Communications in
Analysis and Geometry
Volume 26, Number 4, 955–1008, 2018

Anomaly flows

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The Anomaly flow is a flow which implements the Green-Schwarz anomaly cancellation mechanism originating from superstring theory, while preserving the conformally balanced condition of Hermitian metrics. There are several versions of the flow, depending on whether the gauge field also varies, or is assumed known. A distinctive feature of Anomaly flows is that, in m dimensions, the flow of the Hermitian metric has to be inferred from the flow of its $(m-1)$-th power ω^{m-1}. We show how this can be done explicitly, and we work out the corresponding flows for the torsion and the curvature tensors. The results are applied to produce criteria for the long-time existence of the flow, in the simplest case of zero slope parameter.

Work supported in part by the National Science Foundation Grants DMS-12-66033 and DMS-1605968. Key words: Green-Schwarz anomaly cancellation, quadratic terms in the curvature tensor, torsion constraints, flows, maximum principle.

1. Introduction

Starting with the uniformization theorem, canonical metrics such as Hermitian-Yang-Mills and Kähler-Einstein metrics have played a major role in complex geometry. However, theoretical physics suggests more notions of metrics which should qualify in some sense as canonical. Indeed, the classical canonical metrics are typically defined by a linear constraint in the curvature tensor. But in string theory, the key Green-Schwarz anomaly cancellation mechanism [15] for the consistency of superstring theory is an equation which involves the *square* of the curvature tensor. Furthermore, while the supersymmetry of the heterotic string compactified to 4-dimensional Minkowski space-time required that the intermediate space carry a complex structure [2], it allowed the corresponding Chern unitary connection to have non-vanishing torsion [30]. The resulting condition is known as a Strominger system, and Calabi-Yau manifolds with their Kähler Ricci-flat metrics are only a special solution. What seems to emerge then is an as-yet unexplored area of non-Kähler geometry, where the Kähler condition is replaced by some specific constraint on the torsion, and the canonical metric condition is replaced by an equation on the torsion and possibly higher powers of the curvature. These equations are also novel from the point of view of the theory of partial differential equations, and it is an important problem to develop methods for their solutions.

The goal of the present paper is to develop methods for the study of the following flow of Hermitian metrics on a 3-dimensional complex manifold X,

$$\begin{aligned} &\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar\partial\omega - \alpha'(\mathrm{Tr} Rm \wedge Rm - \Phi(t)) \\ &\omega(0) = \omega_0. \end{aligned} \tag{1.1}$$

Here X is equipped with a nowhere vanishing $(3,0)$ holomorphic form Ω, $\|\Omega\|_\omega$ is the norm of Ω with respect to the Hermitian metric ω, defined by

$$\|\Omega\|_\omega^2 = i\Omega \wedge \bar\Omega\, \omega^{-3}, \tag{1.2}$$

and the expression $\Phi(t)$ is a given closed $(2,2)$-form in the characteristic class $c_2(X)$, evolving with time. The expression Rm is the curvature of the Chern unitary connection of ω, viewed as a $(1,1)$-form valued in the bundle of endomorphisms $End(T^{1,0}(X))$ of $T^{1,0}(X)$. The initial Hermitian form ω_0 is required to satisfy the following *conformally balanced* condition

$$d(\|\Omega\|_{\omega_0} \omega_0^2) = 0. \tag{1.3}$$

The motivation for the flow (1.1) is as follows. In [30], building on the earlier work of Candelas, Horowitz, Strominger, and Witten [2], Strominger identified the following system of equations for a Hermitian metric ω on X and a Hermitian metric $H_{\bar\alpha\beta}$ on a holomorphic vector bundle $E \to X$,

$$F^{2,0} = F^{0,2} = 0, \quad F \wedge \omega^2 = 0 \tag{1.4}$$

$$i\partial\bar\partial\omega - \alpha' \mathrm{Tr}(Rm \wedge Rm - F \wedge F) = 0 \tag{1.5}$$

$$d^\dagger \omega = i(\bar\partial - \partial) \log \|\Omega\|_\omega, \tag{1.6}$$

as conditions for the product of X with 4-dimensional space-time to be a supersymmetric vacuum configuration for the heterotic string. The conditions on F in the first equation above just mean that F is the curvature of the Chern unitary connection of $H_{\bar\alpha\beta}$, and that $H_{\bar\alpha\beta}$ is Hermitian-Yang-Mills with respect to any metric conformal to ω. It is a subsequent, but basic observation of Li and Yau [18] that the third condition on ω above, which is at first sight a *torsion constraints* condition, is equivalent to the condition that ω be conformally balanced

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{1.7}$$

In the special case where (X, ω) is a compact Kähler 3-fold with $c_1(X) = 0$, if we take $E = T^{1,0}(X)$, $H = \omega$, then the anomaly condition is automatically satisfied. The Hermitian-Yang-Mills condition reduces to the condition that ω be Ricci-flat, which can be implemented by Yau's theorem [35]. The norm $\|\Omega\|_\omega$ is then constant, and the torsion constraints follow from the Kähler property of ω. Thus Calabi-Yau 3-folds with their Ricci-flat metrics can be viewed as special solutions of the Strominger system, and they have played a major role ever since in both superstring theory and algebraic geometry [2]. From this point of view, it is natural to think of the pair (ω, H) as a canonical metric for (X, E), and if H happens to be fixed for some reason, of the metric ω itself as a canonical metric in non-Kähler geometry.

Strominger systems are difficult to solve, and the first non-perturbative, non-Kähler solutions to the systems were obtained by Fu-Yau [10, 11], some twenty years after Strominger's original proposal. These solutions were on toric fibrations over $K3$ surfaces constructed earlier by Goldstein and Prokushkin [14]. On such manifolds, Fu-Yau succeeded in reducing the Strominger system to a new complex Monge-Ampère equation on the two-dimensional Kähler base, which they succeeded in solving. Higher dimensional analogues of the Fu-Yau solution were considered by the authors in

[23–25]. Geometric constructions of some special solutions of Strominger systems have been given in e.g. [1, 4–9, 21].

A major problem at the present time is to develop analytical methods for solving the general Strominger system. Even if the curvature F of the bundle metric H were known and we concentrate only on the equations for ω, an immediate difficulty typical of non-Kähler geometry, is that there is no general or convenient way of parametrizing conformally balanced metrics, comparable to the parametrization of Kähler metrics by their potentials which was instrumental in Yau's solution of the Ricci-flat equation. It appears to be a daunting problem to have to deal with the anomaly equation and the conformally balanced equation as a system of equations. A way of bypassing this difficulty was suggested by the authors in [22], which is to introduce the coupled geometric flow

$$\begin{aligned} H^{-1}\,\partial_t H &= -\Lambda F \\ \partial_t(\|\Omega\|_\omega \omega^2) &= i\partial\bar\partial\omega - \alpha' \mathrm{Tr}(Rm\wedge Rm - F\wedge F) \end{aligned} \tag{1.8}$$

with initial conditions $\omega(0) = \omega_0$, $H(0) = H_0$, where H_0 is a given metric on E, and ω_0 is a Hermitian metric on X which satisfies the conformally balanced condition (1.3)[1].

The point of the flow is that, by Chern-Weil theory, the right hand side in the second line above is always closed, and hence the condition $d(\|\Omega\|_\omega\omega^2) = 0$ is preserved by the flow. Thus there is no need to treat the conformally balanced condition as a separate equation, and the stationary points of the flow will automatically satisfy all the equations in the Strominger system. For fixed ω, the flow of the metric $H_{\bar\alpha\beta}$ is just the Donaldson heat flow [3]. If the flow for $H_{\bar\alpha\beta}(t)$ is known, and if we set $\Phi(t) = \mathrm{Tr}(F\wedge F)$, then the flow for ω reduces to the flow (1.1). An understanding of (1.1) appears a necessary preliminary step in an understanding of (1.8). The flow (1.8) was called the Anomaly flow in [22], in reference to the key role played by the right hand side in the Green-Schwarz anomaly cancellation mechanism. We shall use the same generic name for all closely related flows such as (1.1).

Anomaly flows appear to be considerably more complicated than classical flows in geometry of which the Yang-Mills flow and the Ricci flow are well-known examples. A first hurdle is that the flow of metrics $\omega(t)$ has to be deduced from the flow of $(2,2)$-forms $\|\Omega\|_\omega\,\omega^2$. Now the existence in dimension m of an $(m-1)$-th root of a positive $(m-1,m-1)$-form has been

[1]Note that there are ways for constructing individual conformally balanced metrics ω_0 (see e.g. [33]).

shown by Michelsohn [20], and this passage back and forth between positive $(1,1)$-forms and $(m-1,m-1)$-forms has played a major role e.g. in works of Popovici [27] and in the recent proof by Szekelyhidi, Tosatti, and Weinkove [31, 33] of the existence of Gauduchon metrics with prescribed volume form. However, it does not appear possible to use the formalism in these works to deduce the flow of the curvature tensor of ω from the flow of ω^{m-1}. This is one of the main goals of the present paper. What we do is to produce a seemingly new formula for the square root of a $(2,2)$-form, or equivalently, for the Hodge $\star$ operator, without using the antisymmetric symbol ε. With such a formula, and using the very specific torsion constraints resulting from the conformally balanced condition, we obtain the following completely explicit expression for the Anomaly flow:

Theorem 1. *If the initial metric ω_0 is conformally balanced, then the Anomaly flow (1.1) can also be expressed as*

(1.9)
$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega}\left[-\tilde{R}_{\bar{p}q} + g^{\alpha\bar{\beta}}g^{s\bar{r}}T_{\bar{\beta}sq}\bar{T}_{\alpha\bar{r}\bar{p}} - \alpha' g^{s\bar{r}}(R_{[\bar{p}s}{}^{\alpha}{}_{\beta}R_{\bar{r}q]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{p}s\bar{r}q})\right].$$

where $\tilde{R}_{\bar{k}j}$ is the Ricci tensor and $T_{\bar{k}ij}$ is the torsion tensor, as defined in (2.36) and (2.28) below. The brackets $[\,,\,]$ denote anti-symmetrization separately in each of the two sets of barred and unbarred indices.

The above theorem shows that the Anomaly flow can be viewed as generalization of the Ricci flow, with higher order corrections in the curvature tensor proportional to α'. Indeed, the terms $\tilde{R}_{\bar{p}q} - g^{\alpha\bar{\beta}}g^{s\bar{r}}T_{\bar{\beta}sq}\bar{T}_{\alpha\bar{r}\bar{p}}$ reduce to the Ricci curvature $R_{\bar{p}q}$ (see the definition in (2.36)) if the torsion vanishes, and the terms with coefficient α' are the higher order corrections. It is remarkable that this analogy with the Ricci flow is due not to an attempt to generalize the Ricci flow, but rather to the combination of the Green-Schwarz cancellation mechanism, more specifically the de Kalb-Ramond field $i\partial\bar{\partial}\omega$, with the torsion constraints equivalent to the conformally balanced condition. Once the formulation of the flow provided by Theorem 1 is available, it is straightforward to derive the flows of the torsion and curvature tensors. The full results are given in Theorems 4 and 5 below. Here, we note only that they reinforce the same analogy with the Ricci flow. For example, the diffusion operator in the flow for the Ricci curvature is given by

$$\partial_t R_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\left(\Delta R_{\bar{k}j} + 2\alpha' g^{\lambda\bar{\mu}}g^{s\bar{r}}R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha}\nabla_s\nabla_{\bar{\mu}]}R_{\bar{k}j}{}^{\alpha}{}_{\beta}\right) + \cdots \tag{1.10}$$

Up to the factor $(2\|\Omega\|_\omega)^{-1}$, it coincides with the diffusion operator Δ for the Ricci curvature in the Ricci flow, up to a higher order correction in the curvature which is proportional to α'.

The formulation of the Anomaly flow provided by Theorem 1 makes it more amenable to existing techniques for flows, and indeed many flows of metrics with torsion have been studied in the literature (e.g. [13, 19, 28, 29, 32] and others). However, the Anomaly flow still involves a combination of novel features such as the particular torsion constraints, the presence of the factor $\|\Omega\|_\omega$ (which is quite important in string theory as it originates from the dilaton field), and especially the presence of the quadratic terms in the curvature tensor. All this makes a general solution only a remote possibility at this time. Thus we focus on two important special cases. The first case is the Anomaly flow restricted to the Fu-Yau ansatz for solutions of the Strominger system on toric fibrations over Ricci-flat Kähler surfaces. We can show that the Anomaly flow converges in this case, and thus gives another proof of the existence theorem of Fu-Yau. But because of its length and complexity, the full argument will be presented in a companion paper [26] to the present one. The second case is when $\alpha' = 0$. In this case, our main results are as follows.

Theorem 2. *Assume that $\alpha' = 0$. Suppose that $A > 0$ and $\omega(t)$ is a solution to the Anomaly flow (3.1) below, with $t \in [0, \frac{1}{A}]$. Then, for all $k \in \mathbf{N}$, there exists a constant C_k depending on a uniform lower bound of $\|\Omega\|_\omega$ such that, if*

$$|Rm|_\omega + |DT|_\omega + |T|_\omega^2 \leq A, \quad \text{for all } z \in M \text{ and } t \in \left[0, \frac{1}{A}\right], \tag{1.11}$$

then,

$$|D^k Rm(z,t)|_\omega \leq \frac{C_k A}{t^{k/2}}, \quad |D^{k+1} T(z,t)|_\omega \leq \frac{C_k A}{t^{k/2}} \tag{1.12}$$

for all $z \in M$ and $t \in (0, \frac{1}{A}]$.

The estimates given in the above theorem can be viewed as Shi-type derivative estimates for the curvature tensor and torsion tensor along the Anomaly flow (3.1). With this theorem, we can provide a criterion for the long-time existence of the Anomaly flow:

Theorem 3. *Assume that $\alpha' = 0$, and that the Anomaly flow (3.1) exists on an interval $[0,T)$ for some $T > 0$. If $\inf_{t\in[0,T)} \|\Omega\|_\omega > 0$ (or equivalently $\omega^3(t) \le C\,\omega^3(0)$), and if*

$$\sup_{X\times[0,T)}(|Rm|^2_\omega + |DT|^2_\omega + |T|^4_\omega) < \infty \tag{1.13}$$

then the flow can be continued to an interval $[0, T+\epsilon)$ for some $\epsilon > 0$. In particular, the flow exists for all time, unless there is a time $T > 0$ and a sequence (z_j, t_j), with $t_j \to T$, with either $\|\Omega(z_j,t_j)\|_\omega \to 0$, or

$$(|Rm|^2_\omega + |DT|^2_\omega + |T|^4_\omega)(z_j, t_j) \to \infty. \tag{1.14}$$

The paper is organized as follows. In §2, we begin by providing an effective way for recapturing the form $\partial_t\omega$ from the form $\partial_t(\|\Omega\|_\omega\omega^2)$. We then discuss the torsion constraints in the Strominger system, and in particular, how they result in two different notions of Ricci curvature, but a single notion of scalar curvature. We can then prove Theorem 1. With Theorem 1, it is straightforward to derive the flows of the curvature and of the torsion. In §3, we give the proof of Theorem 2. This proof is analogous to the proof for the classical flows, but it is more complicated here due to the non-vanishing torsion and the expression $\|\Omega\|_\omega$. Once we have Theorem 2, it is easy to prove Theorem 3. Finally, we provide a list of conventions in the appendices, together with some basic identities of Hermitian geometry.

2. The flows of the metric, torsion and curvature

The first task in the study of a geometric flow is to derive the flows of the curvature tensor, and in the case of non Levi-Civita connections, of the torsion tensor. In the case of Anomaly flows, this task is complicated by the fact that the flow is defined as a flow of the $(2,2)$-form $\|\Omega\|_\omega\,\omega^2$, and that the flow of ω itself has to be recaptured from there.

2.1. The equation $\varphi \wedge \omega^{m-2} = \Phi$ and the Hodge $\star$ operator

Since $\partial_t\omega^2 = 2\partial_t\omega \wedge \omega$, the flow of ω can be recovered from the flow of ω^2 if we can solve explicitly equations of the form $\varphi \wedge \omega = \Phi$ for a given Φ. We begin by doing this, in general dimension m instead of just $m = 3$, as the resulting formulas for the solution as well as the Hodge $\star$ operator may be of independent interest.

Let $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ and η be a (p,q)-form. We define its components $\eta_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p}$ by

$$\eta = \frac{1}{p!q!} \sum \eta_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p} \, dz^{j_p} \wedge \cdots \wedge dz^{j_1} \wedge d\bar{z}^{k_q} \wedge \cdots \wedge d\bar{z}^{k_1}. \tag{2.1}$$

Lemma 1. *Let Φ be a $(m-1, m-1)$ form on a Hermitian manifold (X, ω) of dimension m. Then the equation*

$$\varphi \wedge \omega^{m-2} = \Phi \tag{2.2}$$

admits a unique solution, given by

$$\varphi_{\bar{j}k} = \frac{1}{\alpha_m} \left\{ i^{-(m-2)} \prod_{p=1}^{m-2} g^{k_p \bar{j}_p} \Phi_{\bar{j}k\bar{j}_1 k_1 \cdots \bar{j}_{m-2} k_{m-2}} - \frac{\beta_m}{(m-1)!^2} (\mathrm{Tr}\,\Phi)\, i\, g_{\bar{j}k} \right\} \tag{2.3}$$

where α_m and β_m are universal constants, depending only on the dimension m, given by

$$\alpha_m = (m-1)!(m-2)! \left(m - 1 - \frac{m^2}{6} \right), \quad \beta_m = \frac{m!(m-2)!}{6}. \tag{2.4}$$

and $\mathrm{Tr}\,\Theta$ for a (p,p)-form Θ is defined by

$$\mathrm{Tr}\,\Theta = \langle \Theta, \omega^p \rangle = i^{-p} \prod_{\ell=1}^{p} g^{k_\ell \bar{j}_\ell} \, \Theta_{\bar{j}_1 k_1 \cdots \bar{j}_p k_p}. \tag{2.5}$$

The traces of φ and Φ are related by

$$\mathrm{Tr}\varphi = \frac{1}{(m-1)!^2} \mathrm{Tr}\,\Phi. \tag{2.6}$$

Proof. In components, the equation $\varphi \wedge \omega^{m-2} = \Phi$ can be expressed as

$$i^{m-2} \, \varphi_{\{\bar{j}k} g_{\bar{j}_1 k_1} \cdots g_{\bar{j}_{m-2} k_{m-2}\}} = \Phi_{\bar{j}k\bar{j}_1 k_1 \cdots \bar{j}_{m-2} k_{m-2}} \tag{2.7}$$

where the bracket $\{,\}$ denote antisymmetrization of all the barred indices as well as of all the unbarred indices. We contract both sides, getting

$$i^{m-2} \prod_{p=1}^{m-2} g^{k_p \bar{j}_p} \varphi_{\{\bar{j}k} g_{\bar{j}_1 k_1} \cdots g_{\bar{j}_{m-2} k_{m-2}\}} = \prod_{p=1}^{m-2} g^{k_p \bar{j}_p} \Phi_{\bar{j}k\bar{j}_1 k_1 \cdots \bar{j}_{m-2} k_{m-2}}. \tag{2.8}$$

We expand the left-hand side by writing down all the terms arising from antisymmetrization of the sub-indices. Carrying out the contractions with $\prod_{p=1}^{m-2} g^{k_p\bar{j}_p}$, it is easy to verify that each term is a constant multiple of $\varphi_{\bar{j}k}$ or of $(\mathrm{Tr}\,\varphi)\,g_{\bar{k}j}$. This shows that we have a relation of the form

$$i^{m-2}\,\alpha_m\varphi_{\bar{k}j} + i^{m-1}\,\beta_m(\mathrm{Tr}\,\varphi)\,g_{\bar{k}j} = \prod_{p=1}^{m-2} g^{k_p\bar{j}_p}\,\Phi_{\bar{j}k\bar{j}_1k_1\cdots\bar{j}_{m-2}k_{m-2}}. \tag{2.9}$$

Next, we have $\varphi\wedge\omega^{m-1} = \Phi\wedge\omega$, which implies

$$\langle\varphi, \star\omega^{m-1}\rangle = \langle\Phi, \star\omega\rangle. \tag{2.10}$$

Recalling that $\star\,\omega^{m-1} = (m-1)!\,\omega$ and $\star\,\omega = \frac{1}{(m-1)!}\,\omega^{m-1}$, we obtain

$$\mathrm{Tr}\,\varphi = \langle\varphi,\omega\rangle = \frac{1}{(m-1)!^2}\,\mathrm{Tr}\,\Phi. \tag{2.11}$$

This establishes the form (2.3).

It is easy to see that $\alpha_m \neq 0$, otherwise we obtain a relation between $g_{\bar{k}j}$ and Φ that cannot hold for an arbitrary $(m-1, m-1)$-form Φ. To determine the precise values of α_m and β_m, we proceed as follows.

First, contracting (2.9) with respect to $g^{k\bar{j}}$ and using the definition of $\mathrm{Tr}\Theta$ in (2.5), we have the following relation between α_m and β_m,

$$i^{m-1}\,\alpha_m\,\mathrm{Tr}\,\varphi + i^{m-1}\,\beta_m m\,\mathrm{Tr}\,\varphi = i^{m-1}\,\mathrm{Tr}\,\Phi = i^{m-1}\,(m-1)!^2\,\mathrm{Tr}\,\varphi, \tag{2.12}$$

and hence

$$\alpha_m + \beta_m m = (m-1)!^2. \tag{2.13}$$

Thus it remains only to determine β_m. We note that the only permutation of indices which can produce a multiple of $(\mathrm{Tr}\,\varphi)\,g_{\bar{k}j}$ is of the form, e.g.,

$$\varphi_{\bar{j}_1k_1}g_{\bar{j}k}\,g_{\bar{j}_2k_2}\cdots g_{\bar{j}_{m-2}k_{m-2}} \tag{2.14}$$

which produces, upon contraction with $\prod_{p=1}^{m-2} g^{k_p\bar{j}_p}$ and antisymmetrization in $j_2,\cdots,j_{n-2}$ and $k_2,\cdots,k_{n-2}$,

$$(\mathrm{Tr}\,\varphi)g_{\bar{k}j}\,\langle\omega^{m-3},\omega^{m-3}\rangle = (\mathrm{Tr}\,\varphi)g_{\bar{k}j}\|\omega^{m-3}\|^2. \tag{2.15}$$

We can compute $\|\omega^{m-3}\|^2$ as follows

$$\omega^{m-3} = (m-3)! \sum_{j<k<\ell} (ie^j \wedge \overline{e^j})^v \cdots (ie^k \wedge \overline{e^k})^v \wedge (ie^\ell \wedge \overline{e^\ell})^v. \tag{2.16}$$

Since the sum in the right hand side consists of exactly $\frac{1}{3!}m(m-1)(m-2)$ terms, we find

$$\|\omega^{m-3}\|^2 = (m-3)!^2 \frac{m(m-1)(m-2)}{6} = \frac{m!(m-3)!}{6}. \tag{2.17}$$

But there are $m-2$ terms of the form (2.14), corresponding to the indices $\bar{j}_1 k_1$ taking successively all values to $\bar{j}_{m-2} k_{m-2}$. Thus we obtain

$$\beta_m = \frac{m!(m-3)!}{6}(m-2) = \frac{m!(m-2)!}{6} \tag{2.18}$$

establishing our claim for β_m. The claim for α_m then follows from the relation (2.13). □

Although the previous lemma suffices for our purpose, it is useful for future considerations to point out that it gives in effect an explicit expression for the Hodge $\star$ operator without the ϵ symbol. This can be seen by comparing it with the following lemma which solves the same equation, but using the Hodge $\star$ operator (and which can also be derived from Proposition 1.2.31 in [16]):

Lemma 2. *Let (X, ω) be a Hermitian manifold of complex dimension $m \geq 2$. Consider the following equation, for a given $(m-1, m-1)$-form Φ,*

$$\psi \wedge \omega^{m-2} = \Phi. \tag{2.19}$$

Then the equation admits a unique solution, given by

$$\psi = -\frac{1}{(m-2)!} \star \Phi + \frac{\langle \Phi, \omega^{m-1} \rangle}{(m-1)!^2}\, \omega. \tag{2.20}$$

Proof. First observe that if ψ_0 is a $(1,1)$-form with $\langle \psi_0, \omega \rangle = 0$, then

$$\star\,(\psi_0 \wedge \omega^{m-2}) = -(m-2)!\psi_0 \tag{2.21}$$

as can be verified by working out $\psi_0 \wedge \omega^{m-2}$. This is equivalent to saying that, if Φ_0 is a $(m-1, m-1)$ form with $\langle \Phi_0, \omega^{m-1} \rangle = 0$, then $\psi_0 =$

$-\frac{1}{(m-1)!}\star\Phi_0$ is the unique solution of the equation $\psi_0\wedge\omega^{m-2}=\Phi_0$. Next, for general Φ, we write

$$\Phi = \frac{\langle\Phi,\omega^{m-1}\rangle}{\|\omega^{m-1}\|^2}\omega^{m-1} + \Phi_0 \tag{2.22}$$

so that $\langle\Phi_0,\omega^{m-1}\rangle = 0$. In view of the previous observation, the $(1,1)$-form

$$\begin{aligned}\psi &= -\frac{1}{(m-2)!}\star\Phi_0 + \frac{\langle\Phi,\omega^{m-1}\rangle}{\|\omega^{m-1}\|^2}\omega \\ &= -\frac{1}{(m-2)!}\star(\Phi - \frac{\langle\Phi,\omega^{m-1}\rangle}{\|\omega^{m-1}\|^2}\omega^{m-1}) + \frac{\langle\Phi,\omega^{m-1}\rangle}{\|\omega^{m-1}\|^2}\omega \\ &= -\frac{1}{(m-2)!}\star\Phi + m\frac{\langle\Phi,\omega^{m-1}\rangle}{\|\omega^{m-1}\|^2}\omega\end{aligned} \tag{2.23}$$

is a solution of the equation (2.19). Here we used the fact that $\star\omega^{m-1} = (m-1)!\omega$. Since $\|\omega^{m-1}\|^2 = m!(m-1)!$, we obtain the desired formula. □

Comparing the previous two lemmas gives the following formula for the Hodge $\star$ operator on $(m-1,m-1)$ forms, on an arbitrary Hermitian m-fold (X,ω),

Lemma 3. *Let (X,ω) be a Hermitian manifold of complex dimension $m\geq 2$. Then for any $(m-1,m-1)$-form Φ, we have*

$$(\star\Phi)_{\bar jk} = \frac{1}{(m-1)!(m^2-6m+6)} \times\left\{6\,i^{-(m-2)}\prod_{p=1}^{m-2} g^{k_p\bar j_p}\Phi_{\bar jk\bar j_1k_1\cdots\bar j_{m-2}k_{m-2}} + (m-6)(\mathrm{Tr}\Phi)\,i\,g_{\bar jk}\right\}.$$

Proof. We equate $\psi=\varphi$, in the notation of the previous two lemmas. Thus

$$\begin{aligned}&-\frac{1}{(m-2)!}(\star\Phi)_{\bar jk} + \frac{\langle\Phi,\omega^{m-1}\rangle}{(m-1)!^2}ig_{\bar jk} \\ &= \left\{\frac{i^{-m+2}}{\alpha_m}\prod_{p=1}^{m-2} g^{k_p\bar j_p}\Phi_{\bar jk\bar j_1k_1\cdots\bar j_{m-2}k_{m-2}} - \frac{\beta_m}{\alpha_m}\frac{i}{(m-1)!^2}(\mathrm{Tr}\,\Phi)g_{\bar jk}\right\}.\end{aligned} \tag{2.24}$$

Since $\mathrm{Tr}\,\Phi = \langle \Phi, \omega^{m-1}\rangle$, we obtain

$$-\frac{1}{(m-2)!} \star \Phi = \frac{i^{-m+2}}{\alpha_m} \prod_{p=1}^{m-2} g^{k_p \bar{j}_p} \Phi_{\bar{j}k\bar{j}_1 k_1 \cdots \bar{j}_{m-2}k_{m-2}} - (1 + \frac{\beta_m}{\alpha_m}) \frac{\mathrm{Tr}\Phi}{(m-1)!^2}\, i\, g_{\bar{j}k}. \tag{2.25}$$

Working out the coefficient $1 + \beta_m/\alpha_m$, we obtain the formula

$$\frac{1}{(m-2)!} \star \Phi = -\frac{i^{-m+2}}{\alpha_m} \prod_{p=1}^{m-2} g^{k_p \bar{j}_p} \Phi_{\bar{j}k\bar{j}_1 k_1 \cdots \bar{j}_{m-2}k_{m-2}} + \frac{(m-1)(m-6)}{m^2 - 6m + 6} \frac{\mathrm{Tr}\Phi}{(m-1)!^2}\, i\, g_{\bar{j}k}.$$

This can be rewritten in turn in the form given in the lemma. □

2.2. Torsion and curvature for conformally balanced metrics

Next, we examine more carefully the implications for the torsion and curvature condition of conformally balanced metrics. Let $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ be a Hermitian metric, viewed as a positive $(1,1)$-form. We define its torsion tensor T and $\bar{T}$ by

$$T = i\partial\omega, \qquad \bar{T} = -i\bar{\partial}\omega \tag{2.26}$$

which are respectively $(2,1)$ and $(1,2)$ forms. Following the conventions for (p,q)-forms given in the appendix, we define the coefficients $T_{\bar{k}jm}$ and $\bar{T}_{j\bar{p}\bar{q}}$ by

$$T = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k, \quad \bar{T} = \frac{1}{2} \bar{T}_{k\bar{j}\bar{m}} d\bar{z}^m \wedge d\bar{z}^j \wedge dz^k, \tag{2.27}$$

and thus

$$T_{\bar{k}jm} = \partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}, \quad \bar{T}_{k\bar{j}\bar{m}} = \partial_{\bar{j}} g_{\bar{m}k} - \partial_{\bar{m}} g_{\bar{j}k}. \tag{2.28}$$

For our later use, it is convenient to introduce

$$T_m = g^{j\bar{k}} T_{\bar{k}jm}, \quad \bar{T}_{\bar{m}} = g^{\bar{j}k} \bar{T}_{k\bar{j}\bar{m}}. \tag{2.29}$$

As noted earlier, the equivalence between (1.6) and (1.7) had been pointed out by Li and Yau [18]. For the convenience of the reader, we provide here the proof, stated in a somewhat more general form.

Lemma 4. *Let (X, ω) be a m dimensional Hermitian manifold equipped with a nowhere vanishing holomorphic $(m,0)$-form Ω. Then the following conditions are equivalent:*

(i) *The metric ω satisfies the conformally balanced condition $d(\|\Omega\|_\omega^a \omega^{m-1}) = 0$ for some constant a;*

(ii) $d^\dagger \omega = i(\bar\partial - \partial) \log \|\Omega\|^a$;

(iii) $T_q = \partial_q \log \|\Omega\|_\omega^a$, $\bar T_{\bar q} = \partial_{\bar q} \log \|\Omega\|_\omega^a$.

Proof. The conformally balanced condition can be written as

$$\partial \log \|\Omega\|_\omega^a \wedge \omega^{m-1} + (m-1)\partial\omega \wedge \omega^{m-2} = 0. \tag{2.30}$$

Now just as $i\theta \wedge \omega^{m-1} = -i(g^{j\bar k}\theta_{\bar k j})\frac{\omega^m}{m}$ for any $(1,1)$-form θ, it is easy to verify that

$$T \wedge \omega^{m-2} = -i(g^{j\bar k} T_{\bar k j m} dz^m) \wedge \frac{\omega^{m-1}}{(m-1)} \tag{2.31}$$

for any $(2,1)$-form T. Substituting $T = i\partial\omega$, and using the previous equation gives

$$(\partial \log \|\Omega\|_\omega^a - T_p dz^p) \wedge \omega^{m-1} = 0, \tag{2.32}$$

which implies $\partial \log \|\Omega\|_\omega^a - T_p dz^p = 0$ and proves the equivalence between (i) and (iii). Finally, the equivalence between (ii) and (iii) follows at once from the expressions of the adjoints of ∂ and $\bar\partial$ on $(1,1)$-forms for a Hermitian metric

$$(\bar\partial^\dagger \Phi)_q = g^{k\bar p}(\nabla_k \Phi_{\bar p q} - T_k\, \Phi_{\bar p q}), \quad (\partial^\dagger \Phi)_{\bar q} = -g^{p\bar j}(\nabla_{\bar j} \Phi_{\bar q p} - \bar T_{\bar j} \Phi_{\bar q p}). \tag{2.33}$$

In particular, when $\Phi = \omega$, $\Phi_{\bar p q} = i g_{\bar p q}$, we obtain

$$(\bar\partial^\dagger \omega)_q = -i T_q, \qquad (\partial^\dagger \omega)_{\bar q} = i \bar T_{\bar q}. \tag{2.34}$$

This implies the equivalence between (ii) and (iii). □

We turn to the notion of Ricci curvature for conformally balanced metrics. Although we have a single notion of Riemann curvature tensor,

$$\begin{aligned} R_{\bar{k}j}{}^{p}{}_{q} &= -\partial_{\bar{k}}(g^{p\bar{\ell}}\partial_j g_{\bar{\ell}q}), \\ Rm &= R_{\bar{k}j}{}^{p}{}_{q} dz^j d\bar{z}^k \in \Lambda^{1,1} \otimes End(T^{1,0}(X)), \end{aligned} \tag{2.35}$$

the lack of the standard symmetries for Levi-Civita connections leads to 4 different notions of Ricci curvature, defined as follows

$$R_{\bar{k}j} = R_{\bar{k}j}{}^{p}{}_{p}, \quad \tilde{R}_{\bar{k}j} = R^{p}{}_{p\bar{k}j}, \quad R'_{\bar{k}j} = R_{\bar{k}}{}^{p}{}_{pj}, \quad R''_{\bar{k}j} = R^{p}{}_{j\bar{k}p}. \tag{2.36}$$

Corresponding to these 4 notions of Ricci curvature are 4 notions of scalar curvature

$$R = g^{j\bar{k}} R_{\bar{k}j}, \quad \tilde{R} = g^{j\bar{k}} \tilde{R}_{\bar{k}j}, \quad R' = g^{j\bar{k}} R'_{\bar{k}j}, \quad R'' = g^{j\bar{k}} R''_{\bar{k}j}. \tag{2.37}$$

With the help from the torsion constraints, we have some nice relation between these different notions of Ricci curvature and scalar curvature. The following lemma is essential for our subsequent calculations:

Lemma 5. *Assume that ω is a conformally balanced metric on an m-fold X, in the sense that the equivalent conditions in Lemma 4 are satisfied. Then*

(i) $\nabla_{\bar{k}} T_j = \nabla_j \bar{T}_{\bar{k}} = \frac{a}{2} R_{\bar{k}j}$.

(ii) $R'_{\bar{k}j} = R''_{\bar{k}j} = (1 - \frac{a}{2}) R_{\bar{k}j}$.

(iii) $\tilde{R}_{\bar{k}j} = (1 - \frac{a}{2}) R_{\bar{k}j} + \nabla^m T_{\bar{k}jm}$.

(iv) $R = \tilde{R}$ *and* $R' = R'' = (1 - \frac{a}{2}) R$.

Proof. By definition, $R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log \omega^m = \partial_j \partial_{\bar{k}} \log \|\Omega\|^2_\omega$, so (i) follows from (iii) in Lemma 4. Next,

$$R'_{\bar{k}j} = R_{\bar{k}p}{}^{p}{}_{j} = R_{\bar{k}j} + \nabla_{\bar{k}} T^{m}{}_{jm} = R_{\bar{k}j} - \nabla_{\bar{k}} T_j = \left(1 - \frac{a}{2}\right) R_{\bar{k}j}, \tag{2.38}$$

which proves a first part of (ii). Similarly,

$$R''_{\bar{k}j} = R_{\bar{k}j}{}^{p}{}_{p} + \nabla_j \bar{T}^{\bar{q}}{}_{\bar{k}\bar{q}} = \left(1 - \frac{a}{2}\right) R_{\bar{k}j} \tag{2.39}$$

which completes the proof of (ii). Next,

$$\tilde{R}_{\bar{k}j} = R_{\bar{k}j} + \nabla_j \bar{T}^{\bar{p}}{}_{\bar{k}\bar{p}} + \nabla_{\bar{p}} T_{\bar{k}jm} g^{m\bar{p}} = \left(1 - \frac{a}{2}\right) R_{\bar{k}j} + \nabla^m T_{\bar{k}jm} \tag{2.40}$$

which proves (iii). Contracting with $g_{\bar{k}j}$, we obtain (iv). □

Note that the identities $R = \tilde{R}$ and $R' = R''$ actually hold for any Hermitian metric.

2.3. Flow of the metric ω and proof of Theorem 1

We can now come back to the derivation of the flow for the metric ω in the Anomaly flow and prove Theorem 1. In the following, we let the dimension m of the manifold X be then 3, and take $a = 1$ in Lemma 4.

It is convenient to denote the right hand side of the Anomaly flow by Ψ,

$$\Psi = i\partial\bar{\partial}\omega - \alpha' \,\mathrm{Tr}(Rm \wedge Rm - \Phi(t)) \tag{2.41}$$

which is then a $(2,2)$-form. As usual, we denote its coefficients by $\Psi_{\bar{p}s\bar{r}q}$, and also introduce the notation $\Psi_{\bar{p}q}$, which can be viewed as the coefficients of a $(1,1)$-form,

$$\Psi = \frac{1}{(2!)^2} \sum \Psi_{\bar{p}s\bar{r}q} dz^q \wedge d\bar{z}^r \wedge dz^s \wedge d\bar{z}^p, \qquad \Psi_{\bar{p}q} = g^{s\bar{r}} \Psi_{\bar{p}s\bar{r}q}. \tag{2.42}$$

We rewrite the Anomaly flow (1.1) as

$$(\partial_t \log \|\Omega\|_\omega \omega + 2\partial_t \omega) \wedge \omega = \frac{1}{\|\Omega\|_\omega} \Psi. \tag{2.43}$$

We apply the second statement in Lemma 1. Since

$$\partial_t \log \|\Omega\|_\omega = -\frac{1}{2} \partial_t \log (\det \omega) = -\frac{1}{2} \mathrm{Tr}(\partial_t \omega) \tag{2.44}$$

we find, in dimension $m = 3$,

$$\mathrm{Tr}(\partial_t \omega) = \frac{1}{2\|\Omega\|_\omega} \mathrm{Tr}\Psi. \tag{2.45}$$

This gives us the flow of the volume form ω^3. Returning once again to the flow (2.43) and applying the first statement in Lemma 1, we find

$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega} g^{s\bar{r}} \Psi_{\bar{p}s\bar{r}q} = \frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}q}. \tag{2.46}$$

It remains to work out the components $\Psi_{\bar{p}s\bar{r}q}$ more explicitly. The first term is

$$i\partial\bar{\partial}\omega = \frac{1}{2^2}\Big\{\partial_{\bar{k}}(\partial_j g_{\bar{\ell}m} - \partial_m g_{\bar{\ell}j}) - \partial_{\bar{\ell}}(\partial_j g_{\bar{k}m} - \partial_m g_{\bar{\ell}j})\Big\} d\bar{z}^\ell \wedge dz^j \wedge dz^m \wedge d\bar{z}^k \tag{2.47}$$

and hence

$$(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = \partial_{\bar{\ell}}(\partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}) - \partial_{\bar{k}}(\partial_j g_{\bar{\ell}m} - \partial_m g_{\bar{\ell}j}). \tag{2.48}$$

On the other hand, the Riemann curvature tensor is given by

$$R_{\bar{k}j}{}^{\ell}{}_{m} = -\partial_{\bar{k}}(g^{\ell\bar{p}}\partial_j g_{\bar{p}m}) = -g^{\ell\bar{p}}\partial_{\bar{k}}\partial_j g_{\bar{p}m} + g^{\ell\bar{r}}\partial_{\bar{k}} g_{\bar{r}s} g^{s\bar{q}}\partial_j g_{\bar{q}m}, \tag{2.49}$$

or, equivalently,

$$R_{\bar{k}j\bar{\ell}m} = -\partial_{\bar{k}}\partial_j g_{\bar{\ell}m} + \partial_{\bar{k}} g_{\bar{\ell}s} g^{s\bar{r}}\partial_j g_{\bar{r}m}. \tag{2.50}$$

Thus we obtain

$$(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = R_{\bar{k}j\bar{\ell}m} - R_{\bar{k}m\bar{\ell}j} + R_{\bar{\ell}m\bar{k}j} - R_{\bar{\ell}j\bar{k}m} + g^{s\bar{r}}\, T_{\bar{r}mj}\bar{T}_{s\bar{k}\bar{\ell}}. \tag{2.51}$$

Applying Lemma 5 on the torsion and Ricci curvatures of conformally balanced metrics gives

$$g^{m\bar{\ell}}(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = \tilde{R}_{\bar{k}j} - g^{s\bar{r}} g^{m\bar{\ell}} T_{\bar{r}mj}\bar{T}_{s\bar{\ell}\bar{k}}. \tag{2.52}$$

We collect the resulting formulas in a lemma:

Lemma 6. *Let the* $(2,2)$*-form* Ψ *be defined by (2.41) and its components* $\Psi_{\bar{p}s\bar{r}q}$*,* $\Psi_{\bar{p}q}$ *by (2.42). Then*

$$\begin{aligned} \Psi_{\bar{k}m\bar{\ell}j} &= R_{\bar{k}m\bar{\ell}j} - R_{\bar{k}j\bar{\ell}m} + R_{\bar{\ell}j\bar{k}m} - R_{\bar{\ell}m\bar{k}j} + g^{s\bar{r}}\, T_{\bar{r}jm}\bar{T}_{s\bar{k}\bar{\ell}} \\ &\quad - \alpha'(R_{[\bar{k}m}{}^{\alpha}{}_{\beta} R_{\bar{\ell}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{k}m\bar{\ell}j}) \\ \Psi_{\bar{k}j} &= -\tilde{R}_{\bar{k}j} + (T\bar{T})_{\bar{k}j} - \alpha' g^{m\bar{\ell}}(R_{[\bar{k}m}{}^{\alpha}{}_{\beta} R_{\bar{\ell}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{k}m\bar{\ell}j}) \end{aligned} \tag{2.53}$$

where the brackets $[\, ,\,]$ *denote anti-symmetrization separately in each of the two sets of barred and unbarred indices and* $(T\bar{T})_{\bar{k}j} := g^{s\bar{r}} g^{m\bar{\ell}} T_{\bar{r}mj}\bar{T}_{s\bar{\ell}\bar{k}}$.

Combining the formula (2.46) for the Anomaly flow, and using the fact that the flow preserves the conformally balanced condition, we obtain Theorem 1.

2.4. Flow of the curvature tensor

The general formula for the flow of the curvature tensor of Chern unitary connections under a flow of metrics is the following

$$\partial_t R_{\bar{k}j}{}^{\mu}{}_{\nu} = -\nabla_{\bar{k}}\nabla_j(g^{\mu\bar{\gamma}}\dot{g}_{\bar{\gamma}\nu}) = -g^{\mu\bar{\gamma}}\nabla_{\bar{k}}\nabla_j\dot{g}_{\bar{\gamma}\nu}. \tag{2.54}$$

To apply this formula to the case of the Anomaly flow, where $\partial_t g_{\bar{\gamma}\nu}$ is given by Theorem 1, we need to work out the covariant derivatives of the curvature tensor for Hermitian metrics. This is done in the following lemma:

Lemma 7. *Let ω be any Hermitian metric (not necessarily conformally balanced). Then we have the following identities*

$$\begin{aligned}
\nabla_{\bar{k}}\nabla_j R_{\bar{\gamma}s\bar{\mu}\lambda} &= \nabla_s\nabla_{\bar{\gamma}} R_{\bar{k}j\bar{\mu}\lambda} + \nabla_{\bar{k}}(T^r{}_{sj}R_{\bar{\gamma}r\bar{\mu}\lambda}) + \nabla_s(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j\bar{\mu}\lambda})\\
&\quad - R_{\bar{k}s\bar{\gamma}}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\mu}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_j R_{\bar{\gamma}\kappa\bar{\mu}\lambda} - R_{\bar{k}s\bar{\mu}}{}^{\bar{\kappa}}R_{\bar{\gamma}j\bar{\kappa}\lambda}\\
&\quad + R_{\bar{k}s}{}^{\kappa}{}_{\lambda}R_{\bar{\gamma}j\bar{\mu}\kappa},\\
\nabla_{\bar{k}}\nabla_j \tilde{R}_{\bar{\mu}\lambda} &= \Delta R_{\bar{k}j\bar{\mu}\lambda} + \nabla_{\bar{k}}(T^r{}_{sj}R^s{}_{r\bar{\mu}\lambda}) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j\bar{\mu}\lambda})\\
&\quad - R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\mu}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_j R^s{}_{\kappa\bar{\mu}\lambda} - R_{\bar{k}s\bar{\mu}}{}^{\bar{\kappa}}R^s{}_{j\bar{\kappa}\lambda}\\
&\quad + R_{\bar{k}s}{}^{\kappa}{}_{\lambda}R^s{}_{j\bar{\mu}\kappa},\\
\nabla_{\bar{k}}\nabla_j \tilde{R} &= \Delta R_{\bar{k}j} + \nabla_{\bar{k}}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j})\\
&\quad - R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j} + R_{\bar{k}s}{}^{\kappa}{}_j R^s{}_{\kappa}.
\end{aligned} \tag{2.55}$$

To clarify the notation: we are writing $\Delta = g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}$ for the 'rough' Laplacian and $\bar{\Delta} = g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j$ for its conjugate. While Δ and $\bar{\Delta}$ agree when acting on functions, they differ by curvature terms when acting on tensors.

Proof. The proof is a straightforward application of the Bianchi identity, beginning with

$$\nabla_{\bar{k}}\nabla_j R_{\bar{\gamma}s\bar{\mu}\lambda} = \nabla_{\bar{k}}(\nabla_s R_{\bar{\gamma}j\bar{\mu}\lambda} + T^r{}_{sj}R_{\bar{\gamma}r\bar{\mu}\lambda}) \tag{2.56}$$

and applying it again, after commuting the covariant derivatives $\nabla_{\bar{k}}$ and ∇_s. $\square$

We return now to the Anomaly flow of conformally balanced metrics. First, we write

$$
\begin{aligned}
(2.57)\quad \partial_t R_{\bar k j}{}^{\rho}{}_{\lambda} &= -\nabla_{\bar k}\nabla_j\left(\frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\Psi_{\bar\mu\lambda}\right)\\
&= -\frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j\Psi_{\bar\mu\lambda} - \nabla_{\bar k}\left(\frac{1}{2\|\Omega\|_\omega}\right)\nabla_j(g^{\rho\bar\mu}\Psi_{\bar\mu\lambda})\\
&\quad - \nabla_j\left(\frac{1}{2\|\Omega\|_\omega}\right)\nabla_{\bar k}(g^{\rho\bar\mu}\Psi_{\bar\mu\lambda}) - \nabla_{\bar k}\nabla_j\left(\frac{1}{2\|\Omega\|_\omega}\right)g^{\rho\bar\mu}\Psi_{\bar\mu\lambda}\\
&= -\frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j\Psi_{\bar\mu\lambda} + \frac{1}{2\|\Omega\|_\omega}\bar T_{\bar k}\nabla_j\Psi^{\rho}{}_{\lambda}\\
&\quad + \frac{1}{2\|\Omega\|_\omega}T_j\nabla_{\bar k}\Psi^{\rho}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega}\left(\frac{1}{2}R_{\bar k j} - T_j\bar T_{\bar k}\right)\Psi^{\rho}{}_{\lambda}
\end{aligned}
$$

where we used (iii) in Lemma 4 to get the last equality.

We concentrate on the first term, which can be written in the following way, using Lemma 6,

$$
\begin{aligned}
(2.58)\qquad & -\frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j\Psi_{\bar\mu\lambda}\\
&= \frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j\tilde R_{\bar\mu\lambda} + \frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}g^{s\bar r}\alpha'\nabla_{\bar k}\nabla_j(R_{[\bar\mu s}{}^{\alpha}{}_{\beta}R_{\bar r\lambda]}{}^{\beta}{}_{\alpha})\\
&\quad - \frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j((T\bar T)_{\bar\mu\lambda} + \alpha'\Phi_{\bar\mu\lambda}).
\end{aligned}
$$

The terms in the second line are lower order terms that we shall leave as they are for the moment, and just collect them at the end. The first term on the right hand side can be rewritten as follows, using Lemma 7,

$$
\begin{aligned}
(2.59)\qquad & \frac{1}{2\|\Omega\|_\omega}g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j\tilde R_{\bar\mu\lambda}\\
&= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar k j}{}^{\rho}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^{r}{}_{sj}R^{s}{}_{r}{}^{\rho}{}_{\lambda}) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k}R_{\bar r j}{}^{\rho}{}_{\lambda})\\
&\quad - R'_{\bar k}{}^{\bar\kappa}R_{\bar\kappa j}{}^{\rho}{}_{\lambda} + R_{\bar k s}{}^{\kappa}{}_{j}R^{s}{}_{\kappa}{}^{\rho}{}_{\lambda} - R_{\bar k s}{}^{\rho\bar\kappa}R^{s}{}_{j\bar\kappa\lambda} + R_{\bar k s}{}^{\kappa}{}_{\lambda}R^{s}{}_{j}{}^{\rho}{}_{\kappa}\Big].
\end{aligned}
$$

It remains only to work out the contribution of the second term on the right hand side,

$$
\begin{aligned}
(2.60)\qquad & \frac{1}{2\|\Omega\|_\omega}\alpha' g^{\rho\bar\mu} g^{s\bar r}\nabla_{\bar k}\nabla_j (R_{[\bar\mu s}{}^{\alpha}{}_{\beta} R_{\bar r\lambda]}{}^{\beta}{}_{\alpha}) \\
&= \frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} 2(\nabla_{\bar k}\nabla_j R_{[\bar\mu s}{}^{\alpha}{}_{\beta}) R_{\bar r\lambda]}{}^{\beta}{}_{\alpha} \\
&\quad + \frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} (\nabla_j R_{[\bar\mu s}{}^{\alpha}{}_{\beta}\nabla_{\bar k} R_{\bar r\lambda]}{}^{\beta}{}_{\alpha} + \nabla_{\bar k} R_{[\bar\mu s}{}^{\alpha}{}_{\beta}\nabla_j R_{\bar r\lambda]}{}^{\beta}{}_{\alpha}).
\end{aligned}
$$

Again the second term on the right hand side contains only lower order terms, which we leave as they are and collect only at the end. Using Lemma 7, the first term can be rewritten as,

$$
\begin{aligned}
(2.61)\qquad & \frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} 2(\nabla_{\bar k}\nabla_j R_{[\bar\mu s}{}^{\alpha}{}_{\beta}) R_{\bar r\lambda]}{}^{\beta}{}_{\alpha} \\
&= \frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} 2R_{[\bar r\lambda}{}^{\beta\bar\delta}\nabla_s\nabla_{\bar\mu]} R_{\bar k j\bar\delta\beta} \\
&\quad + \frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} 2R_{[\bar r[\lambda}{}^{\beta\bar\delta}\Big[\nabla_{\bar k}(T^r{}_{s]j} R_{\bar\mu]r\bar\delta\beta}) + \nabla_{s]}(\bar T^{\bar r}{}_{\bar\mu]\bar k} R_{\bar r j\bar\delta\beta}) \\
&\quad - R_{\bar k s]\bar\mu]}{}^{\bar\kappa} R_{\bar\kappa j\bar\delta\beta} + R_{\bar k s]}{}^{\kappa}{}_j R_{\bar\mu]\kappa\bar\delta\beta} \\
&\quad - R_{\bar k s]\bar\delta}{}^{\bar\kappa} R_{\bar\mu]j\bar\kappa\beta} + R_{\bar k s]}{}^{\kappa}{}_{\beta} R_{\bar\mu]j\bar\delta\kappa}\Big]
\end{aligned}
$$

where we have again anti-symmetrized in the unbarred indices s and λ, and separately in the barred indices $\bar\mu$ and $\bar r$. Whenever there are many indices in the same row and whenever a more explicit designation may be helpful, we have indicated the indices to be anti-symmetrized, either by a symbol [on the left or a symbol] on the right of the relevant index.

We obtain in this way the following theorem:

Theorem 4. *Consider the Anomaly flow (1.1) with an initial metric ω_0 which is conformally balanced. Then the curvature of the metric flows according to the following equation*

$$
\begin{aligned}
(2.62)\qquad \partial_t R_{\bar k j}{}^{\rho}{}_{\lambda} &= \frac{1}{2\|\Omega\|_\omega}\big(\Delta R_{\bar k j}{}^{\rho}{}_{\lambda} + 2\alpha' g^{\rho\bar\mu} g^{s\bar r} R_{[\bar r\lambda}{}^{\beta}{}_{\alpha}\nabla_s\nabla_{\bar\mu]} R_{\bar k j}{}^{\alpha}{}_{\beta}\big) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\bar T_{\bar k}\nabla_j \Psi^{\rho}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega} T_j \nabla_{\bar k}\Psi^{\rho}{}_{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2\|\Omega\|_\omega}\left(\frac{1}{2}R_{\bar{k}j}-T_j\bar{T}_{\bar{k}}\right)\Psi^\rho{}_\lambda-\frac{1}{2\|\Omega\|_\omega}g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j((T\bar{T})_{\bar{\mu}\lambda}+\alpha'\Phi_{\bar{\mu}\lambda})\\
&+\frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar{k}}(T^r{}_{sj}R^s{}_r{}^\rho{}_\lambda)+\nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j}{}^\rho{}_\lambda)\\
&\quad-R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j}{}^\rho{}_\lambda+R_{\bar{k}s}{}^\kappa{}_jR^s{}_\kappa{}^\rho{}_\lambda-R_{\bar{k}s}{}^{\rho\bar{\kappa}}R^s{}_{j\bar{\kappa}\lambda}+R_{\bar{k}s}{}^\kappa{}_\lambda R^s{}_j{}^\rho{}_\kappa\Big]\\
&+\frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}(\nabla_jR_{[\bar{\mu}s}{}^\alpha{}_\beta\nabla_{\bar{k}}R_{\bar{r}\lambda]}{}^\beta{}_\alpha+\nabla_{\bar{k}}R_{[\bar{\mu}s}{}^\alpha{}_\beta\nabla_jR_{\bar{r}\lambda]}{}^\beta{}_\alpha)\\
&+\frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}}\Big[\nabla_{\bar{k}}(T^r{}_{s]j}R_{\bar{\mu}]r\bar{\delta}\beta})+\nabla_{s]}(\bar{T}^{\bar{r}}{}_{\bar{\mu}]\bar{k}}R_{\bar{r}j\bar{\delta}\beta})\\
&\quad-R_{\bar{k}s]\bar{\mu}]}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\delta}\beta}+R_{\bar{k}s]}{}^\kappa{}_jR_{\bar{\mu}]\kappa\bar{\delta}\beta}-R_{\bar{k}s]\bar{\delta}}{}^{\bar{\kappa}}R_{\bar{\mu}]j\bar{\kappa}\beta}+R_{\bar{k}s]}{}^\kappa{}_\beta R_{\bar{\mu}]j\bar{\delta}\kappa}\Big].
\end{aligned}
$$

2.5. Flow of the Ricci curvature

The flow of the Riemann curvature tensor implies immediately that of the Ricci curvature,

$$
\begin{aligned}
\partial_tR_{\bar{k}j}&=\frac{1}{2\|\Omega\|_\omega}\big(\Delta R_{\bar{k}j}+2\alpha' g^{\lambda\bar{\mu}}g^{s\bar{r}}R_{[\bar{r}\lambda}{}^\beta{}_\alpha\nabla_s\nabla_{\bar{\mu}]}R_{\bar{k}j}{}^\alpha{}_\beta\big)\\
&\quad+\frac{1}{2\|\Omega\|_\omega}\bar{T}_{\bar{k}}\nabla_j\Psi^\lambda{}_\lambda+\frac{1}{2\|\Omega\|_\omega}T_j\nabla_{\bar{k}}\Psi^\lambda{}_\lambda+\frac{1}{2\|\Omega\|_\omega}\left(\frac{1}{2}R_{\bar{k}j}-T_j\bar{T}_{\bar{k}}\right)\Psi^\lambda{}_\lambda\\
&\quad-\frac{1}{2\|\Omega\|_\omega}\nabla_{\bar{k}}\nabla_j(|T|^2+\alpha'\Phi^\lambda{}_\lambda)\\
&\quad+\frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar{k}}(T^r{}_{sj}R^s{}_r)+\nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j})-R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j}+R_{\bar{k}s}{}^\kappa{}_jR^s{}_\kappa\Big]\\
&\quad+\frac{\alpha' g^{\lambda\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}\big(\nabla_jR_{[\bar{\mu}s}{}^\alpha{}_\beta\nabla_{\bar{k}}R_{\bar{r}\lambda]}{}^\beta{}_\alpha+\nabla_{\bar{k}}R_{[\bar{\mu}s}{}^\alpha{}_\beta\nabla_jR_{\bar{r}\lambda]}{}^\beta{}_\alpha\big)\\
&\quad+\frac{\alpha' g^{\lambda\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}}\Big[\nabla_{\bar{k}}(T^r{}_{s]j}R_{\bar{\mu}]r\bar{\delta}\beta})+\nabla_{s]}(\bar{T}^{\bar{r}}{}_{\bar{\mu}]\bar{k}}R_{\bar{r}j\bar{\delta}\beta})\\
&\qquad-R_{\bar{k}s]\bar{\mu}]}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\delta}\beta}+R_{\bar{k}s]}{}^\kappa{}_jR_{\bar{\mu}]\kappa\bar{\delta}\beta}-R_{\bar{k}s]\bar{\delta}}{}^{\bar{\kappa}}R_{\bar{\mu}]j\bar{\kappa}\beta}+R_{\bar{k}s]}{}^\kappa{}_\beta R_{\bar{\mu}]j\bar{\delta}\kappa}\Big]
\end{aligned}
\tag{2.63}
$$

with $|T|^2=g^{j\bar{k}}g^{s\bar{r}}g^{m\bar{\ell}}T_{\bar{r}mj}\bar{T}_{s\bar{\ell}\bar{k}}$.

2.6. Flow of the scalar curvature

If we write $R = g^{j\bar{k}} R_{\bar{k}j}$, we obtain

$$\partial_t R = g^{j\bar{k}} \partial_t R_{\bar{k}j} - g^{j\bar{m}} \partial_t g_{\bar{m}q} g^{q\bar{k}} R_{\bar{k}j}. \tag{2.64}$$

Applying the preceding formula for the flow $\partial_t R_{\bar{k}j}$ of the Ricci curvature, we find

$$\begin{aligned}
\partial_t R = {} & \frac{1}{2\|\Omega\|_\omega} \big(\Delta R + 2\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} \tilde{R}^{\alpha}{}_{\beta} \big) \\
& + \frac{1}{2\|\Omega\|_\omega} \left(\bar{T}^j \nabla_j \Psi^{\lambda}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega} T^{\bar{k}} \nabla_{\bar{k}} \Psi^{\lambda}{}_{\lambda} + \left(\frac{1}{2} R - T_j \bar{T}^j \right) \Psi^{\lambda}{}_{\lambda} \right) \\
& - \frac{1}{2\|\Omega\|_\omega} \Delta(|T|^2 + \alpha' \Phi^{\lambda}{}_{\lambda}) - \frac{1}{2\|\Omega\|_\omega} R^{q\bar{m}} \Psi_{\bar{m}q} \\
& + \frac{1}{2\|\Omega\|_\omega} \left(\nabla_{\bar{k}} (T^{r}{}_{sj} R^{s}{}_{r}) + \nabla^{\bar{\gamma}} (\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}} R_{\bar{r}j}) \right) \\
& + \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega} (\nabla_j R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla^j R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} + \nabla_{\bar{k}} R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla^{\bar{k}} R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}) \\
& + \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega} 2 R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}} \Big[\nabla^j (T^{\gamma}{}_{s]j} R_{\bar{\mu}]\gamma\bar{\delta}\beta}) + \nabla_{s]} (\bar{T}^{\bar{\gamma}}{}_{\bar{\mu}]}{}^{j} R_{\bar{\gamma}j\bar{\delta}\beta}) \\
& - R^{j}{}_{s]\bar{\mu}]}{}^{\bar{\kappa}} R_{\bar{\kappa}j\bar{\delta}\beta} + \tilde{R}_{s]}{}^{\kappa} R_{\bar{\mu}]\kappa\bar{\delta}\beta} - R^{j}{}_{s]\bar{\delta}}{}^{\bar{\kappa}} R_{\bar{\mu}]j\bar{\kappa}\beta} + R^{j}{}_{s]}{}^{\kappa}{}_{\beta} R_{\bar{\mu}]j\bar{\delta}\kappa} \Big].
\end{aligned} \tag{2.65}$$

2.7. Flow of the torsion tensor

We differentiate the coefficients $T_{\bar{p}jq}$ of the torsion tensor,

$$\begin{aligned}
\partial_t T_{\bar{p}jq} &= \partial_j \dot{g}_{\bar{p}q} - \partial_q \dot{g}_{\bar{p}j} \\
&= \partial_j \left(\frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}q} \right) - \partial_q \left(\frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}j} \right) \\
&= \frac{1}{2\|\Omega\|_\omega} (\nabla_j \Psi_{\bar{p}q} - \nabla_q \Psi_{\bar{p}j} + T^{m}{}_{jq} \Psi_{\bar{p}m}) \\
&\quad - \frac{1}{2\|\Omega\|_\omega} (T_j \Psi_{\bar{p}q} - T_q \Psi_{\bar{p}j}).
\end{aligned} \tag{2.66}$$

Once again, we concentrate on the leading term, which is

$$
\begin{aligned}
(2.67)\qquad & \frac{1}{2\|\Omega\|_\omega}(\nabla_j \Psi_{\bar p q} - \nabla_q \Psi_{\bar p j}) \\
&= \frac{1}{2\|\Omega\|_\omega}(\nabla_j(-\tilde R_{\bar p q} + (T\bar T)_{\bar p q}) - \nabla_q(-\tilde R_{\bar p j} + (T\bar T)_{\bar p j}) \\
&\quad - \frac{1}{2\|\Omega\|_\omega}\alpha' g^{s\bar r}\nabla_j (R_{[\bar p s}{}^{\alpha}{}_{\beta} R_{\bar r q]}{}^{\beta}{}_{\alpha} - \Phi_{\bar p s \bar r q}) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\alpha' g^{s\bar r}\nabla_q (R_{[\bar p s}{}^{\alpha}{}_{\beta} R_{\bar r j]}{}^{\beta}{}_{\alpha}) - \Phi_{\bar p s \bar r j}).
\end{aligned}
$$

Although this is not apparent at first sight, the key diffusion term $\Delta T_{\bar p j q}$ can be extracted from the right hand side. The basic identity in this case is the following:

Lemma 8. *Let ω be any Hermitian metric (not necessarily conformally balanced). Then*

$$
(2.68)\qquad (\Delta T)_{\bar p j q} = \nabla_q \tilde R_{\bar p j} - \nabla_j \tilde R_{\bar p q} + T^r{}_{q\lambda} R^\lambda{}_{r \bar p j} - T^r{}_{j\lambda} R^\lambda{}_{r\bar p q}.
$$

Proof. We compute the components of the left hand side, using the Bianchi identities,

$$
\begin{aligned}
(2.69)\qquad (\Delta T)_{\bar p j q} &= g^{\lambda\bar\mu}\nabla_\lambda \nabla_{\bar\mu} T_{\bar p j q} \\
&= g^{\lambda\bar\mu}\nabla_\lambda (R_{\bar\mu q \bar p j} - R_{\bar\mu j \bar p q}) \\
&= g^{\lambda\bar\mu}(\nabla_q R_{\bar\mu\lambda\bar p j} - \nabla_j R_{\bar\mu\lambda\bar p q} + T^r{}_{q\lambda} R_{\bar\mu r \bar p j} - T^r{}_{j\lambda} R_{\bar\mu r\bar p q}). \\
&= \nabla_q \tilde R_{\bar p j} - \nabla_j \tilde R_{\bar p q} + T^r{}_{q\lambda} R^\lambda{}_{r\bar p j} - T^r{}_{j\lambda} R^\lambda{}_{r\bar p q}.
\end{aligned}
$$

This proves the lemma. □

Comparing this identity with the previous expression that we derived for $\partial_t T_{\bar p j q}$, we obtain the following theorem:

Theorem 5. *Consider the Anomaly flow (1.1) with an initial metric ω_0 which is conformally balanced. Then the flow of the torsion $T = i\partial\omega$ is given* *by*

$$\begin{aligned}\partial_t T_{\bar{p}jq} = {} & \frac{1}{2\|\Omega\|_\omega}\Big[\Delta T_{\bar{p}jq} - \alpha' g^{s\bar{r}}(\nabla_j(R_{[\bar{p}s}{}^{\alpha}{}_{\beta}R_{\bar{r}q]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{p}s\bar{r}q}) \\ & + \alpha' g^{s\bar{r}}\nabla_q(R_{[\bar{p}s}{}^{\alpha}{}_{\beta}R_{\bar{r}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{p}s\bar{r}j}))\Big] \\ & + \frac{1}{2\|\Omega\|_\omega}(T^m{}_{jq}\Psi_{\bar{p}m} - T_j\Psi_{\bar{p}q} + T_q\Psi_{\bar{p}j} + \nabla_j(T\bar{T})_{\bar{p}q} - \nabla_q(T\bar{T})_{\bar{p}j}) \\ & - \frac{1}{2\|\Omega\|_\omega}(T^r{}_{q\lambda}R^{\lambda}{}_{r\bar{p}j} - T^r{}_{j\lambda}R^{\lambda}{}_{r\bar{p}q}). \end{aligned} \tag{2.70}$$

3. A model problem: $\alpha' = 0$

A first model which is simpler than the full Anomaly flow and whose study could be instructive, is obtained by setting $\alpha' = 0$. While this special case eliminates the quadratic terms in the curvature tensor in (1.1), it still presents some new difficulties relative to the well-known Ricci flow and Donaldson heat flow because of the evolving torsion. More precisely, we shall consider the flow

$$\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega. \tag{3.1}$$

The stationary points of the flow are then given by the equivalent equations

$$i\partial\bar{\partial}\omega = 0 \tag{3.2}$$

for a Hermitian metric satisfying the conformally balanced condition $d(\|\Omega\|_\omega \omega^2) = 0$. Such a Hermitian metric must be Kähler and Ricci-flat [17], since contracting (2.52) and applying Lemma 5 shows that such a metric must satisfy

$$g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log\|\Omega\|^2 = \tilde{R} = |T|^2. \tag{3.3}$$

By the maximum principle, $|T|^2 = 0$ and $\log\|\Omega\|^2$ is constant. Thus the Anomaly flow with $\alpha' = 0$ can be used to determine whether a conformally balanced manifold is actually Kähler.

We note that the flow (3.1) is also related to the problem of prescribing metrics in a balanced class raised in the recent survey of Garcia-Fernandez [12], so our results in this section can be viewed as a first step towards an eventual solution.

3.1. Flow of the curvature and the torsion

For convenience, we summarize here the main formulas for the Anomaly flow (3.1). They can be obtained from the general formulas obtained earlier by setting $\alpha' = 0$. Let us still use Ψ to denote the right hand side of the flow, that is $\Psi_{\bar p q} = -\tilde R_{\bar p q} + (T\bar T)_{\bar p q}$. Then the flow of the metric is given by

$$\partial_t g_{\bar p q} = \frac{1}{2\|\Omega\|_\omega}\Psi_{\bar p q} = \frac{1}{2\|\Omega\|_\omega}\Big[- \tilde R_{\bar p q} + (T\bar T)_{\bar p q}\Big] \tag{3.4}$$

while the flows of the curvature tensors are given by

$$\begin{aligned}
\partial_t R_{\bar k j}{}^\rho{}_\lambda &= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar k j}{}^\rho{}_\lambda - \frac{1}{2\|\Omega\|_\omega} g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j (T\bar T)_{\bar\mu\lambda} \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla_j + T_j\nabla_{\bar k} + (\frac{1}{2}R_{\bar k j} - T_j \bar T_{\bar k})\Big)\Psi^\rho{}_\lambda \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^r{}_{sj} R^s{}_r{}^\rho{}_\lambda) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k} R_{\bar r j}{}^\rho{}_\lambda) \\
&\qquad - R'_{\bar k}{}^{\bar\kappa} R_{\bar\kappa j}{}^\rho{}_\lambda + R_{\bar k s}{}^\kappa{}_j R^s{}_\kappa{}^\rho{}_\lambda - R_{\bar k s}{}^{\rho\bar\kappa} R^s{}_{j\bar\kappa\lambda} + R_{\bar k s}{}^\kappa{}_\lambda R^s{}_j{}^\rho{}_\kappa\Big]
\end{aligned}$$

$$\begin{aligned}
\partial_t R_{\bar k j} &= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar k j} - \frac{1}{2\|\Omega\|_\omega}\nabla_{\bar k}\nabla_j |T|^2 \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla_j + T_j\nabla_{\bar k} + (\frac{1}{2}R_{\bar k j} - T_j\bar T_{\bar k})\Big)(-R + |T|^2) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k}R_{\bar r j}) - R'_{\bar k}{}^{\bar\kappa}R_{\bar\kappa j} + R_{\bar k s}{}^\kappa{}_j R^s{}_\kappa\Big]
\end{aligned}$$

$$\begin{aligned}
\partial_t R &= \frac{1}{2\|\Omega\|_\omega}\Delta R - \frac{1}{2\|\Omega\|_\omega}\Delta|T|^2 - \frac{1}{2\|\Omega\|_\omega}R^{j\bar k}\Psi_{\bar k j} \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla^{\bar k} + T_j\nabla^j + (\frac{1}{2}R - T_j\bar T^j)\Big)(-R + |T|^2) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\nabla_{\bar k}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k}R_{\bar r j})\Big)
\end{aligned} \tag{3.5}$$

and the flow of the torsion is given by

$$\begin{aligned}
\partial_t T_{\bar p j q} &= \frac{1}{2\|\Omega\|_\omega}\Delta T_{\bar p j q} - \frac{1}{2\|\Omega\|_\omega}(T^r{}_{q\lambda}R^\lambda{}_{r\bar p j} - T^r{}_{j\lambda}R^\lambda{}_{r\bar p q}) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}(T^m{}_{jq}\Psi_{\bar p m} - T_j\Psi_{\bar p q} + T_q\Psi_{\bar p j} + \nabla_j(T\bar T)_{\bar p q} - \nabla_q(T\bar T)_{\bar p j}).
\end{aligned} \tag{3.6}$$

For later use, we also record here the flow of the norm $\|\Omega\|_\omega$,

$$\partial_t \|\Omega\|_\omega = \frac{1}{4}(R - |T|^2). \tag{3.7}$$

3.2. Estimates for derivatives of curvature and torsion

The goal in this section is to prove Theorem 2. We shall use D to denote the derivative when we do not distinguish between ∇ and $\bar\nabla$. For example, $|DT|$ would include both $|\nabla T|$ and $|\bar\nabla T|$, and

$$|D^k T|^2 = \sum_{i+j=k} |\nabla^i \bar\nabla^j T|^2. \tag{3.8}$$

The proof of Theorem 2 is by induction on k. The idea is find a suitable test function $G_k(z,t)$ for each k, similar to the Ricci flow, and apply the maximum principle.

We will first prove the estimate (1.12) for $k = 1$ case. Then, we assume that, for any $0 \le j \le k-1$,

$$|D^j Rm(z,t)|_\omega \le \frac{C_j A}{t^{j/2}}, \qquad |D^{j+1} T(z,t)|_\omega \le \frac{C_j A}{t^{j/2}} \tag{3.9}$$

for all $z \in M$ and $t \in (0, \frac{1}{A}]$ and show the estimate also holds for $j = k$.

We already have the flows of the curvature and of the torsion, as given above. To prove the theorem, we shall also need the flows of their covariant derivatives. They are given in the following lemmas.

Lemma 9. *Under the induction assumption (3.9) and $|T|^2 \le A$, we have*

$$\begin{aligned} \partial_t |D^k Rm|^2 \le \frac{1}{2\|\Omega\|_\omega} \Big\{ & \frac{1}{2}\Delta_{\mathbf{R}} |D^k Rm|^2 - \frac{3}{4}|D^{k+1} Rm|^2 \\ & + CA^{\frac{1}{2}} \left(|D^{k+1} Rm| + |D^{k+2} T| \right) \cdot |D^k Rm| \\ & + CA \left(|D^k Rm| + |D^{k+1} T| \right) \cdot |D^k Rm| \\ & + C A^2 t^{-\frac{k}{2}} \cdot |D^k Rm| + CA^3 t^{-k} \Big\} \end{aligned} \tag{3.10}$$

where we write $\Delta_{\mathbf{R}} = \Delta + \bar\Delta$ and $\Delta = g^{\bar q p} \nabla_p \nabla_{\bar q}$.

Proof. First, we observe that the flow of the curvature tensor can be expressed as

$$\partial_t Rm = \frac{1}{2\|\Omega\|_\omega}\bigg\{\frac{1}{2}\Delta_{\mathbf{R}} Rm + \nabla\bar\nabla(T*\bar T) + \bar\nabla(T*Rm) + \nabla(\bar T*Rm) + Rm*Rm + (\bar\nabla T - \bar T*T)*\Psi + \bar T*\nabla\Psi + T*\bar\nabla\Psi\bigg\}. \tag{3.11}$$

To clarify notation: if E and F are tensors, we write $E*F$ for any linear combination of products of the tensors E and F formed by contractions on $E_{i_1\cdots i_k}$ and $F_{j_1\cdots j_l}$ using the metric g.

Let the terms in the large bracket be denoted by H, that is

$$\partial_t Rm = \frac{1}{2\|\Omega\|_\omega} H. \tag{3.12}$$

In general, the Chern unitary connection of a Hermitian metric $g_{\bar k j}$ evolves by

$$\partial_t A^j_{\bar k m} = 0, \qquad \partial_t A^j_{km} = g^{j\bar p}\nabla_k(\partial_t g_{\bar p m}). \tag{3.13}$$

This implies

$$\partial_t(\nabla^m\bar\nabla^\ell Rm) = \nabla^m\bar\nabla^\ell(\partial_t Rm) + \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^{m-i}\bar\nabla^{\ell-j}Rm*\nabla^i\bar\nabla^j(\partial_t g). \tag{3.14}$$

Using the evolution equation of Rm, we get

$$\partial_t(\nabla^m\bar\nabla^\ell Rm) = \sum_{i=1}^{m}\sum_{j=1}^{\ell}\nabla^{m-i}\bar\nabla^{\ell-j}Rm*\nabla^i\bar\nabla^j(\partial_t g) + \frac{1}{2\|\Omega\|_\omega}\nabla^m\bar\nabla^\ell H + \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^{m-i}\bar\nabla^{\ell-j}H*\nabla^i\bar\nabla^j\left(\frac{1}{2\|\Omega\|_\omega}\right). \tag{3.15}$$

We compute the second term,

$$\begin{aligned}\nabla^m\bar\nabla^\ell H = {} & \frac{1}{2}\nabla^m\bar\nabla^\ell\Delta_{\mathbf{R}}Rm + \nabla^m\bar\nabla^\ell\nabla\bar\nabla(T*T) + \nabla^m\bar\nabla^{\ell+1}(T*Rm)\\ & + \nabla^m\bar\nabla^\ell\nabla(\bar T*Rm) + \nabla^m\bar\nabla^\ell(Rm*Rm) + \nabla^m\bar\nabla^{\ell+1}(T*\Psi)\\ & + \nabla^m\bar\nabla^\ell(\Psi*\bar T*T) + \nabla^m\bar\nabla^\ell(\nabla\Psi*\bar T) + \nabla^m\bar\nabla^\ell(T*\bar\nabla\Psi).\end{aligned} \tag{3.16}$$

In view of the commutation identity given in the appendix,

$$\begin{aligned}\nabla^m\bar{\nabla}^\ell\Delta_{\mathbf{R}}Rm &= \Delta_{\mathbf{R}}(\nabla^m\bar{\nabla}^\ell Rm) + \sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^i\bar{\nabla}^j Rm * \nabla^{m-i}\bar{\nabla}^{\ell-j}Rm \\ &\quad + \sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^i\bar{\nabla}^j T * \nabla^{m-i}\bar{\nabla}^{\ell+1-i}Rm \\ &\quad + \sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^i\bar{\nabla}^j T * \nabla^{m+1-i}\bar{\nabla}^{\ell-j}Rm. \end{aligned} \tag{3.17}$$

We obtain

$$\begin{aligned} &\partial_t(\nabla^m\bar{\nabla}^\ell Rm) \\ &= \frac{1}{2\|\Omega\|_\omega}\Bigg\{\frac{1}{2}\Delta_{\mathbf{R}}(\nabla^m\bar{\nabla}^\ell Rm) + \sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^i\bar{\nabla}^j Rm * \nabla^{m-i}\bar{\nabla}^{\ell-j}Rm \\ &\quad + \sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^i\bar{\nabla}^j T * \left(\nabla^{m-i}\bar{\nabla}^{\ell+1-i}Rm + \nabla^{m+1-i}\bar{\nabla}^{\ell-j}Rm\right) \\ &\quad + \nabla^m\bar{\nabla}^\ell\nabla\bar{\nabla}(T*T) + \nabla^m\bar{\nabla}^{\ell+1}(T*Rm) \\ &\quad + \nabla^m\bar{\nabla}^\ell\nabla(\bar{T}*Rm) + \nabla^m\bar{\nabla}^\ell(Rm*Rm) \\ &\quad + \nabla^m\bar{\nabla}^{\ell+1}(T*\Psi) + \nabla^m\bar{\nabla}^\ell(\Psi*\bar{T}*T) + \nabla^m\bar{\nabla}^\ell(\nabla\Psi*\bar{T})\Bigg\} \\ &\quad + \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^{m-i}\bar{\nabla}^{\ell-j}Rm * \nabla^i\bar{\nabla}^j(\partial_t g) \\ &\quad + \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}\nabla^{m-i}\bar{\nabla}^{\ell-j}H * \nabla^i\bar{\nabla}^j\left(\frac{1}{2\|\Omega\|_\omega}\right). \end{aligned} \tag{3.18}$$

Next we compute

$$\begin{aligned} &\partial_t|\nabla^m\bar{\nabla}^\ell Rm|^2 \\ &\le \langle\partial_t\nabla^m\bar{\nabla}^\ell Rm,\, \nabla^m\bar{\nabla}^\ell Rm\rangle + \langle\nabla^m\bar{\nabla}^\ell Rm,\, \partial_t\nabla^m\bar{\nabla}^\ell Rm\rangle \\ &\quad + \frac{C}{2\|\Omega\|_\omega}|\nabla^m\bar{\nabla}^\ell Rm|^2\cdot|\Psi|. \end{aligned} \tag{3.19}$$

We also compute

$$
\begin{aligned}
&\Delta_{\mathbf{R}}|\nabla^m\bar\nabla^\ell Rm|^2\\
&= \langle \Delta_{\mathbf{R}}\nabla^m\bar\nabla^\ell Rm,\, \nabla^m\bar\nabla^\ell Rm\rangle + \langle \nabla^m\bar\nabla^\ell Rm,\, \Delta_{\mathbf{R}}\nabla^m\bar\nabla^\ell Rm\rangle\\
&\quad + 2|\nabla^{m+1}\bar\nabla^\ell Rm|^2 + 2|\bar\nabla\nabla^m\bar\nabla^\ell Rm|^2\\
&= \langle \Delta_{\mathbf{R}}\nabla^m\bar\nabla^\ell Rm,\, \nabla^m\bar\nabla^\ell Rm\rangle + \langle \nabla^m\bar\nabla^\ell Rm,\, \Delta_{\mathbf{R}}\nabla^m\bar\nabla^\ell Rm\rangle\\
&\quad + 2|\nabla^{m+1}\bar\nabla^\ell Rm|^2 + 2|\nabla^m\bar\nabla^{\ell+1} Rm|^2\\
&\quad + 2\left(|\bar\nabla\nabla^m\bar\nabla^\ell Rm|^2 - |\nabla^m\bar\nabla^{\ell+1}Rm|^2\right).
\end{aligned} \tag{3.20}
$$

We can estimate the last term by a commutation identity.

$$
\bar\nabla\nabla^m\bar\nabla^\ell Rm - \nabla^m\bar\nabla\bar\nabla^\ell Rm = \sum_{i=0}^{m-1}\nabla^i Rm * \nabla^{m-1-i}\bar\nabla^\ell Rm. \tag{3.21}
$$

It follows that

$$
\begin{aligned}
&|\bar\nabla\nabla^m\bar\nabla^\ell Rm|^2 - |\nabla^m\bar\nabla^{\ell+1}Rm|^2\\
&\geq -C|\nabla^m\bar\nabla^{\ell+1}Rm|\cdot\sum_{i=0}^{m-1}|\nabla^i Rm * \nabla^{m-1-i}\bar\nabla^\ell Rm|\\
&\quad - C\sum_{i=0}^{m-1}|\nabla^i Rm * \nabla^{m-1-i}\bar\nabla^\ell Rm|^2\\
&\geq -C_1|\nabla^m\bar\nabla^{\ell+1}Rm|\cdot\sum_{i=0}^{m-1}|D^i Rm|\cdot|D^{m+\ell-1-i}Rm|\\
&\quad - C\sum_{i=0}^{m-1}|D^i Rm|^2\cdot|D^{m+\ell-1-i}Rm|^2.
\end{aligned} \tag{3.22}
$$

Putting all the computation together, we arrive at

$$
\begin{aligned}
&\partial_t|\nabla^m\bar\nabla^\ell Rm|^2\\
&\leq \frac{1}{2\|\Omega\|_\omega}\left\{\frac{1}{2}\Delta_{\mathbf{R}}|\nabla^m\bar\nabla^\ell Rm|^2 - |\nabla^{m+1}\bar\nabla^\ell Rm|^2 - |\nabla^m\bar\nabla^{\ell+1}Rm|^2\right.
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&+ C_1|\nabla^m\bar\nabla^{\ell+1}Rm|\cdot\sum_{i=0}^{m-1}|D^iRm|\cdot|D^{m+\ell-1-i}Rm|\\
&+ C\sum_{i=0}^{m-1}|D^iRm|^2\cdot|D^{m+\ell-1-i}Rm|^2\\
&+ C|\nabla^m\bar\nabla^\ell Rm|\cdot\Bigg[\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^i\bar\nabla^j Rm|\cdot|\nabla^{m-i}\bar\nabla^{\ell-j}Rm|\\
&+ |\nabla^{m+1}\bar\nabla^{\ell+1}(T*T)|\\
&+ \sum_{i=0}^{m}\sum_{j=0}^{\ell-1}|\nabla^i\bar\nabla^j Rm|\cdot|\nabla^{m-i}\bar\nabla^{\ell-j}(T*T)| + |\nabla^m\bar\nabla^{\ell+1}(T*Rm)|\\
&+ \sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^i\bar\nabla^j T|\cdot\Big(\nabla^{m-i}\bar\nabla^{\ell+1-j}Rm| + |\nabla^{m+1-i}\bar\nabla^{\ell-j}Rm|\Big)\\
&+ |\nabla^{m+1}\bar\nabla^\ell(\bar T*Rm)| + \sum_{i=0}^{m}\sum_{j=0}^{\ell-1}|\nabla^i\bar\nabla^j Rm|\cdot|\nabla^{m-i}\bar\nabla^{\ell-1-j}(\bar T*Rm)|\\
&+ |\nabla^m\bar\nabla^\ell(Rm*Rm)| + |\nabla^m\bar\nabla^{\ell+1}(T*\Psi)|\\
&+ |\nabla^m\bar\nabla^\ell(\Psi*\bar T*T)| + |\nabla^m\bar\nabla^\ell(\nabla\Psi*\bar T)|\\
&+ \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^{m-i}\bar\nabla^{\ell-j}H|\cdot\left|\nabla^i\bar\nabla^j\left(\frac{1}{2\|\Omega\|_\omega}\right)\right|\\
&+ \sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^{m-i}\bar\nabla^{\ell-j}Rm|\cdot|\nabla^i\bar\nabla^j(\partial_t g)|\Bigg]\Bigg\}\\
&+ \frac{C}{2\|\Omega\|_\omega}|\nabla^m\bar\nabla^\ell Rm|^2\cdot|\Psi|
\end{aligned}
$$

where we used commutating identities for terms $\nabla^m\bar\nabla^\ell\nabla\bar\nabla(T*T)$ and $\nabla^m\bar\nabla^\ell\nabla(\bar T*Rm)$ in the evolution equation $\partial_t\nabla^k\bar\nabla^\ell Rm$. Next, we use the non-standard notation D introduced at the beginning of this section. Note that, for a tensor E,

$$|\nabla^i\bar\nabla^j E| \le |D^{i+j}E|. \tag{3.24}$$

Let $k = m+\ell$. We have

$$
\begin{aligned}
&\partial_t|\nabla^m\bar\nabla^\ell Rm|^2 \\
&\le \frac{1}{2\|\Omega\|_\omega}\Bigg\{\frac{1}{2}\Delta_{\mathbf{R}}|\nabla^m\bar\nabla^\ell Rm|^2 - |\nabla^{m+1}\bar\nabla^\ell Rm|^2 - |\nabla^m\bar\nabla^{\ell+1}Rm|^2
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&+ C_1|\nabla^m\bar\nabla^{\ell+1}Rm|\cdot\sum_{i=0}^{k-1}|D^iRm|\cdot|D^{k-1-i}Rm|\\
&+ C\sum_{i=0}^{k-1}|D^iRm|^2\cdot|D^{k-1-i}Rm|^2\\
&+ C|\nabla^m\bar\nabla^\ell Rm|\cdot\Bigg[\sum_{i=0}^{k}|D^iRm|\cdot|D^{k-i}Rm| + \sum_{i=0}^{k}|D^iT|\cdot|D^{k+1-i}Rm|\\
&+ |D^{k+2}(T*T)| + \sum_{i=0}^{k-1}|D^iRm|\cdot|D^{k-i}(T*T)|\\
&+ |D^{k+1}(T*Rm)| + |D^{k+1}(\bar T*Rm)| + \sum_{i=0}^{k-1}|D^iRm|\cdot|D^{k-1-i}(\bar T*Rm)|\\
&+ |D^k(Rm*Rm)| + |D^{k+1}(T*\Psi)| + |D^k(\Psi*T*T)| + |D^k(\nabla\Psi*T)|\\
&+ \sum_{i=1}^{k}|D^{k-i}H|\cdot\left|D^i\left(\frac{1}{2\|\Omega\|_\omega}\right)\right| + \sum_{i=1}^{k}|D^{k-i}Rm|\cdot|D^i(\partial_t g)|\Bigg]\Bigg\}\\
&+ \frac{C}{2\|\Omega\|_\omega}|\nabla^m\bar\nabla^\ell Rm|^2\cdot|\Psi|.
\end{aligned}
$$

Recall that

$$|D^kRm|^2 = \sum_{m+\ell=k}|\nabla^m\bar\nabla^\ell Rm|^2 \tag{3.26}$$

$$|\nabla^m\bar\nabla^{\ell+1}Rm| \le |D^{k+1}Rm|, \quad |\nabla^m\bar\nabla^\ell Rm| \le |D^kRm| \tag{3.27}$$

and we also have

$$
\begin{aligned}
|D^{k+1}Rm|^2 &= \sum_{m+q=k+1}|\nabla^m\bar\nabla^q Rm|^2\\
&= \sum_{m+q-1=k,\,q\ge1}|\nabla^m\bar\nabla^q Rm|^2 + |\nabla^{k+1}Rm|^2\\
&= \sum_{m+\ell=k,\,m\ge0,\,\ell\ge0}|\nabla^m\bar\nabla^{\ell+1}Rm|^2 + |\nabla^{k+1}Rm|^2\\
&\le \sum_{m+\ell=k}|\nabla^m\bar\nabla^{\ell+1}Rm|^2 + \sum_{m+\ell=k}|\nabla^{m+1}\bar\nabla^\ell Rm|^2.
\end{aligned}
\tag{3.28}
$$

Using these inequalities, we get

$$
\begin{aligned}
(3.29)\qquad & \partial_t |D^k Rm|^2 \\
\le & \frac{1}{2\|\Omega\|_\omega} \Bigg\{ \frac{1}{2}\Delta_{\mathbf{R}} |D^k Rm|^2 - |D^{k+1} Rm|^2 \\
& + C_1 |D^{k+1} Rm| \cdot \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i} Rm| \\
& + C \sum_{i=0}^{k-1} |D^i Rm|^2 \cdot |D^{k-1-i} Rm|^2 \\
& + C |D^k Rm| \cdot \Bigg[\sum_{i=0}^{k} |D^i Rm| \cdot |D^{k-i} Rm| + \sum_{i=0}^{k} |D^i T| \cdot |D^{k+1-i} Rm| \\
& + |D^{k+2}(T * T)| + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-i}(T * T)| \\
& + |D^{k+1}(T * Rm)| + |D^{k+1}(\bar{T} * Rm)| \\
& + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i}(\bar{T} * Rm)| \\
& + |D^k (Rm * Rm)| + |D^{k+1}(T * \Psi)| + |D^k(\Psi * T * T)| \\
& + |D^k(\nabla \Psi * T)| \\
& + \sum_{i=1}^{k} |D^{k-i} H| \cdot \left| D^i \left(\frac{1}{2\|\Omega\|_\omega} \right) \right| + \sum_{i=1}^{k} |D^{k-i} Rm| \cdot |D^i(\partial_t g)| \Bigg] \Bigg\} \\
& + \frac{C}{2\|\Omega\|_\omega} |D^k Rm|^2 \cdot |\Psi|.
\end{aligned}
$$

We estimate the terms on right hand side one by one. Recall that we have

$$ (3.30)\qquad |D^j Rm| \le \frac{C\,A}{t^{j/2}}, \quad 0 \le j \le k-1 $$

$$ (3.31)\qquad |D^{j+1} T| \le \frac{C\,A}{t^{j/2}}, \quad 0 \le j \le k-1 $$

$$ (3.32)\qquad |T|^2 \le C\,A; $$

and the unknown terms are $|D^{k+1}Rm|, |D^k Rm|, |D^{k+2}T|$ and $|D^{k+1}T|$.

- Estimate for $|D^{k+1}Rm|\cdot\sum_{i=0}^{k-1}|D^iRm|\cdot|D^{k-1-i}Rm|$:

$$\begin{aligned}(3.33)\qquad & |D^{k+1}Rm|\cdot\sum_{i=0}^{k-1}|D^iRm|\cdot|D^{k-1-i}Rm| \\ &\leq |D^{k+1}Rm|\cdot\sum_{i=0}^{k-1}\frac{CA}{t^{i/2}}\cdot\frac{CA}{t^{(k-1-i)/2}} \\ &\leq |D^{k+1}Rm|\cdot CA^2\,t^{-\frac{k-1}{2}} \\ &\leq \theta|D^{k+1}Rm|^2+C(\theta)\,A^3\,t^{-k}.\end{aligned}$$

where θ is a small positive number such that $C_1\theta<\frac{1}{4}$. To obtain the last inequality, we used Cauchy-Schwarz inequality and the fact that $A\,t<1$.

- Estimate for $\sum_{i=0}^{k-1}|D^iRm|^2\cdot|D^{k-1-i}Rm|^2$:

$$\begin{aligned}(3.34)\qquad \sum_{i=0}^{k-1}|D^iRm|^2\cdot|D^{k-1-i}Rm|^2 &\leq \sum_{i=0}^{k-1}\left(\frac{CA}{t^{i/2}}\right)^2\cdot\left(\frac{CA}{t^{(k-1-i)/2}}\right)^2 \\ &\leq CA^4\,t^{-(k-1)} \\ &\leq CA^3\,t^{-k}.\end{aligned}$$

- Estimate for $\sum_{i=0}^{k}|D^iRm|\cdot|D^{k-i}Rm|$:

$$\begin{aligned}(3.35)\qquad \sum_{i=0}^{k}|D^iRm|\cdot|D^{k-i}Rm| &= 2|D^kRm|\cdot|Rm| \\ &\quad+\sum_{i=1}^{k-1}|D^iRm|\cdot|D^{k-i}Rm| \\ &\leq CA\,|D^kRm|+CA^2\,t^{-\frac{k}{2}}.\end{aligned}$$

- Estimate for $\sum_{i=0}^{k}|D^iT|\cdot|D^{k+1-i}Rm|$:

$$\begin{aligned}(3.36)\quad \sum_{i=0}^{k}|D^iT|\cdot|D^{k+1-i}Rm| &= |T|\cdot|D^{k+1}Rm|+|DT|\cdot|D^kRm| \\ &\quad+\sum_{i=2}^{k}|D^iT|\cdot|D^{k+1-i}Rm| \\ &\leq CA^{\frac{1}{2}}\,|D^{k+1}Rm|+CA\,|D^kRm|+CA^2\,t^{-\frac{k}{2}}.\end{aligned}$$

• Estimate for $|D^{k+2}(T*T)|$:

$$
\begin{aligned}
|D^{k+2}(T*T)| &\leq \sum_{i=0}^{k+2} |D^i T|\cdot|D^{k+2-i}T| \\
&= 2|T|\cdot|D^{k+2}T| + 2|DT|\cdot|D^{k+1}T| \\
&\quad + \sum_{i=2}^{k} |D^i T|\cdot|D^{k+2-i}T| \\
&\leq CA^{\frac{1}{2}}\,|D^{k+2}T| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}}.
\end{aligned}
\tag{3.37}
$$

• Estimate for $\sum_{i=0}^{k-1}|D^i Rm|\cdot|D^{k-i}(T*T)|$:

$$
\begin{aligned}
&\sum_{i=0}^{k-1} |D^i Rm|\cdot|D^{k-i}(T*T)| \\
&\leq 2\sum_{i=0}^{k-1} |D^i Rm|\cdot|T|\cdot|D^{k-i}T| \\
&\quad + \sum_{i=0}^{k-1}\sum_{j=1}^{k-i} |D^i Rm|\cdot|D^j T|\cdot|D^{k-i-j}T| \\
&\leq CA^2\,t^{-\frac{k}{2}}.
\end{aligned}
\tag{3.38}
$$

• Estimate for $|D^{k+1}(T*Rm)|$:

$$
\begin{aligned}
&|D^{k+1}(T*Rm)| \\
&\leq |T|\cdot|D^{k+1}Rm| + |DT|\cdot|D^k Rm| + |D^{k+1}T|\cdot|Rm| \\
&\quad + \sum_{i=2}^{k} |D^i T|\cdot|D^{k+1-i}Rm| \\
&\leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,|D^k Rm| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}}.
\end{aligned}
\tag{3.39}
$$

• Estimate $|D^{k+1}(\bar{T}*Rm)|$:

$$
\begin{aligned}
&|D^{k+1}(\bar{T}*Rm)| \\
&\leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,|D^k Rm| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}}.
\end{aligned}
\tag{3.40}
$$

• Estimate for $\sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i}(\bar{T} * Rm)|$:

$$\begin{aligned}(3.41) \qquad & \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i}(\bar{T} * Rm)| \\ & \leq \sum_{i=0}^{k-1} |D^i Rm| \cdot |T| \cdot |D^{k-1-i} Rm| \\ & \quad + \sum_{i=0}^{k-1} \sum_{j=1}^{k-1-i} |D^i Rm| \cdot |D^j T| \cdot |D^{k-1-i-j} Rm| \\ & \leq C A^2 \, t^{-\frac{k}{2}}.\end{aligned}$$

• Estimate for $|D^k(Rm * Rm)|$:

$$\begin{aligned}(3.42) \qquad |D^k(Rm * Rm)| & \leq 2|Rm| \cdot |D^k Rm| + \sum_{i=1}^{k-1} |D^i Rm| \cdot |D^{k-i} Rm| \\ & \leq CA \, |D^k Rm| + CA^2 \, t^{-\frac{k}{2}}.\end{aligned}$$

• Estimate for $|D^{k+1}(T * \Psi)|$:
Recall that $\Psi_{\bar{p}q} = -\tilde{R}_{\bar{p}q} + g^{s\bar{r}} \, g^{m\bar{n}} \, T_{\bar{n}sq} \, \bar{T}_{m\bar{r}\bar{p}}$, we have

$$(3.43) \qquad |D^{k+1}(\Psi * T)| \leq |D^{k+1}(Rm * T)| + |D^{k+1}(T * T * T)|.$$

The first term is the same as (3.39). We only need to estimate the second term.

$$\begin{aligned}(3.44) \quad |D^{k+1}(T * T * T)| & \leq |D^{k+1} T| \cdot |T|^2_\omega + \sum_{p+q=k+1; p,q>0} |D^p T| \cdot |D^q T| \cdot |T| \\ & \quad + \sum_{p+q+r=k+1; p,q,r>0} |D^p T| \cdot |D^q T| \cdot |D^r T| \\ & \leq CA \, |D^{k+1} T| + CA^{\frac{5}{2}} \, t^{-\frac{(k-1)}{2}} + CA^3 \, t^{-\frac{(k-2)}{2}} \\ & \leq CA \, |D^{k+1} T| + CA^2 \, t^{-\frac{k}{2}}.\end{aligned}$$

It follows that

$$\begin{aligned}(3.45) \qquad |D^{k+1}(\Psi * T)| & \leq CA^{\frac{1}{2}} \, |D^{k+1} Rm| \\ & \quad + CA \, (|D^k Rm| + |D^{k+1} T|) + CA^2 \, t^{-\frac{k}{2}}.\end{aligned}$$

• Estimate for $|D^k(\Psi * T * T)|$:

$$|D^k(\Psi * T * T)| \le |D^k(Rm * T * T)| + |D^k(T * T * T * T)|. \tag{3.46}$$

We use the same trick as above to estimate these two terms. For the first term, we have

$$\begin{aligned} &|D^k(Rm * T * T)| \\ &\le |D^k Rm| \cdot |T|_\omega^2 + \sum_{p+q=k;q>0} |D^p Rm| \cdot |D^q T| \cdot |T| \\ &\quad + \sum_{p+q+r=k;q,r>0} |D^p Rm| \cdot |D^q T| \cdot |D^r T| \\ &\le CA\,|D^k Rm| + CA^{\frac{5}{2}}\, t^{-\frac{k-1}{2}} + CA^3\, t^{-\frac{k-2}{2}} \\ &\le CA\,|D^k Rm| + CA^2\, t^{-\frac{k}{2}}. \end{aligned} \tag{3.47}$$

For the second term, we have

$$\begin{aligned} &|D^k(T * T * T * T)| \\ &\le 4|D^k T| \cdot |T|^3 + \sum_{p+q=k;\, p,q>0} |D^p T| \cdot |D^q T| \cdot |T|_\omega^2 \\ &\quad + \sum_{p+q+r=k;\, p,q,r>0} |D^p T| \cdot |D^q T| \cdot |D^r T| \cdot |T|_\omega \\ &\quad + \sum_{p+q+r+s=k;\, p,q,r,s>0} |D^p T| \cdot |D^q T| \cdot |D^r T| \cdot |D^s T| \\ &\le CA^{\frac{5}{2}}\, t^{-\frac{k-1}{2}} + CA^3\, t^{-\frac{k-2}{2}} + CA^{\frac{7}{2}}\, t^{-\frac{k-3}{2}} + CA^4\, t^{-\frac{k-4}{2}} \\ &\le CA^2\, t^{-\frac{k}{2}}. \end{aligned} \tag{3.48}$$

Thus, we have

$$|D^k(\Psi * T * T)| \le CA\,|D^k Rm| + CA^2\, t^{-\frac{k}{2}}. \tag{3.49}$$

• Estimate for $|D^k(\nabla\Psi * T)|$:

$$|D^k(\nabla\Psi * T)| \le |D^k(\nabla Rm * T)| + |D^k(\nabla(T * T) * T)| \tag{3.50}$$

$$\leq |D^{k+1}Rm| \cdot |T| + |D^k Rm| \cdot |DT| + \sum_{i=2}^{k} |D^{k+1-i}Rm| \cdot |D^i T|$$
$$+ |D^{k+1}(T * T)| \cdot |T| + \sum_{i=1}^{k} |D^{k+1-i}(T * T)| \cdot |D^i T|$$
$$\leq CA^{\frac{1}{2}} |D^{k+1}Rm| + CA |D^k Rm| + CA |D^{k+1}T| + CA^2 t^{-\frac{k}{2}}.$$

• Estimate for $\sum_{i=1}^{k} |D^{k-i}H| \cdot \left| D^i \left(\frac{1}{2\|\Omega\|_\omega} \right) \right|$:
Recall that

$$H = \frac{1}{2}\Delta_{\mathbf{R}} Rm + \nabla\bar{\nabla}(T * \bar{T}) + \bar{\nabla}(T * Rm) + \nabla(\bar{T} * Rm)$$
$$+ Rm * Rm + (\bar{\nabla}T - \bar{T} * T) * \Psi + \bar{T} * \nabla\Psi + T * \bar{\nabla}\Psi. \tag{3.51}$$

and we also compute, for any m,

$$\nabla^m \left(\frac{1}{2\|\Omega\|_\omega} \right) = \nabla^{m-1}\nabla \left(\frac{1}{2\|\Omega\|_\omega} \right) = -\nabla^{m-1} \left(\frac{1}{2\|\Omega\|_\omega} T \right)$$
$$= -\nabla^{m-1} \left(\frac{1}{2\|\Omega\|_\omega} \right) * T - \frac{1}{2\|\Omega\|_\omega} \nabla^{m-1} T$$
$$= \frac{1}{2\|\Omega\|_\omega} \sum_{j=1}^{m} \nabla^{m-j} T * T^{j-1}. \tag{3.52}$$

where $T^{j-1} = T * T * \cdots * T$ with $(j-1)$ factors. Again keep in mind that the unknown terms are $|D^{k+1}Rm|$, $|D^k Rm|$,$|D^{k+2}T|$ and $|D^{k+1}T|$. Notice that these terms only appear for $i = 1, 2$ in the summation.

$$\sum_{i=1}^{k} |D^{k-i}H| \cdot \left| D^i \left(\frac{1}{2\|\Omega\|_\omega} \right) \right| \tag{3.53}$$
$$= |D^{k-1}H| \cdot \left| D \left(\frac{1}{2\|\Omega\|_\omega} \right) \right| + |D^{k-2}H| \cdot \left| D^2 \left(\frac{1}{2\|\Omega\|_\omega} \right) \right|$$
$$+ \sum_{i=3}^{k} |D^{k-i}H| \cdot \left| D^i \left(\frac{1}{2\|\Omega\|_\omega} \right) \right|.$$

Using (3.51) and (3.52), we can estimate the terms on the right hand side one by one and obtain

$$\sum_{i=1}^{k}|D^{k-i}H|\cdot\left|D^i\left(\frac{1}{2\|\Omega\|_\omega}\right)\right| \le CA^{\frac{1}{2}}|D^{k+1}Rm| + CA\left(|D^kRm| + |D^{k+1}T|\right) + CA^2\,t^{-\frac{k}{2}}. \tag{3.54}$$

• Estimate for $\sum_{i=1}^{k}|D^{k-i}Rm|\cdot|D^i(\partial_t g)|$:

$$|D^i(\partial_t g)| = \left|D^i\left(\frac{1}{2\|\Omega\|_\omega}\Psi\right)\right| = \sum_{j=0}^{i}\left|D^j\left(\frac{1}{2\|\Omega\|_\omega}\right)\right|\cdot|D^{i-j}\Psi|. \tag{3.55}$$

By the definition of Ψ and the computation (3.52), we know that the only unknown term appeared in the summation is when $j = i = k$. Thus, we arrive the following estimate

$$\sum_{i=1}^{k}|D^{k-i}Rm|\cdot|D^i(\partial_t g)| \le CA\,|D^kRm| + CA^2\,t^{-\frac{k}{2}}. \tag{3.56}$$

• Estimate for the last term $|D^kRm|^2\cdot|\Psi|$:

$$|D^kRm|^2\cdot|\Psi| \le CA\,|D^kRm|^2. \tag{3.57}$$

Finally, putting all the above estimates together, we obtain the lemma. □

Following the same strategy, we can also prove the following lemma on estimates for the derivatives of the torsion.

Lemma 10. *Under the same assumption as in Lemma 9, we have*

$$\begin{aligned}\partial_t|D^{k+1}T|^2 \le \frac{1}{2\|\Omega\|_\omega}\Big\{&\frac{1}{2}\Delta_{\mathbf{R}}|D^{k+1}T|^2 - \frac{3}{4}\,|D^{k+2}T|^2\\ &+ CA^{\frac{1}{2}}\Big(|D^{k+2}T| + |D^{k+1}Rm|\Big)\cdot|D^{k+1}T|\\ &+ CA\Big(|D^{k+1}T| + |D^kRm|\Big)\cdot|D^{k+1}T|\\ &+ CA^2\,t^{-\frac{k}{2}}\,|\nabla^{k+1}T| + CA^3t^{-k}\Big\}.\end{aligned} \tag{3.58}$$

Now we return to the proof of Theorem 2:

We first prove the estimate (1.12) for the case $k=1$. To obtain the desired estimate, we apply the maximum principle to the function

$$G_1(z,t) = t\left(|DRm|^2 + |D^2T|^2\right) + \Lambda\left(|Rm|^2 + |DT|^2\right). \tag{3.59}$$

Using Lemma 9 and Lemma 10 with $k=1$, we have

$$\begin{aligned}
&\partial_t\left(|DRm|^2 + |D^2T|^2\right) \\
&\le \frac{1}{2\|\Omega\|_\omega}\bigg\{\frac{1}{2}\Delta_{\mathbf{R}}\left(|DRm|^2 + |D^2T|^2\right) - \frac{3}{4}\left(|D^2Rm|^2 + |D^3T|^2\right) \\
&\quad + CA^{\frac{1}{2}}\left(|D^2Rm| + |D^3T|\right)\cdot\left(|DRm| + |D^2T|\right) \\
&\quad + CA\left(|DRm| + |D^2T|\right)^2 + CA^2\,t^{-\frac{1}{2}}\left(|DRm| + |D^2T|\right) \\
&\quad + CA^3\,t^{-1}\bigg\} \\
&\le \frac{1}{2\|\Omega\|_\omega}\bigg\{\frac{1}{2}\Delta_{\mathbf{R}}\left(|DRm|^2 + |D^2T|^2\right) - \frac{1}{2}\left(|D^2Rm|^2 + |D^3T|^2\right) \\
&\quad + CA\left(|DRm|^2 + |D^2T|^2\right) + CA^3\,t^{-1}\bigg\}.
\end{aligned} \tag{3.60}$$

where we used the Cauchy-Schwarz inequality in the last inequality.

Recall the evolution equation

$$\begin{aligned}
&\partial_t(|DT|^2 + |Rm|^2) \\
&\le \frac{1}{2\|\Omega\|_\omega}\bigg\{\frac{1}{2}\Delta_{\mathbf{R}}(|DT|^2 + |Rm|^2) - \frac{1}{2}(|D^2T|^2 + |DRm|^2) \\
&\quad + CA^{\frac{3}{2}}(|DRm| + |D^2T|) + CA^3\bigg\}.
\end{aligned} \tag{3.61}$$

It follows that

$$\begin{aligned}
\partial_t G_1 \le \frac{1}{4\|\Omega\|_\omega}&\Big\{\Delta_{\mathbf{R}}G_1 - t\left(|D^2Rm|^2 + |D^3T|^2\right) - \Lambda\left(|D^2T|^2 + |DRm|^2\right) \\
&+ CA\,t\,(|DRm|^2 + |D^2T|^2) + CA^3 \\
&+ CA^{\frac{3}{2}}\,\Lambda(|DRm| + |D^2T|) + CA^3\,\Lambda\Big\} + (|DRm|^2 + |D^2T|^2).
\end{aligned} \tag{3.62}$$

Again, using Cauchy-Schwarz inequality,

$$C\,A^{\frac{3}{2}}\,\Lambda(|DRm| + |D^2T|) \le CA^3\,\Lambda + \Lambda(|DRm|^2 + |D^2T|^2). \tag{3.63}$$

Putting these estimates together, we have

$$\partial_t G \leq \frac{1}{4\|\Omega\|_\omega} \Big\{ \Delta_{\mathbf{R}} G - t\left(|D^2 Rm|^2 + |D^3 T|^2\right) + (\|\Omega\|_\omega - \Lambda + CAt)\left(|D^2 T|^2 + |DRm|^2\right) + CA^3 \Big\}. \tag{3.64}$$

By $At \leq 1$ and choosing Λ large enough,

$$\partial_t G \leq \frac{1}{4\|\Omega\|_\omega} \Big\{ 2\Delta_{\mathbf{R}} G + CA^3\Lambda \Big\}. \tag{3.65}$$

We note that the choice of constant Λ depends on the upper bound of $\|\Omega\|_\omega$. However, with the assumption (1.11), we can get the uniform C^0 bound of the metric depending on the uniform lower bound of $\|\Omega\|_\omega$. Consequently, we obtain the upper bound of $\|\Omega\|_\omega$, which also depends on the uniform lower bound of $\|\Omega\|_\omega$.

To finish the proof for $k = 1$, observing that when $t = 0$,

$$G(0) = \frac{\Lambda}{2}(|DT|^2 + |Rm|^2) \leq C\Lambda A^2. \tag{3.66}$$

Thus, applying the maximum principle to the above inequality implies that

$$G(t) \leq C\Lambda A^2 + CA^3\Lambda\, t \leq CA^2. \tag{3.67}$$

It follows

$$|DRm| + |D^2 T| \leq \frac{CA}{t^{1/2}}. \tag{3.68}$$

This establishes the estimate (1.12) when $k = 1$. Next, we use induction on k to prove the higher order estimates.

Using Lemma 9 and Lemma 10 again, we have

$$
\begin{aligned}
(3.69)\qquad & \partial_t \left(|D^k Rm|^2 + |D^{k+1} T|^2 \right) \\
\le{} & \frac{1}{2\|\Omega\|_\omega} \left\{ \frac{1}{2} \Delta_{\mathbf{R}} \left(|D^k Rm|^2 + |D^{k+1} T|^2 \right) \right. \\
& - \frac{3}{4} \left(|D^{k+1} Rm|^2 + |D^{k+2} T|^2 \right) \\
& + C A^{\frac{1}{2}} \left(|D^{k+1} Rm| + |D^{k+2} T| \right) \cdot \left(|D^k Rm| + |D^{k+1} T| \right) \\
& + C A \left(|D^k Rm| + |D^{k+1} T| \right)^2 \\
& \left. + C A^2 \, t^{-\frac{k}{2}} \left(|D^k Rm| + |D^{k+1} T| \right) + C A^3 \, t^{-k} \right\} \\
\le{} & \frac{1}{2\|\Omega\|_\omega} \left\{ \frac{1}{2} \Delta_{\mathbf{R}} \left(|D^k Rm|^2 + |D^{k+1} T|^2 \right) \right. \\
& - \frac{1}{2} \left(|D^{k+1} Rm|^2 + |D^{k+2} T|^2 \right) \\
& \left. + C A \left(|D^k Rm|^2 + |D^{k+1} T|^2 \right) + C A^3 \, t^{-k} \right\}.
\end{aligned}
$$

Denote

$$
f_j(z,t) = |D^j Rm|^2 + |D^{j+1} T|^2. \tag{3.70}
$$

Then,

$$
\partial_t f_k \le \frac{1}{4\|\Omega\|_\omega} \left(\Delta_{\mathbf{R}} f_k - f_{k+1} + C A \, f_k + C A^3 \, t^{-k} \right). \tag{3.71}
$$

Next, we apply the maximum principle to the test function

$$
G_k(z,t) = t^k f_k + \sum_{i=1}^{k} \Lambda_i \, B_i^k \, t^{k-i} \, f_{k-i} \tag{3.72}
$$

where $\Lambda_i \, (1 \le i \le k)$ are large numbers to be determined and $B_i^k = \frac{(k-1)!}{(k-i)!}$. We note that, for $1 \le i < k$, we still have an inequality similar to (3.70) for f_{k-i}.

$$
\begin{aligned}
(3.73)\qquad \partial_t f_{k-i} &\le \frac{1}{4\|\Omega\|_\omega} \left(2\Delta_{\mathbf{R}} f_{k-i} - f_{k-i+1} + C A \, f_{k-i} + C A^3 \, t^{-(k-i)} \right) \\
&\le \frac{1}{4\|\Omega\|_\omega} \left(2\Delta_{\mathbf{R}} f_{k-i} - f_{k-i+1} + C A^3 \, t^{-(k-i)} \right)
\end{aligned}
$$

where we used the induction condition (3.9) for the term f_{k-i} when $1 \le i < k$. From (3.70) and (3.73), we deduce

$$
\begin{aligned}
(3.74)\quad \partial_t G_k &= kt^{k-1}\, f_k + t^k \partial_t f_k + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} \\
&\quad + \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, \partial_t f_{k-i} \\
&= kt^{k-1}\, f_k + \frac{1}{4\|\Omega\|_\omega}\, t^k \left(2\Delta_{\mathbf{R}} f_k - f_{k+1} + CA\, f_k + CA^3\, t^{-k}\right) \\
&\quad + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} \\
&\quad + \frac{1}{4\|\Omega\|_\omega} \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i} \left(2\Delta_{\mathbf{R}} f_{k-i} - f_{k-i+1} + CA^3\, t^{-(k-i)}\right) \\
&= \frac{1}{4\|\Omega\|_\omega}\, 2\Delta_{\mathbf{R}} G_k - \frac{1}{4\|\Omega\|_\omega}\, t^k\, f_{k+1} + t^{k-1} f_k \left(k + \frac{CA\, t}{4|\Omega|_\omega}\right) \\
&\quad + \frac{1}{4\|\Omega\|_\omega}\, CA^3 \left(1 + \sum_{i=1}^{k} \Lambda_i\, B_i^k\right) + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} \\
&\quad - \frac{1}{4\|\Omega\|_\omega} \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1} \\
&\le \frac{1}{4\|\Omega\|_\omega}\, 2\Delta_{\mathbf{R}} G_k + t^{k-1} f_k \left(k + \frac{CA\, t}{4\|\Omega\|_\omega} - \frac{\Lambda_1\, B_1^k}{4\|\Omega\|_\omega}\right) \\
&\quad + \frac{1}{4\|\Omega\|_\omega}\, CA^3 + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} \\
&\quad - \frac{1}{4\|\Omega\|_\omega} \sum_{i=2}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1}.
\end{aligned}
$$

We note that the last two terms can be re-written as

$$
\begin{aligned}
(3.75)\qquad & \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} - \frac{1}{4\|\Omega\|_\omega} \sum_{i=2}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1} \\
&= \sum_{i=1}^{k-1} \left(\Lambda_i\, B_i^k\, (k-i) - \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}\, B_{i+1}^k\right) t^{k-i-1}\, f_{k-i} \\
&= \sum_{i=1}^{k-1} \left(\Lambda_i - \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}\right) B_{i+1}^k\, t^{k-i-1}\, f_{k-i}.
\end{aligned}
$$

Thus, we obtain

$$\partial_t G_k \leq \frac{1}{4\|\Omega\|_\omega} 2\Delta_{\mathbf{R}} G_k + t^{k-1} f_k \left(k + \frac{CA\,t}{4\|\Omega\|_\omega} - \frac{\Lambda_1 B_1^k}{4\|\Omega\|_\omega} \right) + \frac{1}{4\|\Omega\|_\omega} CA^3 + \sum_{i=1}^{k-1} \left(\Lambda_i - \frac{1}{4\|\Omega\|_\omega} \Lambda_{i+1} \right) B_{i+1}^k t^{k-i-1} f_{k-i}. \tag{3.76}$$

Choosing Λ_1 large enough and $\Lambda_i \leq \frac{1}{4\|\Omega\|_\omega} \Lambda_{i+1}$ for $1 \leq i \leq k-1$, we have

$$\partial_t G_k \leq \frac{1}{4\|\Omega\|_\omega} (2\Delta_{\mathbf{R}} G_k + CA^3). \tag{3.77}$$

Note that

$$\max_{z \in M} G(z,0) = \Lambda_k B_k^k f_0 = \frac{(k-1)!}{2} \Lambda_k (|Rm|^2 + |DT|^2) \leq CA^2. \tag{3.78}$$

Applying the maximum principle to the inequality satisfied by G_k, we have

$$\max_{z \in M} G(z,t) \leq CA^2 + CA^3\, t \leq CA^2. \tag{3.79}$$

Finally, we get

$$|D^k Rm| + |D^{k+1} T| \leq CA\, t^{-\frac{k}{2}}. \tag{3.80}$$

The proof of Theorem 2 is complete. □

3.3. Doubling estimates for the curvature and torsion

Let

$$f(z,t) = |DT|^2_\omega + |Rm|^2_\omega + |T|^4_\omega \tag{3.81}$$

and denote $f(t) = \max_{z \in M} f(z,t)$. We can derive a doubling-time estimate for $f(t)$, which roughly says that $f(t)$ cannot blow up quickly.

Proposition 1. *There is a constant C depending on a lower bound for $\|\Omega\|_\omega$ such that*

$$\max_M \left(|DT|^2 + |Rm|^2 + |T|^4 \right)(t) \leq 4 \max_M \left(|DT|^2 + |Rm|^2 + |T|^4 \right)(0) \tag{3.82}$$

for all $t \in \left[0, \frac{1}{4Cf^{\frac{1}{2}}(0)} \right]$.

Proof. The proof is standard and we apply the maximum principle to $f(z,t)$. Recall the evolution equations, by taking $k=0$ in (3.29),

$$\begin{aligned}(3.83)\qquad \partial_t|Rm|^2 \le \frac{1}{2\|\Omega\|_\omega}\Big\{&\frac{1}{2}\Delta_{\mathbf{R}}|Rm|^2 - |DRm|^2 + C|D^2T|\cdot|Rm|\cdot|T| \\ &+ C|DRm|\cdot|Rm|\cdot|T| + C|DT|^2\cdot|Rm| \\ &+ C|DT|\cdot|Rm|^2 + C|Rm|^3 + C|DT|\cdot|Rm|\cdot|T|^2 \\ &+ C|Rm|^2\cdot|T|^2 + C|Rm|\cdot|T|^4\Big\}.\end{aligned}$$

We apply the Young's inequalities and get

$$\begin{aligned}(3.84)\qquad \partial_t|Rm|^2 \le \frac{1}{2\|\Omega\|_\omega}\Big\{&\frac{1}{2}\Delta_{\mathbf{R}}|Rm|^2 - \frac{1}{2}|DRm|^2 + \frac{1}{2}|D^2T|^2 \\ &+ C\left(|DT|^3 + |Rm|^3 + |T|^6\right)\Big\}.\end{aligned}$$

Similarly, considering the evolution equation for $|DT|^2$ and $|T|^2$, we can derive

$$\begin{aligned}(3.85)\qquad \partial_t|\nabla T|^2 \le \frac{1}{2\|\Omega\|_\omega}\Big\{&\frac{1}{2}\Delta_{\mathbf{R}}|DT|^2 - \frac{1}{2}|D^2T|^2 + \frac{1}{2}|DRm|^2 \\ &+ C\left(|DT|^3 + |Rm|^3 + |T|^6\right)\Big\}\end{aligned}$$

and

$$(3.86)\qquad \partial_t|T|^4 \le \frac{1}{2\|\Omega\|_\omega}\left\{\frac{1}{2}\Delta_{\mathbf{R}}|T|^4 + C\left(|DT|^3 + |Rm|^3 + |T|^6\right)\right\}.$$

Putting the above evolution equations together, we have

$$\begin{aligned}(3.87)\qquad \partial_t f(z,t) &\le \frac{1}{2\|\Omega\|_\omega}\left\{\frac{1}{2}\Delta_{\mathbf{R}} f + C\left(|DT|^3 + |Rm|^3 + |T|^6\right)\right\} \\ &\le \frac{1}{2\|\Omega\|_\omega}\left(\frac{1}{2}\Delta_{\mathbf{R}} f + C f^{\frac{3}{2}}\right).\end{aligned}$$

Finally, by the maximum principle, we have

$$(3.88)\qquad \partial_t f(t) \le \frac{C}{2\|\Omega\|_\omega} f^{\frac{3}{2}}$$

which implies that

$$f(t) \leq \frac{f(0)}{\left(1 - 2C\, f^{\frac{1}{2}}(0)\, t\right)^2}. \tag{3.89}$$

Thus, as long as the flow exists and $t \leq \frac{1-\frac{1}{A}}{2Cf^{\frac{1}{2}}(0)}$, we have $f(t) \leq A^2 f(0)$. □

3.4. A criterion for the long-time existence of the flow

We can give now the proof of Theorem 3. We begin by observing that, under the given hypotheses, the metrics $\omega(t)$ are uniformly equivalent for $t \in (T-\delta, T)$. Our goal is to show that the metrics are uniformly bounded in C^∞ for some interval $t \in (T-\delta, T)$. This would imply the existence of the limit $\omega(T)$ of a subsequence $\omega(t_j)$ with $t_j \to T$. By the short-time existence theorem for the Anomaly flow proved in [22], it follows that the flow extends to $[0, T+\epsilon)$ for some $\epsilon > 0$.

3.4.1. C^1 bounds for the metric. We need to establish the C^∞ convergence of (subsequence of) the metrics $g_{\bar{k}j}(t)$ as $t \to T$. We have already noted the C^0 uniform boundedness of $g_{\bar{k}j}(t)$. In this section, we establish the C^1 bounds. For this, we fix a reference metric $\hat{g}_{\bar{k}j}$ and introduce the relative endomorphism

$$h^j{}_m(t) = \hat{g}^{j\bar{p}} g_{\bar{p}m}(t). \tag{3.90}$$

The uniform C^0 bound of $g_{\bar{k}j}(t)$ is equivalent to the C^0 bound of $h(t)$. We need to estimate the derivatives of $h(t)$. For this, recall the curvature relation between two different metrics $g_{\bar{k}j}(t)$ and $\hat{g}_{\bar{k}j}$,

$$R_{\bar{k}j}{}^p{}_m = \hat{R}^p_{\bar{k}jm} - \partial_{\bar{k}}(h^p{}_q \hat{\nabla}_j h^p{}_m) \tag{3.91}$$

where $\hat{\nabla}$ denotes the covariant derivative with respect to $\hat{g}_{\bar{k}j}$. This relation can be viewed as a second order PDE in h, with bounded right hand sides because the curvature $R_{\bar{k}j}{}^p{}_m$ is assumed to be bounded, and which is uniformly elliptic because the metrics $g_{\bar{k}j}(t)$ are uniformly equivalent (and hence the relative endomorphisms $h(t)$ are uniformly bounded away from 0 and ∞). It follows that

$$\|h\|_{C^{1,\alpha}} \leq C. \tag{3.92}$$

3.4.2. C^k bounds for the metric. We will use the notation G_k for the summation of norms squared of all combinations of $\hat{\nabla}^m\overline{\hat{\nabla}^\ell}$ acting on g such that $m+\ell=k$. For example,

$$G_2 = |\hat{\nabla}\hat{\nabla}g|^2 + |\hat{\nabla}\overline{\hat{\nabla}}g|^2 + |\overline{\hat{\nabla}\hat{\nabla}}g|^2. \tag{3.93}$$

We introduce the tensor

$$\Theta^k{}_{ij} = -g^{k\bar{\ell}}\hat{\nabla}_i g_{\bar{\ell}j}, \tag{3.94}$$

which is the difference of the background connection and the evolving connection: $\Theta = \Gamma_0 - \Gamma$. We will use the notation S_k for the summation of norms squared of all combinations of $\nabla^m\overline{\nabla^\ell}$ acting on Θ such that $m+\ell=k$. For example,

$$S_2 = |\nabla\nabla\Theta|^2 + |\nabla\overline{\nabla}\Theta|^2 + |\overline{\nabla\nabla}\Theta|^2. \tag{3.95}$$

Our evolution equation is

$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega}\,\Psi_{\bar{p}q}, \tag{3.96}$$

where $\Psi_{\bar{p}q} = -\tilde{R}_{\bar{p}q} + g^{\alpha\bar{\beta}}g^{s\bar{r}}T_{\bar{\beta}sq}\bar{T}_{\alpha\bar{r}\bar{p}}$.

Proposition 2. *Suppose all covariant derivatives of curvature and torsion of $g(t)$ with respect to the evolving connection ∇ are bounded on $[0,T)$. Then all covariant derivatives of $\frac{\Phi_{\bar{p}q}}{2\|\Omega\|_\omega}$ with respect to the evolving connection ∇ are bounded on $[0,T)$.*

Proof. Compute

$$\nabla^m\overline{\nabla}^\ell\left(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}\right) = \frac{1}{2}\sum_{i\le m}\sum_{j\le\ell}\nabla^i\overline{\nabla}^j\left(\frac{1}{\|\Omega\|_\omega}\right)\nabla^{m-i}\overline{\nabla}^{\ell-j}\Psi_{\bar{p}q}. \tag{3.97}$$

We have

$$\begin{aligned}\nabla^i\overline{\nabla}^j\left(\frac{1}{\|\Omega\|_\omega}\right) &= -\nabla^i\overline{\nabla}^{j-1}\left(\frac{\overline{T}}{\|\Omega\|_\omega}\right)\\ &= \frac{1}{\|\Omega\|_\omega}\sum\nabla^{i_1}\overline{\nabla}^{i_2}T^{i_3} * \nabla^{i_4}\overline{\nabla}^{i_5}\overline{T}^{i_6} * T^{i_7} * \overline{T}^{i_8}.\end{aligned} \tag{3.98}$$

Since Ψ is written in terms of curvature and torsion, and $\|\Omega\|_\omega$ has a lower bound, the proposition follows. $\square$

Proposition 3. *Suppose all covariant derivatives of curvature and torsion of $g(t)$ with respect to the evolving connection ∇ are bounded on $[0,T)$. If $G_i \leq C$ and $S_{i-1} \leq C$ for all non-negative integers $i \leq k$, then $G_{k+1} \leq C$ and $S_k \leq C$ on $[0,T)$.*

Proof. By the previous proposition, all covariant derivatives of $\frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega}$ with respect to the evolving connection ∇ are bounded on $[0,T)$. Let $m+\ell = k+1$, and compute

$$\begin{aligned}\hat\nabla^m \overline{\hat\nabla^\ell} \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega} &= (\nabla + \Theta)^m (\overline{\nabla} + \overline{\Theta})^\ell \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega} \\ &= \nabla^m \overline{\nabla}^{\ell-1} \left(\overline{\Theta} \, \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega} \right) + O(1) \\ &= \nabla^m \overline{\nabla}^{\ell-1} \overline{\Theta} \cdot \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega} + O(1),\end{aligned} \tag{3.99}$$

where $O(1)$ represents terms which involve evolving covariant derivatives of $\frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega}$ and up to $(k-1)$th order evolving covariant derivatives of Θ, which are bounded by assumption. If $\ell = 0$, the right-hand side is replaced by $\nabla^{m-1}\Theta \cdot \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega}$. Next, we compute

$$\begin{aligned}\nabla^m \overline{\nabla}^{\ell-1} \overline{\Theta}_{\bar i \bar j}^{\bar k} &= -g^{\ell \bar k} \nabla^m \overline{\nabla}^{\ell-1} \hat\nabla_{\bar i} g_{\bar j \ell} \\ &= -g^{\ell \bar k} (\hat\nabla - \Theta)^m (\overline{\hat\nabla} - \overline{\Theta})^{\ell-1} \hat\nabla_{\bar i} g_{\bar j \ell} \\ &= -g^{\ell \bar k} \hat\nabla^m \overline{\hat\nabla^{\ell-1}} \, \hat\nabla_{\bar i} g_{\bar j \ell} + O(1).\end{aligned} \tag{3.100}$$

It follows that

$$\left| \hat\nabla^m \overline{\hat\nabla^\ell} \, \frac{\Psi_{\bar p q}}{2\|\Omega\|_\omega} \right| \leq C \left(1 + |\hat\nabla^m \overline{\hat\nabla^\ell} \, g| \right). \tag{3.101}$$

By differentiating the evolution equation and using the above estimate, we have

$$\partial_t |\hat\nabla^m \overline{\hat\nabla^\ell} g|^2_{\hat g} \leq C \left(1 + |\hat\nabla^m \overline{\hat\nabla^\ell} \, g|^2_{\hat g} \right), \tag{3.102}$$

hence $|\hat\nabla^m \overline{\hat\nabla^\ell} \, g|$ has exponential growth. This proves $G_{k+1} \leq C$. Then $S_k \leq C$ now follows from (3.100), since $\nabla^m \overline{\nabla}^\ell \Theta = \overline{\overline{\nabla}^m \nabla^\ell \overline{\Theta}}$ and we can exchange evolving covariant derivatives up to bounded terms. □

By the C^1 bound on the metric, we have $G_1 \leq C$. We see that $S_0 = |\Theta| \leq C$ by definition of Θ. Hence we can apply the previous proposition to deduce any estimate of the form

$$|\hat{\nabla}^m \overline{\hat{\nabla}^\ell} g| \leq C. \tag{3.103}$$

By differentiating the evolution equation with respect to time, we obtain

$$\partial_t^i \hat{\nabla}^m \overline{\hat{\nabla}^\ell} g = \hat{\nabla}^m \overline{\hat{\nabla}^\ell} \partial_t^i \left(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right). \tag{3.104}$$

Time derivatives of $\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}$ can be expressed as time derivatives of connections, curvature and torsion, which in previous sections have been written as covariant derivatives of curvature and torsion. It follows that $\hat{\nabla}^m \overline{\hat{\nabla}^\ell} \partial_t^i \big(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \big)$ can be written in terms of evolving covariant derivatives of curvature and torsion, and hence is bounded. Therefore

$$\left| \partial_t^i \hat{\nabla}^m \overline{\hat{\nabla}^\ell} g \right| \leq C, \tag{3.105}$$

on $[0, T)$.

4. Appendix

Appendix A. Conventions for differential forms

Let φ be a (p,q)-form on the manifold X. We define its components $\varphi_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p}$ by

$$\varphi = \frac{1}{p!q!} \sum \varphi_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p} \, dz^{j_p} \wedge \cdots \wedge dz^{j_1} \wedge d\bar{z}^{k_q} \wedge \cdots \wedge d\bar{z}^{k_1}. \tag{A.1}$$

Although ϕ can be expressed in several ways under the above form, we reserve the notation $\varphi_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p}$ for the uniquely defined coefficients $\varphi_{\bar{k}_1 \cdots \bar{k}_q j_1 \cdots j_p}$ which are anti-symmetric under permutation of any two of the barred indices or any two of the unbarred indices.

To each Hermitian metric $g_{\bar{k}j}$ corresponds a Hermitian, positive $(1,1)$-form defined by

$$\omega = i g_{\bar{k}j} \, dz^j \wedge d\bar{z}^k. \tag{A.2}$$

The Hermitian property $\overline{g_{\bar{k}j}} = g_{\bar{j}k}$ is then equivalent to the condition $\overline{\omega} = \omega$.

Appendix B. Conventions for Chern unitary connections

Let $E \to X$ be a holomorphic vector bundle over a complex manifold X. Let $H_{\bar{\alpha}\beta}$ be a Hermitian metric on E. The Chern unitary connection is defined by

$$\nabla_{\bar{k}} V^\alpha = \partial_{\bar{k}} V^\alpha, \qquad \nabla_k V^\alpha = H^{\alpha\bar{\gamma}} \partial_k (H_{\bar{\gamma}\beta} V^\beta) \tag{B.1}$$

for V^α any section of E. Its curvature tensor is then defined by

$$[\nabla_j, \nabla_{\bar{k}}] V^\alpha = F_{\bar{k}j}{}^\alpha{}_\beta V^\beta. \tag{B.2}$$

Explicitly, we have

$$\nabla_k V^\alpha = \partial_k V^\alpha + A^\alpha_{k\beta} V^\beta, \qquad A^\alpha_{k\beta} = H^{\alpha\bar{\gamma}} \partial_k H_{\bar{\gamma}\beta} \tag{B.3}$$

and

$$F_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}} A^\alpha_{j\beta} = -\partial_{\bar{k}} (H^{\alpha\bar{\gamma}} \partial_j H_{\bar{\gamma}\beta}). \tag{B.4}$$

In particular, when $E = T^{1,0}(X)$, and $g_{\bar{k}j}$ is a Hermitian metric on X, we have the corresponding formulas

$$\begin{aligned} &\nabla_{\bar{k}} V^p = \partial_{\bar{k}} V^p, \qquad \nabla_k V^p = g^{p\bar{m}} \partial_k (g_{\bar{m}q} V^q) \\ &[\nabla_j, \nabla_{\bar{k}}] V^p = R_{\bar{k}j}{}^p{}_q V^q \\ &R_{\bar{k}j}{}^p{}_q = -\partial_{\bar{k}} A^p_{jq} = -\partial_{\bar{k}} (g^{p\bar{m}} \partial_j g_{\bar{m}q}). \end{aligned} \tag{B.5}$$

Our convention for the curvature form Rm is

$$Rm = R_{\bar{k}j}{}^p{}_q dz^j \wedge d\bar{z}^k. \tag{B.6}$$

It is the same as in [10, 11], but it differs from that of [30] by a factor of i.

When the metric on X has torsion, the commutator identities $[\nabla_j, \nabla_k]$ for the Chern connections on any holomorphic vector bundle are given by Hence for any tensor A, we have

$$[\nabla_j, \nabla_k] A = T^\lambda{}_{jk} \nabla_\lambda A, \qquad [\nabla_{\bar{j}}, \nabla_{\bar{k}}] A = \bar{T}^{\bar{\lambda}}{}_{\bar{j}\bar{k}} \nabla_{\bar{\lambda}} A. \tag{B.7}$$

Some useful examples are

$$\begin{aligned}\nabla_c\nabla_a\nabla_{\bar b}A_{\bar i j\bar k\ell} &= \nabla_a\nabla_c\nabla_{\bar b}A_{\bar i j\bar k\ell} - T^\lambda{}_{ca}\nabla_\lambda\nabla_{\bar b}A_{\bar i j\bar k\ell}\\ &= \nabla_a\nabla_{\bar b}\nabla_c A_{\bar i j\bar k\ell} - T^\lambda{}_{ca}\nabla_\lambda\nabla_{\bar b}A_{\bar i j\bar k\ell}\\ &\quad + \nabla_a(R_{\bar b c\bar i}{}^{\bar\lambda}A_{\bar\lambda j\bar k\ell} + R_{\bar b c\bar k}{}^{\bar\lambda}A_{\bar i j\bar\lambda\ell} - R_{\bar b c}{}^{\lambda}{}_j A_{\bar i\lambda\bar k\ell} - R_{\bar b c}{}^{\lambda}{}_\ell A_{\bar i j\bar k\lambda})\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}\nabla_c\nabla_{\bar d}\nabla_a\nabla_{\bar b}A &= \nabla_c\nabla_a\nabla_{\bar d}\nabla_{\bar b}A + \nabla(Rm * \bar\nabla A)\\ &= \nabla_a\nabla_c\nabla_{\bar d}\nabla_{\bar b}A + T * \nabla\bar\nabla\bar\nabla A + \nabla(Rm * \bar\nabla A)\\ &= \nabla_a\nabla_c\nabla_{\bar b}\nabla_{\bar d}A + \nabla\nabla(\bar T * \bar\nabla A) + T * \nabla\bar\nabla\bar\nabla A + \nabla(Rm * \bar\nabla A)\\ &= \nabla_a\nabla_{\bar b}\nabla_c\nabla_{\bar d}A + \nabla\nabla(\bar T * \bar\nabla A) + T * \nabla\bar\nabla\bar\nabla A + \nabla(Rm * \bar\nabla A).\end{aligned} \tag{B.9}$$

The general pattern is

$$\begin{aligned}\nabla^{(k)}\overline{\nabla}^{(\ell)}\nabla_a\nabla_{\bar b}A &= \nabla_a\nabla_{\bar b}\nabla^{(k)}\overline{\nabla}^{(\ell)}A\\ &\quad + \sum_{\nu+\lambda=k}\sum_{\mu+\rho=\ell}\nabla^{(\nu)}\overline{\nabla}^{(\mu)}Rm * \nabla^{(\lambda)}\overline{\nabla}^{(\rho)}A\\ &\quad + \sum_{\nu+\lambda=k}\sum_{\mu+\rho=\ell+1}\nabla^{(\nu)}\overline{\nabla}^{(\mu)}T * \nabla^{(\lambda)}\overline{\nabla}^{(\rho)}A\\ &\quad + \sum_{\nu+\lambda=k+1}\sum_{\mu+\rho=\ell}\nabla^{(\nu)}\overline{\nabla}^{(\mu)}\overline{T} * \nabla^{(\lambda)}\overline{\nabla}^{(\rho)}A.\end{aligned} \tag{B.10}$$

Appendix C. Identities for non-Kähler metrics

When the metric is not Kähler, the integration by parts formula becomes

$$\int_X \nabla_j V^j\,\omega^n = \int_X (A^p_{pj} - A^p_{jp})V^j\,\omega^n = \int_X g^{p\bar q}T_{\bar q pj}\,V^j\,\omega^n. \tag{C.1}$$

It is convenient to introduce

$$T_j = g^{p\bar q}T_{\bar q pj} \tag{C.2}$$

so that the above equation becomes

$$\int_X \nabla_j V^j\,\omega^n = \int_X T_j V^j\,\omega^n. \tag{C.3}$$

C.1. The adjoints $\bar\partial^\dagger$ and $\partial^\dagger$ with torsion

Since the signs are crucial, we work out in detail the operators $\bar\partial^\dagger$ and $\partial^\dagger$ on the space $\Lambda^{1,1}$ of $(1,1)$-forms.

Consider first the operator $\bar\partial : \Lambda^{1,0} \to \Lambda^{1,1}$. Explicitly,

$$\bar\partial(f_j dz^j) = \partial_{\bar k} f_j d\bar z^k \wedge dz^j = -\partial_{\bar k} f_j dz^j \wedge d\bar z^k \tag{C.4}$$

which means that

$$(\bar\partial f)_{\bar k j} = -\partial_{\bar k} f_j. \tag{C.5}$$

Let $\Phi = \Phi_{\bar p q} dz^q \wedge d\bar z^p$ be a $(1,1)$-form. The adjoint $\bar\partial^\dagger$ is characterized by the equation

$$\langle \bar\partial f, \Phi\rangle = \langle f, \bar\partial^\dagger \Phi\rangle \tag{C.6}$$

which is equivalent to

$$\int_X (-\partial_{\bar k} f_j)\overline{\Phi_{\bar p q}} g^{p\bar k} g^{j\bar q} \frac{\omega^n}{n!} = \int_X f_j \overline{(\bar\partial^\dagger \Phi)_q} g^{j\bar q} \frac{\omega^n}{n!}. \tag{C.7}$$

Integrating by parts, we find

$$(\bar\partial^\dagger \Phi)_q = g^{k\bar p}(\nabla_k \Phi_{\bar p q} - T^j{}_{kj}\Phi_{\bar p q}) = g^{k\bar p}(\nabla_k \Phi_{\bar p q} - T_k \Phi_{\bar p q}). \tag{C.8}$$

Similarly, we work out $\partial^\dagger$. For $f = f_{\bar k} d\bar z^k$, we have $\partial f = \partial_j f_{\bar k} dz^j \wedge d\bar z^k$, so that $(\partial f)_{\bar k j} = \partial_j f_{\bar k}$. Thus, the equation $\langle \partial f, \Phi\rangle = \langle f, \partial^\dagger \Phi\rangle$ becomes

$$\int_X \partial_j f_{\bar k} \overline{\Phi_{\bar p q}} g^{p\bar k} g^{j\bar q} \frac{\omega^n}{n!} = \int_X f_{\bar k} \overline{(\partial^\dagger \Phi)_{\bar p}} g^{p\bar k} \frac{\omega^n}{n!}. \tag{C.9}$$

This results now into

$$(\partial^\dagger \Phi)_{\bar q} = -g^{p\bar j}(\nabla_{\bar j} \Phi_{\bar q p} - \bar T_{\bar j} \Phi_{\bar q p}). \tag{C.10}$$

C.2. Bianchi identities for non-Kähler metrics

It is well-known that the Riemann curvature tensor of Kähler metrics satisfies the following important identities

$$\begin{aligned} &R_{\bar\ell m \bar k j} = R_{\bar k m \bar\ell j} = R_{\bar k j \bar\ell m} \\ &\nabla_q R_{\bar\ell m}{}^k{}_j = \nabla_m R_{\bar\ell q}{}^k{}_j, \qquad \nabla_{\bar p} R_{\bar\ell m}{}^k{}_j = \nabla_{\bar\ell} R_{\bar p m}{}^k{}_j. \end{aligned} \tag{C.11}$$

For general Hermitian metrics, these identities become

$$R_{\bar\ell m\bar k j} = R_{\bar\ell j\bar k m} + \nabla_{\bar\ell} T_{\bar k jm}$$
$$R_{\bar\ell m\bar k j} = R_{\bar k m\bar\ell j} + \nabla_m \bar T_{j\bar k\bar\ell}$$

and

$$\nabla_m R_{\bar k j}{}^p{}_q = \nabla_j R_{\bar k m}{}^p{}_q + T^r{}_{jm} R_{\bar k r}{}^p{}_q,$$
$$\nabla_m R_{\bar k j\bar p q} = \nabla_j R_{\bar k m\bar p q} + T^r{}_{jm} R_{\bar k r\bar p q} \tag{C.12}$$
$$\nabla_{\bar m} R_{\bar k j}{}^p{}_q = \nabla_{\bar k} R_{\bar m j}{}^p{}_q + \bar T^{\bar r}{}_{\bar k\bar m} R_{\bar r j}{}^p{}_q,$$
$$\nabla_{\bar m} R_{\bar k j\bar p q} = \nabla_{\bar k} R_{\bar m j\bar p q} + \bar T^{\bar r}{}_{\bar k\bar m} R_{\bar r j\bar p q}.$$

Observe that to interchange, say m and q in the second Bianchi identity for non-Kähler metrics, we have to use the first Bianchi identity and differentiate, resulting into

$$\nabla_m R_{\bar k j\bar p q} - \nabla_q R_{\bar k j\bar p m} = \nabla_q \nabla_{\bar k} T_{\bar p mj} + \nabla_m \nabla_{\bar k} T_{\bar p qj} + T^r{}_{qm} R_{\bar k r\bar p j}. \tag{C.13}$$

The occurrence of D^2T on the right hand side is a source of potential difficulties, so it is desirable not to exchange this type of pairs of indices.

Acknowledgements. The authors would like to thank the referees for a particularly careful reading of their paper, and in particular for pointing out several typos that could have been quite confusing to the reader.

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Received May 27, 2017