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Bishop and Laplacian comparison theorems on Sasakian manifolds

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We prove a Bishop volume comparison theorem and a Laplacian comparison theorem for a natural sub-Riemannian structure defined on Sasakian manifolds. This generalizes to arbitrary dimensions the corresponding three-dimensional results in [1, 5, 6].

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1. Introduction

Bishop volume comparison theorem and Laplacian comparison theorem are basic tools in Riemannian geometry and geometric analysis. For Bishop volume comparison theorem, one estimates the volume of a ball in a Riemannian manifold with the Ricci curvature bounded from below by that of the corresponding space form. Similarly, the Laplacian comparison theorem compares the Laplacian of the Riemannian distance function on a Riemannian manifold with a Ricci curvature lower bound to that of the corresponding space form. Various results for Riemannian manifolds with Ricci curvature bounded from below are based on these two comparison theorems. In this paper, we prove analogues of these results for a natural sub-Riemannian structure defined on a Sasakian manifold.

Recall that a Sasakian manifold is a $(2n+1)$-dimensional manifold M equipped with an almost contact structure $(\mathbf{J}, \alpha_0, v_0)$ and a Riemannian metric $\langle\cdot,\cdot\rangle$ satisfying certain compatibility conditions (see Section 3 for the definitions). The restriction of the Riemannian metric on the distribution $\mathcal{D} := \ker \alpha_0$ defines a sub-Riemannian structure. Let $B_x(R)$ be the sub-Riemannian ball of radius R centered at x and let η be the Riemannian volume form of the Riemannian metric $\langle\cdot,\cdot\rangle$. The Heisenberg group and the complex Hopf fibration are well-known Sasakian manifolds (see Section 7 for more detail). Their volume forms are denoted, respectively, by η_0 and η_H. We also denote their sub-Riemannian balls by and $B_0(R)$ and $B_H(R)$, respectively. The following Bishop type volume comparison theorems generalize the earlier three dimensional case in [1, 5, 6].

Theorem 1.1. *Assume that the Tanaka-Webster curvature Rm^* of the Sasakian manifold satisfies*

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle \geq 0$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle \geq 0$,

where v is any vector in $\mathcal{D}$ and $w_1, \ldots, w_{2n-2}$ is an orthonormal frame of $\{v_0, v, \boldsymbol{J}v\}^\perp$. Then

$$\eta(B_x(R)) \leq \eta_0(B_0(R)).$$

Moreover, equality holds only if

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle = 0$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle = 0$,

on $B_x(R)$.

Theorem 1.2. *Assume that the Tanaka-Webster curvature* Rm^* *of the Sasakian manifold satisfies*

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle \geq 4|v|^4$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle \geq (2n-2)|v|^2$,

where v *is any vector in* $\mathcal{D}$ *and* $w_1, \ldots, w_{2n-2}$ *is an orthonormal frame of* $\{v_0, v, \boldsymbol{J}v\}^{\perp}$*. Then*

$$\eta(B_x(R)) \leq \eta_H(B_H(R)).$$

Moreover, equality holds only if

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle = 4|v|^4$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle = (2n-2)|v|^2$,

on $B_x(R)$*.*

A Laplacian type comparison theorem generalizing the one in [1] also holds. Recall that sub-Laplacian Δ_H is defined by

$$\Delta f = \sum_{i=1}^{2n} \langle \nabla_{v_i} \nabla f, v_i\rangle,$$

where $v_1, \ldots, v_{2n}$ is an orthonormal frame in $\mathcal{D}$.

Theorem 1.3. *Let* x_0 *be a point in* M *and let* $d(x) := d(x_0, x)$ *be the sub-Riemannian distance from the point* x_0*. Assume that the Tanaka-Webster curvature* Rm^* *of the Sasakian manifold satisfies*

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle \geq k_1|v|^4$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle \geq (2n-2)k_2|v|^2$,

for some constants k_1 *and* k_2*, where* v *is any vector in* $\mathcal{D}$ *and* $w_1, \ldots, w_{2n-2}$ *is an orthonormal frame of* $\{v_0, v, \boldsymbol{J}v\}^{\perp}$*. Then*

$$\Delta_H d \leq h(d, v_0(d)),$$

where $\mathfrak{k}_1(r,z) = z^2 + k_1 r^2$, $\mathfrak{k}_2(r,z) = \frac{1}{4}z^2 + k_2 r^2$, $\mathfrak{s}_1 = \sqrt{|\mathfrak{k}_1|}$, $\mathfrak{s}_2 = \sqrt{|\mathfrak{k}_2|}$ *and*

$$h(r,z) = \frac{\mathfrak{s}_1(\sin(\mathfrak{s}_1 - \mathfrak{s}_1\cos(\mathfrak{s}_1))}{r(2 - 2\cos(\mathfrak{s}_1) - \mathfrak{s}_1\sin(\mathfrak{s}_1))} + \frac{(2n-2)\mathfrak{s}_2\cot(\mathfrak{s}_2)}{r}$$

if $\mathfrak{k}_1 \geq 0$ *and* $\mathfrak{k}_2 \geq 0$,

$$h(r,z) = \frac{\mathfrak{s}_1(\mathfrak{s}_1 \cosh(\mathfrak{s}_1)) - \sinh(\mathfrak{s}_1))}{r(2 - 2\cosh(\mathfrak{s}_1) + \mathfrak{s}_1 \sinh(\mathfrak{s}_1))} + \frac{(2n-2)\mathfrak{s}_2 \cot(\mathfrak{s}_2)}{r}$$

if $\mathfrak{k}_1 \leq 0$ *and* $\mathfrak{k}_2 \geq 0$,

$$h(r,z) = \frac{\mathfrak{s}_1(\sin(\mathfrak{s}_1 - \mathfrak{s}_1 \cos(\mathfrak{s}_1))}{r(2 - 2\cos(\mathfrak{s}_1) - \mathfrak{s}_1 \sin(\mathfrak{s}_1))} + \frac{(2n-2)\mathfrak{s}_2 \coth(\mathfrak{s}_2)}{r}$$

if $\mathfrak{k}_1 \geq 0$ *and* $\mathfrak{k}_2 \leq 0$,

$$h(r,z) = \frac{\mathfrak{s}_1(\mathfrak{s}_1 \cosh(\mathfrak{s}_1)) - \sinh(\mathfrak{s}_1))}{r(2 - 2\cosh(\mathfrak{s}_1) + \mathfrak{s}_1 \sinh(\mathfrak{s}_1))} + \frac{(2n-2)\mathfrak{s}_2 \coth(\mathfrak{s}_2)}{r}$$

if $\mathfrak{k}_1 \leq 0$ *and* $\mathfrak{k}_2 \leq 0$.

We also have the following special case of Theorem 1.3.

Corollary 1.4. *Let x_0 be a point in M and let $d(x) := d(x_0, x)$ be the sub-Riemannian distance from the point x_0. Assume that the Tanaka-Webster curvature Rm^* of the Sasakian manifold satisfies*

1) $\langle Rm^*(\boldsymbol{J}v, v)v, \boldsymbol{J}v\rangle \geq 0$,
2) $\sum_{i=1}^{2n-2} \langle Rm^*(w_i, v)v, w_i\rangle \geq 0$,

where v is any vector in $\mathcal{D}$ and $w_1, \dots, w_{2n-2}$ is an orthonormal frame of $\{v_0, v, \boldsymbol{J}v\}^{\perp}$. Then

$$\Delta_H d \leq \frac{2n+2}{d}.$$

A version of Hessian comparison theorem as in [1] also hold. The proof is very similar to and simpler than that of Theorem 1.3. We omit the statement since it is rather lengthy.

The paper is organized as follows. In Section 2, we recall the construction of the canonical frame introduced in [8]. In Section 3, we recall the definition of Sasakian manifolds. We also recall the definition of parallel adapted frame introduced in [7]. It will be used to simplify some tedious calculations in a way very similar to the use of geodesic normal coordinates in Riemannian geometry. The canonical frame and the corresponding curvature are computed in Section 5. Unlike the approach in [9], the computation in this paper does not rely on any symmetry and the method can be used to deal with more general situations.

In Section 6, we prove a first conjugate time estimate under the lower bounds on the Tanaka-Webster curvature. In Section 7, we discuss the Heisenberg group, the complex Hopf fibration, and their sub-Riemannian cut locus. The volume estimate and the proof of Theorem 1.1 and 1.2 are done in Section 8. Finally, Section 9 is devoted to the proof of Theorem 1.3.

2. Canonical frames and curvatures of a Jacobi curve

In this section, we recall how to construct canonical frames and define the curvature of a curve in Lagrangian Grassmannian. We will only do the construction in our simplified setting. For the most general discussion, see [8]. For completeness, we will also include the full proof of the results in our case.

Let $t \mapsto J(t)$ be a curve in the Lagrangian Grassmannian of a symplectic vector space $\mathfrak{V}$. Let g_t^0 be the bilinear form on $J(t)$ defined by

$$g_t^0(e,e) = \omega(\dot{e}(t), e),$$

where $e(\cdot)$ is any curve in J such that $e(t) = e$.

Assume that the curve J is monotone which means that g_t^0 is non-negative definite for each t. Let J^{-1}, J^1, and J^2 be defined by

$$\begin{aligned}
&J^{-2}(t) = \{e(t)|\dot{e}(t), \ddot{e}(t) \in J(t)\},\\
&J^{-1}(t) = \{e(t)|\dot{e}(t) \in J(t)\},\\
&J^{1}(t) = \mathbf{span}\{e(t), \dot{e}(t)|e(\cdot) \in J\} = (J^{-1})^{\angle}\\
&J^{2}(t) = \mathbf{span}\{e(t), \dot{e}(t), \ddot{e}(t)|e(\cdot) \in J\} = (J^{-2})^{\angle}
\end{aligned}$$

where the superscript $W^{\angle}$ denotes the symplectic complement of the subspace W.

We will consider the case $J^1 \neq \mathfrak{V}$ and $J^2 = \mathfrak{V}$. Assume that J and J^{-1} have dimensions N and k, respectively.

Theorem 2.1. *[8] Under the above assumptions, there exists a family of frames*

$$\begin{aligned}
&E^1(t) = (E^1_1(t), \dots, E^1_k(t))^T, \quad E^2(t) = (E^2_1(t), \dots, E^2_k(t))^T,\\
&E^3(t) = (E^3_1(t), \dots, E^3_{N-2k}(t))^T,\\
&F^1(t) = (F^1_1(t), \dots, F^1_k(t))^T, \quad F^2(t) = (F^2_1(t), \dots, F^2_k(t))^T,\\
&F^3(t) = (F^3_1(t), \dots, F^3_{N-2k}(t))^T
\end{aligned}$$

such that

1) $E(t) = (E^1(t), E^2(t), E^3(t))^T, F(t) = (F^1(t), F^2(t), F^3(t))^T$ *is a symplectic basis for each* t,
2) $E^1(t)$ *is a basis of* $J^{-1}(t)$,
3) $\dot{E}(t) = C_1 E(t) + C_2 F(t), \quad \dot{F}(t) = -R(t)E(t) - C_1^T F(t),$

where

$$C_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$
$$R(t) = \begin{pmatrix} R^{11}(t) & 0 & R^{13}(t) \\ 0 & R^{22}(t) & R^{23}(t) \\ R^{31}(t) & R^{32}(t) & R^{33}(t) \end{pmatrix},$$

and $R(t)$ *is symmetric.*

The frame $(E^1, E^2, E^3, F^1, F^2, F^3)$ is called a canonical frame of the curve J and the coefficients R^{ij} are the curvatures of the curve J. We also write the above equations as

$$\begin{aligned} &\dot{E}^1(t) = E^2(t), \quad \dot{E}^2(t) = F^2(t), \quad \dot{E}^3(t) = F^3(t), \\ &\dot{F}^1(t) = -R^{11}(t)E^1(t) - R^{13}(t)E^3(t), \\ &\dot{F}^2(t) = -R^{22}(t)E^2(t) - R^{23}(t)E^3(t) - F^1(t), \\ &\dot{F}^3(t) = -R^{31}(t)E^1(t) - R^{32}(t)E^2(t) - R^{33}(t)E^3(t). \end{aligned} \tag{2.1}$$

3. Sasakian manifolds and parallel adapted frames

In this section, we recall the definition of Sasakian manifolds and introduce the parallel adapted frames. For the part on Sasakian manifolds, we mainly follow [3]. Parallel adapted frames were introduced in [7]. It will be used to simplify some tedious calculations in a way very similar to the use of geodesic normal coordinates in Riemannian geometry.

Recall that a manifold M of dimension $2n+1$ has an almost contact structure $(\mathbf{J}, v_0, \alpha_0)$ if $\mathbf{J} : TM \to TM$ is a $(1,1)$ tensor, v_0 is a vector field, and α_0 is a 1-form satisfying

$$\mathbf{J}^2(v) = -v + \alpha_0(v)v_0 \quad \text{and} \quad \alpha_0(v_0) = 1$$

for all tangent vector v in TM.

An almost contact structure is normal if the following tensor vanishes

$$(v,w) \mapsto [\mathbf{J},\mathbf{J}](v,w) + d\alpha_0(v,w)v_0,$$

where $[\mathbf{J},\mathbf{J}]$ is defined by

$$[\mathbf{J},\mathbf{J}](v,w) = \mathbf{J}^2[v,w] + [\mathbf{J}v,\mathbf{J}w] - \mathbf{J}[\mathbf{J}v,w] - \mathbf{J}[v,\mathbf{J}w].$$

A Riemannian metric $\langle\cdot,\cdot\rangle$ is compatible with a given almost contact manifold if

$$\langle \mathbf{J}v,\mathbf{J}w\rangle = \langle v,w\rangle - \alpha_0(v)\alpha_0(w)$$

for all tangent vectors v and w in TM.

If, in addition, the Riemannian metric satisfies the condition

$$\langle v,\mathbf{J}w\rangle = d\alpha_0(v,w),$$

then we say that the metric is associated to the given almost contact structure.

Finally, a Sasakian manifold is a normal almost contact manifold with an associated Riemannian metric. The following results can be found in [3]. Since the sign conventions in [3] is different, we include the proof in the appendix.

Theorem 3.1. *The followings hold on a Sasakian manifold* $(\boldsymbol{J}, v_0, \alpha_0, g = \langle\cdot,\cdot\rangle)$

1) $\mathcal{L}_{v_0}(\boldsymbol{J}) = 0$,
2) $\nabla_{v_0} v_0 = 0$,
3) $\mathcal{L}_{v_0} g = 0$,
4) $\boldsymbol{J} = -2\nabla v_0$,

where ∇ *denotes the Levi-Civita connection.*

Theorem 3.2. *An almost contact metric manifold* $(\boldsymbol{J}, v_0, \alpha_0, \langle\cdot,\cdot\rangle)$ *is Sasakian if and only if it satisfies*

$$(\nabla_v \boldsymbol{J})w = \frac{1}{2}\langle v,w\rangle v_0 - \frac{1}{2}\alpha_0(w)v$$

for all tangent vectors v *and* w.

Let Rm denotes the Riemann curvature tensor.

Theorem 3.3. *Assume that the almost contact metric manifold* $(\boldsymbol{J}, v_0, \alpha_0, \langle\cdot,\cdot\rangle)$ *is Sasakian. Then*

$$Rm(X,Y)v_0 = \frac{1}{4}\alpha_0(Y)X - \frac{1}{4}\alpha_0(X)Y.$$

The Tanaka connection ∇^* is defined by

$$\nabla^*_X Y = \nabla_X Y + \frac{1}{2}\alpha_0(X)\mathbf{J}Y - \alpha_0(Y)\nabla_X v_0 + \nabla_X \alpha_0(Y) v_0.$$

The corresponding curvature operator is denoted by Rm^* and we call it Tanaka-Webster curvature.

Theorem 3.4. *Assume that the tangent vectors X, Y, and Z are contained in* $\ker\alpha_0$. *Then*

$$Rm^*(X,Y)Z = (Rm(X,Y)Z)^h + \langle Z, \nabla_Y v_0\rangle \nabla_X v_0 - \langle Z, \nabla_X v_0\rangle \nabla_Y v_0,$$

where the superscript X^h denotes the the component of X in $\ker\alpha_0$.

If the manifold is Sasakian, then

$$Rm^*(X,Y)Z = (Rm(X,Y)Z)^h + \frac{1}{4}\langle Z, \boldsymbol{J}Y\rangle \boldsymbol{J}X - \frac{1}{4}\langle Z, \boldsymbol{J}X\rangle \boldsymbol{J}Y.$$

Finally, we introduce the parallel adapted frames. The proofs of the following lemmas are done in the last appendix.

Lemma 3.5. *Let v_0 be a vector field in a Riemannian manifold M. Let $\gamma : [0,T] \to M$ be a curve in the Riemannian manifold M and let $v_0, \ldots, v_{2n}$ be an orthonormal frame at $x := \gamma(0)$. Then there is a orthonormal frame $v_0(t) := v_0(\gamma(t)), v_1(t), \ldots, v_{2n}(t)$ such that*

1) *$v_i(0) = v_i$ and*

2) *$\dot{v}_i(t)$ is contained in $\mathbb{R}v_0$ for each t,*

where $\dot{v}_i(t)$ denotes the covariant derivative of $v(\cdot)$ along $\gamma(\cdot)$ and $i = 1, \ldots, 2n$.

The moving frame defined in Lemma 3.5 is called parallel adapted frame introduced in [7]. Using this frame, we obtain the following convenient local frame.

Lemma 3.6. *Suppose that* $(\boldsymbol{J}, v_0, \alpha_0)$ *defines an almost contact structure on* M *and let* $\langle\cdot,\cdot\rangle$ *be an associated Riemannian metric. For each point* x *in* M*, there is orthonormal frame* $v_0, v_1, \dots, v_{2n}$ *defined in a neighborhood of* x *such that the following conditions hold at* x*.*

1) $\nabla_{v_i} v_j = -\langle \nabla_{v_i} v_0, v_j\rangle v_0$,
2) $\nabla_{v_i} v_0 = \sum_{j\neq 0} \langle \nabla_{v_i} v_0, v_j\rangle v_j$,
3) $\nabla_{v_0} v_i = \nabla_{v_0} v_0 = 0$,

where $i, j = 1, \dots, 2n$*.*

If, in addition, the manifold M *together with* $(\boldsymbol{J}, v_0, \alpha_0)$ *is Sasakian, then the followings hold at* x*.*

1) $\nabla_{v_i} v_j = \frac{1}{2}\langle \boldsymbol{J} v_i, v_j\rangle v_0$,
2) $\nabla_{v_i} v_0 = -\frac{1}{2}\boldsymbol{J} v_i$,
3) $\nabla_{v_0} v_i = \nabla_{v_0} v_0 = 0$.

The following will be useful for the later sections.

Lemma 3.7. *Assume that* $(M, \boldsymbol{J}, v_0, \alpha_0, \langle\cdot,\cdot\rangle)$ *is Sasakian. Let* $v_0, v_1, \dots, v_{2n}$ *be a frame defined by Lemma 3.6, let* $\boldsymbol{J}_{ij} = \langle \boldsymbol{J} v_i, v_j\rangle$*, and let* $\Gamma_{ij}^k = \langle \nabla_{v_i} v_j, v_k\rangle$*. Then the following holds at* x

1) $\Gamma_{00}^i = \Gamma_{0i}^0 = \Gamma_{i0}^0 = 0$,
2) $\Gamma_{ij}^0 = -\Gamma_{ji}^0 = \frac{1}{2}\boldsymbol{J}_{ij}$,
3) $v_k \boldsymbol{J}_{ij} = 0$ *if* $i, j, k \neq 0$,
4) $Rm(v_i, v_j)v_k = \sum_{s\neq 0}\left((v_i\Gamma_{jk}^s) - (v_j\Gamma_{ik}^s) - \frac{1}{4}\boldsymbol{J}_{jk}\boldsymbol{J}_{is} + \frac{1}{4}\boldsymbol{J}_{ik}\boldsymbol{J}_{js}\right) v_s$
if $i, j, k \neq 0$.

4. Sub-Riemannian geodesic flows and Jacobi curves

In this section, we give a quick review on some basic notions in sub-Riemannian geometry. In particular, we will introduce Jacobi curves corresponding to the sub-Riemannian geodesic flow and its induced geometric structures.

A sub-Riemannian manifold is a triple $(M, \mathcal{D}, \langle\cdot,\cdot\rangle)$, where M is a manifold of dimension n, $\mathcal{D}$ is a distribution (sub-bundle of the tangent bundle TM), and $\langle\cdot,\cdot\rangle$ is a sub-Riemannian metric (smoothly varying inner product defined on $\mathcal{D}$). Assuming that the manifold M is connected and the distribution $\mathcal{D}$ satisfies the Hörmander condition (the sections of $\mathcal{D}$ and their iterated Lie brackets span each tangent space, also called "bracket-generating" condition). Then, by Chow-Rashevskii Theorem, any two given points on the manifold M can be connected by a horizontal curve (a curve which is almost everywhere tangent to $\mathcal{D}$). Therefore, we can define the sub-Riemannian distance d as

$$d(x_0, x_1) = \inf_{\gamma\in\Gamma} l(\gamma), \tag{4.1}$$

where the infimum is taken over the set Γ of all horizontal paths $\gamma : [0,1] \to M$ satisfying $\gamma(0) = x_0$ and $\gamma(1) = x_1$. The minimizers of (4.1) are called length minimizing geodesics (or simply geodesics). As in the Riemannian case, reparametrizations of a geodesic are also geodesics. Therefore, we assume that all geodesics have constant speed. These constant speed geodesics are also minimizers of the kinetic energy functional

$$\inf_{\gamma\in\Gamma} \int_0^1 \frac{1}{2}|\dot\gamma(t)|^2 dt, \tag{4.2}$$

where $|\cdot|$ denotes the norm w.r.t. the sub-Riemannian metric.

Let $H : T^*M \to \mathbb{R}$ be the Hamiltonian defined by the Legendre transform:

$$H(x,p) = \sup_{v\in\mathcal{D}} \left(p(v) - \frac{1}{2}|v|^2\right)$$

and let

$$\vec{H} = \sum_{i=1}^{n} (H_{p_i}\partial_{x_i} - H_{x_i}\partial_{p_i})$$

be the Hamiltonian vector field. Assume, through out this paper, that the vector field $\vec{H}$ defines a complete flow which is denoted by $e^{t\vec{H}}$. The projections of the trajectories of $e^{t\vec{H}}$ to the manifold M give minimizers of (4.2).

In this paper, we assume that the sub-Riemannian structure is given by a Sasakian manifold. More precisely, assume that the almost contact structure $(\mathbf{J}, v_0, \alpha_0)$ together with the Riemannian structure $\langle\cdot,\cdot\rangle$ form a Sasakian manifold. The distribution $\mathcal{D}$ is given by $\mathcal{D} = \ker\alpha_0$ and the sub-Riemannian metric is given by the restriction of the Riemannian metric to $\mathcal{D}$. In this case all minimizers of (4.2) are given by the projections of the trajectories of $e^{t\vec{H}}$ (see [10] for more detail).

Next, we discuss a sub-Riemannian analogue of Jacobi fields. Let ω be the symplectic form on the cotangent bundle T^*M defined in local coordinates $(x_1, \dots, x_{2n+1}, p_1, \dots, p_{2n+1})$ by

$$\omega = \sum_{i=1}^{2n+1} dp_i \wedge dx_i.$$

Let $\pi : T^*M \to M$ be the canonical projection and let $\mathcal{V}$ be the vertical sub-bundle of the cotangent bundle T^*M defined by

$$\mathcal{V}_{(x,p)} = \{v \in T_{(x,p)}T^*M | \pi_*(v) = 0\}.$$

The family of Lagrangian subspaces

$$\mathfrak{J}_{(x,p)}(t) := e_*^{-t\vec{H}}(\mathcal{V}_{e^{t\vec{H}}(x,p)}) \tag{4.3}$$

defined a curve in the Lagrangian Grassmannian of $T_{(x,p)}T^*M$, called the Jacobi curve at (x,p) of the flow $e^{t\vec{H}}$.

Assuming that the manifold is Sasakian. Then Theorem 2.1 applies and we let $E^1(t), E^2(t), E^3(t), F^1(t), F^2(t), F^3(t)$ be a canonical frame of $\mathfrak{J}_{(x,p)}$. This defines a splitting of the vertical space $\mathcal{V}_{(x,p)}$ and the cotangent space $T_{(x,p)}T^*M$. More precisely, let

$$\begin{aligned} &\mathcal{V}_1 = \text{span}\{E^1(0)\}, \quad \mathcal{V}_2 = \text{span}\{E^2(0)\}, \quad \mathcal{V}_3 = \text{span}\{E^3(0)\} \\ &\mathcal{H}_1 = \text{span}\{F^1(0)\}, \quad \mathcal{H}_2 = \text{span}\{F^2(0)\}, \quad \mathcal{H}_3 = \text{span}\{F^3(0)\}. \end{aligned}$$

Then $\mathcal{V}_{(x,p)} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ and $T_{(x,p)}T^*M = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$. Note that $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are all 1-dimensional. $\mathcal{V}_3$ and $\mathcal{H}_3$ are $(2n-2)$-dimensional. Let α and h be, respectively, a 1-form and a function on T^*M. Let $\vec{\alpha}$ and $\vec{h}$ be the vector fields defined, respectively, by

$$\omega(\vec{\alpha}, \cdot) = -\alpha \quad \text{and} \quad \omega(\vec{h}, \cdot) = -dh.$$

The proof of the following theorem is done in the appendix.

Theorem 4.1. *For each point x in M, the above splitting of the cotangent bundle is given by the followings*

1) $\mathcal{V}_1 = span\{\vec{\alpha}_0\}$,
2) $\mathcal{V}_2 = span\{\sum_{k,l\neq 0} h_k \boldsymbol{J}_{kl}\vec{\alpha}_l\}$,

3) $\mathcal{V}_3 = span\{\sum_b a_b \vec{\alpha}_b | \sum_{j,k\neq 0} a_k h_j \boldsymbol{J}_{kj} = 0 \text{ and } a_0 = \frac{h_0}{2H}\sum_{k\neq 0} a_k h_k\}$,

4) $\mathcal{H}_1 = span\{2H\vec{h}_0 - h_0\vec{H}\}$,

5) $\mathcal{H}_2 = span\{h_0 \sum_{k\neq 0} h_k \vec{\alpha}_k - \sum_{j,k\neq 0} h_j \boldsymbol{J}_{jk} \vec{h}_k - H\vec{\alpha}_0$
$- \sum_{j,k,l\neq 0} h_j h_l \Gamma_{0l}^k \boldsymbol{J}_{jk} \vec{\alpha}_0 - \sum_{j,k,l,s\neq 0} h_j h_l \boldsymbol{J}_{js} \Gamma_{kl}^s \vec{\alpha}_k\}$,

6) $\mathcal{H}_3 = \{\sum_{i\neq 0} a_i \vec{h}_i + \sum_a c_a \vec{\alpha}_a | \sum_{j,k\neq 0} a_k h_j \boldsymbol{J}_{kj} = 0$,
$a_0 = \frac{h_0}{2H}\sum_{k\neq 0} a_k h_k, c_0 = \sum_{i,j\neq 0} a_i h_j \Gamma_{0j}^i$,
$c_k = \sum_{j\neq 0}\left(\frac{1}{2}a_j \boldsymbol{J}_{jk} h_0 - \frac{1}{2}a_0 h_j \boldsymbol{J}_{jk} + \sum_{i\neq 0} a_i h_j \Gamma_{kj}^i\right)\}$,

where $v_0, v_1, \ldots, v_{2n}$ is a local frame defined in a neighborhood of a point x by Lemma 3.6, $\boldsymbol{J}_{ij} = \langle \boldsymbol{J}v_i, v_j\rangle$.

The vertical splitting can be written in a coordinate free way. For this, we identify the tangent bundle TM with the vertical bundle $\mathcal{V}$ using the Riemannian metric via

$$v \in TM \to \alpha(\cdot) = \langle v, \cdot\rangle \in T^*M \to -\vec{\alpha} \in ver.$$

Under this identification, we have

Theorem 4.2. *For each point x in M, the above splitting of the cotangent bundle is given by the followings*

1) $\mathcal{V}_1 = \mathbb{R}v_0$,
2) $\mathcal{V}_2 = \mathbb{R}\boldsymbol{J}p^h$,
3) $\mathcal{V}_3 = \mathbb{R}(p^h + p(v_0)v_0) \oplus \{v| \langle v, p^h\rangle = \langle v, \boldsymbol{J}p^h\rangle = \langle v, v_0\rangle = 0\}$.
4) $\pi_*\mathcal{H}_1 = \mathbb{R}(|p^h|^2 v_0 - p(v_0)p^h)$,
5) $\pi_*\mathcal{H}_2 = \mathbb{R}\boldsymbol{J}p^h$,
6) $\pi_*\mathcal{H}_3 = \{X| \langle X, \boldsymbol{J}p^h\rangle = \langle X, v_0\rangle = 0\}$,

where p^h is the vector in $\ker\alpha_0$ *defined by $p(v) = \langle p^h, v\rangle$ and v ranges over vectors in* $\ker\alpha_0$.

Under the above identification, we can also define a volume form $\mathfrak{m}$ on $\mathcal{V}$ by $\mathfrak{m}(v_0, \ldots, v_{2n}) = 1$. The Riemannian volume on M is denoted by η. The proof of Theorem 4.1 also gives

Theorem 4.3. *The volume forms $\mathfrak{m}$ and η satisfy*

1) $\mathfrak{m}(E(0)) = \frac{1}{|p^h|}$,
2) $\eta(\pi_* F(0)) = |p^h|$.

5. Curvatures of sub-Riemannian geodesic flows

In this section, we will focus on the computation of the curvature $R^{ij}(0)$, where the Jacobi curve is given by the sub-Riemannian geodesic flow. For this, let $\mathcal{R}^{ij} : \mathcal{V}_i \to \mathcal{V}_j$ be the operator for which the matrix representation with respect to bases $E^i(0)$ and $E^j(0)$ of $\mathcal{V}_i$ and $\mathcal{V}_j$, respectively, is given by $R^{ij}(0)$. More precisely,

$$\mathcal{R}^{ij}(E^i_k(0)) = \sum_l R^{ij}_{kl}(0) E^j_l(0),$$

where $R^{ij}_{kl}(0)$ is the kl-th entry of $R^{ij}(0)$.

For a vector v in the vertical space $\mathcal{V}$, we denote the component of v in $\mathcal{V}_i$ by $v_{\mathcal{V}_i}$. We also denote by $\mathcal{R} : \mathcal{V} \to \mathcal{V}$ the operator

$$\mathcal{R}(v_{\mathcal{V}_i})_{\mathcal{V}_j} = \mathcal{R}^{ij}(v_{\mathcal{V}_i}).$$

Theorem 5.1. *Assume that the manifold is Sasakian. Then, under the identifications of Theorem 4.2, $\mathcal{R}$ is given by*

1) $\mathcal{R}(v) = 0$ *for all v in $\mathcal{V}_1$,*
2) $\mathcal{R}(v)_{\mathcal{V}_2} = (Rm(\boldsymbol{J}p^h, p^h)p^h)_{\mathcal{V}_2} + \left(\frac{1}{4}|p^h|^2 + p(v_0)^2\right) \boldsymbol{J}p^h$
$= (Rm^*(\boldsymbol{J}p^h, p^h)p^h)_{\mathcal{V}_2} + p(v_0)^2 \boldsymbol{J}p^h$ *for all v in $\mathcal{V}_2$,*
3) $\mathcal{R}(v)_{\mathcal{V}_3} = (Rm(\boldsymbol{J}p^h, p^h)p^h)_{\mathcal{V}_3} = (Rm^*(\boldsymbol{J}p^h, p^h)p^h)_{\mathcal{V}_3}$ *for all v in $\mathcal{V}_2$,*
4) $\mathcal{R}(v)_{\mathcal{V}_1} = 0$ *for all v in $\mathcal{V}_3$,*
5) $\mathcal{R}(v)_{\mathcal{V}_2} = (Rm(v^h, p^h)p^h)_{\mathcal{V}_2} = (Rm^*(v^h, p^h)p^h)_{\mathcal{V}_2}$ *for all v in $\mathcal{V}_3$,*
6) $\mathcal{R}(p^h + p(v_0)v_0) = 0$*,*
7) $\mathcal{R}(v)_{\mathcal{V}_3} = (Rm(v^h, p^h)p^h)_{\mathcal{V}_3} + \frac{1}{4}p(v_0)^2 v^h = (Rm^*(v^h, p^h)p^h)_{\mathcal{V}_3}$
$+ \frac{1}{4}p(v_0)^2 v^h$ *for all v in $\mathcal{V}_3$ satisfying $\langle v^h, p^h \rangle = 0$.*

Proof. For $i \neq j$, let $\Lambda_{\mathcal{V}_i\mathcal{H}_j} : \mathcal{V}_i \to \mathcal{H}_j$ be the operator defined by

$$\Lambda_{\mathcal{V}_i\mathcal{H}_j}(V) = [\vec{H}, V]_{\mathcal{H}_j},$$

where V is a section in $\mathcal{V}_i$ and the subscript $\mathcal{H}_j$ denotes the $\mathcal{H}_j$-component of the vector.

It follows from (2.1) that $\Lambda_{\mathcal{V}_i\mathcal{H}_j}$ is tensorial and so well-defined. We also define operators $\Lambda_{\mathcal{V}_i\mathcal{V}_j}$, $\Lambda_{\mathcal{H}_i\mathcal{V}_j}$, and $\Lambda_{\mathcal{H}_i\mathcal{H}_j}$ in a similar way. By (2.1), we have

Lemma 5.2. *The following relations hold.*

1) $\mathcal{R}^{11} = \Lambda_{\mathcal{H}_1\mathcal{V}_1} \circ \Lambda_{\mathcal{H}_2\mathcal{H}_1} \circ \Lambda_{\mathcal{V}_2\mathcal{H}_2} \circ \Lambda_{\mathcal{V}_1\mathcal{V}_2}$,
2) $\mathcal{R}^{13} = \Lambda_{\mathcal{H}_1\mathcal{V}_3} \circ \Lambda_{\mathcal{H}_2\mathcal{H}_1} \circ \Lambda_{\mathcal{V}_2\mathcal{H}_2} \circ \Lambda_{\mathcal{V}_1\mathcal{V}_2}$,
3) $\mathcal{R}^{22} = -\Lambda_{\mathcal{H}_2\mathcal{V}_2} \circ \Lambda_{\mathcal{V}_2\mathcal{H}_2}$,
4) $\mathcal{R}^{23} = -\Lambda_{\mathcal{H}_2\mathcal{V}_3} \circ \Lambda_{\mathcal{V}_2\mathcal{H}_2}$,
5) $\mathcal{R}^{31} = -\Lambda_{\mathcal{H}_3\mathcal{V}_1} \circ \Lambda_{\mathcal{V}_3\mathcal{H}_3}$,
6) $\mathcal{R}^{32} = -\Lambda_{\mathcal{H}_3\mathcal{V}_2} \circ \Lambda_{\mathcal{V}_3\mathcal{H}_3}$,
7) $\mathcal{R}^{33} = -\Lambda_{\mathcal{H}_3\mathcal{V}_3} \circ \Lambda_{\mathcal{V}_3\mathcal{H}_3}$.

Clearly, $\Lambda_{\mathcal{H}_1\mathcal{V}_1} \equiv 0$ and $\Lambda_{\mathcal{H}_1\mathcal{V}_3} \equiv 0$. For the rest, we need a lemma for which the proof is given in the appendix.

Lemma 5.3. *The following holds at x*

1) $[\vec{h}_k, \vec{h}_i] = \boldsymbol{J}_{ki}\vec{h}_0 + \sum_a b^a_{ki}\vec{\alpha}_a$,
2) $\sum_{k\neq 0} h_k b^0_{ki} = \sum_{k,s\neq 0} h_k h_s v_k(\Gamma^s_{0i})$ *if* $k, i \neq 0$,
3) $\sum_{k\neq 0} h_k b^l_{ki} = -\sum_{s,k\neq 0} h_s h_k[v_k\Gamma^s_{il} - v_k\Gamma^s_{li} - v_i\Gamma^s_{kl}]$ *if* $k, i, l \neq 0$,
4) $[\vec{H}, \vec{h}_i] = \sum_{k\neq 0} h_k\boldsymbol{J}_{ki}\vec{h}_0 - \sum_{k\neq 0} h_0\boldsymbol{J}_{ik}\vec{h}_k + \sum_{k\neq 0,a} h_k b^a_{ki}\vec{\alpha}_a$.

Let $a_i\vec{h}_i + c_a\vec{\alpha}_a$ be a vector in $\mathcal{H}_3$. A computation shows that the followings hold at x.

$$\begin{aligned}&[\vec{H}, a_i\vec{h}_i + c_a\vec{\alpha}_a]\\ &= (\vec{H}a_k)\vec{h}_k - \frac{1}{2}(a_j\mathbf{J}_{jk}h_0 + a_0h_j\mathbf{J}_{jk})\vec{h}_k + (\vec{H}c_i)\vec{\alpha}_i + a_ih_kb^j_{ki}\vec{\alpha}_j.\end{aligned}$$

On the other hand, we have

$$\frac{h_0}{2H}\left(\vec{H}a_k - \frac{1}{2}a_j\mathbf{J}_{jk}h_0 - \frac{1}{2}a_0h_j\mathbf{J}_{jk}\right)h_k = \frac{h_0}{2H}(\vec{H}a_k)h_k$$

and

$$\frac{1}{2}\left(\vec{H}a_i - \frac{1}{2}a_j\mathbf{J}_{ji}h_0 - \frac{1}{2}a_0h_j\mathbf{J}_{ji}\right)\mathbf{J}_{ik}h_0 - \frac{1}{2}\frac{h_0}{2H}(\vec{H}a_i)h_ih_j\mathbf{J}_{jk}$$
$$= \frac{1}{2}(\vec{H}a_i)\mathbf{J}_{ik}h_0 + \frac{1}{4}a_kh_0^2 + \frac{1}{4}a_0h_kh_0 - \frac{h_0}{4H}(\vec{H}a_i)h_ih_j\mathbf{J}_{jk}$$

at x.

Therefore,

$$[\vec{H}, a_i\vec{h}_i + c_a\vec{\alpha}_a]_{\mathcal{V}} = -\frac{1}{2}(\vec{H}a_i)\mathbf{J}_{ik}h_0\vec{\alpha}_k - \frac{1}{4}a_kh_0^2\vec{\alpha}_k - \frac{1}{4}a_0h_kh_0\vec{\alpha}_k + \frac{h_0}{4H}(\vec{H}a_i)h_ih_j\mathbf{J}_{jk}\vec{\alpha}_k + (\vec{H}c_k)\vec{\alpha}_k + a_ih_jb^k_{ji}\vec{\alpha}_k.$$

Another computation shows that

$$\vec{H}c_k = \frac{1}{2}(\vec{H}a_j)\mathbf{J}_{jk}h_0 - \frac{h_0}{4H}(\vec{H}a_l)h_lh_j\mathbf{J}_{jk} + \frac{1}{2}a_0h_0h_k + a_ih_jh_l(v_l\Gamma^i_{kj}).$$

Hence,

$$[\vec{H}, a_i\vec{h}_i + c_a\vec{\alpha}_a]_{\mathcal{V}} = -\frac{1}{4}h_0(a_kh_0 - a_0h_k)\vec{\alpha}_k - a_ih_sh_l\mathrm{Rm}_{ilsk}\vec{\alpha}_k.$$

where $\mathrm{Rm}_{ijks} = \langle \mathrm{Rm}(v_i, v_j)v_k, v_s\rangle$.

This finishes the proof of the last three assertions. Let

$$h_j\mathbf{J}_{jk}\vec{h}_k - h_0h_k\vec{\alpha}_k + H\vec{\alpha}_0 + h_jh_l\Gamma^k_{0l}\mathbf{J}_{jk}\vec{\alpha}_0 + h_jh_l\mathbf{J}_{js}\Gamma^s_{kl}\vec{\alpha}_k$$

be a section of the bundle $\mathcal{H}_2$. Then a tedious calculation shows that

$$\left[\vec{H}, h_j\mathbf{J}_{jk}\vec{h}_k - h_0h_k\vec{\alpha}_k + H\vec{\alpha}_0 + h_jh_l\Gamma^k_{0l}\mathbf{J}_{jk}\vec{\alpha}_0 + h_jh_l\mathbf{J}_{js}\Gamma^s_{kl}\vec{\alpha}_k\right]_{\mathcal{V}}$$
$$= -\left(h_0^2 + \frac{1}{2}H\right)h_i\mathbf{J}_{ik}\vec{\alpha}_k - h_jh_kh_s\mathbf{J}_{ji}\mathrm{Rm}_{kils}\vec{\alpha}_l.$$

This finishes the proof of (3) and (4). □

6. Conjugate time estimates and Bonnet-Myer's type theorem

In this section, we give estimates for the first conjugate time under certain curvature lower bound. Let $\psi_t : T_x^*M \to M$ be the map defined by $\psi_t(x,p) =$

$\pi(e^{t\vec{H}}(x,p))$, where $\pi : T^*M \to M$ is the projection. Let us fix a covector (x,p). The first conjugate time is the smallest $t_0 > 0$ such that the linear map $(d\psi_{t_0})_{(x,p)}$ is not bijective. The curve $t \mapsto \psi_t(x,p)$ is no longer minimizing if $t > t_0$ (see [2]).

Theorem 6.1. *Assume that the Tanaka-Webster curvature Rm^* of the Sasakian manifold satisfies*

1) $\left\langle Rm^*(\boldsymbol{J}p^h, p^h)p^h, \boldsymbol{J}p^h\right\rangle \geq k_1|p^h|^4$,
2) $\sum_{i=1}^{2n-2}\left\langle Rm^*(w_i, p^h)p^h, w_i\right\rangle \geq (2n-2)k_2|p^h|^2$,

for some non-negative constants k_1 and k_2, where $w_1, \dots, w_{2n-2}$ is an orthonormal frame of $\{p^h, \boldsymbol{J}p^h, v_0\}^\perp$. Then the first conjugate time of the geodesic $t \mapsto \psi_t(x,p)$ is less than or equal to $\frac{2\pi}{\sqrt{p(v_0)^2+k_1|p^h|^2}}$ and $\frac{2\pi}{\sqrt{p(v_0)^2+4k_2|p^h|^2}}$.

Moreover, if

1) $\left\langle Rm^*(\boldsymbol{J}p^h, p^h)p^h, \boldsymbol{J}p^h\right\rangle = k_1|p^h|^4$,
2) $\sum_{i=1}^{2n-2}\left\langle Rm^*(w_i, p^h)p^h, w_i\right\rangle = (2n-2)k_2|p^h|^2$,

then the first conjugate time of the geodesic $t \mapsto \psi_t(x,p)$ is equal to the minimum of $\frac{2\pi}{\sqrt{p(v_0)^2+k_1|p^h|^2}}$ and $\frac{2\pi}{\sqrt{p(v_0)^2+4k_2|p^h|^2}}$.

Proof. Let $E(t) = (E^1(t), E^2(t), E^3(t)), F(t) = (F^1(t), F^2(t), F^3(t))$ be a canonical frame of the Jacobi curve $\mathfrak{J}_{(x,p)}(t)$. Let $A(t)$ and $B(t)$ be matrices defined by

$$E(0) = A(t)E(t) + B(t)F(t). \tag{6.1}$$

On the other hand, if we differentiate the equation (6.1) with respect to t, then

$$\begin{aligned} 0 &= \dot{A}(t)E(t) + A(t)\dot{E}(t) + \dot{B}(t)F(t) + B(t)\dot{F}(t) \\ &= \dot{A}(t)E(t) + A(t)C_1E(t) + A(t)C_2F(t) \\ &\quad + \dot{B}(t)F(t) - B(t)R(t)E(t) - B(t)C_1^TF(t). \end{aligned}$$

It follows that

$$\begin{aligned} &\dot{A}(t) + A(t)C_1 - B(t)R(t) = 0 \\ &\dot{B}(t) + A(t)C_2 - B(t)C_1^T = 0 \end{aligned} \tag{6.2}$$

with initial conditions $B(0) = 0$ and $A(0) = I$.

If we set $S(t) = B(t)^{-1}A(t)$, then $S(t)$ satisfies the following Riccati equation

$$\dot{S}(t) - S(t)C_2S(t) + C_1^TS(t) + S(t)C_1 - R(t) = 0. \tag{6.3}$$

Let us choose $E^3_{2n-1}(0) = p^h + p(v_0)v$ and let

$$S(t) = \begin{pmatrix} S_1(t) & S_2(t) & S_3(t) \\ S_2(t)^T & S_4(t) & S_5(t) \\ S_3(t)^T & S_5(t)^T & S_6(t) \end{pmatrix},$$

where $S_1(t)$ is a 2×2 matrix and $S_6(t)$ is 1×1. Then

$$\begin{aligned} &\dot{S}_1(t) - S_1(t)\tilde{C}_2S_1(t) - S_2(t)S_2(t)^T \\ &- S_3(t)S_3(t)^T + \tilde{C}_1^TS_1(t) + S_1(t)\tilde{C}_1 - \tilde{R}^1(t) = 0, \\ &\dot{S}_4(t) - S_4(t)^2 - S_5(t)S_5(t)^T - S_2(t)^T\tilde{C}_2S_2(t) - \tilde{R}^2(t) = 0, \\ &\dot{S}_6(t) - S_6(t)^2 - S_5(t)^TS_5(t) - S_3(t)^T\tilde{C}_2S_3(t) = 0, \end{aligned} \tag{6.4}$$

where $\tilde{C}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $\tilde{R}^1(t) = \begin{pmatrix} 0 & 0 \\ 0 & R^{22}(t) \end{pmatrix}$. $\tilde{R}^2(t)$ is the $(2n-2) \times (2n-2)$ matrix with ij-th entry equal to $R^{33}_{ij}(t)$.

Note that $U(t) = S(t)^{-1}$ also satisfies $U(0) = 0$ and the Riccati equation

$$\dot{U}(t) + C_2 - U(t)C_1^T - C_1U(t) + U(t)R(t)U(t) = 0.$$

This gives

$$U(t) = -tC_2 - \frac{t^2}{2}(C_1 + C_1^T) - \frac{t^3}{6}(C_1C_1^T + C_2R(0)C_2) + O(t^4).$$

By using this expansion and $S(t)U(t) = I$, we obtain

$$S_1(t) = \begin{pmatrix} -\frac{12}{t^3} + O(1/t^2) & \frac{6}{t^2} + O(1/t) \\ \frac{6}{t^2} + O(1/t) & -\frac{4}{t} + O(1) \end{pmatrix},$$

$$\mathbf{tr}(S_4(t)) = -\frac{2n-2}{t} + O(1), \quad S_6(t) = -\frac{1}{t} + O(1).$$

(For instance, one can take the dot product of the first row

$$s(t) = (S_{1,1}(t), \dots, S_{1,2n+1}(t))$$

of $S(t)$ with the third, fourth, $\dots$, $2n$-th columns of $U(t)$. This gives the order of the dominating terms of $(S_{1,3}(t), \dots, S_{1,2n+1}(t))$ in terms of that of

$S_{1,2}(t)$. By taking the dot product of $s(t)$ with the first and second column of $U(t)$, we obtain the leading order terms of $S_{1,1}(t)$ and $S_{1,2}(t)$. Similar procedure works for other entries of $S(t)$.)

By applying the comparison principle of Riccati equations in [12] to $S(t)$, we have $S_1(t) \geq \Gamma_1(t)$, where $\Gamma_1(t)$ is a solution of the following Riccati equation

$$\dot{\Gamma}_1(t) - \Gamma_1(t)\tilde{C}_2\Gamma_1(t) + \tilde{C}_1^T\Gamma_1(t) + \Gamma_1(t)\tilde{C}_1 - K_1 = 0$$

with the initial condition $\lim_{t\to 0}\Gamma_1^{-1}(t) = 0$. (Of course, one needs to apply the comparison principle to $S(t)$ and $\Gamma(t+\epsilon)$ and let ϵ to zero as usual). Here $K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{k}_1 \end{pmatrix}$ and $\mathfrak{k}_1 = p(v_0)^2 + k_1|p^h|^2$. Thus

$$\begin{aligned} \mathbf{tr}(\tilde{C}_2S_1(t)) &\geq \mathbf{tr}(\tilde{C}_2\Gamma_1(t)) \\ &= \frac{\sqrt{\mathfrak{k}_1}(\sqrt{\mathfrak{k}_1}t\cos(\sqrt{\mathfrak{k}_1}t) - \sin(\sqrt{\mathfrak{k}_1}t))}{(2 - 2\cos(\sqrt{\mathfrak{k}_1}t) - \sqrt{\mathfrak{k}_1}t\sin(\sqrt{\mathfrak{k}_1}t))}. \end{aligned} \tag{6.5}$$

For the term $S_4(t)$, we can take the trace and obtain

$$\frac{d}{dt}\mathbf{tr}(S_4(t)) \geq \frac{1}{2n-2}\mathbf{tr}(S_4(t))^2 + (2n-2)\mathfrak{k}_2,$$

where $\mathfrak{k}_2 = \frac{1}{4}p(v_0)^2 + k_2|p^h|^2$.

Now applying the comparison principle in [12] again we have

$$\mathbf{tr}(S_4(t)) \geq -\sqrt{\mathfrak{k}_2}(2n-2)\cot(\sqrt{\mathfrak{k}_2}t)). \tag{6.6}$$

Finally, for the term $S_6(t)$, we have

$$\dot{S}_6(t) \geq S_6(t)^2.$$

which implies

$$S_6(t) \geq -\frac{1}{t}.$$

By combining this with (6.5) and (6.6), we obtain

$$\begin{aligned} \mathbf{tr}(C_2S(t)) \geq& -\sqrt{\mathfrak{k}_2}(2n-2)\cot\left(\sqrt{\mathfrak{k}_2}t\right) - \frac{1}{t} \\ &+ \frac{\sqrt{\mathfrak{k}_1}(\sqrt{\mathfrak{k}_1}t\cos(\sqrt{\mathfrak{k}_1}t) - \sin(\sqrt{\mathfrak{k}_1}t))}{(2 - 2\cos(\sqrt{\mathfrak{k}_1}t) - \sqrt{\mathfrak{k}_1}t\sin(\sqrt{\mathfrak{k}_1}t))}. \end{aligned} \tag{6.7}$$

Therefore,

$$\frac{d}{dt}\log|\det B(t)| = \mathrm{tr}(C_1^T - S(t)C_2) = -\mathrm{tr}(C_2S(t))$$
$$\leq \sqrt{\mathfrak{k}_2}(2n-2)\cot\left(\sqrt{\mathfrak{k}_2}t\right)$$
$$+\frac{1}{t} - \frac{\sqrt{\mathfrak{k}_1}(\sqrt{\mathfrak{k}_1}t\cos(\sqrt{\mathfrak{k}_1}t) - \sin(\sqrt{\mathfrak{k}_1}t))}{(2-2\cos(\sqrt{\mathfrak{k}_1}t) - \sqrt{\mathfrak{k}_1}t\sin(\sqrt{\mathfrak{k}_1}t))}$$

and hence

$$|\det B(t)| \leq Ca(t)$$

where $C = \lim_{t_0\to 0}\frac{|\det B(t_0)|}{a(t_0)}$ and

$$a(t) = t\sin^{2n-2}(\sqrt{\mathfrak{k}_2}t)(2-2\cos(\sqrt{\mathfrak{k}_1}t) - \sqrt{\mathfrak{k}_1}t\sin(\sqrt{\mathfrak{k}_1}t)).$$

Using (6.2) and the definition of determinant, we see that $B(t) = -C_2t + \frac{1}{2}(C_1 - C_1^T)t^2 + \frac{1}{6}(C_2R(0)C_2 + C_1C_1^T)t^3 + O(t^4)$ and $|\det B(t)| = \frac{1}{12}t^{2n+3} + O(t^{2n+4})$.

Therefore,

$$|\det B(t)| \leq \frac{t\sin^{2n-2}(\sqrt{\mathfrak{k}_2}t)(2-2\cos(\sqrt{\mathfrak{k}_1}t) - \sqrt{\mathfrak{k}_1}t\sin(\sqrt{\mathfrak{k}_1}t))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}.$$

The first assertion follows. Let $S^{k_1,k_2}(t)$ be a solution of (6.3) with $R(t)$ replaced by

$$R^{k_1,k_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathfrak{k}_1 & 0 & 0 \\ 0 & 0 & \mathfrak{k}_2 I_{2n-2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the initial condition $\lim_{t\to 0}(S_t^{k_1,k_2})^{-1} = 0$.

A calculation similar to that of Theorem 6.1 shows that

$$S^{k_1,k_2}(t) = \begin{pmatrix} \frac{-(k_1)^{3/2}\sin(\tau_t)}{s(t)} & \frac{k_1(1-\cos(\tau_t))}{s(t)} & 0 & 0 \\ \frac{k_1(1-\cos(\tau_t))}{s(t)} & \frac{(k_1)^{1/2}(\tau_t\cos(\tau_t)-\sin(\tau_t))}{s(t)} & 0 & 0 \\ 0 & 0 & -\sqrt{\mathfrak{k}_2}\cot(\sqrt{\mathfrak{k}_2}t)I_{2n-2} & 0 \\ 0 & 0 & 0 & -\frac{1}{t} \end{pmatrix},$$

where $\tau_t = \sqrt{\mathfrak{k}_1}t$ and $s(t) = 2-2\cos(\tau_t) - \tau_t\sin(\tau_t)$.

The rest follows as the proof of the previous assertion (with all inequalities replaced by equalities). □

7. Model cases

In this section, we discuss two examples, the Heisenberg group and the complex Hopf fibration which are relevant to the later sections. First, we consider a Sasakian manifold $(M, \mathbf{J}, v_0, \alpha_0, g = \langle \cdot, \cdot \rangle)$ for which the quotient of M by the flow of v_0 is a manifold B. Since $\mathcal{L}_{v_0}\mathbf{J} = 0$ and $\mathcal{L}_{v_0} g = 0$, they descend to a complex structure $\mathbf{J}_B$ and a Riemannian metric g_B on B. Moreover, by Theorem 3.2, they form a Kähler manifold. Moreover, the Tanaka-Webster curvature Rm^* on M and the Riemann curvature tensor Rm^B of B are related by

Lemma 7.1. *The curvature tensors Rm^* and Rm^B are related by*

$$Rm^*(\bar{X}, \bar{Y})\bar{Z} = \overline{Rm^B(X, Y)Z},$$

where $\bar{X}$ denotes the vector orthogonal to v_0 which project to the vector X.

Proof. Since $M \to B$ is a Riemannian submersion, we have (see [11])

$$\nabla^*_{\bar{X}}\bar{Y} = (\nabla_{\bar{X}}\bar{Y})^h = \overline{\nabla_X Y}.$$

Since $\bar{Z}$ projects to Z, we also have

$$\nabla^*_{v_0}\bar{Z} = (\nabla_{v_0}\bar{Z})^h + \frac{1}{2}\mathbf{J}\bar{Z} = (\nabla_{\bar{Z}} v_0)^h + \frac{1}{2}\mathbf{J}\bar{Z} = 0.$$

Therefore,

$$\begin{aligned} &\mathrm{Rm}^*(\bar{X}, \bar{Y})\bar{Z} = \nabla^*_{\bar{X}}\nabla^*_{\bar{Y}}\bar{Z} - \nabla^*_{\bar{Y}}\nabla^*_{\bar{X}}\bar{Z} - \nabla^*_{[\bar{X},\bar{Y}]}\bar{Z} \\ &= \overline{\nabla_X\nabla_Y Z} - \overline{\nabla_Y\nabla_X Z} - \overline{\nabla_{[X,Y]}Z} - \alpha_0([\bar{X}, \bar{Y}])\nabla^*_{v_0}\bar{Z} \\ &= \overline{\mathrm{Rm}^B(X, Y)Z}. \end{aligned}$$

□

The first example is the Heisenberg group. In this case the manifold M is the Euclidean space $\mathbb{R}^{2n+1}$. If we fix a coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$, then the 1-form α_0 and the vector field v_0, are given, respectively, by

$$\alpha_0 = dz - \frac{1}{2}\sum_{i=1}^n x_i dy_i + \frac{1}{2}\sum_{i=1}^n y_i dx_i \quad \text{and} \quad v_0 = \partial_z.$$

The Riemannian metric is the one for which the frame

$$X_i = \partial_{x_i} - \frac{1}{2}y_i\partial_z, \quad Y_i = \partial_{y_i} + \frac{1}{2}x_i\partial_z, \quad \partial_z$$

is orthonormal. The tensor $\mathbf{J}$ is defined by

$$\mathbf{J}(X_i) = Y_i, \quad \mathbf{J}(Y_i) = -X_i, \quad \mathbf{J}(\partial_z) = 0.$$

The quotient B is $\mathbb{C}^n$ equipped with the standard complex structure and Euclidean inner product.

Let (x, p) be a covector with $|p^h| = 1$. Assume that $t \mapsto \psi(x, t\epsilon p)$ is length minimizing between its endpoints for some $\epsilon > 0$. Then, we define the cut time of (x, p) to be the largest such ϵ. The following is well-known. We give the proof for completeness.

Theorem 7.2. *On the Heisenberg group equipped with the above sub-Riemannian structure, the cut time coincides with the first conjugate time.*

Proof. Let $P_{X_i} = p_{x_i} - \frac{1}{2}y_i p_z$ and $P_{Y_i} = p_{y_i} + \frac{1}{2}x_i p_z$. A computation as in [10] shows that

$$\begin{aligned}
P_j(t) &:= P_{X_j}(t) + iP_{Y_j}(t) = P_j(0)e^{itp_z},\\
w_j(t) &:= x_j(t) + iy_j(t) = w_j(0) - \frac{iP_j(0)}{p_z}(e^{itp_z} - 1),\\
z(t) &:= z(0) + \frac{1}{2}\sum_{k=1}^{n}\int_0^t \mathrm{Im}(\bar{w}_k(s)\dot{w}_k(s))ds.
\end{aligned}$$

If (w, z) and $(\tilde{w}, \tilde{z})$ are unit speed geodesics with the same length L and end-points, then

$$\frac{\tilde{P}_j(0)}{\tilde{p}_z}(e^{iL\tilde{p}_z} - 1) = \frac{P_j(0)}{p_z}(e^{iLp_z} - 1).$$

By taking the norms, it follows that

$$\frac{1 - \cos(L\tilde{p}_z)}{\tilde{p}_z^2} = \frac{1 - \cos(Lp_z)}{p_z^2}.$$

Using $w_j(0) = \tilde{w}_j(0)$ and $w_j(L) = \tilde{w}_j(L)$, we also have

$$\frac{e^{i\tilde{\theta}}}{\tilde{p}_z}(e^{iL\tilde{p}_z} - 1) = \frac{e^{i\theta}}{p_z}(e^{iLp_z} - 1),$$

where $P_j(0) = e^{i\theta}$ and $\tilde{P}_j(0) = e^{i\tilde{\theta}}$. Therefore,

$$\frac{\cos(\theta + Lp_z) - \cos(\theta)}{p_z} = \frac{\cos(\tilde{\theta} + L\tilde{p}_z) - \cos(\tilde{\theta})}{\tilde{p}_z},$$
$$\frac{\sin(\theta + Lp_z) - \sin(\theta)}{p_z} = \frac{\sin(\tilde{\theta} + L\tilde{p}_z) - \sin(\tilde{\theta})}{\tilde{p}_z}.$$

Finally, since $z(L) = \tilde{z}(L)$, a computation together with the above implies that

$$\frac{L\tilde{p}_z - \sin(L\tilde{p}_z)}{\tilde{p}_z^2} = \frac{Lp_z - \sin(Lp_z)}{p_z^2}.$$

By investigating the graph of $\frac{1-\cos(x)}{x^2}$ and $\frac{x-\sin(x)}{x^2}$, we have $p_z = \tilde{p}_z$. Therefore, if $L < \frac{2\pi}{p_z}$, then $P_j(0) = \tilde{P}_j(0)$ and the two geodesics coincide. Hence, the result follows from Theorem 6.1. □

The second example is the complex Hopf fibration. We follow the discussion in [4]. In this case, the manifold is given by the sphere $S^{2n+1} = \{z \in \mathbb{C}^{n+1} | |z| = 1\}$. The 1-form α_0 and the vector field v_0 are given, respectively, by

$$\alpha_0 = \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$$

and

$$v_0 = 2\sum_{i=1}^{n}(-y_i\partial_{x_i} + x_i\partial_{y_i})$$

where $z_j = x_j + iy_j$.

The tangent space of S^{2n+1} is the direct sum of $\ker\alpha_0$ and $\mathbb{R}v_0$. The Riemannian metric is defined in such a way that v_0 has length one, v_0 is orthogonal to $\ker\alpha_0$, and the restriction of the metric to $\ker\alpha_0$ coincides with the Euclidean one. The (1,1)-tensor $\mathbf{J}$ is defined analogously by the conditions $\mathbf{J}v_0 = 0$ and the restriction of $\mathbf{J}$ to $\ker\alpha_0$ coincides with the standard complex structure on $\mathbb{C}^n$. The base manifold B is the complex projective space $\mathbb{CP}^n$ and the induced Riemannian metric is given by the Fubini-Study metric. It follows from Lemma 7.1 that

$$\langle \mathrm{Rm}^*(\mathbf{J}X, X)X, \mathbf{J}X\rangle = 4 \text{ and } \langle \mathrm{Rm}^*(v, X)X, v\rangle = 1$$

for all v in the orthogonal complement of $\{X, JX\}$.

Theorem 7.3. *On the complex Hopf fibration equipped with the above sub-Riemannian structure, the cut time coincides with the first conjugate time.*

Proof. The sub-Riemannian geodesic flow is given by

$$\left(a\cos(|v|t) + \frac{v}{|v|}\sin(|v|t)\right) e^{-it\langle v_0, v\rangle},$$

where a is the initial point of the geodesic and v is the initial (co)vector (see [4, 10]).

By the choice of the complex coordinate system, we can assume $a = (1, 0, \dots, 0)$. Let $v = (v_1, \dots, v_n)$. Then the real part of v_1 equal 0. Moreover, $v^h = (0, v_2, \dots, v_n)$ is the horizontal part of v. Assume that $|v^h| = 1$ and let w be another such covector such that the corresponding geodesic has the same end point and the same length L as that of v.

Under the above assumptions, we have

$$|v|^2 - \frac{1}{4}(\mathrm{Im}(v_1))^2 = 1 = |w|^2 - \frac{1}{4}(\mathrm{Im}(w_1))^2$$

and

$$\begin{aligned}&\left(a\cos(||v||L) + \frac{v}{||v||}\sin(||v||L)\right) e^{-\frac{iL}{2}\mathrm{Im}(v_1)}\\ &= \left(a\cos(||w||L) + \frac{w}{||w||}\sin(||w||L)\right) e^{-\frac{iL}{2}\mathrm{Im}(w_1)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\left(\cos(|v|L) + \frac{v_1}{|v|}\sin(|v|L)\right) e^{-\frac{iL}{2}\mathrm{Im}(v_1)}\\ &= \left(\cos(|w|L) + \frac{w_1}{|w|}\sin(|w|L)\right) e^{-\frac{iL}{2}\mathrm{Im}(w_1)}\end{aligned}$$

and

$$\left(\frac{v_i}{|v|}\sin(|v|L)\right) e^{-\frac{iL}{2}\mathrm{Im}(v_1)} = \left(\frac{w_i}{|w|}\sin(|w|L)\right) e^{-\frac{iL}{2}\mathrm{Im}(w_1)}.$$

for all $i \neq 1$.

By taking the norm of the second equation, we obtain

$$\frac{|v_i|^2}{|v|^2}\sin^2(|v|L) = \frac{|w_i|^2}{|w|^2}\sin^2(|w|L).$$

If we sum over $i \neq 1$, then we have

$$\frac{\sin^2(|v|L)}{|v|^2} = \frac{\sin^2(|w|L)}{|w|^2}.$$

If both $|v|$ and $|w|$ are less than or equal to $\frac{\pi}{L}$, then $|v| = |w|$. It follows that $\mathrm{Im}(v_1) = \pm\mathrm{Im}(w_1)$.

If $\mathrm{Im}(v_1) = \mathrm{Im}(w_1)$, then either $v_i = w_i$ for all i which implies that the two geodesics coincide or $\sin(L|v|) = 0 = \sin(L|w|)$. In this case $|v| = |w| = \frac{\pi}{L}$.

If $\mathrm{Im}(v_1) = -\mathrm{Im}(w_1)$, then

$$\left(\cos(|v|) + \frac{v_1}{|v|}\sin(|v|)\right) e^{iL\mathrm{Im}(v_1)} = \left(\cos(|v|) - \frac{v_1}{|v|}\sin(|v|)\right).$$

It follows that

$$\frac{\tan(|v|)}{|v|} = \frac{\tan(\mathrm{Im}(v_1)/2)}{\mathrm{Im}(v_1)/2}.$$

Since $|v| > \frac{1}{2}\mathrm{Im}(v_1)$, we have a contradiction. Therefore, the result follows from this and Theorem 6.1. □

8. Volume growth estimates

In this section, we prove a volume growth estimate and the proof of Theorem 1.1 and Theorem 1.2. Let Ω be the set of points (x, p) in the cotangent space T_x^*M such that the curve $t \in [0, 1] \mapsto \psi_t(x, p)$ is a length minimizing. Let

$$\Sigma = \{p \in \Omega | |p^h| = 1 \ \text{ and } \ \epsilon p \in \Omega \text{ for some } \epsilon > 0\}.$$

For each p in Σ, we let $T(p)$ be the cut time which is the maximal time T such that $t \in [0, T] \mapsto \psi_t(x, p)$ is length minimizing. Finally, let us denote the ball centered at x of radius R with respect to the sub-Riemannian distance by $B_R(x)$ and the Riemannian volume form by η.

Theorem 8.1. *Assume that the Tanaka-Webster curvature Rm^* of the Sasakian manifold satisfies*

1) $\left\langle Rm^*(\boldsymbol{J}p^h, p^h)p^h, \boldsymbol{J}p^h \right\rangle \geq k_1|p^h|^4$,

2) $\sum_{i=1}^{2n-2}\left\langle Rm^*(w_i,p^h)p^h,w_i\right\rangle \geq (2n-2)k_2|p^h|^2$,

for some constants k_1 *and* k_2*, where* $w_1,\dots,w_{2n-2}$ *is an orthonormal frame of* $\boldsymbol{span}\{p^h,\boldsymbol{J}p^h,v_0\}^\perp$*. Then*

$$\int_{B_R(x)} d\eta \leq \int_0^{\min\{T(p),R\}}\int_\Sigma k(r,z)d\mathfrak{m}(r,z)$$

where (r,z) *denotes the cylindrical coordinates defined by* $r=|p^h|$ *and* $z=p(v_0)$*,* $\mathfrak{k}_1(r,z)=z^2+k_1r^2$*,* $\mathfrak{k}_2(r,z)=\frac{1}{4}z^2+k_2r^2$*. The function* k *is defined by*

$$k(r,z)=r^2\left[\frac{\sin^{2n-2}(\sqrt{\mathfrak{k}_2})(2-2\cos(\sqrt{\mathfrak{k}_1})-\sqrt{\mathfrak{k}_1}\sin(\sqrt{\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right]$$

if $\mathfrak{k}_1\geq 0$ *and* $\mathfrak{k}_2\geq 0$*,*

$$k(r,z)=r^2\left[\frac{\sinh^{2n-2}(\sqrt{-\mathfrak{k}_2})(2-2\cos(\sqrt{\mathfrak{k}_1})-\sqrt{\mathfrak{k}_1}\sin(\sqrt{\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right]$$

if $\mathfrak{k}_1\geq 0$ *and* $\mathfrak{k}_2\leq 0$*,*

$$k(r,z)=r^2\left[\frac{\sin^{2n-2}(\sqrt{\mathfrak{k}_2})(2-2\cosh(\sqrt{-\mathfrak{k}_1})+\sqrt{-\mathfrak{k}_1}\sinh(\sqrt{-\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right]$$

if $\mathfrak{k}_1\leq 0$ *and* $\mathfrak{k}_2\geq 0$*,*

$$k(r,z)=r^2\left[\frac{\sinh^{2n-2}(\sqrt{-\mathfrak{k}_2})(2-2\cosh(\sqrt{-\mathfrak{k}_1})+\sqrt{-\mathfrak{k}_1}\sinh(\sqrt{-\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right]$$

if $\mathfrak{k}_1\leq 0$ *and* $\mathfrak{k}_2\leq 0$*.*

Proof. We use the same notations as in the proof of Theorem 6.1.

Let $\rho_t:T_x^*M\to\mathbb{R}$ be the function defined by $\psi_t^*\eta=\rho_t\mathfrak{m}$. It follows from Theorem 4.3 that

$$\rho_t=|p^h|^2|\det B(t)|. \tag{8.1}$$

Next, we replace the matrix $R(t)$ in (6.2) by R^{k_1,k_2} and denote the solutions by $A^{k_1,k_2}(t)$ and $B^{k_1,k_2}(t)$. Then

$$\frac{\frac{d}{dt}\det B(t)}{\det B(t)}=-\mathrm{tr}(S(t)C_2)\leq-\mathrm{tr}(S^{k_1,k_2}(t)C_2)=\frac{\frac{d}{dt}\det B^{k_1,k_2}(t)}{\det B^{k_1,k_2}(t)}.$$

It follows that $\frac{\det B(t)}{\det B^{k_1,k_2}(t)}$ is non-increasing.

It follows that from the proof of Theorem 6.1 that

$$\begin{aligned}\int_{B_R(x)} d\eta &= \int_\Sigma \int_0^{\min\{T(p),R\}} \rho_t d\mathfrak{m}\\ &\le \int_{\Omega_R} |p^h|^2 \left[\frac{\sin^{2n-2}(\sqrt{\mathfrak{k}_2})(2-2\cos(\sqrt{\mathfrak{k}_1})-\sqrt{\mathfrak{k}_1}\sin(\sqrt{\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right] d\mathfrak{m}(p).\end{aligned}$$

□

Proof of Theorems 1.1 and 1.2. By the proof of Theorem 6.1 and Theorem 7.3, the volume of sub-Riemannian ball of radius R in the Complex Hopf fibration is given by

$$\int_{\Omega_R} |p^h|^2 \left[\frac{\sin^{2n-2}(\sqrt{\mathfrak{k}_2})(2-2\cos(\sqrt{\mathfrak{k}_1})-\sqrt{\mathfrak{k}_1}\sin(\sqrt{\mathfrak{k}_1}))}{\mathfrak{k}_1^2\mathfrak{k}_2^{2n-2}}\right] d\mathfrak{m}(p).$$

Therefore, the result follows from Theorem 8.1. □

9. Laplacian comparison theorem

In this section, we define a version of Hessian following [1] and prove Theorem 1.3.

Let $f : M \to \mathbb{R}$ be a smooth function. The graph G of the differential df defines a sub-manifold of the manifold T^*M. Let v be a tangent vector in T_xM. Then there is a vector X in the tangent space of G at df_x such that $\pi_*(X) = v$, where $\pi : T^*M \to M$ is the projection. The sub-Riemannian Hessian $\mathbf{Hess}\, f$ at x is defined by $\mathbf{Hess}\, f(v) = X_{\mathcal{V}}$. Recall that $X_{\mathcal{V}}$ is the component of X in $\mathcal{V}$ with respect to the splitting $TT^*M = \mathcal{V} \oplus \mathcal{H}$.

Lemma 9.1. *Under the identification in Theorem 4.2, the sub-Riemannian Hessian is given by*

1) $\boldsymbol{Hess}\, f(v) = \nabla_v \nabla f$ *if v is contained in the orthogonal complement of* $\{\nabla f^h, \boldsymbol{J}\nabla f, v_0\}$,
2) $\boldsymbol{Hess}\, f(\nabla f^h) = \nabla_{\nabla f^h} \nabla f - \frac{1}{2}\langle \nabla f, v_0\rangle \boldsymbol{J}\nabla f^h$,
3) $\boldsymbol{Hess}\, f(\boldsymbol{J}\nabla f) = \nabla_{\boldsymbol{J}\nabla f} \nabla f - \frac{1}{2}\langle \nabla f, v_0\rangle \nabla f^h + \frac{1}{2}|\nabla f^h|^2 v_0$,
4) $\boldsymbol{Hess}\, f(v) = \nabla_v \nabla f + \frac{|\nabla f|^2}{2}\boldsymbol{J}\nabla f$ *if* $v = |\nabla f^h|^2 v_0 - (v_0 f)\nabla f^h$.

Proof. Let $\{v_0, \dots, v_{2n}\}$ be a frame defined as in Lemma 3.7 around a point x. Since $\pi_*(\vec{h}_i) = v_i$, we have

$$(df)_*(k_a v_a) = k_a \vec{h}_a + \bar{k}_a \vec{\alpha}_a.$$

It follows that

$$\begin{aligned}\bar{k}_c + k_a dh_a(\vec{h}_c) &= \omega(\vec{h}_c, (df)_*(k_a v_a)) = -dh_c((df)_*(k_a v_a)) \\ &= -k_a(v_a v_c f) = -k_a \langle \nabla_{v_a} \nabla f, v_c \rangle - k_a \langle \nabla f, \nabla_{v_a} v_c \rangle .\end{aligned}$$

Therefore, we have the following at x.

$$\begin{aligned}\bar{k}_i &= -k_a \langle \nabla_{v_a} \nabla f, v_i \rangle - k_a \langle \nabla f, \nabla_{v_a} v_i \rangle - k_a dh_a(\vec{h}_i) \\ &= -k_a \langle \nabla_{v_a} \nabla f, v_i \rangle - \frac{k_j}{2} \mathbf{J}_{ji} v_0 f - k_0 dh_0(\vec{h}_i) - k_j dh_j(\vec{h}_i) \\ &= -k_a \langle \nabla_{v_a} \nabla f, v_i \rangle + \frac{k_j}{2} \mathbf{J}_{ji} v_0 f + \frac{k_0}{2} \mathbf{J}_{ik} v_k f\end{aligned}$$

and

$$\begin{aligned}\bar{k}_0 &= -k_a \langle \nabla_{v_a} \nabla f, v_0 \rangle - k_i \langle \nabla f, \nabla_{v_i} v_0 \rangle - k_i dh_i(\vec{h}_0) \\ &= -k_a \langle \nabla_{v_a} \nabla f, v_0 \rangle + \frac{k_i}{2} \left\langle \mathbf{J} v_i, \nabla f^h \right\rangle - \frac{1}{2} k_i \mathbf{J}_{ij} h_j = -k_a \langle \nabla_{v_a} \nabla f, v_0 \rangle .\end{aligned}$$

Hence, if $v := k_a v_a$ is contained in $\pi_* \mathcal{H}_3$, then

$$((df)_*(k_i v_i))_{\mathcal{V}} = -\left(\frac{1}{2} k_j \mathbf{J}_{ji} v_0 f - \frac{(v_0 f)(v_s f) k_s}{2 |\nabla f^h|^2} (v_j f) \mathbf{J}_{ji} \right) \vec{\alpha}_i + \bar{k}_a \vec{\alpha}_a.$$

If v is contained in $\pi_* \mathcal{H}_3$ and the orthogonal complement of ∇f^h, then

$$((df)_*(k_i v_i))_{\mathcal{V}} = - \langle \nabla_{k_i v_i} \nabla f, v_a \rangle \vec{\alpha}_a.$$

If $v = \nabla f^h$, then

$$((df)_*(\nabla f^h))_{\mathcal{V}} = - \left\langle \nabla_{\nabla f^h} \nabla f, v_a \right\rangle \vec{\alpha}_a + \frac{1}{2} \left\langle \mathbf{J} \nabla f^h, v_i \right\rangle \langle \nabla f, v_0 \rangle \vec{\alpha}_i.$$

If $v = \mathbf{J} \nabla f^h$, then

$$\begin{aligned}((df)_*(\mathbf{J} \nabla f))_{\mathcal{V}} &= [(v_j f) \mathbf{J}_{ji} \vec{h}_i]_{\mathcal{V}} - \langle \nabla_{\mathbf{J} \nabla f} \nabla f, v_0 \rangle \vec{\alpha}_0 \\ &\quad - \langle \nabla_{\mathbf{J} \nabla f} \nabla f, v_i \rangle \vec{\alpha}_i - \frac{v_i f}{2} (v_0 f) \vec{\alpha}_i \\ &= - \langle \nabla_{\mathbf{J} \nabla f} \nabla f, v_a \rangle \vec{\alpha}_a + \frac{v_i f}{2} (v_0 f) \vec{\alpha}_i - \frac{1}{2} |\nabla f^h|^2 \vec{\alpha}_0.\end{aligned}$$

Finally, if $v = |\nabla f^h|^2 v_0 - (v_0 f)\nabla f^h$, then we have

$$((df)_*(v))_{\mathcal{V}} = -\langle \nabla_v \nabla f, v_a \rangle \vec{\alpha}_a - \frac{|\nabla f|^2}{2} \langle \mathbf{J}\nabla f, v_i \rangle \vec{\alpha}_i.$$

□

Proof of Theorem 1.3. Let $f(x) = -\frac{1}{2}d^2(x, x_0)$. Then the curve $t \in [0,1] \mapsto \pi e^{t\vec{H}}(df_x)$ is the geodesic which starts from x and ends at x_0. Let $E(t) = (E^1(t), E^2(t), E^3(t)), F(t) = (F^1(t), F^2(t), F^3(t))$ be a canonical frame of the Jacobi curve $\mathfrak{J}_{(x,df_x)}(t)$. Let

$$\mathcal{E} = (\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3_1, \dots, \mathcal{E}^3_{2n-1})^T, \mathcal{F} = (\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3_1, \dots, \mathcal{F}^3_{2n-1})^T$$

be a symplectic basis of $T_{(x_0,p)}T^*M$ such that $\mathcal{E}^i$ is contained in $\mathcal{V}_i$ and $\mathcal{F}^i$ is contained in $\mathcal{H}_i$, where $(x_0, p) = e^{1\cdot\vec{H}}(df_x)$. Let

$$v = (v^1, v^2, v^3_1, \dots, v^3_{2n-1})^T$$

be a basis of T_xM such that $e^{t\vec{H}}_*(df_x)_*(v) = \mathcal{E}$. Let $A(t)$ and $B(t)$ be matrices such that

$$(df_x)_*(v) = A(t)E(t) + B(t)F(t).$$

By construction, we have $B(1) = 0$. We can also pick $E(t)$ such that $A(1) = I$.

By the definition of $\mathbf{Hess}\, f$, we also have

$$\mathbf{Hess}\, f(B(0)\pi_*F(0)) = \mathbf{Hess}\, f(v) = A(0)E(0).$$

Therefore, if we let $S(t) = B(t)^{-1}A(t)$, then

$$\mathbf{Hess}\, f(\pi_*F(0)) = S(0)E(0).$$

A computation as in the proof of Theorem 6.1 shows that

$$\dot{S}(t) - S(t)C_2S(t) + C_1^T S(t) + S(t)C_1 - R(t) = 0.$$

Therefore, by applying similar computation as in the proof of Theorem 6.1 to $S(1-t)$, we obtain estimates for $S(0)$. Since $\Delta_H f(x) = \mathbf{tr}(C_2S(0))$, the result follows. □

10. Appendix I

In this section, we give the proof of various known results in Section 3.

Proof of Lemma 3.1. Since the almost contact manifold is normal, we have

$$0 = [\mathbf{J},\mathbf{J}](v,v_0) + d\alpha_0(v,v_0)v_0 = \mathbf{J}^2[v,v_0] - \mathbf{J}[\mathbf{J}v,v_0] = \mathbf{J}\mathcal{L}_{v_0}(\mathbf{J})v.$$

It follows that $\mathcal{L}_{v_0}(\mathbf{J}) = 0$.

Since the metric is associated to the almost contact structure,

$$\begin{aligned}0 &= \mathcal{L}_{v_0}\alpha_0(v) = \mathcal{L}_{v_0}(\langle v_0, v\rangle) - \alpha_0([v_0,v])\\ &= \langle \nabla_{v_0}v_0, v\rangle + \langle v_0, \nabla_{v_0}v\rangle - \langle v_0, \nabla_{v_0}v\rangle + \langle v_0, \nabla_v v_0\rangle\\ &= \langle \nabla_{v_0}v_0, v\rangle.\end{aligned}$$

Since the metric is associated to the almost contact structure and $\mathcal{L}_{v_0}(\mathbf{J}) = 0$, we also have

$$\mathcal{L}_{v_0}g(v,\mathbf{J}w) = (\mathcal{L}_{v_o}d\alpha_0)(v,w) = 0.$$

Therefore, $\mathcal{L}_{v_0}g = 0$ as claimed.

By Lemma 3.7, we have

$$\langle \mathbf{J}(v_j), v_i\rangle = \mathbf{J}_{ji} = 2\Gamma^0_{ji} = -2\left\langle \nabla_{v_j}v_0, v_i\right\rangle.$$

Therefore, $\mathbf{J} = -2\nabla v_0$. □

Proof of Theorem 3.2. Let $v_0, v_1, \dots, v_{2n}$ be a local frame defined by Lemma 3.6. Then

$$\begin{aligned}0 &= \mathcal{L}_{v_0}(\mathbf{J})(v_i) = [v_0, \mathbf{J}v_i] - \mathbf{J}[v_0, v_i]\\ &= \nabla_{v_0}(\mathbf{J}v_i) - \nabla_{\mathbf{J}v_i}(v_0) - \mathbf{J}\nabla_{v_0}v_i + \mathbf{J}\nabla_{v_i}v_0\\ &= (\nabla_{v_0}\mathbf{J})v_i - \nabla_{\mathbf{J}v_i}(v_0) + \mathbf{J}\nabla_{v_i}v_0\\ &= (\nabla_{v_0}\mathbf{J})v_i + \frac{1}{2}\mathbf{J}^2v_i - \frac{1}{2}\mathbf{J}^2v_i = (\nabla_{v_0}\mathbf{J})v_i\end{aligned}$$

Since $\mathbf{J}v_0 = 0$,

$$(\nabla_{v_i}\mathbf{J})v_0 = -\mathbf{J}\nabla_{v_i}v_0 = \frac{1}{2}\mathbf{J}\mathbf{J}v_i = -\frac{1}{2}v_i.$$

Since $\nabla_{v_0}v_0 = 0$, we also have $(\nabla_{v_0}\mathbf{J})v_0 = -\mathbf{J}(\nabla_{v_0}v_0) = 0$.

Finally, we need to show $(\nabla_{v_i}\mathbf{J})v_j = \frac{1}{2}\delta_{ij}v_0$. First, by the properties of the frame $v_1, \dots, v_n$, we have

$$\langle(\nabla_{v_i}\mathbf{J})v_j, v_0\rangle = -\langle\mathbf{J}v_j, \nabla_{v_i}v_0\rangle = \frac{1}{2}\langle\mathbf{J}v_j, \mathbf{J}v_i\rangle = \frac{1}{2}\delta_{ij}$$

at x

By normality and properties of the frame $v_1, \dots, v_{2n}$, we have

$$0 = (\nabla_{\mathbf{J}v_i}\mathbf{J})v_j - (\nabla_{\mathbf{J}v_j}\mathbf{J})v_i + \mathbf{J}(\nabla_{v_j}\mathbf{J})v_i - \mathbf{J}(\nabla_{v_i}\mathbf{J})v_j + d\alpha_0(v_i, v_j)v_0.$$

It follows from Lemma 3.7 that

$$\begin{aligned}
0 &= \langle(\nabla_{\mathbf{J}v_i}\mathbf{J})v_j, v_k\rangle - \left\langle(\nabla_{\mathbf{J}v_j}\mathbf{J})v_i, v_k\right\rangle + \left\langle\mathbf{J}(\nabla_{v_j}\mathbf{J})v_i, v_k\right\rangle - \langle\mathbf{J}(\nabla_{v_i}\mathbf{J})v_j, v_k\rangle \\
&= -\langle(\nabla_{v_k}\mathbf{J})\mathbf{J}v_i, v_j\rangle - \left\langle(\nabla_{v_j}\mathbf{J})v_k, \mathbf{J}v_i\right\rangle + \langle(\nabla_{v_k}\mathbf{J})\mathbf{J}v_j, v_i\rangle \\
&\quad + \langle(\nabla_{v_i}\mathbf{J})v_k, \mathbf{J}v_j\rangle + \left\langle\mathbf{J}(\nabla_{v_j}\mathbf{J})v_i, v_k\right\rangle - \langle\mathbf{J}(\nabla_{v_i}\mathbf{J})v_j, v_k\rangle \\
&= -\langle(\nabla_{v_k}\mathbf{J})\mathbf{J}v_i, v_j\rangle + \left\langle(\nabla_{v_j}\mathbf{J})\mathbf{J}v_i, v_k\right\rangle + \langle\mathbf{J}(\nabla_{v_k}\mathbf{J})v_i, v_j\rangle \\
&\quad - \langle(\nabla_{v_i}\mathbf{J})\mathbf{J}v_j, v_k\rangle + \left\langle\mathbf{J}(\nabla_{v_j}\mathbf{J})v_i, v_k\right\rangle - \langle\mathbf{J}(\nabla_{v_i}\mathbf{J})v_j, v_k\rangle .
\end{aligned}$$

Since $\mathbf{J}^2v_j = -v_j$, we also have $\langle(\nabla_{v_i}\mathbf{J})\mathbf{J}v_j, v_k\rangle = -\langle\mathbf{J}(\nabla_{v_i}\mathbf{J})v_j, v_k\rangle$. Therefore, the above equation simplifies to

$$0 = -2\langle(\nabla_{v_k}\mathbf{J})v_i, \mathbf{J}v_j\rangle .$$

□

Proof of Theorem 3.3. Since the manifold is Sasakian, we have

$$\begin{aligned}
\mathrm{Rm}(X,Y)v_0 &= \nabla_X\nabla_Y v_0 - \nabla_Y\nabla_X v_0 - \nabla_{[X,Y]}v_0 \\
&= \frac{1}{2}(-\nabla_X(\mathbf{J}(Y)) + \nabla_Y(\mathbf{J}(X)) + \mathbf{J}[X,Y]) \\
&= \frac{1}{2}(-\nabla_X\mathbf{J}(Y) + \nabla_Y\mathbf{J}(X)) \\
&= \frac{1}{4}\alpha_0(Y)X - \frac{1}{4}\alpha_0(X)Y.
\end{aligned}$$

□

Proof of Theorem 3.4. Let ∇^* be the Tanaka connection defined by

$$\nabla^*_X Y = \nabla_X Y + \alpha_0(X)\mathbf{J}Y - \alpha_0(Y)\nabla_X v_0 + \nabla_X\alpha_0(Y)v_0$$

Assume that X and Y are horizontal. Then

$$\nabla^*_X Y = \nabla_X Y - \langle\nabla_X Y, v_0\rangle v_0.$$

Therefore,

$$\begin{aligned}\nabla^*_X\nabla^*_Y Z &= \nabla_X(\nabla_Y Z - \langle\nabla_Y Z, v_0\rangle\, v_0) - \langle\nabla_X(\nabla_Y Z - \langle\nabla_Y Z, v_0\rangle\, v_0), v_0\rangle\, v_0\\ &= \nabla_X\nabla_Y Z - \langle\nabla_X\nabla_Y Z, v_0\rangle\, v_0 - \langle\nabla_Y Z, v_0\rangle\,\nabla_X v_0\end{aligned}$$

Let Rm^* be the curvature corresponding to ∇^*. Assume that X, Y, Z are horizontal. Then

$$\begin{aligned}\mathrm{Rm}^*(X,Y)Z &= \nabla^*_X\nabla^*_Y Z - \nabla^*_Y\nabla^*_X Z - \nabla^*_{[X,Y]}Z\\ &= \nabla_X\nabla_Y Z - \langle\nabla_X\nabla_Y Z, v_0\rangle\, v_0 - \langle\nabla_Y Z, v_0\rangle\,\nabla_X v_0 - \nabla_Y\nabla_X Z\\ &\quad + \langle\nabla_Y\nabla_X Z, v_0\rangle\, v_0 + \langle\nabla_X Z, v_0\rangle\,\nabla_Y v_0\\ &\quad - \nabla_{[X,Y]}Z + \left\langle\nabla_{[X,Y]}Z, v_0\right\rangle v_0\\ &= (\mathrm{Rm}(X,Y)Z)^h + \langle Z, \nabla_Y v_0\rangle\,\nabla_X v_0 - \langle Z, \nabla_X v_0\rangle\,\nabla_Y v_0.\end{aligned}$$

□

11. Appendix II

This appendix is devoted to the proofs of Theorem 4.1 and Lemma 5.3.

Proof of Theorem 4.1. Let $v_0, v_1, \dots, v_{2n}$ be the local frame defined in a neighborhood of x by Lemma 3.6. Let Γ^c_{ab} and $\mathbf{J}_{ij}$ be defined by

$$\nabla_{v_a} v_b = \Gamma^c_{ab} v_c \quad \text{and} \quad \mathbf{J}_{ij} = \langle \mathbf{J} v_i, v_j\rangle\,,$$

respectively. From now on, we sum over repeated indices. The indices i, j, k, s, l ranges over $1, \dots, 2n$ and a, b, c, d ranges over $0, \dots, 2n$.

It is clear that $\Gamma^c_{ab} = -\Gamma^b_{ac}$ wherever it is defined. We also have $\Gamma^i_{00} = \Gamma^0_{0i} = \Gamma^0_{i0} = 0$. Indeed, since $d\alpha_0(v_0, v_i) = 0$, we have

$$0 = \alpha_0([v_0, v_i]) = \Gamma^0_{0i} - \Gamma^0_{i0} = \Gamma^0_{0i} = -\Gamma^i_{00}.$$

Since $\langle \mathbf{J} v_i, v_j\rangle = -2\,\langle\nabla_{v_i} v_0, v_j\rangle$, we have $\mathbf{J}_{ij} = -2\Gamma^j_{i0} = 2\Gamma^0_{ij}$. Let $\alpha_0, \dots, \alpha_{2n}$ be the dual frame of $v_0, \dots, v_{2n}$ and let $h_i(x,p) = p(v_i)$. Then $\pi^*\alpha_0, \dots, \pi^*\alpha_n, dh_0, \dots, dh_n$ forms a local co-frame of the cotangent bundle. We will also denote $\pi^*\alpha_i$ simply by α_i.

The proofs of the following two lemmas are done after the proof of Theorem 4.1.

Lemma 11.1. *One has the following relations on the Lie bracket of the vector fields introduced above.*

1) $\alpha_a(\vec{h}_b) = \delta_{ab}$,
2) $[\vec{\alpha}_a, \vec{\alpha}_b] = 0$,
3) $dh_b(\vec{h}_c) = \sum_a (\Gamma^a_{cb} - \Gamma^a_{bc}) h_a$,
4) $[\vec{\alpha}_a, \vec{h}_b] = \sum_c (\Gamma^a_{bc} - \Gamma^a_{cb}) \vec{\alpha}_c$,
5) $[\vec{H}, \vec{\alpha}_i] = \vec{h}_i + \sum_{j\neq 0,a} h_j (\Gamma^i_{aj} - \Gamma^i_{ja}) \vec{\alpha}_a$ *if* $i \neq 0$,
6) $[\vec{H}, \vec{\alpha}_0] = \sum_{j,k\neq 0} h_j (\Gamma^0_{kj} - \Gamma^0_{jk}) \vec{\alpha}_k = -\sum_{j,k\neq 0} h_j \boldsymbol{J}_{jk} \vec{\alpha}_k$,
7) $[\vec{H}, \vec{h}_i] = \sum_{k\neq 0} h_k [\vec{h}_k, \vec{h}_i] - \sum_{k\neq 0,a} h_a (\Gamma^a_{ik} - \Gamma^a_{ki}) \vec{h}_k$,
8) $[\vec{H}, [\vec{H}, \vec{\alpha}_0]] = h_0 \sum_{k\neq 0} h_k \vec{\alpha}_k - \sum_{k,j\neq 0} h_j \boldsymbol{J}_{jk} \vec{h}_k$
$- H\vec{\alpha}_0 - \sum_{j,l,k\neq 0} h_j h_l \Gamma^k_{0l} \boldsymbol{J}_{jk} \vec{\alpha}_0 - \sum_{j,l,s,k\neq 0} h_j h_l \boldsymbol{J}_{js} \Gamma^s_{kl} \vec{\alpha}_k$,
9) $[\vec{H}, [\vec{H}, \vec{\alpha}_i]] = 2\sum_{l,k\neq 0} h_l \Gamma^k_{li} \vec{h}_k + \sum_{l\neq 0} h_l \boldsymbol{J}_{li} \vec{h}_0 - \sum_{k\neq 0} h_0 \boldsymbol{J}_{ik} \vec{h}_k$
(mod vertical) when $i \neq 0$,
10) $[\vec{H}, [\vec{H}, [\vec{H}, \vec{\alpha}_0]]] = h_0 \vec{H} - 2H\vec{h}_0$ *(mod vertical).*

Here, the phrase "mod vertical" means the that the difference of the two vectors is contained in the vertical bundle $\mathcal{V}$.

The relations reduce to the following ones at x

Lemma 11.2. *One has the following relations at* x.

1) $dh_j(\vec{h}_i) = \boldsymbol{J}_{ij} h_0$ *if* $i \neq 0 \neq j$,
2) $dh_j(\vec{h}_0) = \frac{1}{2} \sum_{k\neq 0} \boldsymbol{J}_{jk} h_k$ *if* $j \neq 0$,
3) $[\vec{\alpha}_i, \vec{h}_j] = \frac{1}{2} \boldsymbol{J}_{ij} \vec{\alpha}_0$ *if* $i \neq 0 \neq j$,
4) $[\vec{\alpha}_i, \vec{h}_0] = \frac{1}{2} \sum_{k\neq 0} \boldsymbol{J}_{ki} \vec{\alpha}_k$ *if* $i \neq 0$,
5) $[\vec{\alpha}_0, \vec{h}_j] = \sum_{k\neq 0} \boldsymbol{J}_{jk} \vec{\alpha}_k$ *if* $j \neq 0$,
6) $[\vec{H}, \vec{\alpha}_i] = \vec{h}_i + \sum_{j\neq 0} h_j \boldsymbol{J}_{ji} \vec{\alpha}_0$ *when* $i \neq 0$,
7) $[\vec{H}, \vec{\alpha}_0] = -\sum_{j,k\neq 0} h_j \boldsymbol{J}_{jk} \vec{\alpha}_k$,
8) $[\vec{H}, [\vec{H}, \vec{\alpha}_0]] = h_0 \sum_{k\neq 0} h_k \vec{\alpha}_k - \sum_{j,k\neq 0} h_j \boldsymbol{J}_{jk} \vec{h}_k - H\vec{\alpha}_0$,

Now, we apply the above lemmas to prove the theorem. Since $[\vec{H}, \vec{\alpha}_0]$ is vertical, $\vec{\alpha}_0$ is in $J^{-1}(0)$. Therefore, $\vec{\alpha}_0 = fE^1(0)$ for some function f on the cotangent bundle. It follows from Theorem 2.1 that

1) $fE^2(0) = [\vec{H}, \vec{\alpha}_0] - (\vec{H}f)E^1(0)$,

2) $fF^2(0) = [\vec{H}, [\vec{H}, \vec{\alpha}_0]] - (\vec{H}^2 f)E^1(0) - 2(\vec{H}f)E^2(0)$,

3) $f\dot{F}^2(0) = [\vec{H}, [\vec{H}, [\vec{H}, \vec{\alpha}_0]]] - (\vec{H}^3 f)E_1 - 3(\vec{H}^2 f)E_2 - 3(\vec{H}f)F_2$.

By Lemma 11.1, we have

$$f^2 = \omega(fF^2(0), fE^2(0)) = \sum_{i,l,j,k\neq 0} h_i h_j \mathbf{J}_{il}\mathbf{J}_{jk}\omega(\vec{h}_l, \vec{\alpha}_k) = 2H.$$

It follows from this and Lemma 11.1 that

1) $fE^2(0) = -\sum_{k,l\neq 0} h_k \mathbf{J}_{kl}\vec{\alpha}_l$,

2) $fF^2(0) = h_0 \sum_{j,k,l\neq 0} h_k\vec{\alpha}_k - \sum_{j,k\neq 0} h_j \mathbf{J}_{jk}\vec{h}_k - H\vec{\alpha}_0$
$- \sum_{j,k,l\neq 0} h_j h_l \Gamma^k_{0l}\mathbf{J}_{jk}\vec{\alpha}_0 - \sum_{j,k,l,s\neq 0} h_j h_l \mathbf{J}_{js}\Gamma^s_{kl}\vec{\alpha}_k$,

3) $-fF^1(0) = f\dot{F}^2(0) = h_0\vec{H} - 2H\vec{h}_0$ (mod vertical).

This gives the characterizations of $\mathcal{V}_1$, $\mathcal{V}_2$, and $\mathcal{H}_2$.

Suppose that $a_b\vec{\alpha}_b$ is contained in $\mathcal{V}_3$. Since $\mathcal{V}_3$ and $\mathcal{H}_2$ are skew-orthogonal,

$$-\sum_{j,k\neq 0} a_k h_j \mathbf{J}_{kj} = \omega\left(a_b\vec{\alpha}_b, h_j\mathbf{J}_{ij}\vec{h}_i\right) = 0. \tag{11.1}$$

Since $\mathcal{V}_3$ and $\mathcal{H}_1$ are skew-orthogonal, we also have

$$0 = -\omega\left(a_b\vec{\alpha}_b, h_0\vec{H} - 2H\vec{h}_0\right) = h_0 h_k a_k - 2Ha_0 \tag{11.2}$$

This gives the characterizations of $\mathcal{V}_3$.

It also follows that

$$\begin{aligned}
&[\vec{H}, a_0\vec{\alpha}_0 + a_i\vec{\alpha}_i] \\
&= (\vec{H}a_0)\vec{\alpha}_0 + a_0[\vec{H}, \vec{\alpha}_0] + (\vec{H}a_i)\vec{\alpha}_i + a_i[\vec{H}, \vec{\alpha}_i] \\
&= (\vec{H}a_0)\vec{\alpha}_0 - a_0 h_j\mathbf{J}_{jk}\vec{\alpha}_k + (\vec{H}a_i)\vec{\alpha}_i + a_i\vec{h}_i + a_i h_j(\Gamma^i_{aj} - \Gamma^i_{ja})\vec{\alpha}_a.
\end{aligned}$$

It follows from the structural equation that $[\vec{H}, a_0\vec{\alpha}_0 + a_i\vec{\alpha}_i]$ is contained in $\mathcal{V}_3 \oplus \mathcal{H}_3$. Moreover, if X_1 and X_2 are the $\mathcal{V}_3$ and $\mathcal{H}_3$ parts of $[\vec{H}, a_0\vec{\alpha}_0 + a_i\vec{\alpha}_i]$, respectively, then

$$\pi_*[\vec{H}, X_1] = \pi_*[\vec{H}, X_2].$$

Suppose that $a_i\vec{h}_i + c_a\vec{\alpha}_a$ is contained in $\mathcal{H}_3$. Then it follows from Lemma 11.1 and the characterization of $\mathcal{V}_3$ that

$$\begin{aligned}&\pi_*[\vec{H}, a_i\vec{h}_i + c_a\vec{\alpha}_a]\\ &= (\vec{H}a_i)v_i + a_ih_j(\Gamma^k_{ji} + \Gamma^j_{ki})v_k - a_i\mathbf{J}_{ik}h_0v_k + c_iv_i\end{aligned}$$

and

$$\begin{aligned}&\pi_*[\vec{H}, (\vec{H}a_0)\vec{\alpha}_0 - a_0h_j\mathbf{J}_{jk}\vec{\alpha}_k + (\vec{H}a_i)\vec{\alpha}_i + a_ih_j(\Gamma^i_{aj} - \Gamma^i_{ja})\vec{\alpha}_a - c_a\vec{\alpha}_a]\\ &= -a_0h_j\mathbf{J}_{jk}v_k + (\vec{H}a_i)v_i + a_ih_j(\Gamma^i_{kj} - \Gamma^i_{jk})v_k - c_iv_i\end{aligned}$$

It follows that

$$c_k = a_ih_j\Gamma^i_{kj} + \frac{1}{2}(a_j\mathbf{J}_{jk}h_0 - a_0h_j\mathbf{J}_{jk}).$$

It also follows from this that

$$\begin{aligned}&(\vec{H}a_0 - c_0)\vec{\alpha}_0 - a_0h_j\mathbf{J}_{jk}\vec{\alpha}_k + (\vec{H}a_i)\vec{\alpha}_i + a_ih_j(\Gamma^i_{0j} - \Gamma^i_{j0})\vec{\alpha}_0\\ &+ a_ih_j(\Gamma^i_{kj} - \Gamma^i_{jk})\vec{\alpha}_k - \left(\frac{1}{2}a_j\mathbf{J}_{jk}h_0 - \frac{1}{2}a_0h_j\mathbf{J}_{jk} + a_ih_j\Gamma^i_{kj}\right)\vec{\alpha}_k\\ &= (\vec{H}a_0 - c_0 + a_ih_j\Gamma^i_{0j})\vec{\alpha}_0 + (\vec{H}a_i)\vec{\alpha}_i - a_ih_j\Gamma^i_{jk}\vec{\alpha}_k\\ &- \frac{1}{2}\left(a_j\mathbf{J}_{jk}h_0 + a_0h_j\mathbf{J}_{jk}\right)\vec{\alpha}_k\end{aligned}$$

is contained in $\mathcal{V}_3$. Therefore,

$$\begin{aligned}&2H\left(\vec{H}a_0 - c_0 + a_ih_j\Gamma^i_{0j}\right)\\ &= h_0\left(\vec{H}a_k - \frac{1}{2}a_jh_0\mathbf{J}_{jk} - \frac{1}{2}a_0h_j\mathbf{J}_{jk} - a_ih_j\Gamma^i_{jk}\right)h_k\\ &= h_0\left(\vec{H}a_k\right)h_k - h_0a_i\Gamma^i_{jk}h_jh_k\end{aligned}$$

On the other hand, it follows from (11.2) that

$$h_0h_lh_s\Gamma^s_{lk}a_k + h_0h_k\vec{H}a_k - 2H\vec{H}a_0 = 0.$$

Therefore, $c_0 = a_ih_j\Gamma^i_{0j}$ and this finishes the characterization of $\mathcal{H}_3$.

By the tenth relation in Lemma 11.1 and the structural equation, we can choose a vector in $\mathcal{H}_1$ of the form

$$2H\vec{h}_0 - h_0\vec{H} + r_a\vec{\alpha}_a.$$

Since $\mathcal{H}_1$ is in the skew orthogonal complement of $\mathcal{H}_3$, we have

$$0 = \omega\left(a_i\vec{h}_i + c_a\vec{\alpha}_a, 2H\vec{h}_0 - h_0\vec{H} + r_a\vec{\alpha}_a\right) = r_i a_i.$$

Therefore, by (11.1), we have $r_i = r\mathbf{J}_{ij}h_j$ for some r, where $i = 1, \ldots, 2n$.

Since $\mathcal{H}_2$ is also skew orthogonal to $\mathcal{H}_1$, a tedious computation shows that

$$\begin{aligned} 0 = \omega\Big(&h_0h_k\vec{\alpha}_k - h_j\mathbf{J}_{jk}\vec{h}_k - H\vec{\alpha}_0 - h_jh_l\Gamma^k_{0l}\mathbf{J}_{jk}\vec{\alpha}_0 \\ &- h_jh_l\mathbf{J}_{js}\Gamma^s_{kl}\vec{\alpha}_k, 2H\vec{h}_0 - h_0\vec{H} + r_0\vec{\alpha}_0 + r\mathbf{J}_{ij}h_j\vec{\alpha}_i\Big) = 2rH. \end{aligned}$$

Therefore $r = 0$. Finally, since $2H\vec{h}_0 - h_0\vec{H} + r_0\vec{\alpha}_0$ is in $\mathcal{H}_1$, it follows from the structural equation that

$$\begin{aligned} 0 &= \omega([\vec{H}, 2H\vec{h}_0 - h_0\vec{H} + r_0\vec{\alpha}_0], 2h_0h_k\vec{\alpha}_k - h_j\mathbf{J}_{jk}\vec{h}_k - 2H\vec{\alpha}_0) \\ &= r_0\omega([\vec{H}, \vec{\alpha}_0], 2h_0h_k\vec{\alpha}_k - h_j\mathbf{J}_{jk}\vec{h}_k - 2H\vec{\alpha}_0). \end{aligned}$$

Hence, $r_0 = 0$ and this gives $\mathcal{H}_1$. □

Proof of Lemma 11.1. By the definition of $\vec{h}_a$, we have $\pi_*(\vec{h}_a) = v_a$. Therefore, the first relation follows. The second relation follows from $\pi_*\vec{\alpha}_a = 0$.

Let θ be the tautological 1-form defined by $\theta = p_a dx_a$. Note that $\theta(\vec{h}_a) = h_a$ and $\omega = d\theta$. The third relation follows from

$$\begin{aligned} dh_b(\vec{h}_a) &= d\theta(\vec{h}_a, \vec{h}_b) \\ &= \vec{h}_a(\theta(\vec{h}_b)) - \vec{h}_b(\theta(\vec{h}_a)) - \theta([\vec{h}_a, \vec{h}_b]) \\ &= 2dh_b(\vec{h}_a) - (\Gamma^c_{ab} - \Gamma^c_{ba})h_c. \end{aligned}$$

It is clear that $[\vec{\alpha}_a, \vec{h}_b]$ is vertical. The fourth relation follows from

$$dh_c([\vec{\alpha}_a, \vec{h}_b]) = \vec{\alpha}_a(dh_c(\vec{h}_b)) = (\Gamma^d_{bc} - \Gamma^d_{cb})dh_d(\vec{\alpha}_a) = \Gamma^a_{cb} - \Gamma^a_{bc}.$$

The fifth and sixth relations follow from the fourth one and $\vec{H} = h_i\vec{h}_i$. The seventh follows from the third.

The eighth relation follows from the fifth and the sixth. Indeed,

$$\begin{aligned}
&[\vec{H},[\vec{H},\vec{\alpha}_0]] = -\vec{H}(h_j\mathbf{J}_{jk})\vec{\alpha}_k - h_j\mathbf{J}_{jk}[\vec{H},\vec{\alpha}_k]\\
&= -h_l dh_j(\vec{h}_l)\mathbf{J}_{jk}\vec{\alpha}_k - h_l h_j(v_l\mathbf{J}_{jk})\vec{\alpha}_k - h_j\mathbf{J}_{jk}\left(\vec{h}_k + h_l(\Gamma_{al}^k - \Gamma_{la}^k)\vec{\alpha}_a\right)\\
&= -h_l h_j\Gamma_{ls}^j\mathbf{J}_{sk}\vec{\alpha}_k + h_0 h_k\vec{\alpha}_k - h_l h_j(\Gamma_{lj}^s\mathbf{J}_{sk} + \Gamma_{lk}^s\mathbf{J}_{js})\vec{\alpha}_k\\
&\quad - h_j\mathbf{J}_{jk}\vec{h}_k - H\vec{\alpha}_0 - h_j h_l\Gamma_{0l}^k\mathbf{J}_{jk}\vec{\alpha}_0 - h_j h_l\mathbf{J}_{js}(\Gamma_{kl}^s - \Gamma_{lk}^s)\vec{\alpha}_k\\
&= h_0 h_k\vec{\alpha}_k - h_j\mathbf{J}_{jk}\vec{h}_k - H\vec{\alpha}_0 - h_j h_l\Gamma_{0l}^k\mathbf{J}_{jk}\vec{\alpha}_0 - h_j h_l\mathbf{J}_{js}\Gamma_{kl}^s\vec{\alpha}_k.
\end{aligned}$$

By the fifth relation, we have

$$[\vec{H},[\vec{H},\vec{\alpha}_i]] = [\vec{H},\vec{h}_i] + h_j(\Gamma_{kj}^i - \Gamma_{jk}^i)\vec{h}_k \quad \text{(mod vertical)}$$

Since $\pi_*[\vec{h}_j,\vec{h}_k] = [v_j,v_k]$, the above equation becomes

$$\begin{aligned}
&[\vec{H},[\vec{H},\vec{\alpha}_i]]\\
&= h_l(\Gamma_{li}^a - \Gamma_{il}^a)\vec{h}_a - h_a(\Gamma_{ik}^a - \Gamma_{ki}^a)\vec{h}_k + h_l(\Gamma_{kl}^i - \Gamma_{lk}^i)\vec{h}_k \quad \text{(mod vertical)}\\
&= 2h_l\Gamma_{li}^k\vec{h}_k + h_l\mathbf{J}_{li}\vec{h}_0 - h_0\mathbf{J}_{ik}\vec{h}_k \quad \text{(mod vertical)}.
\end{aligned}$$

Finally, by the sixth relation, we have

$$\begin{aligned}
&[\vec{H},[\vec{H},[\vec{H},\vec{\alpha}_0]]]\\
&= -2\vec{H}(h_j\mathbf{J}_{jk})\vec{h}_k - h_j\mathbf{J}_{jk}[\vec{H},[\vec{H},\vec{\alpha}_k]]\text{(mod vertical)}\\
&= -2h_l dh_j(\vec{h}_l)\mathbf{J}_{jk}\vec{h}_k - 2h_l h_j(v_l\mathbf{J}_{jk})\vec{h}_k\\
&\quad - 2h_l h_j\mathbf{J}_{jk}\Gamma_{lk}^i\vec{h}_i - 2H\vec{h}_0 - h_0\vec{H}\text{(mod vertical)}\\
&= -2h_i h_l\Gamma_{lj}^i\mathbf{J}_{jk}\vec{h}_k - 2h_l h_j(v_l\mathbf{J}_{jk})\vec{h}_k\\
&\quad - 2h_l h_j\mathbf{J}_{jk}\Gamma_{lk}^i\vec{h}_i - 2H\vec{h}_0 + h_0\vec{H}\text{(mod vertical)}\\
&= -2h_l h_j(\mathbf{J}_{ik}\Gamma_{li}^j + \mathbf{J}_{ji}\Gamma_{li}^k + v_l\mathbf{J}_{jk})\vec{h}_k - 2H\vec{h}_0 + h_0\vec{H}\text{(mod vertical)}
\end{aligned}$$

Since the manifold is Sasakian, we have

$$[\vec{H},[\vec{H},[\vec{H},\vec{\alpha}_0]]] = -2H\vec{h}_0 + h_0\vec{H}\text{(mod vertical)}.$$

□

Proof of Lemma 5.3. Since $\pi_*\vec{h}_j = v_j$, $[\vec{h}_k,\vec{h}_i]$ is of the form

$$[\vec{h}_k,\vec{h}_i] = (\Gamma_{ki}^a - \Gamma_{ik}^a)\vec{h}_a + b_{ki}^a\vec{\alpha}_a = \mathbf{J}_{ki}\vec{h}_0 + b_{ki}^a\vec{\alpha}_a$$

at x. By applying both sides by dh_l, we obtain

$$\begin{aligned}
&- b_{ki}^0 = dh_0[\vec{h}_k, \vec{h}_i] \\
&= \vec{h}_k(dh_0(\vec{h}_i)) - \vec{h}_i(dh_0(\vec{h}_k)) \\
&= \vec{h}_k[(\Gamma_{i0}^s - \Gamma_{0i}^s)h_s] - \vec{h}_i[(\Gamma_{k0}^s - \Gamma_{0k}^s)h_s] \\
&= h_0[\mathbf{J}_{ks}\mathbf{J}_{is} - \mathbf{J}_{is}\mathbf{J}_{ks}] + h_s[v_k(\Gamma_{i0}^s - \Gamma_{0i}^s) - v_i(\Gamma_{k0}^s - \Gamma_{0k}^s)] \\
&= h_s[v_k(\Gamma_{i0}^s - \Gamma_{0i}^s) - v_i(\Gamma_{k0}^s - \Gamma_{0k}^s)] = h_s[v_i\Gamma_{0k}^s - v_k\Gamma_{0i}^s]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\mathbf{J}_{ki}\mathbf{J}_{ls}h_s - b_{ki}^l = dh_l[\vec{h}_k, \vec{h}_i] \\
&= \vec{h}_k(dh_l(\vec{h}_i)) - \vec{h}_i(dh_l(\vec{h}_k)) \\
&= \vec{h}_k[(\Gamma_{il}^a - \Gamma_{li}^a)h_a] - \vec{h}_i[(\Gamma_{kl}^a - \Gamma_{lk}^a)h_a] \\
&= -\frac{1}{2}\mathbf{J}_{il}\mathbf{J}_{ks}h_s + \frac{1}{2}\mathbf{J}_{kl}\mathbf{J}_{is}h_s + h_s[v_k(\Gamma_{il}^s - \Gamma_{li}^s) - v_i(\Gamma_{kl}^s - \Gamma_{lk}^s)]
\end{aligned}$$

at x.

It also follows that

$$h_k b_{ki}^0 = h_k h_s v_k(\Gamma_{0i}^s),$$

and

$$h_k b_{ki}^l = -h_s h_k[v_k(\Gamma_{il}^s) - v_k(\Gamma_{li}^s) - v_i(\Gamma_{kl}^s)]$$

at x.

Finally,

$$[\vec{H}, \vec{h}_i] = h_k\mathbf{J}_{ki}\vec{h}_0 - h_0\mathbf{J}_{ik}\vec{h}_k + h_k b_{ki}^a\vec{\alpha}_a.$$

□

12. Appendix III

In this appendix, we provide the proof of Lemmas 3.5, 3.6, and 3.7.

Proof of Lemma 3.5. Let $w_0(t) := v_0(\gamma(t)), w_1(t), \ldots, w_n(t)$ be an orthonormal frame defined along $\gamma(\cdot)$. Let $O(\cdot)$ be a family of $2n \times 2n$ orthogonal matrices and let $K_{ij} = \langle \dot{w}_i(t), w_j(t)\rangle$, and let $v_i(t) := \sum_{j=1}^{2n} O_{ij}(t)w_j(t)$. By differentiating with respect to time t, we have

$$\langle \dot{v}_i(t), v_j(t)\rangle = \sum_{k,l}\left(\dot{O}_{ik}(t) + O_{il}(t)K_{lk}(t)\right) O_{jk}(t).$$

Therefore, by setting $\dot{O}(t) + O(t)K(t) = 0$, we have that $\dot{v}_i$ is vertical. □

Proof of Lemma 3.6. We fix a neighborhood of x on which any point in it can be connected to x by a unique geodesic. We then define v_i to be the vector field on this neighborhood such that $v_i(\gamma(t))$ is a parallel adapted frame along each geodesic $\gamma(\cdot)$ with $\gamma(0) = x$. It follows immediately that $\nabla_{v_k} v_i$ is vertical, where $i = 1, \dots, 2n$ and $k = 0, \dots, 2n$. Therefore,

$$\nabla_{v_k} v_i = \langle \nabla_{v_k} v_i, v_0\rangle\, v_0 = -\langle v_i, \nabla_{v_k} v_0\rangle\, v_0.$$

If $k = 0$, then

$$0 = d\alpha_0(v_0, v_i) = -\alpha_0([v_0, v_i]) = \langle v_0, \nabla_{v_0} v_i\rangle - \langle v_0, \nabla_{v_i} v_0\rangle .$$

Since $|v_0| = 1$, we also have

$$\langle v_0, \nabla_{v_0} v_i\rangle = \langle \nabla_{v_i} v_0, v_0\rangle = 0$$

and hence $\nabla_{v_0} v_i = 0$.

It also follows that $\langle \nabla_{v_0} v_0, v_i\rangle = -\langle v_0, \nabla_{v_0} v_i\rangle = 0$. Therefore, $\nabla_{v_0} v_0 = 0$. The second part follows from $\langle \nabla_{v_i} v_0, v_j\rangle = -\langle \mathbf{J} v_i, v_j\rangle$ for Sasakian manifolds. □

Proof of Lemma 3.7. It is clear that $\Gamma^0_{i0} = 0$. Since $\nabla_{v_0} v_0 = 0$,

$$0 = \langle \nabla_{v_0} v_0, v_i\rangle = \Gamma^i_{00} = -\Gamma^0_{0i} = 0.$$

Since $\mathcal{L}_{v_0} g = 0$,

$$0 = \mathcal{L}_{v_0} g(v_i, v_j) = -\langle v_i, [v_0, v_j]\rangle - \langle [v_0, v_i], v_j\rangle = -\Gamma^0_{ji} - \Gamma^0_{ij}.$$

Since the Riemannian metric is associated to the almost contact structure,

$$\mathbf{J}_{ji} = \langle v_i, \mathbf{J} v_j\rangle = d\alpha_0(v_i, v_j) = -\alpha_0([v_i, v_j]) = -(\Gamma^0_{ij} - \Gamma^0_{ji}) = 2\Gamma^0_{ji}.$$

The third relation follows from the property of the frame $v_0, \dots, v_{2n}$ and Theorem 3.2.

Finally, we have

$$\begin{aligned}\mathrm{Rm}(v_i, v_j)v_k &= \nabla_{v_i}\nabla_{v_j}v_k - \nabla_{v_j}\nabla_{v_i}v_k - \nabla_{[v_i,v_j]}v_k \\ &= \sum_l (v_i\Gamma^l_{jk})v_l + \sum_{l,s}\Gamma^l_{jk}\Gamma^s_{il}v_s - \sum_l (v_j\Gamma^l_{ik})v_l \\ &\quad - \sum_{l,s}\Gamma^l_{ik}\Gamma^s_{jl}v_s - \sum_{l,s}\Gamma^l_{ij}\Gamma^s_{lk}v_s + \sum_{l,s}\Gamma^l_{ji}\Gamma^s_{lk}v_s \\ &= \sum_{s\neq 0}\left((v_i\Gamma^s_{jk}) - (v_j\Gamma^s_{ik}) - \frac{1}{4}\mathbf{J}_{jk}\mathbf{J}_{is} + \frac{1}{4}\mathbf{J}_{ik}\mathbf{J}_{js}\right)v_s\end{aligned}$$

□

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