follows. □

Now define

$$(6.8) \quad \mu := e^{4a(n-1)t}(\kappa - \lambda)^2$$

for $a \in \mathbb{R}$. Then Lemma 6.2 yields

$$(6.9) \quad \begin{aligned} \frac{\partial \mu}{\partial t} &= \Delta \mu - e^{4a(n-1)t} |\nabla(\kappa - \lambda)|^2 + [n - 1 + \kappa + (n - 2)\lambda] r \frac{\partial \mu}{\partial r} \\ &\quad + 4 \left[2(n - 1)\lambda - \frac{n}{r^2} + (n - 1)(1 + a) \right] \mu \\ &\leq \Delta \mu + [n - 1 + \kappa + (n - 2)\lambda] r \frac{\partial \mu}{\partial r} + 4(n - 1)(1 + a + 2\lambda)\mu, \end{aligned}$$

where the inequality follows since by definition we have $\mu \geq 0$. Writing out the Laplacian in coordinates, we obtain

$$(6.10) \quad \frac{\partial \mu}{\partial t} \leq \frac{1}{f^2} \frac{\partial^2 \mu}{\partial r^2} + \left((n - 1)r + \frac{(n - 1)}{r} - r\lambda \right) \frac{\partial \mu}{\partial r} + 4(n - 1)(1 + a + 2\lambda)\mu.$$

The $1/r$ multiplying the first derivative produces a singularity, but only of the mild sort expected from our choice of polar coordinate system.

Nonetheless, we will have to handle it, and since we do not have *a priori* uniform control over μ at $r = 0$, we follow the technique used in the last section and define

$$(6.11) \quad \mu_\alpha := \frac{r^\alpha}{(r^\alpha + \alpha)}\mu, \quad \alpha \in (0, 1),$$

so that $\mu_\alpha(t, 0) = 0$. Using the straightforward expressions

$$(6.12) \quad \frac{\partial \mu}{\partial r} = \left(1 + \frac{\alpha}{r^\alpha}\right) \frac{\partial \mu_\alpha}{\partial r} - \frac{\alpha^2}{r^{\alpha+1}}\mu_\alpha$$

and

$$(6.13) \quad \frac{\partial^2 \mu}{\partial r^2} = \left(1 + \frac{\alpha}{r^\alpha}\right) \frac{\partial^2 \mu_\alpha}{\partial r^2} - \frac{2\alpha^2}{r^{\alpha+1}} \frac{\partial \mu_\alpha}{\partial r} + \frac{\alpha^2(\alpha + 1)}{r^{\alpha+2}}\mu_\alpha,$$

and simplifying a term using the definition of λ (3.13), we obtain an inequality for the evolution of μ_α :

$$(6.14) \quad \begin{aligned} \frac{\partial \mu_\alpha}{\partial t} \leq & \frac{1}{f^2} \frac{\partial^2 \mu_\alpha}{\partial r^2} + \left[(n-1)r + \frac{(n-1)}{r} - r\lambda - \frac{2\alpha^2}{r(r^\alpha + \alpha)f^2} \right] \frac{\partial \mu_\alpha}{\partial r} \\ & + \left[4(n-1)(1+a+2\lambda) \right. \\ & \left. - \frac{\alpha^2}{(r^\alpha + \alpha)} \left(\frac{(n-2-\alpha)}{r^2} + \alpha\lambda + n-1 \right) \right] \mu_\alpha. \end{aligned}$$

Using $n \geq 3$, taking $\alpha \in (0, 1)$, and using that $\mu_\alpha \geq 0$ by its definition, this simplifies to

$$(6.15) \quad \begin{aligned} \frac{\partial \mu_\alpha}{\partial t} \leq & \frac{1}{f^2} \frac{\partial^2 \mu_\alpha}{\partial r^2} + \left[(n-1)r + \frac{(n-1)}{r} - r\lambda - \frac{2\alpha^2}{r(r^\alpha + \alpha)f^2} \right] \frac{\partial \mu_\alpha}{\partial r} \\ & + \hat{c}(t, r)\mu_\alpha, \end{aligned}$$

where

$$(6.16) \quad \hat{c}(t, r) := 4(n-1)(1+a+2\lambda) - \frac{\alpha^3\lambda}{(r^\alpha + \alpha)}.$$

6.3. Evolution and convergence of μ

We can now prove that $\mu(t) \rightarrow 0$ as $t \rightarrow \infty$, provided that $\lambda(0) < 0$. As before, when $\lambda(0) \leq -1$ we will find rapid convergence, whereas in the more

general case the convergence may be slower. Even more generally, if we allow that $\lambda(0) \leq 0$, we will see that μ remains bounded at any finite time, but we will be unable to show convergence unless $\lambda(0) < 0$.

Proposition 6.3. *Given $\lambda(t, r) \leq -1$ for all $(t, r) \in [0, T_{\max}) \times [0, \infty)$, then*

$$(6.17) \quad |\kappa(t, r) - \lambda(t, r)| \leq e^{-2b(n-1)t} \max_{r \in [0, \infty)} |\kappa(0, r) - \lambda(0, r)|$$

for any $b < 1$.

Proof. Choose any $0 < T < T_{\max}$. As we have that $\mu_\alpha(t, 0) = 0$ and $\mu_\alpha(t, r) \rightarrow 0$ as $r \rightarrow \infty$, Proposition 4.1 implies that either μ_α is identically zero (and then μ is identically zero, and the Bianchi identity implies the manifold is hyperbolic space), or the maximum of $\mu_\alpha(t, r)$ on $[0, T] \times [0, \infty)$ is positive and occurs at $t = 0$, or the maximum of $\mu_\alpha(t, r)$ is positive and occurs at a point $(t^*, r^*) \in (0, T] \times (0, \infty)$ where $\hat{c}(t^*, r^*) \geq 0$. Since $\lambda(t, r) \leq -1$ we find $0 \leq \frac{-\alpha^3 \lambda}{r^\alpha + \alpha} \leq -\alpha^2 \lambda$, and then (6.16) yields

$$(6.18) \quad \begin{aligned} \hat{c}(t, r) &\leq 4(n-1) \left[1 + a + \left(2 - \frac{\alpha^2}{4(n-1)} \right) \lambda \right] \\ &\leq 4(n-1) \left[a + \left(1 - \frac{\alpha^2}{4(n-1)} \right) \lambda \right]. \end{aligned}$$

Pick α sufficiently small, say $0 < \alpha < 1$. Since $\lambda \leq -1$, the right-hand side of (6.18) is strictly negative if $a < 1 - \frac{\alpha^2}{4(n-1)}$. As we can choose $\alpha > 0$ as close to zero as we wish, any $a < 1$ suffices to make $\hat{c} < 0$. Therefore, for sufficiently small α , the minimum of μ_α occurs at $t = 0$. Now take $\alpha \searrow 0$ and then take $T \nearrow T_{\max}$. Finally, to express the decay in the form of (6.17), take a square root. □

We now weaken the condition $\lambda(0) \leq -1$. We begin with the following time-dependent bound which implies a long-time existence result.

Proposition 6.4. *Given $\lambda(0, \cdot) \leq 0$ for all $(t, r) \in [0, \infty) \times [0, \infty)$, then $\kappa(t, \cdot) + 1$ is bounded on $[0, T]$ for any $T > 0$.*

Proof. First, from Propositions 5.1 and 5.2, $\lambda(t, r)$ is bounded above by 0 and bounded below by a function that decays to -1 exponentially. Thus we can find some $K > 0$ which depends only on the initial sectional curvature $\lambda(0, \cdot)$ such that $0 \geq \lambda(t, r) \geq -4(n-1)K$. Then estimating (6.16) with $\alpha \in (0, 1)$

we have

$$(6.19) \quad \hat{c}(t, r) \leq 4(n - 1) \left[1 + a + \left(2 - \frac{\alpha^2}{4(n - 1)} \right) \lambda \right] \\ \leq 4(n - 1)(1 + a) + 4(n - 1)K,$$

Choose $a = -K - 2$. Then $\hat{c}(t, r) \leq -4(n - 1) < 0$, and thus each μ_α governed by (6.15) is bounded above by an α -independent bound which depends only on $\mu(0, \cdot)$. As before, we can therefore take $\alpha \searrow 0$, thereby obtaining a bound for $\mu(t, r)$. Call the bound C^2 . Then from (6.8) we have

$$(6.20) \quad |\kappa - \lambda| := e^{-2a(n-1)t} \sqrt{\mu} \leq C e^{2(n-1)(K+2)t}.$$

Since $\lambda + 1$ is bounded on $[0, T]$ for any $T > 0$ and since $|\kappa - \lambda|$ is also bounded on $[0, T]$ for any $T > 0$, then so is $\kappa + 1$. □

Now we are able to show that if $\lambda(0, \cdot) < 0$, then $\kappa(t)$ decays to -1 . The idea is that Proposition 6.4 would have yielded the requisite decay of κ if we could have chosen a positive value for a in (6.19). Since (6.19) is obtained from (6.16) by estimating λ , we return to (6.16) and observe that when λ is sufficiently close to -1 and α is sufficiently small, there are positive choices for a which will make $c < 0$. If λ is not initially close enough to -1 , we can invoke Propositions 6.4 and 5.3, which tell us that eventually $\lambda(t, \cdot)$ will be close enough to -1 .

Proposition 6.5. *Assume that $\lambda(0, r) < 0$ for all $r \in [0, \infty)$. Then there are constants $\gamma > 0$ and $C > 0$ depending only on $g(0, \cdot)$ and n such that $|\kappa(t, r) + 1| < C e^{-\gamma t}$ for all $(t, r) \in [0, \infty) \times [0, \infty)$.*

Proof. Take $\lambda(0, r) \leq -M < 0$ for some $M > 0$ (since $\lambda(0, r) < 0$ and $\lambda(0, r) \rightarrow -1$ as $r \rightarrow \infty$). As in the preamble, by Proposition 6.4 the flow will exist for all $t > 0$. By Proposition 5.3 there will be a time $T(M, \epsilon) \geq 0$ after which $|\lambda(t) + 1| < \epsilon$, where we take ϵ to be small enough for our purposes, say $\epsilon = 1/10$. During the interval $[0, T(M, \epsilon)]$, the magnitude of $\kappa + 1$ may have increased, but is bounded by the estimate in Proposition 6.4.

Then equation (6.16) yields

$$(6.21) \quad \hat{c}(t, r) \leq 4(n - 1) \left[1 + a + \left(2 - \frac{\alpha^2}{4(n - 1)} \right) \lambda \right] \\ < 4(n - 1) \left(a - \frac{4}{5} + \frac{9\alpha^2}{40(n - 1)} \right),$$

and once more choosing α near 0 allows us to make a positive choice for a yielding $\hat{c}(t, r) < 0$. The rest of the argument is as before, yielding that on the domain $[T(M, 1/10), \infty)$ μ decays as

$$(6.22) \quad \mu \leq C e^{-2a(n-1)t},$$

for some constant C which depends on $\mu(T(M, 1/10))$ and n , and thus only on M and n . Finally, because both μ and $\lambda + 1$ decay exponentially, so does $\kappa + 1$. \square

7. Proofs of the main theorems

Since Theorem 1.1 was proved in Section 2, it remains to prove Theorems 1.2 and 1.4. In what follows, it will help to recall that

$$(7.1) \quad |\text{Rm} + K|^2 = 4(n-1) \left[(\kappa + 1)^2 + \frac{1}{2}(n-2)(\lambda + 1)^2 \right]$$

in rotational symmetry. For the purposes of Theorem 1.1, we interpret the left-hand side as being evaluated at $F_t^{-1}(r, \dots)$; i.e., we evaluate the left-hand side along the normalized Ricci flow obtained by pulling back our diffeomorphism-modified flow (3.9) by the diffeomorphisms F_t of the preceding section.

7.1. Proof of Theorem 1.2

Proof. Statement (a) follows from Theorem 1.1. By Propositions 4.4 and 5.1, the combination $\lambda + 1$ as defined in terms of equation (3.13) remains bounded on any finite interval of time. Then by Proposition 6.4, so does the combination $\kappa + 1$ with κ given by (3.12). But by equation (7.1) and the unique evolution posited by Theorem 1.1 then the solution $g(t)$ of the corresponding normalized Ricci flow exists for all time, so statement (b.i) is proved. Moreover, the conclusion of Corollary 4.4 holds under pullback to the normalized Ricci flow, so (b.ii) is also proved. The decay estimate (b.iii) similarly follows from Propositions 5.3 and 6.5. \square

7.2. Proof of Theorem 1.4

Proof. Pick any increasing, divergent sequence of times $\{t_k | k = 1, 2, \dots, t_1 > 0\}$ along the flow, and set $g_k = g(t_k)$. By Propositions 5.3 and 6.5 the

sectional curvatures are uniformly bounded along any such sequence. By Corollary 2.4, all covariant derivatives of the curvatures are also uniformly bounded.

Since (\mathbb{R}^n, g_k) is rotationally symmetric, choose the origin of symmetry and call it O_k . There can be no geodesic loops through O_k since geodesics through the origin of a rotationally symmetric manifold are necessarily radial, and radial geodesics for a rotationally symmetric metric on \mathbb{R}^n do not close. Then $\text{inj}_{O_k} = \text{conj}_{O_k}$ for each k . But $\text{conj}_{O_k} = \infty$, for if a point p at distance $R > 0$ from O_k were conjugate to O_k , then by rotational symmetry the sphere of radius R about O_k would consist entirely of points conjugate to O_k , and no geodesic through O_k could minimize beyond this sphere, contradicting the Hopf-Rinow theorem on \mathbb{R}^n .

Thus by [11, Theorem 2.3], we can find a convergent subsequence. Furthermore, by Propositions 5.3 and 6.5 the limit metric must have constant sectional curvature -1 . \square

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