

# Operations between functions

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A structural theory of operations between real-valued (or extended-real-valued) functions on a nonempty subset  $A$  of  $\mathbb{R}^n$  is initiated. It is shown, for example, that any operation  $*$  on a cone of functions containing the constant functions, which is pointwise, positively homogeneous, monotonic, and associative, must be one of 40 explicitly given types. In particular, this is the case for operations between pairs of arbitrary, or continuous, or differentiable functions. The term *pointwise* means that  $(f * g)(x) = F(f(x), g(x))$ , for all  $x \in A$  and some function  $F$  of two variables. Several results in the same spirit are obtained for operations between convex functions or between support functions. For example, it is shown that ordinary addition is the unique pointwise operation between convex functions satisfying the identity property, i.e.,  $f * 0 = 0 * f = f$ , for all convex  $f$ , while other results classify  $L_p$  addition. The operations introduced by Volle via monotone norms, of use in convex analysis, are shown to be, with trivial exceptions, precisely the pointwise and positively homogeneous operations between nonnegative convex functions. Several new families of operations are discovered. Some results are also obtained for operations that are not necessarily pointwise. Orlicz addition of functions is introduced and a characterization of the Asplund sum is given. With one exception, a full set of examples is provided showing that none of the assumptions made can be omitted.

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## 1. Introduction

Throughout mathematics and wherever it finds applications, there is a need to combine two or more functions to produce a new function. The four basic arithmetic operations, together with composition, are so fundamental that there seems no need to question their existence or utility, for example in calculus. In more advanced mathematics, other operations make their appearance, but are still sometimes tied to simpler operations, as is the case for convolution, which via the Fourier transform becomes multiplication. Of course, a myriad of different operations have been found useful. One such is  $L_p$  addition  $+_p$ , defined for  $f$  and  $g$  in a suitable class of nonnegative functions by

$$(1) \quad (f +_p g)(x) = (f(x)^p + g(x)^p)^{1/p},$$

for  $0 < p < \infty$ , and by  $(f +_{\infty} g)(x) = \max\{f(x), g(x)\}$ . Particularly for  $1 \leq p \leq \infty$ ,  $L_p$  addition is of paramount significance in functional analysis and its many applications. It is natural to extend  $L_p$  addition to  $-\infty \leq p < 0$  by defining

$$(f +_p g)(x) = \begin{cases} (f(x)^p + g(x)^p)^{1/p}, & \text{if } f(x)g(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

when  $-\infty < p < 0$ , and  $(f +_{-\infty} g)(x) = \min\{f(x), g(x)\}$ .

But what is so special about  $L_p$  addition, or, for that matter, ordinary addition? This is a motivating question for our investigation, which focuses on operations  $*$ :  $\Phi(A)^m \rightarrow \Phi(A)$ ,  $m \geq 2$ , where  $\Phi(A)$  is a class of real-valued (or extended-real-valued) functions on a nonempty subset  $A$  of  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ . (See Section 2 for basic definitions and notation.) We offer a variety of answers, usually stating that an operation  $*$  satisfying just a few natural properties must belong to rather special class of operations. What emerges is the beginning of a structural theory of operations between functions.

Our most general results need no restriction on  $\Phi(A)$  other than that it is a cone (i.e.,  $rf \in \Phi(A)$  whenever  $f \in \Phi(A)$  and  $r \geq 0$ ) of real-valued functions, a property enjoyed by the classes of arbitrary, or continuous, or differentiable functions, among many others. For example, in Theorem 7.6, we prove that if  $m = 2$ ,  $\Phi(A)$  is a cone containing the constant functions, and  $*$  is pointwise, positively homogeneous, monotonic, and associative, then  $*$  must be one of 40 types of operations. Of the properties assumed, the last three are familiar; see Section 5 for precise definitions of these and other properties. The first property, *pointwise*, means that there is a function  $F: E \subset \mathbb{R}^m \rightarrow \mathbb{R}$  such that

$$(2) \quad (*f_1, \dots, f_m)(x) = F(f_1(x), \dots, f_m(x)),$$

for all  $f_1, \dots, f_m$  in  $\Phi(A)$  and all  $x \in A$ , where  $*(f_1, \dots, f_m)$  denotes the result of combining the functions  $f_1, \dots, f_m$  via the operation  $*$ . The pointwise property is, to be sure, a quite restrictive one, immediately eliminating composition, for example. Nonetheless, (2) is general enough to admit a huge assortment of operations, and it is surprising that with just three other assumptions the possibilities can be narrowed to a relatively small number.

Behind Theorem 7.6 is a contribution to the theory of associative real-valued functions of two variables: A positively homogeneous, monotonic, and associative function  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  must be one of 40 types explicitly given in

Theorem 4.7. Study of the associativity equation goes back at least to Abel's pioneering work and, with the widespread interest in triangular norms and copulas, and their applications, has generated an extensive literature; see, for example, [3–5, 18]. More specifically, Theorem 4.7 is in a line of results beginning with Bohnenblust's celebrated paper [6] on  $L_p$  spaces and related ones of Aczél [1] on functional equations and Pearson [22] on semiring theory. None of these known results suffice for our purposes, since the latter two assume the continuity of  $F$  and all concern only positively homogeneous and associative functions  $F$  on  $[0, \infty)^2$ . The monotonicity assumption in Theorem 4.7 is crucial, since it appears to be difficult to characterize positively homogeneous, continuous, and associative functions  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

For nonnegative functions, the situation is easier. Theorem 7.4 implies that if  $\Phi(A)$  is a cone of nonnegative functions and  $*$  is pointwise, positively homogeneous, monotonic (or pointwise continuous), and associative, then  $*$  must be one of six types of functions (or three types of functions, respectively), including  $L_p$  addition, for some  $-\infty \leq p \neq 0 \leq \infty$ . Here we are able to use Pearson's results in [22] without too much extra work. Except for  $L_p$  addition, the various types of functions listed in Theorem 7.4 are either rather trivial or trivial modifications of  $L_p$  addition.

The key Lemmas 7.10 and 7.11 provide a link to various results concerning classes of convex functions. Convex functions are of wide interest in their own right and the recent text [7] of Borwein and Vanderwerff is devoted entirely to them. They are the focus of convex analysis, a vast subject that arose from the pioneering work of Fenchel and others and found its proper place with the masterly exposition of Rockafellar [25]. The extraordinarily large number of citations to [25] testifies to the many applications of convex analysis in diverse fields of mathematics, engineering, and economics. Of the key lemmas just mentioned, we confine attention here to Lemma 7.11, which states that a pointwise operation  $* : \Phi(A)^m \rightarrow \text{Cvx}(A)$  must be monotonic, with an associated function  $F$  that is increasing in each variable, when  $\Phi(A)$  is  $\text{Cvx}(A)$ ,  $\text{Cvx}^+(A)$ ,  $\text{Supp}(\mathbb{R}^n)$ , or  $\text{Supp}^+(\mathbb{R}^n)$ . Here  $\text{Cvx}(A)$  is the class of real-valued convex functions on a nontrivial convex set  $A$  in  $\mathbb{R}^n$ ,  $\text{Supp}(\mathbb{R}^n)$  is the class of support functions of nonempty compact convex sets in  $\mathbb{R}^n$ , and the superscript  $+$  denotes the nonnegative functions in these classes. This has several fundamental consequences for convex analysis, which we now describe.

In Theorem 7.15 we prove that  $* : \text{Cvx}(A)^m \rightarrow \text{Cvx}(A)$  is pointwise and positively homogeneous if and only if there is a nonempty compact convex

set  $M \subset [0, \infty)^m$  such that

$$(3) \quad (* (f_1, \dots, f_m))(x) = h_M (f_1(x), \dots, f_m(x)),$$

for all  $f_1, \dots, f_m \in \text{Cvx}(A)$  and all  $x \in A$ , where  $h_M$  is the support function of  $M$ . We call an operation defined by (3), for an arbitrary nonempty compact convex subset  $M$  of  $\mathbb{R}^m$ , *M-addition*, and denote it by  $\oplus_M$ . As far as we know, the operations  $* : \text{Cvx}(A)^m \rightarrow \text{Cvx}(A)$  in Theorem 7.15 have not been considered before. The same theorem also characterizes the pointwise and positively homogeneous operations  $* : \text{Cvx}^+(A)^m \rightarrow \text{Cvx}^+(A)$  as those satisfying (3) for some 1-unconditional compact convex set  $M$  in  $\mathbb{R}^m$ , but in this case, at least for  $m = 2$ , such operations were first introduced by Volle [29]. He observed that if  $\|\cdot\|$  is a monotone norm on  $\mathbb{R}^2$  (i.e.,  $\|(x_1, y_1)\| \leq \|(x_2, y_2)\|$  whenever  $x_1 \leq x_2$  and  $y_1 \leq y_2$ ), then the operation  $(f +_{\|\cdot\|} g)(x) = \|(f(x), g(x))\|$  still preserves the convexity of nonnegative real-valued functions, where (1) corresponds to the  $L_p$  norm. From Theorem 7.15 it is an easy step to Theorem 7.16, which characterizes Volle's operations as being, with trivial exceptions, precisely the pointwise and positively homogeneous operations  $* : \text{Cvx}^+(A)^2 \rightarrow \text{Cvx}^+(A)$ . Moreover, in Theorem 7.17 it is shown, again with trivial exceptions, that any pointwise, positively homogeneous, and associative operation  $* : \text{Cvx}^+(A)^2 \rightarrow \text{Cvx}^+(A)$  must be  $L_p$  addition, for some  $1 \leq p \leq \infty$ . Theorem 7.18 completely characterizes the pointwise, positively homogeneous, and associative operations  $* : \text{Cvx}(A)^2 \rightarrow \text{Cvx}(A)$ ; with trivial exceptions, they are either ordinary addition or defined by

$$(4) \quad (f * g)(x) = \begin{cases} (f +_p g)(x), & \text{if } f(x), g(x) \geq 0, \\ f(x), & \text{if } f(x) \geq 0, g(x) < 0, \\ g(x), & \text{if } f(x) < 0, g(x) \geq 0, \\ -(|f| +_q |g|)(x), & \text{if } f(x), g(x) < 0, \end{cases}$$

for all  $f, g \in \text{Cvx}(A)$  and  $x \in A$  and for some  $1 \leq p \leq \infty$  and  $-\infty \leq q \leq 0$ . Here Theorem 4.7 is used in an essential way, the function  $F$  associated with the operation  $*$  defined by (4) (with  $m = 2$ ) being one of the 40 listed in that result. Again, it appears that these operations have not been considered before. In Theorem 7.21 we give a somewhat surprising characterization of ordinary addition by proving it to be the unique pointwise operation  $* : \text{Cvx}(A)^2 \rightarrow \text{Cvx}(A)$  satisfying the identity property, i.e.,  $f * 0 = 0 * f = f$ , for all  $f \in \text{Cvx}(A)$ . Remark 7.23 explains why there does not seem to be a natural version of Theorem 7.21 that applies to the class  $\text{Cvx}^+(A)$ .

All the results in the previous paragraph have counterparts for operations  $* : \text{Supp}(\mathbb{R}^n)^m \rightarrow \text{Supp}(\mathbb{R}^n)$  or  $* : \text{Supp}^+(\mathbb{R}^n)^m \rightarrow \text{Supp}^+(\mathbb{R}^n)$ ; indeed, the same results hold verbatim, if the condition of positive homogeneity is omitted. Such operations can be transferred in a natural manner to operations between compact convex sets, so they are in part anticipated by work of Gardner, Hug, and Weil [12], of which the present paper can be regarded as a sequel. Even in this context, however, parts (c) and (d) of Theorem 7.20 are new, giving a partial answer to the still unresolved question of the role of associativity in classifying operations between arbitrary compact convex sets.

In convex analysis it is essential to work not only with real-valued functions but also with extended-real-valued functions. We undertake this task in Section 9. Basically, all our main results from Section 7 go through for operations between functions in the classes  $\overline{\text{Cvx}}(A)$ ,  $\overline{\text{Cvx}}^+(A)$ ,  $\overline{\text{Supp}}(\mathbb{R}^n)$ , and  $\overline{\text{Supp}}^+(\mathbb{R}^n)$ , though of course further operations do arise in the analysis. Here  $A$  is a nontrivial convex subset of  $\mathbb{R}^n$ ,  $\overline{\text{Cvx}}(A)$  represents the class of extended-real-valued convex functions  $f : A \rightarrow (-\infty, \infty]$ ,  $\overline{\text{Supp}}(\mathbb{R}^n)$  is the class of extended-real-valued support functions of nonempty closed convex sets in  $\mathbb{R}^n$ , and the superscript  $+$  signifies the nonnegative members of these classes.

We stress that for each of the previously described results, and indeed those throughout the paper (with one exception; see the comment after Corollary 10.9), we provide a full set of examples showing that none of the assumptions we make can be omitted. In particular, the assumption that the operations are pointwise is essential. Nevertheless, in certain circumstances it is possible to classify operations that are not necessarily pointwise, *provided they are associative*. Our inspiration here is the work of Milman and Rotem [20] on operations between closed convex sets, and we lean heavily on their methods to achieve our results. In Theorem 10.5, we prove that any operation  $* : \overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$  or  $* : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$  that is monotonic, associative, weakly homogeneous, and has the identity and  $\delta$ -finite properties, must be  $L_p$  addition, for some  $1 \leq p \leq \infty$ . The  $\delta$ -finite property is a weak technical condition defined in (55) and (56) below. Weak homogeneity is introduced here for the first time and serves two purposes: It directly relates to (and is much weaker than) positive homogeneity and it allows us to avoid the slightly artificial “homothety” property used in [20]. Indeed, Theorem 10.3 states that in the presence of monotonicity and the identity property, the homothety property implies weak homogeneity. In Theorem 10.8 we establish a corresponding result for

operations  $*$  :  $\text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$ , which immediately yields Corollary 10.9, a characterization of  $L_p$  addition as an operation between closed convex (or compact convex) sets containing the origin, that strengthens [20, Theorems 2.2 and 6.1]. As in [20], the proofs of the latter results require in addition the weak assumption that  $*$  is (pointwise) continuous from below when dealing with compact convex sets.

Returning to pointwise operations, we briefly mention two other contributions. The first is the introduction in Section 8 of Orlicz addition between functions. This is motivated by recent developments in the Brunn-Minkowski theory, the heart of convex geometry, in which Orlicz addition of sets, a generalization of  $L_p$  addition of sets, was recently discovered; see [13, 14, 30, 31]. Orlicz addition, denoted by  $+_\varphi$  and defined by (42) below, is an operation  $+_\varphi : \Phi(A)^m \rightarrow \Phi(A)$  in several useful instances, for example when  $\Phi(A)$  is the class of nonnegative Borel or nonnegative continuous functions on  $A$ , or  $\text{Cvx}^+(A)$ , or  $\text{Supp}^+(\mathbb{R}^n)$ . It has the remarkable features that the function  $F$  associated with  $+_\varphi$  (as in (2)) is implicit and that when  $m = 2$ ,  $+_\varphi$  is in general neither commutative nor associative.

The second contribution referred to above is Theorem 11.1, a characterization of the so-called Asplund sum among operations between log-concave functions on  $\mathbb{R}^n$ .

The paper is organized as follows. After Sections 2 and 3 on preliminaries and background results, Section 4 is primarily devoted to proving Theorem 4.7, our classification of positively homogeneous, monotonic, and associative functions  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ . The proof is of necessity long and technical and we recommend skipping it on a first reading. Most of the properties of operations we employ are listed and defined in Section 5, while Section 6 defines the most important operations for our investigation. Pointwise operations between real-valued functions are the focus of Section 7, where many of the results described above can be found. Section 8 introduces Orlicz addition of functions. In Section 9, we characterize pointwise operations between extended-real-valued functions. Section 10 is concerned with operations that are not necessarily pointwise and the short Section 11 records our characterization of the Asplund sum.

## 2. Preliminaries

As usual,  $S^{n-1}$  denotes the unit sphere and  $o$  the origin in Euclidean  $n$ -space  $\mathbb{R}^n$ . The standard orthonormal basis for  $\mathbb{R}^n$  will be  $\{e_1, \dots, e_n\}$ . Otherwise, we usually denote the coordinates of  $x \in \mathbb{R}^n$  by  $x_1, \dots, x_n$ . We write  $[x, y]$  for the line segment with endpoints  $x$  and  $y$ .

The *extended real numbers* are the elements of the set  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ .

If  $X$  is a set in  $\mathbb{R}^n$ , we denote by  $\text{cl } X$  and  $\text{relint } X$  the *closure* and *relative interior* of  $X$  (with respect to its affine hull). If  $S$  is a subspace of  $\mathbb{R}^n$ , then  $X|S$  is the (orthogonal) projection of  $X$  on  $S$ .

If  $t \in \mathbb{R}$ , then  $tX = \{tx : x \in X\}$ . The set  $-X = (-1)X$  is the *reflection* of  $X$  in the origin.

A set  $X$  is a *cone* if  $rx \in X$  whenever  $x \in X$  and  $r \geq 0$ .

A set is *o-symmetric* if it is centrally symmetric, with center at the origin. We shall call a set in  $\mathbb{R}^n$  *1-unconditional* if it is symmetric with respect to each coordinate hyperplane; this is traditional in convex geometry for compact convex sets.

Let  $\mathcal{K}^n$  be the class of nonempty compact convex subsets of  $\mathbb{R}^n$ , let  $\mathcal{K}_s^n$  denote the class of *o*-symmetric members of  $\mathcal{K}^n$  and let  $\mathcal{K}_o^n$  be the class of members of  $\mathcal{K}^n$  containing the origin. A set  $K \in \mathcal{K}^n$  is called a *convex body* if its interior is nonempty and *nontrivial* if it contains more than one point. We shall also need the notation  $\mathcal{CC}^n$  and  $\mathcal{CC}_o^n$  for the classes of nonempty closed (not necessarily bounded) convex sets in  $\mathbb{R}^n$  and those containing the origin.

If  $K \in \mathcal{CC}^n$ , then

$$(5) \quad h_K(x) = \sup\{x \cdot y : y \in K\},$$

for  $x \in \mathbb{R}^n$ , is its *support function*, which determines  $K$  uniquely. Support functions are *positively homogeneous*, that is,

$$h_K(rx) = rh_K(x),$$

for all  $x \in \mathbb{R}^n$  and  $r \geq 0$ , and are therefore often regarded as functions on  $S^{n-1}$ . They are also *subadditive*, i.e.,

$$h_K(x + y) \leq h_K(x) + h_K(y),$$

for all  $x, y \in \mathbb{R}^n$ . Any real-valued function on  $\mathbb{R}^n$  that is *sublinear*, that is, both positively homogeneous and subadditive, is the support function of a unique compact convex set. Proofs of these facts can be found in [27]. Gruber's book [15] is also a good general reference for convex sets.

A set  $K$  in  $\mathbb{R}^n$  is *star-shaped at o* if  $o \in K$  and for each  $x \in \mathbb{R}^n \setminus \{o\}$ , the intersection  $K \cap \{cx : c \geq 0\}$  is a (possibly degenerate) compact line segment. If  $K$  is star-shaped at  $o$ , we define its *radial function*  $\rho_K$  for  $x \in$



$\mathbb{R}^n \setminus \{o\}$  by

$$\rho_K(x) = \max\{c \geq 0 : cx \in K\}.$$

This definition is a slight modification of [11, (0.28)]; as defined here, the domain of  $\rho_K$  is always  $\mathbb{R}^n \setminus \{o\}$ . Radial functions are *homogeneous of degree  $-1$* , that is,

$$\rho_K(rx) = r^{-1}\rho_K(x),$$

for all  $x \in \mathbb{R}^n \setminus \{o\}$  and  $r > 0$ , and are therefore often regarded as functions on the unit sphere  $S^{n-1}$ . Conversely, any nonnegative and homogeneous of degree  $-1$  function on  $\mathbb{R}^n \setminus \{o\}$  is the radial function of a unique subset of  $\mathbb{R}^n$  that is star-shaped at  $o$ .

A *star set* in  $\mathbb{R}^n$  is a bounded Borel set that is star-shaped at  $o$ . We denote the class of star sets in  $\mathbb{R}^n$  by  $\mathcal{S}^n$  and the class of  $o$ -symmetric members of  $\mathcal{S}^n$  by  $\mathcal{S}_s^n$ . Note that  $\mathcal{S}^n$  is closed under finite unions, countable intersections, and intersections with subspaces. Also, if a set  $K$  in  $\mathbb{R}^n$  is star-shaped at  $o$ , then  $K \in \mathcal{S}^n$  if and only if  $\rho_K$ , restricted to  $S^{n-1}$ , is a bounded Borel-measurable function. Our definitions and notation differ from those used elsewhere, such as [11, Section 0.7].

We denote the class of real-valued convex functions on a set  $A$  by  $\text{Cvx}(A)$ , while  $\overline{\text{Cvx}}(A)$  represents the class of extended-real-valued convex functions  $f : A \rightarrow (-\infty, \infty]$ . In these contexts, it will always be assumed that  $A$  is a nontrivial convex set in  $\mathbb{R}^n$ . The class of support functions of sets in  $\mathcal{K}^n$  (or extended-real-valued support functions of sets in  $\mathcal{CC}^n$ ) will be denoted by  $\text{Supp}(\mathbb{R}^n)$  (or  $\overline{\text{Supp}}(\mathbb{R}^n)$ , respectively). A superscript  $+$  signifies the nonnegative members of these classes, so that  $\text{Supp}^+(\mathbb{R}^n)$  is the class of support functions of sets in  $\mathcal{K}_o^n$ .

We define a number of functions from  $\mathbb{R}^2$  to  $\mathbb{R}$  that will play an important role. The coordinate projections are denoted by

$$\Pi_1(s, t) = s \quad \text{and} \quad \Pi_2(s, t) = t.$$

We set

$$M_p(s, t) = \text{sgn} \left( \text{sgn}(s)|s|^p + \text{sgn}(t)|t|^p \right) \left| \text{sgn}(s)|s|^p + \text{sgn}(t)|t|^p \right|^{1/p}$$

when  $0 < p < \infty$ , and

$$M_p(s, t) = \begin{cases} \frac{1}{2}(\text{sgn}(s) + \text{sgn}(t))(|s|^p + |t|^p)^{1/p}, & \text{if } st \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

when  $-\infty < p < 0$ ,  $s, t \in \mathbb{R}$ . Let

$$M_\infty(s, t) = \begin{cases} s, & \text{if } -|s| \leq t \leq |s| \\ t, & \text{otherwise,} \end{cases}$$

and

$$M_{-\infty}(s, t) = \frac{1}{2}(\operatorname{sgn}(s) + \operatorname{sgn}(t)) \min\{|s|, |t|\},$$

for all  $s, t \in \mathbb{R}$ . Finally, it will be convenient to define  $M_0(s, t) = 0$ , though this notation is often used for the geometric mean of  $s$  and  $t$ .

Note that on  $[0, \infty)^2$ , we have  $M_p(s, t) = (s^p + t^p)^{1/p}$ , for  $p > 0$ ,  $M_\infty(s, t) = \max\{s, t\}$ , and  $M_{-\infty}(s, t) = \min\{s, t\}$ . For  $p < 0$ , the first equality still holds on  $(0, \infty)^2$ . On  $(-\infty, 0]^2$ , we have  $M_p(s, t) = -(|s|^p + |t|^p)^{1/p}$ , for  $p > 0$ ,  $M_\infty(s, t) = \min\{s, t\}$ , and  $M_{-\infty}(s, t) = \max\{s, t\}$ . For  $p < 0$ , the previous formula for  $M_p$  holds on  $(-\infty, 0)^2$ .

### 3. Background results

The following properties of functions  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  have been considered by other authors (see, for example, [12]):

- (i) (Positive homogeneity)  $F(rs, rt) = rF(s, t)$ , for  $r, s, t \geq 0$ .
- (ii) (Increasing in each variable)  $F(s, t) \leq F(s', t')$ , for  $0 \leq s \leq s'$  and  $0 \leq t \leq t'$ .
- (iii) (Symmetry)  $F(s, t) = F(t, s)$ .
- (iv)  $F(0, 1) = 1$ ;
- (v) (Associativity)  $F(s, F(t, u)) = F(F(s, t), u)$ , for  $s, t, u \geq 0$ .

Here we introduce the following weaker version of (i):

- (vi) (Weak homogeneity) There is a set  $Q$ , dense in  $[0, \infty)$  and containing 0, such that  $F(rs, rt) = rF(s, t)$ , for  $r \geq 0$  and  $s, t \in Q$ .

We use the same labels for the corresponding properties of functions  $F : (0, \infty)^2 \rightarrow \mathbb{R}$ . The associativity equation (v) has generated a large literature; see, for example, [3], [4], and [5]. The following result is due to Bohnenblust [6].

**Proposition 3.1.** *If  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  satisfies properties (i)–(v), then  $F = M_p$  on  $[0, \infty)^2$ , for some  $0 < p \leq \infty$ .*

In [9, Theorem 4], Fleming states: *The conclusion of Bohnenblust's theorem remains true even with condition (iii) of the hypotheses removed.* He

means to say also that (iv) should be replaced by  $F(1, 0) = F(0, 1) = 1$  (or else the function  $F(s, t) = t$  for all  $s, t \geq 0$  would be a counterexample). Fleming ascribes this result to B. Randrianantoanina in a personal communication. See also [10, Theorem 9.5.3].

For  $s, t \geq 0$ , let

$$\begin{aligned} G_1(s, t) &= \log(e^s + e^t - 1); \\ G_2(s, t) &= \begin{cases} \min\{s, t\}, & \text{if } s > 0 \text{ and } t > 0, \\ \max\{s, t\}, & \text{if } s = 0 \text{ or } t = 0; \end{cases} \\ G_3(s, t) &= t = \Pi_2(s, t); \\ G_4(s, t) &= \min\{s, t\} = M_{-\infty}(s, t); \\ G_5(s, t) &= s + t + \sqrt{st}. \end{aligned}$$

Then one can check that for  $i = 1, \dots, 5$ , the function  $G_i(s, t)$  satisfies all but the  $i$ th of properties (i)–(v).

There are some related results that weaken some of the above assumptions but require that  $F$  is continuous. One of them is due to Aczél [2, Theorem 2]. He shows that if  $F$  is continuous and satisfies only (i), (ii) (but with *strict* inequalities), and (v), then  $F = M_p$  on  $[0, \infty)^2$ , for some  $0 < p \leq \infty$ .

The following result, stronger than Aczél’s, was proved by Pearson [22, Theorem 2] in a paper on topological semigroups.

**Proposition 3.2.** *Let  $F : [0, \infty)^2 \rightarrow [0, \infty)$  be a continuous function satisfying (i) and (v). Then on  $[0, \infty)^2$ , either  $F = \Pi_1$ , or  $F = \Pi_2$ , or  $F = M_p$ , for some  $-\infty \leq p \leq \infty$ .*

The functions  $G_1$  and  $G_5$  above are continuous and show that (i) and (v) are necessary in the previous proposition, while  $G_2$  shows that the continuity hypothesis cannot be omitted. Note that Proposition 3.2 also implies that any function  $F : [0, \infty)^2 \rightarrow [0, \infty)$  that satisfies (i) and (v), but not (ii), cannot be continuous. Indeed, if such a function were continuous, it would have to be one of the possibilities given by Proposition 3.2, but each of these satisfies (ii).

The first conclusion of the following proposition (see [12, p. 3307]) shows that Proposition 3.2 implies Aczél’s result mentioned above, since it implies that if  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  satisfies (i) (and hence (vi)) and (ii), then  $F$  is non-negative.

**Proposition 3.3.** *Suppose that  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  satisfies (ii) and (vi) (or  $F : (0, \infty)^2 \rightarrow [0, \infty)$  satisfies (ii) and (vi) for  $r > 0$ ). Then*

- (a)  $F \geq 0$  on  $[0, \infty)^2$ .
- (b)  $F$  is continuous on  $(0, \infty)^2$ , and positive on  $(0, \infty)^2$  unless it is identically zero.
- (c)  $F$  satisfies (i) (or (i) for  $r > 0$ , respectively).

*Proof.* Suppose that  $F : [0, \infty)^2 \rightarrow \mathbb{R}$  satisfies (ii) and (vi).

(a) If  $s \in Q$ , then by (vi),  $F(0, 0) = F(0 \cdot s, 0 \cdot s) = 0 \cdot F(s, s) = 0$ . From (ii), it then follows that  $F \geq 0$ .

(b) Let  $s_0, t_0 > 0$  and fix  $0 < \varepsilon_0 < \min\{s_0, t_0\}$ . Choose  $s'_0 \in (s_0 - \varepsilon_0/2, s_0) \cap Q$  and  $t'_0 \in (t_0 - \varepsilon_0/2, t_0) \cap Q$ . Let  $0 \leq \varepsilon < \varepsilon_0$ ,  $s \in (s_0 - \varepsilon/2, s_0 + \varepsilon/2)$ , and  $t \in (t_0 - \varepsilon/2, t_0 + \varepsilon/2)$ . Using (ii), the fact that  $\max\{1/s'_0, 1/t'_0\} \leq 2/\varepsilon_0$ , and (vi), we obtain

$$\begin{aligned} F(s, t) &\leq F(s'_0 + \varepsilon, t'_0 + \varepsilon) \leq F(s'_0(1 + 2\varepsilon/\varepsilon_0), t'_0(1 + 2\varepsilon/\varepsilon_0)) \\ &= (1 + 2\varepsilon/\varepsilon_0)F(s'_0, t'_0) \leq (1 + 2\varepsilon/\varepsilon_0)F(s_0, t_0). \end{aligned}$$

Similarly,  $F(s, t) \geq (1 - 2\varepsilon/\varepsilon_0)F(s_0, t_0)$  and hence

$$|F(s, t) - F(s_0, t_0)| \leq \frac{2}{\varepsilon_0} F(s_0, t_0) \varepsilon.$$

Therefore  $F$  is continuous at  $(s_0, t_0)$ .

To prove the remaining claim in (b), suppose that  $F(s_1, t_1) = 0$  for some  $s_1, t_1 > 0$ . By (ii), there are  $s'_1 \in (0, s_1) \cap Q$  and  $t'_1 \in (0, t_1) \cap Q$  with  $F(s'_1, t'_1) = 0$ . Let  $s, t \geq 0$  and choose  $r > 0$  so that  $rs \leq s'_1$  and  $rt \leq t'_1$ . Then, by (ii) and (vi), we have  $F(s, t) \leq F(s'_1/r, t'_1/r) = F(s'_1, t'_1)/r = 0$ . Therefore  $F$  is identically zero.

(c) As  $F$  is continuous on  $(0, \infty)^2$  by (b) and  $Q \cap (0, \infty)$  is dense in  $(0, \infty)$ , (vi) implies  $F(rs, rt) = rF(s, t)$ , for all  $r, s, t > 0$ . With arguments similar to those used to prove (b), it can be shown that  $s \mapsto F(s, 0)$  is continuous on  $(0, \infty)$ , where  $0 \in Q$  is used. This and (vi) show that  $F(rs, 0) = rF(s, 0)$ , for all  $r, s > 0$ . Similarly,  $F(0, rt) = rF(0, t)$ , for all  $r, t > 0$ . Together with the fact  $F(0, 0) = 0$ , shown in (a), this yields the positive homogeneity of  $F$ .

Now suppose that  $F : (0, \infty)^2 \rightarrow [0, \infty)$  satisfies (ii) and (vi) for  $r > 0$ . Part (a) is trivially true, and (b) and (c) follow from the proofs given above.  $\square$

Define  $F : [0, \infty)^2 \rightarrow [0, \infty)$  by  $F(s, t) = s = \Pi_1(s, t)$  on  $(0, \infty)^2$  and  $F(s, t) = 0$ , otherwise. Then  $F$  satisfies (i), (ii), and (v), but it is not continuous on the positive  $s$ -axis. This shows that it is not possible to conclude from the hypotheses of Proposition 3.3 that  $F : [0, \infty)^2 \rightarrow [0, \infty)$  is continuous, even if it is also assumed that  $F$  is associative.

Pearson [22, Theorem 1] also proved the following result.

**Proposition 3.4.** *Let  $F : (0, \infty)^2 \rightarrow (0, \infty)$  be a continuous function satisfying (i) and (v) for  $r, s, t, u > 0$ . Then on  $(0, \infty)^2$ , either  $F = \Pi_1$ , or  $F = \Pi_2$ , or  $F = M_p$ , for some  $-\infty \leq p \neq 0 \leq \infty$ .*

Proposition 3.4 implies Bohnenblust’s Proposition 3.1, in its stronger form with (iii) removed and (iv) replaced by  $F(1, 0) = F(0, 1) = 1$ . To see this, let  $F : [0, \infty)^2 \rightarrow [0, \infty)$  satisfy the hypotheses of this version of Proposition 3.1. Since  $F$  is not identically zero, Proposition 3.3(b) shows that the restriction of  $F$  to  $(0, \infty)^2$  is continuous and positive, and it follows from Proposition 3.4 that this restriction must be of one of the forms given there. Then the continuity of  $F$  on  $[0, \infty)^2$  and  $F(1, 0) = F(0, 1) = 1$  eliminate all the functions listed in Proposition 3.4 except for those satisfying  $F = M_p$ , for some  $0 < p \leq \infty$ .

We conclude that if  $F : (0, \infty)^2 \rightarrow (0, \infty)$  satisfies (i) and (v), then (ii) is equivalent to the continuity of  $F$ . This follows from Proposition 3.4 and the arguments that lead to continuity in Proposition 3.3(b). The example given after Proposition 3.3 shows that the continuity of a function  $F : [0, \infty)^2 \rightarrow [0, \infty)$  satisfying (i) and (v) is no longer equivalent to (ii).

We close this section with another result of Aczél [1] (see also [3, Section 6.2]), which, with stronger other assumptions, allows the positive homogeneity to be dropped.

**Proposition 3.5.** *Let  $D$  be an interval in  $\mathbb{R}$ , finite or infinite, which is open on one side. A function  $F : D^2 \rightarrow D$  is continuous, strictly increasing in each variable, and associative if and only if there exists a continuous and strictly monotonic function  $\phi : D \rightarrow \mathbb{R}$  such that for all  $s, t \in D$ ,*

$$(6) \quad F(s, t) = \phi^{-1}(\phi(s) + \phi(t)).$$

#### 4. Positively homogeneous, increasing, and associative functions

Properties (i)–(v) are as listed at the beginning of Section 3.

**Lemma 4.1.** *If  $F : [0, \infty)^2 \rightarrow [0, \infty)$  satisfies (i) and (v), then either  $F(s, 0) = 0$ , for all  $s \geq 0$ , or  $F(s, 0) = s$ , for all  $s \geq 0$ . Also, either  $F(0, t) = 0$ , for all  $t \geq 0$ , or  $F(0, t) = t$ , for all  $t \geq 0$ .*

*Proof.* If  $F(s, 0) = 0$ , for some  $s > 0$ , then (i) implies  $F(s, 0) = 0$ , for all  $s \geq 0$ . Hence we may assume that  $F(s, 0) = a$ , for some  $a > 0$  and  $s > 0$ . By (i), we have  $F(0, 0) = 0$  and (v) yields

$$F(a, 0) = F(F(s, 0), 0) = F(s, F(0, 0)) = F(s, 0) = a.$$

By (i) again, it follows that  $F(s, 0) = s$ , for all  $s \geq 0$ . The argument for  $F(0, t)$  is similar.  $\square$

**Corollary 4.2.** *Let  $F : [0, \infty)^2 \rightarrow [0, \infty)$  satisfy (i), (ii), and (v). Then on  $[0, \infty)^2$ ,  $F$  is one of the following six types of functions:  $\Pi_1$ , or  $\Pi_2$ , or  $M_p$ , for some  $-\infty \leq p \leq \infty$ , or*

$$F(s, t) = \begin{cases} \Pi_1(s, t), & \text{if } t > 0, \\ 0, & \text{if } t = 0, \end{cases} \quad \text{or} \quad F(s, t) = \begin{cases} \Pi_2(s, t), & \text{if } s > 0, \\ 0, & \text{if } s = 0, \end{cases}$$

or

$$F(s, t) = \begin{cases} M_p(s, t), & \text{if } s, t > 0, \\ 0, & \text{if } s = 0 \text{ or } t = 0, \end{cases}$$

for some  $0 < p \leq \infty$ .

*Proof.* Assume  $F \neq M_0$ , i.e.,  $F$  is nonzero on  $[0, \infty)^2$ . By Proposition 3.3(b), the restriction of  $F$  to  $(0, \infty)^2$  is positive and continuous, so by Proposition 3.4, it must be of one of the forms listed there. Also, by Lemma 4.1, either  $F(s, 0) = 0$ , for all  $s \geq 0$ , or  $F(s, 0) = s$ , for all  $s \geq 0$  and in addition either  $F(0, t) = 0$ , for all  $t \geq 0$ , or  $F(0, t) = t$ , for all  $t \geq 0$ . Considering all possibilities and bearing in mind that  $F$  must be increasing in each variable and associative, we conclude that  $F$  must be one of the functions listed in the statement of the corollary.  $\square$

The functions  $G_1$ ,  $G_2$ , and  $G_5$  defined after Proposition 3.1 show that none of the assumptions (i), (ii), and (v) in the previous corollary can be omitted.

Pearson [22, Theorem 3] also proves a version of Proposition 3.2 for continuous, homogeneous functions  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  that satisfy the associativity property (v) on  $\mathbb{R}$ . Here homogeneous means that  $F(rs, rt) = rF(s, t)$  for all  $s, t \in \mathbb{R}$  and all  $r \in \mathbb{R}$ . He proves that such a function must be one of 10

types of functions, essentially variants of those given in Proposition 3.2. We prove a variant of this result, where the associative function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is assumed to be increasing in each variable, but not necessarily continuous, and the homogeneity condition is replaced by positive homogeneity. More precisely, we work with the following conditions:

- (i')  $F(rs, rt) = rF(s, t)$ , for  $r \geq 0$  and  $s, t \in \mathbb{R}$ ;
- (ii')  $F(s, t) \leq F(s', t')$  for  $s \leq s'$  and  $t \leq t'$ ;
- (v')  $F(s, F(t, u)) = F(F(s, t), u)$  for  $s, t, u \in \mathbb{R}$ .

The functions  $\Pi_1, \Pi_2$ , and  $M_p$ ,  $-\infty \leq p \leq \infty$ , all satisfy (i'), (ii'), and (v'). However, there are also non-continuous functions satisfying (i'), (ii'), and (v'). A complete list will be given in Theorem 4.7, the proof of which requires several lemmas. It will be convenient to denote by  $Q_i$ ,  $i = 1, \dots, 4$ , the  $i$ th open quadrant of  $\mathbb{R}^2$ .

**Lemma 4.3.** *Suppose that  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies (i'), (ii'), and (v'). Then on  $Q_1$ ,  $F$  is either  $\Pi_1$ , or  $\Pi_2$ , or  $M_p$ , for some  $-\infty \leq p \leq \infty$ . Moreover, on  $Q_3$ ,  $F$  is either  $\Pi_1$ , or  $\Pi_2$ , or  $M_q$ , for some  $-\infty \leq q \leq \infty$ .*

*Proof.* In view of (ii'), the restriction of  $F$  to  $\text{cl } Q_1$  is bounded from below by  $F(0, 0) = 0$ . Hence, by Proposition 3.3(b), on  $Q_1$ ,  $F$  is either  $M_0$  or it is a continuous mapping into  $(0, \infty)$  satisfying (i) and (v). Proposition 3.4 now yields the desired result.

The second statement is obtained by applying this to  $-F(-s, -t)$  and using  $-\Pi_i(-s, -t) = \Pi_i(s, t)$ ,  $i = 1, 2$ , and  $-M_p(-s, -t) = M_p(s, t)$ , for all  $s, t > 0$ . □

**Lemma 4.4.** *Let  $i \in \{1, \dots, 4\}$ . If  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies (i'), (ii'), and (v'), and  $F = 0$  on  $Q_i$ , then  $F = 0$  on  $\text{cl } Q_i$ .*

*Proof.* Suppose that  $F = 0$  on  $Q_1$ . Then, for  $s \geq 0$ , we have by (i') and (ii') that

$$0 = F(0, 0) \leq F(s, 0) \leq F(s, 1) = 0.$$

Hence,  $F(s, 0) = 0$ , for all  $s \geq 0$ . In a similar way one shows  $F(0, t) = 0$  for all  $t \geq 0$ . This proves the lemma for  $i = 1$ . A similar argument disposes of the case  $i = 3$ .

Suppose that  $F = 0$  on  $Q_4$ . If  $F = 0$  on  $Q_1$ , then  $F(s, 0) = 0$ , for all  $s \geq 0$ , as we have seen. Otherwise, by Lemma 4.3, we have  $F(s, 1) > 0$  for

all  $s > 0$ . Therefore

$$F(s, 0) = F(s, F(1, -1)) = F(F(s, 1), -1) = 0,$$

for all  $s \geq 0$ . This, together with the same argument applied to  $-F(-t, -s)$ , proves the lemma for  $i = 4$ , and the case  $i = 2$  follows similarly.  $\square$

Next, we determine  $F$  in the part of the fourth quadrant where it is positive, based on its behavior in the first quadrant. Applying the following lemma to  $F(t, s)$ , analogous results are obtained for the second quadrant. In a similar way, an application to  $-F(-s, -t)$  yields properties of  $F$  on the second and fourth quadrant given its values in the third quadrant.

**Lemma 4.5.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy (i'), (ii'), and (v'), and let  $s > 0$ .*

- (a) *If  $F = M_0$  on  $Q_1$ , then  $F(s, t) = 0$ , for all  $t \leq 0$ , or  $F(s, t) = t$ , for all  $t \leq 0$ .*
- (b) *If  $F = M_p$  on  $Q_1$ , for some  $0 < p < \infty$ , and  $F > 0$  somewhere in  $Q_4$ , then there is a constant  $c \geq 0$ , independent of  $s$ , with*

$$(7) \quad F(s, t) = (s^p - c(-t)^p)^{1/p},$$

*for all  $t \leq 0$  such that  $F(s, t) \geq 0$ .*

- (c) *If  $F \neq M_p$  on  $Q_1$  for any  $0 \leq p < \infty$ , and  $F > 0$  somewhere in  $Q_4$ , then  $F(s, t) = s$ , for all  $t \leq 0$  such that  $F(s, t) \geq 0$ .*

*Proof.* By (i'), we may assume without loss of generality that  $s = 1$ .

To prove (a), we apply Lemma 4.1 to  $-F(-s, -t)$  on  $[0, \infty)^2$  to conclude that either  $F(0, t) = t$ , for all  $t \leq 0$ , or  $F(0, t) = 0$  for all  $t \leq 0$ . In the latter case, we obtain

$$0 = F(0, t) \leq F(1, t) \leq F(1, 1) = 0,$$

and thus  $F(1, t) = 0$ , for all  $t \leq 0$ . Suppose that  $F(0, t) = t$ , for all  $t \leq 0$ . If  $t \leq 0$ , then (i'), (v'),  $F(1, t) \leq F(1, 1) = 0$ , and Lemma 4.4 with  $i = 1$  imply that

$$\begin{aligned} -F(1, t)^2 &= (-F(1, t)) F(1, t) = F(-F(1, t), F(1, t)(-t)) \\ &= F(-F(1, t), F(-t, -t^2)) = F(F(-F(1, t), -t), -t^2) \\ &= F(0, -t^2) = -t^2. \end{aligned}$$

As  $F(1, t) \leq 0$ , this gives  $F(1, t) = t$ , for all  $t \leq 0$ , as required.



To prove (b), suppose that  $F = M_p$  on  $Q_1$ , for some  $0 < p < \infty$ . If  $F > 0$  somewhere in  $Q_4$ , Lemma 4.1 and (ii') imply that  $F = M_p$  on  $(0, \infty) \times [0, \infty)$ . This yields (7) when  $t = 0$ . For  $t < 0$  such that  $F(1, t) \geq 0$  and fixed  $s > 0$  it gives

$$(s^p + F(1, t)^p)^{1/p} = F(s, F(1, t)) = F(F(s, 1), t) = F(s, 1)F(1, F(s, 1)^{-1}t).$$

Taking  $s = ((t/t')^p - 1)^{1/p}$ , for some  $t < t' < 0$ , we obtain

$$(8) \quad F(1, t) = (1 - h(t')(-t)^p)^{1/p},$$

where  $h(t') = (1 - F(1, t')^p)/|t'|^p$ . As this holds whenever  $t < t' < 0$ ,  $h$  cannot depend on  $t'$  and must be a constant,  $c$ , say. By Lemma 4.1,  $c \geq (1 - F(1, 0)^p)/|t'|^p \geq 0$ . Therefore (b) follows from (8).

For (c), note that by Lemma 4.3, the assumption  $F \neq M_p$  on  $Q_1$ , for any  $0 \leq p < \infty$ , implies that on  $Q_1$ ,  $F$  is either  $\Pi_1$ , or  $\Pi_2$ , or  $M_\infty$ , or  $M_p$ , for some  $-\infty \leq p < 0$ . Hence  $F(1, 1) \leq 1$ . If  $t \leq 0$  is such that  $F(1, t) \geq 0$ , this gives

$$F(1, 0) \geq F(1, t) \geq F(F(1, 1), t) = F(1, F(1, t)) \geq F(1, 0).$$

By Lemma 4.1 and the assumption that  $F > 0$  somewhere in  $Q_4$ , we have  $F(1, 0) = 1$  and hence  $F(1, t) = 1$ . □

**Lemma 4.6.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy (i'), (ii'), and (v').*

- (a) *If  $F \geq 0$  on  $Q_4$ , then on  $Q_4$ ,  $F = M_0$  or  $F = \Pi_1$ .*
- (b) *If  $F \leq 0$  on  $Q_4$ , then on  $Q_4$ ,  $F = M_0$  or  $F = \Pi_2$ .*
- (c) *If  $F$  attains positive, negative, and zero values on  $Q_4$ , then there is a  $0 < p < \infty$  and a  $t_0 < 0$  such that*

$$(9) \quad F(1, t) = \begin{cases} (1 - (t/t_0)^p)^{1/p}, & \text{if } t \geq t_0, \\ -((t/t_0)^p - 1)^{1/p}, & \text{if } t < t_0, \end{cases}$$

*for all  $t < 0$ . Also,  $F = M_p$  on  $Q_1$  and  $Q_3$ .*

- (d) *If  $F$  attains positive and negative values on  $Q_4$  and  $F \neq 0$  on  $Q_4$ , then there is a  $t_0 < 0$  such that*

$$(10) \quad F(1, t) = \begin{cases} 1, & \text{if } t_0 < t < 0, \\ t, & \text{if } t < t_0, \end{cases}$$

*and  $F(1, t_0) \in \{1, t_0\}$ . Also,  $F = M_\infty$  on  $Q_1$  and  $Q_3$ .*

*Proof.* To prove (a), suppose that  $F \geq 0$  and  $F \neq M_0$  on  $Q_4$ , so that  $F > 0$  somewhere in  $Q_4$ . By parts (a), (b), and (c) of Lemma 4.5, there is a  $0 < p < \infty$  and a constant  $c \geq 0$  such that

$$F(s, t) = (s^p - c(-t)^p)^{1/p},$$

for all  $(s, t) \in Q_4$ . This function is well defined only if  $s^p - c(-t)^p \geq 0$ , for all  $t < 0$ . It follows that  $c = 0$  and hence that  $F = \Pi_1$  on  $Q_4$ .

Part (b) follows from (a) applied to  $-F(-t, -s)$ .

For (c) and (d), suppose that  $F$  attains positive and negative values on  $Q_4$ . By (ii'), there are  $t_0 \leq t'_0 < 0$  such that

$$F(1, t) \begin{cases} < 0, & \text{for } t < t_0, \\ = 0, & \text{for } t_0 < t < t'_0, \\ > 0, & \text{for } t'_0 < t < 0. \end{cases}$$

By (ii'),  $F(1, 0) > 0$  and hence by (ii') again and Proposition 3.4, on  $Q_1$ ,  $F = \Pi_1$  or  $F = M_p$  for some  $0 < p \leq \infty$ . Similarly, on  $Q_3$ ,  $F = \Pi_2$  or  $F = M_q$ , for some  $0 < q \leq \infty$ .

We now consider two cases, assuming first that  $F \neq \Pi_1, \Pi_2$ , or  $M_\infty$  on at least one of  $Q_1$  and  $Q_3$ . Without loss of generality, assume that on  $Q_3$ ,  $F = M_q$ , for some  $0 < q < \infty$ . Lemma 4.5(b), applied to  $-F(-t, -s)$ , provides a constant  $c' \geq 0$  such that

$$F(1, t) = -((-t)^q - c')^{1/q},$$

for all  $t < t'_0$ . By Lemma 4.5(b), (c), there is  $0 < p < \infty$  and a constant  $c \geq 0$  such that

$$F(1, t) = (1 - c(-t)^p)^{1/p},$$

for all  $t_0 < t < 0$ . If  $c > 0$ , then on  $Q_1$ ,  $F = M_p$ . The last two displayed equations can only hold when  $t_0 = t'_0$ , and we obtain

$$F(1, t) = \begin{cases} (1 - c(-t)^p)^{1/p}, & \text{if } t_0 < t < 0, \\ -((-t)^q - c')^{1/q}, & \text{if } t < t_0. \end{cases}$$

Let  $s > 0$  satisfy  $1 < -st_0 < 2^{1/q}$ . As  $-1/s > t_0 > -2^{1/q}/s$ , we have

$$F(s, -1) = sF\left(1, -\frac{1}{s}\right) = (s^p - c)^{1/p}$$

and

$$F(s, F(-1, -1)) = F(s, -2^{1/q}) = -(2 - c's^q)^{1/q}.$$

The previous expression is negative and therefore equals

$$F(F(s, -1), -1) = F((s^p - c)^{1/p}, -1) = -(1 - c'(s^p - c)^{q/p})^{1/q}.$$

As  $s$  can vary in an open interval, we must have  $p = q$  and  $cc' = 1$ . Since  $c \leq (-t_0)^{-p}$  and  $c' \leq (-t_0)^q$ , we obtain (9), where the case  $t = t_0$  follows from (ii'). Now  $c > 0$ , so  $F = M_p$  on  $Q_1 \cup Q_3$ . Note that  $F = 0$  somewhere in  $Q_4$ .

It remains to consider the case where  $F = \Pi_1$  or  $F = M_\infty$  on  $Q_1$ , and  $F = \Pi_2$  or  $F = M_\infty$  on  $Q_3$ . Then  $F \neq 0$  on  $Q_4$ . Indeed, if  $t < 0$  is such that  $F(1, t) = 0$ , then Lemma 4.1 and the fact that  $F > 0$  somewhere in  $Q_4$  yield

$$1 = F(1, 0) = F(1, F(1, t)) = F(F(1, 1), t) = F(1, t) = 0,$$

a contradiction. Applying Lemma 4.5(c), both to  $F$  and to  $-F(-t, -s)$ , we obtain

$$F(1, t) = \begin{cases} 1, & \text{if } t > t_0, \\ t, & \text{if } t < t_0, \end{cases}$$

for  $t < 0$ , and  $F(1, t_0) \in \{1, t_0\}$ . It remains to show that  $F \neq \Pi_1$  on  $Q_1$  and  $F \neq \Pi_2$  on  $Q_3$ . If the former holds, then  $F(1, 2) = 1$  and therefore

$$\begin{aligned} 1 = F(1, 2) &= F(1, 2F(1, t)) = F(1, F(2, 2t)) \\ &= F(F(1, 2), 2t) = F(1, 2t) = 2t, \end{aligned}$$

for all  $t$  such that  $2t < t_0 < t < 0$ , a contradiction. A similar argument shows that  $F \neq \Pi_2$  on  $Q_3$ . □

**Theorem 4.7.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy conditions (i'), (ii'), and (v'). Then  $F$  is of one of the following 40 types of functions:*

$$(11) \quad F(s, t) = \begin{cases} G_1(s, t), & \text{if } s > 0, t > 0, \\ G_2(s, t), & \text{if } s < 0, t < 0, \\ 0, & \text{otherwise,} \end{cases}$$

where  $G_1 \in \{\Pi_1, \Pi_2, M_p\}$ ,  $-\infty \leq p \leq \infty$ , and  $G_2 \in \{\Pi_1, \Pi_2, M_q\}$ ,  $-\infty \leq q \leq \infty$ ;

$$(12) \quad \begin{aligned} F(s, t) &= \begin{cases} \Pi_1(s, t), & \text{if } s \geq 0, \\ G(s, t), & \text{if } s < 0, t < 0, \\ 0, & \text{otherwise,} \end{cases} \\ \text{or } F(s, t) &= \begin{cases} \Pi_2(s, t), & \text{if } t \geq 0, \\ G(s, t), & \text{if } s < 0, t < 0, \\ 0, & \text{otherwise,} \end{cases} \\ \text{or } F(s, t) &= \begin{cases} \Pi_1(s, t), & \text{if } s \leq 0, \\ G(s, t), & \text{if } s > 0, t > 0, \\ 0, & \text{otherwise,} \end{cases} \\ \text{or } F(s, t) &= \begin{cases} \Pi_2(s, t), & \text{if } t \leq 0, \\ G(s, t), & \text{if } s > 0, t > 0, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where  $G \in \{\Pi_1, \Pi_2, M_p\}$ ,  $-\infty \leq p \leq \infty$ ;

$$(13) \quad \begin{aligned} F(s, t) &= \begin{cases} M_p(s, t), & \text{if } s \geq 0, t \geq 0, \\ \Pi_1(s, t), & \text{if } s \geq 0, t < 0, \\ \Pi_2(s, t), & \text{if } s < 0, t \geq 0, \\ G(s, t), & \text{otherwise,} \end{cases} \\ \text{or } F(s, t) &= \begin{cases} G(s, t), & \text{if } s > 0, t > 0, \\ \Pi_1(s, t), & \text{if } s \leq 0, t > 0, \\ \Pi_2(s, t), & \text{if } s > 0, t \leq 0, \\ M_p(s, t), & \text{otherwise,} \end{cases} \end{aligned}$$

for some  $0 < p \leq \infty$  and  $G \in \{\Pi_1, \Pi_2, M_q\}$ ,  $-\infty \leq q \leq \infty$ ;

$$(14) \quad F(s, t) = \begin{cases} M_p(s, t), & \text{if } s > 0, t > 0, \\ \Pi_1(s, t), & \text{if } t \leq 0, \\ \Pi_2(s, t), & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
 \text{or } F(s, t) &= \begin{cases} M_p(s, t), & \text{if } s > 0, t > 0, \\ \Pi_2(s, t), & \text{if } s \leq 0, \\ \Pi_1(s, t), & \text{otherwise,} \end{cases} \\
 \text{or } F(s, t) &= \begin{cases} \Pi_2(s, t), & \text{if } s \geq 0, t < 0, \\ \Pi_1(s, t), & \text{if } t \geq 0, \\ M_p(s, t), & \text{otherwise,} \end{cases} \\
 \text{or } F(s, t) &= \begin{cases} \Pi_1(s, t), & \text{if } s < 0, t \geq 0, \\ \Pi_2(s, t), & \text{if } s \geq 0, \\ M_p(s, t), & \text{otherwise,} \end{cases}
 \end{aligned}$$

for some  $0 < p \leq \infty$ ;

$$\begin{aligned}
 (15) \quad F(s, t) &= \begin{cases} M_p(s, t), & \text{if } s \geq 0, t \geq 0, \\ \Pi_1(s, t), & \text{if } s > 0, t < 0, \\ \Pi_2(s, t), & \text{if } s < 0, t > 0, \\ M_q(s, t), & \text{otherwise,} \end{cases} \\
 \text{or } F(s, t) &= \begin{cases} M_p(s, t), & \text{if } s \geq 0, t \geq 0, \\ \Pi_1(s, t), & \text{if } s < 0, t > 0, \\ \Pi_2(s, t), & \text{if } s > 0, t < 0, \\ M_q(s, t), & \text{otherwise,} \end{cases}
 \end{aligned}$$

for some  $0 < p \leq \infty$  and  $0 < q \leq \infty$ ;

$$(16) \quad F(s, t) = \Pi_1(s, t) \quad \text{or} \quad F(s, t) = \Pi_2(s, t);$$

$$(17) \quad F(s, t) = \begin{cases} M_p(s, t), & \text{if } st \geq 0, \\ (s^p - (t/t_0)^p)^{1/p}, & \text{if } s > 0, st_0 \leq t < 0, \\ -((-t)^p - (-t_0s)^p)^{1/p}, & \text{if } s > 0, st_0 \geq t, \\ (t^p - (s/t_0)^p)^{1/p}, & \text{if } t > 0, tt_0 < s < 0, \\ -((-s)^p - (-t_0t)^p)^{1/p}, & \text{if } t > 0, tt_0 \geq s, \end{cases}$$

for some  $t_0 < 0$  and  $0 < p < \infty$ ;

$$(18) \quad F(s, t) = \begin{cases} M_\infty(s, t), & \text{if } st \geq 0, \\ s, & \text{if } (s > 0, st_0 < t < 0) \text{ or } (t > 0, s < tt_0), \\ G(s, t), & \text{if } (s > 0, st_0 = t) \text{ or } (t > 0, s = tt_0), \\ t, & \text{otherwise,} \end{cases}$$

for some  $t_0 < 0$  and  $G \in \{\Pi_1, \Pi_2\}$ , or  $G(s, t) = \begin{cases} t, & \text{if } s \geq 0, \\ s, & \text{otherwise,} \end{cases}$  or

$$G(s, t) = \begin{cases} s, & \text{if } s \geq 0, \\ t, & \text{otherwise.} \end{cases}$$

*Proof.* We consider six cases, depending on the restrictions of  $F$  to  $Q_2$  and  $Q_4$ .

*Case 1:* Suppose that  $F = M_0$  on  $Q_2 \cup Q_4$ . By Lemma 4.4, we have  $F = M_0$  on  $\text{cl } Q_2 \cup \text{cl } Q_4$  and hence  $F$  is of the form (11), by Propositions 3.3(b) and 3.4 applied to  $F$  and  $-F(-s, -t)$ .

*Case 2:* Suppose that  $F = M_0$  on exactly one of  $Q_2$  and  $Q_4$ , and that  $F$  does not change sign on the other. By considering one of  $F$ ,  $F(t, s)$ ,  $-F(-s, -t)$ , or  $-F(-t, -s)$ , we may assume that  $F = M_0$  on  $Q_2$  and  $0 \neq F \geq 0$  on  $Q_4$ . We have  $F = M_0$  on  $\text{cl } Q_2$  by Lemma 4.4 and  $F = \Pi_1$  on  $Q_4$  by Lemma 4.6(a). Hence,

$$F(1, 2) = F(F(1, -1), 2) = F(1, F(-1, 2)) = F(1, 0).$$

The only function in Lemma 4.3 that satisfies this requirement together with  $F(1, 0) \geq F(1, -1) = 1$  is  $\Pi_1$ . By (ii'),  $F(s, t) = \Pi_1(s, t)$  for all  $s > 0$ ,  $t \in \mathbb{R}$ . As

$$F(0, t) = F(F(-1, 1), t) = F(-1, F(1, t)) = F(-1, 1) = 0,$$

for all  $t < 0$ , we get  $F(s, t) = \Pi_1(s, t)$  for all  $s \geq 0$  and  $t \in \mathbb{R}$ . Taking all the possibilities into account, we conclude that  $F$  is of the form (12).

*Case 3:* Suppose that  $F \neq M_0$  on  $Q_2$  and on  $Q_4$ , and  $F$  does not change sign on  $Q_2 \cup Q_4$ . Assume first that  $F > 0$  somewhere in  $Q_2$  and somewhere in  $Q_4$ . By Lemma 4.6(a) applied to  $F$  and  $F(t, s)$ , we have  $F = \Pi_1$  on  $Q_4$  and  $F = \Pi_2$  on  $Q_2$ . On  $Q_1$ , we have  $F(s, t) \geq \max\{F(s, -1), F(-1, t)\} = \max\{s, t\}$ , due to (ii'). Therefore, by Propositions 3.3(b) and 3.4, on  $Q_1$ ,  $F = M_p$ , for some  $0 < p \leq \infty$ . Moreover, this holds on  $\text{cl } Q_1$  due to (ii'). If  $F = 0$  on both negative half-axes, then  $F$  is the first function in (13). The second function in (13) is obtained when  $F \leq 0$  on  $Q_2 \cup Q_4$ . If  $F = 0$  on the negative  $t$ -axis, and  $F \neq 0$  on the negative  $s$ -axis, then Lemma 4.1 applied to  $-F(-s, -t)$  implies that  $F(s, 0) = s$ , for all  $s < 0$ . Then

$$F(s, -1) = F(F(s, 0), -1) = F(s, F(0, -1)) = F(s, 0) = s,$$

for  $s < 0$ , so  $F = \Pi_1$  on  $Q_3$ . Therefore  $F$  is the first function in (14). If  $F = 0$  on the negative  $s$ -axis, and  $F \neq 0$  on the negative  $t$ -axis, similar arguments

show that  $F$  is the second function in (14). The remaining two functions in (14) are obtained the same way when  $F \leq 0$  on  $Q_2 \cup Q_4$ . Finally, if  $F \not\equiv 0$  on both negative half-axes, then Lemma 4.1 applied to  $-F(-s, -t)$  implies that  $F(s, 0) = s$ , for all  $s < 0$ , and  $F(0, t) = t$ , for all  $t < 0$ . By (ii'), on  $Q_3$ , we have  $F(s, t) \leq \min\{F(s, 0), F(0, t)\} = \min\{s, t\}$ , so Propositions 3.3(b) and 3.4, applied to  $-F(-s, -t)$ , show that on  $Q_3$ ,  $F = M_q$ , for some  $0 < q \leq \infty$ . Consequently,  $F$  is of the form (15).

*Case 4:* Suppose that  $F \neq M_0$  on  $Q_2$  and on  $Q_4$ , and  $F \leq 0$  on  $Q_2$  and  $F \geq 0$  on  $Q_4$ . Then  $F = \Pi_1$  on  $Q_2 \cup Q_4$ , by Lemma 4.6(a) applied to  $F$  and Lemma 4.6(b) applied to  $-F(-t, -s)$ . For  $t > 0$ ,

$$F(1, t) = F(F(1, -1), t) = F(1, F(-1, t)) = F(1, -1) = 1,$$

so  $F = \Pi_1$  on  $Q_1$ . The same argument applied to  $-F(-s, -t)$  shows that  $F = \Pi_1$  on  $Q_3$ . By (ii'),  $F = \Pi_1$ . Similarly, if instead  $F \geq 0$  on  $Q_2$  and  $F \leq 0$  on  $Q_4$ , then  $F = \Pi_2$ .

*Case 5:* Suppose that  $F \neq M_0$  on  $Q_2$  and on  $Q_4$ , and on at least one of these quadrants, say  $Q_4$ ,  $F$  changes sign and is zero somewhere. By Lemma 4.6(c), there is a  $0 < p < \infty$  and a  $t_0 < 0$  such that  $F = M_p$  on  $Q_1 \cup Q_3$  and (9) holds. Applying Lemma 4.6(a),(b),(c) to the function  $F(t, s)$ , we see that on  $Q_2$ , either  $F = \Pi_1$ , or  $F = \Pi_2$ , or

$$(19) \quad F(s, 1) = \begin{cases} (1 - (s/t_1)^r)^{1/r} & \text{if } s \geq t_1, \\ -((-s)^r - (-t_1)^r)^{1/r} & \text{if } s < t_1, \end{cases}$$

for  $s < 0$  and some  $0 < r < \infty$  and  $t_1 < 0$ . Note that Lemma 4.6(d) does not apply, as  $F \neq M_\infty$  on  $Q_1 \cup Q_3$ .

For  $t_0 < t < 0$ , we have  $F(1, t) > 0$  by (9) and hence

$$(20) \quad (F(1, t)^p + 1)^{1/p} = F(F(1, t), 1) = F(1, F(t, 1)).$$

If  $F = \Pi_1$  or  $F = \Pi_2$  on  $Q_2$ , the right-hand side of (20) would be  $F(1, t)$  or  $2^{1/p}$ , respectively. In either case, this is different from the left-hand side of (20), so  $F$  is of the form (19) on  $Q_2$ . Equation (20) with  $\max\{t_0, t_1\} < t < 0$  gives  $p = r$  and  $t_0 = t_1$ . This yields the function in (17). The assumption that  $F \neq M_0$  on  $Q_4$  and  $F$  changes sign and is zero somewhere on  $Q_2$  yields the same function.

*Case 6:* Suppose that  $F \neq 0$  on  $Q_2$  and on  $Q_4$ , and on at least one of these quadrants,  $Q_4$  say,  $F$  changes sign but is never zero. By Lemma 4.6(d), there is a  $t_0 < 0$  such that (10) holds,  $F(1, t_0) \in \{1, t_0\}$ , and  $F = M_\infty$  on  $Q_1 \cup Q_3$ .

Applying Lemma 4.6(a),(b),(d) to  $F(t, s)$ , we see that on  $Q_2$ , either  $F = \Pi_1$ , or  $F = \Pi_2$ , or

$$(21) \quad F(s, 1) = \begin{cases} 1 & \text{if } 0 > s > t_2, \\ s & \text{if } s < t_2, \end{cases}$$

for  $s < 0$  and some  $t_2 < 0$ , and also  $F(t_2, 1) \in \{t_2, 1\}$ . Note that Lemma 4.6(c) does not apply, as  $F \neq M_p$  on  $Q_1 \cup Q_3$  for any  $0 < p < \infty$ .

It is not possible that  $F = \Pi_1$  on  $Q_2$ , since otherwise, for any  $t_0 < t < 0$  and  $s > -1/t > 0$ , we would have

$$st = \min\{-1, st\} = F(-1, st) = F(F(-1, s), st) = F(-1, F(s, st)) = -1,$$

a contradiction. For  $-1/t_0 > s > 0$  and  $t_0 > t > -1/s$ , we have

$$(22) \quad F(F(-1, s), st) = F(-1, F(s, st)) = F(-1, sF(1, t)) = F(-1, st) = -1.$$

If  $F = \Pi_2$  on  $Q_2$ , (22) simplifies to  $st = -1$ , so  $F$  must satisfy (21). If  $t_0 > t_2$ , we could find  $-1/t_0 > s > -1/t_2$  and  $t_0 > t > -1/s$ , for which (22) again gives  $st = -1$ . Hence  $t_0 \leq t_2$ . Applying this argument to  $F(t, s)$  shows that  $t_0 \geq t_2$ , so  $t_0 = t_2$ . This yields the functions in (18). Note that for  $t_0 = -1$  and  $G = \Pi_1$ , the function  $F$  in (18) coincides with  $M_\infty$ . □

It is straightforward, though tedious, to check that all the functions listed in Theorem 4.7 satisfy (i'), (ii'), and (v'), so the result characterizes all functions with these properties. For  $(s, t) \in \mathbb{R}^2$ , let  $H_1(s, t) = 1$ ,

$$H_2(s, t) = \begin{cases} \min\{s, t\}, & \text{if } st \neq 0, \\ t, & \text{if } s = 0, \\ s, & \text{if } t = 0, \end{cases}$$

and  $H_3(s, t) = 2s + t$ . Then  $H_1$ ,  $H_2$ , and  $H_3$  do not satisfy (i'), (ii'), and (v'), respectively, but they satisfy all other conditions on  $F$  in Theorem 4.7. Therefore none of the conditions (i'), (ii'), and (v') can be omitted.

The following corollary provides a version of the result of Pearson [22, Theorem 3] mentioned earlier, where the continuity assumption is replaced by the increasing property.

**Corollary 4.8.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfy  $F(rs, rt) = rF(s, t)$  for all  $s, t \in \mathbb{R}$  and all  $r \in \mathbb{R}$ , (ii'), and (v'). Then  $F$  is one of the following seven types of*



functions:

$$(23) \quad F(s, t) = \begin{cases} G(s, t), & \text{if } st > 0, \\ 0, & \text{otherwise,} \end{cases}$$

with  $G \in \{\Pi_1, \Pi_2, M_p\}$ , for some  $-\infty \leq p \leq \infty$ ;

$$F(s, t) = \Pi_1(s, t), \quad F(s, t) = \Pi_2(s, t), \quad \text{or} \quad F(s, t) = M_p(s, t),$$

for some  $0 < p \leq \infty$ ;

$$F(s, t) = \begin{cases} s, & \text{if } -|s| < t < |s|, \\ t, & \text{otherwise.} \end{cases}$$

*Proof.* The functions in the statement of the corollary are exactly those listed in Theorem 4.7 which also satisfy  $F(s, t) = -F(-s, -t)$ , for all  $s, t \in \mathbb{R}$ . Note that the last function given is  $M_\infty$ , with values on the line  $\{(s, -s) : s \in \mathbb{R}\}$  changed. □

### 5. Properties of operations between functions

For certain classes  $\Phi(A) \subset \Psi(A)$  of real-valued functions on a nonempty subset  $A$  of  $\mathbb{R}^n$ , we consider natural properties to impose on an arbitrary  $m$ -ary operation  $*$  :  $\Phi(A)^m \rightarrow \Psi(A)$ , where  $m \geq 2$ . When  $m = 2$ , we will write  $f * g$  rather than  $*(f, g)$ . In the following list, it is assumed that  $\Phi(A)$  and  $\Psi(A)$  are appropriate classes for the property under consideration. The properties are supposed to hold for all appropriate  $f, g, h, f_j, g_j, f_{ij} \in \Phi(A)$ .

1. (Commutativity)  $f * g = g * f$ .
2. (Associativity)  $f * (g * h) = (f * g) * h$ .
3. (Homothety) There exists a function  $\xi : \mathbb{N} \rightarrow [0, \infty)$  such that

$$(24) \quad m \odot f = f * \cdots * f = \xi(m)f,$$

where the operation  $*$  is taken  $m$  times and is independent of the order of evaluation.

4. (Positive homogeneity)  $*(rf_1, \dots, rf_m) = r*(f_1, \dots, f_m)$ , for all  $r \geq 0$ .

5. (Weak homogeneity) There is a set  $Q$ , dense in  $\{f(x) : x \in A, f \in \Phi(A)\}$  and containing 0, such that

$$*(rq_1f, \dots, rq_mf) = r*(q_1f, \dots, q_mf),$$

for all  $r \geq 0$  and  $q_1, \dots, q_m \in Q$ .

6. (Identity)  $*(0, \dots, 0, f_j, 0, \dots, 0) = f_j$ .  
 7. (Continuity)

$$f_{ij} \rightarrow f_{0j}, j = 1, \dots, m \Rightarrow *(f_{i1}, \dots, f_{im}) \rightarrow *(f_{01}, \dots, f_{0m})$$

as  $i \rightarrow \infty$ .

8. (Monotonicity)  $f_j \leq g_j, j = 1, \dots, m \Rightarrow *(f_1, \dots, f_m) \leq *(g_1, \dots, g_m)$ .  
 9. (Strict monotonicity)  $*$  is monotonic and  $f_j < g_j, j = 1, \dots, m \Rightarrow *(f_1, \dots, f_m) < *(g_1, \dots, g_m)$ .  
 10. (Pointwise) There is a function  $F : E \subset \mathbb{R}^m \rightarrow \mathbb{R}$  associated with  $*$  such that

$$(25) \quad *(f_1, \dots, f_m)(x) = F(f_1(x), \dots, f_m(x)),$$

for all  $x \in A$ .

Properties 3 and 5 will only be used in Section 10. Property 5 is much weaker than Property 4, since the functions  $q_i f$  in Property 5 are all multiples of the same function  $f$ .

Note that for  $\Phi(A)$  to be appropriate for positive or weak homogeneity (Properties 4 and 5), it is necessary that  $\Phi(A)$  be a cone, i.e.,  $rf \in \Phi(A)$  whenever  $f \in \Phi(A)$  and  $r \geq 0$ . Hence  $\{f(x) : x \in A, f \in \Phi(A)\}$  is a cone in  $\mathbb{R}$ , so for a nontrivial class  $\Phi(A)$ , the set  $Q$  in Property 5 is dense in  $[0, \infty)$ ,  $(-\infty, 0]$ , or  $\mathbb{R}$ , depending on the class. Weak homogeneity is slightly weaker than the assumption that  $*$  is positively homogeneous on all one-dimensional cones  $\{rf : r \geq 0\}, f \in \Phi(A)$ .

Of course, continuity (Property 7) is with respect to some suitable topology. We shall call  $*$  *pointwise continuous* if the convergence is pointwise, and *pointwise continuous from below* if it satisfies the continuity property for all pointwise convergent, increasing sequences of functions.

## 6. Examples of pointwise operations

The usual arithmetic operations between functions are of course pointwise operations, whereas composition of functions is not.

For  $-\infty \leq p \neq 0 \leq \infty$ , the *p*th sum of nonnegative real-valued (or extended-real-valued) functions  $f$  and  $g$  on a nonempty subset  $A$  of  $\mathbb{R}^n$  is defined by

$$(26) \quad (f +_p g)(x) = (f(x)^p + g(x)^p)^{1/p},$$

if  $-\infty < p \neq 0 < \infty$ ,

$$(27) \quad (f +_\infty g)(x) = \max \{f(x), g(x)\},$$

and

$$(28) \quad (f +_{-\infty} g)(x) = \min \{f(x), g(x)\}.$$

In the context of convex analysis, Seeger [28] appears to have been the first to consider the *p*th sum, noting that if  $A$  is a nontrivial convex set, Minkowski's inequality implies that if  $p \geq 1$ , then the *p*th sum of nonnegative real-valued (or extended-real-valued) convex functions is again such a function. For  $1 \leq p \leq \infty$ , the operation  $+_p$ , which we shall call *L<sub>p</sub> addition*, has all the properties listed in Section 5, where continuity is taken to be pointwise continuity.

Suppose that  $\|\cdot\|$  is a norm on  $\mathbb{R}^2$ . One can define

$$(29) \quad (f +_{\|\cdot\|} g)(x) = \|(f(x), g(x))\|.$$

Then (26) and (27) correspond to the  $L_p$  norm and  $L_\infty$  norm, respectively. Volle [29] noticed that when  $\|\cdot\|$  is a *monotone norm*, i.e.,  $\|(x_1, y_1)\| \leq \|(x_2, y_2)\|$  whenever  $x_1 \leq x_2$  and  $y_1 \leq y_2$ , then the operation  $+_{\|\cdot\|}$  still preserves the convexity of nonnegative real-valued (or extended-real-valued) functions.

Still more generally, let  $M$  be an arbitrary nonempty compact convex set in  $\mathbb{R}^m$ ,  $m \geq 2$ , and define the *M-sum* of functions  $f_1, \dots, f_m$  by

$$(30) \quad \oplus_M (f_1, \dots, f_m)(x) = h_M (f_1(x), \dots, f_m(x)).$$

Then (29) corresponds to the special case when  $M$  is a *o*-symmetric convex body in  $\mathbb{R}^2$ , since every such set generates a norm on  $\mathbb{R}^2$ , and monotone

norms derive from sets  $M$  such that the support function  $h_M$  is increasing in each variable.

Of course, the operations (26)–(30) still make sense as operations between arbitrary functions, provided the functions involved are replaced by their absolute values where appropriate. However, even  $L_p$  addition does not in general preserve convexity if  $p > 1$  and the functions are not nonnegative. In view of this, we can extend the definition of  $L_p$  addition to arbitrary functions on  $\mathbb{R}^n$ , by analogy with [12, Example 6.7], by setting

$$(31) \quad (f +_p g)(x) = \begin{cases} (\max\{f(x), 0\}^p + \max\{g(x), 0\}^p)^{1/p}, & \text{if } 1 < p < \infty, \\ \max\{\max\{f(x), 0\}, \max\{g(x), 0\}\}, & \text{if } p = \infty. \end{cases}$$

Then, since  $\max\{f, 0\}$  is a nonnegative convex function whenever  $f$  is convex, the operation  $+_p$  defined by (31) preserves convexity. It also has all the properties listed in Section 5 (again where continuity is taken to be pointwise continuity) except Property 6, the identity property.

For the remainder of this section, we assume that the dimension  $n$  is at least 2.

As an operation between support functions, (30) was effectively introduced in [12]. To explain this, we begin by recalling from [12, Section 6] that if  $M$  is an arbitrary subset of  $\mathbb{R}^m$ ,  $m \geq 2$ , the  $M$ -sum  $\oplus_M(K_1, \dots, K_m)$  of arbitrary sets  $K_1, \dots, K_m$  in  $\mathbb{R}^n$  is defined by

$$\begin{aligned} \oplus_M(K_1, \dots, K_m) &= \left\{ \sum_{j=1}^m a_j x^{(j)} : x^{(j)} \in K_j, (a_1, \dots, a_m) \in M \right\} \\ &= \bigcup \{a_1 K_1 + \dots + a_m K_m : (a_1, \dots, a_m) \in M\}. \end{aligned}$$

In [12, Theorem 6.5], it is proved that

$$(32) \quad h_{\oplus_M(K_1, \dots, K_m)}(x) = h_M(h_{K_1}(x), \dots, h_{K_m}(x)),$$

for  $x \in \mathbb{R}^n$ , holds either if  $K_1, \dots, K_m \in \mathcal{K}_s^n$  and  $M$  is a 1-unconditional compact convex subset of  $\mathbb{R}^m$ , or if  $K_1, \dots, K_m \in \mathcal{K}^n$  and  $M$  is a compact convex subset of  $[0, \infty)^m$ . It follows that under these assumptions, the operation  $\oplus_M$  defined by (30) preserves support functions of compact convex sets.

It appears that  $M$ -addition was first introduced, for centrally symmetric compact convex sets  $K$  and  $L$  and a 1-unconditional convex body  $M$  in  $\mathbb{R}^2$ , by Protasov [23, 24], motivated by work on the joint spectral radius in the theory of normed algebras.

An operation  $* : (\mathcal{K}^n)^m \rightarrow \mathcal{K}^n$  is called *projection covariant* if

$$*(K_1, \dots, K_m)|S = *(K_1|S, \dots, K_m|S),$$

for any subspace  $S$  in  $\mathbb{R}^n$ . Important examples are Minkowski addition and  $L_p$  addition for  $p > 1$ . By [13, Theorem 3.3 and Corollary 3.4], an operation  $* : (\mathcal{K}_s^n)^m \rightarrow \mathcal{K}^n$  is continuous and  $GL(n)$ -covariant if and only if it is projection covariant, and such operations are precisely those defined for all  $K_1, \dots, K_m \in \mathcal{K}_s^n$  by  $*(K_1, \dots, K_m) = \oplus_M(K_1, \dots, K_m)$ , where  $M$  is a 1-unconditional compact convex subset of  $\mathbb{R}^m$ . In view of (30) and (32), we have

$$\begin{aligned} h_{*(K_1, \dots, K_m)}(x) &= h_{\oplus_M(K_1, \dots, K_m)}(x) \\ &= h_M(h_{K_1}(x), \dots, h_{K_m}(x)) = \oplus_M(h_{K_1}, \dots, h_{K_m})(x), \end{aligned}$$

for all  $K_1, \dots, K_m \in \mathcal{K}_s^n$  and all  $x \in \mathbb{R}^n$ . In other words, a projection covariant operation between  $o$ -symmetric compact convex sets corresponds to the pointwise operation of taking  $M$ -sums of their support functions. Moreover, the equation  $h_{*(K_1, \dots, K_m)} = *(h_{K_1}, \dots, h_{K_m})$  allows an operation between compact convex sets to be transferred to one between their support functions, and vice versa. In this context, [13, Lemma 3.2] says that pointwise operations between support functions of  $o$ -symmetric compact convex sets, with associated positively homogeneous function  $F$ , correspond precisely to projection covariant operations between the sets themselves.

Without origin symmetry, the picture is more complicated. By [13, Theorem 3.5 and Corollary 3.6], an operation  $* : (\mathcal{K}^n)^m \rightarrow \mathcal{K}^n$  (or  $* : (\mathcal{K}_o^n)^m \rightarrow \mathcal{K}^n$ ) is continuous and  $GL(n)$ -covariant if and only if it is projection covariant, and the latter holds if and only if there is an  $M \in \mathcal{CC}^{2m}$  such that

$$(33) \quad h_{*(K_1, \dots, K_m)}(x) = h_M(h_{-K_1}(x), h_{K_1}(x), \dots, h_{-K_m}(x), h_{K_m}(x)),$$

for all  $K_1, \dots, K_m \in \mathcal{K}^n$  (or all  $K_1, \dots, K_m \in \mathcal{K}_o^n$ , respectively) and  $x \in \mathbb{R}^n$ . Note that in this case the right-hand side of (33) corresponds to a pointwise operation that involves the support functions of the reflections in the origin of the sets concerned as well.

However, suppose that an operation  $* : (\mathcal{K}_o^n)^m \rightarrow \mathcal{K}^n$  is both projection covariant and a pointwise operation between support functions with associated function  $F$  when defined by  $*(h_{K_1}, \dots, h_{K_m}) = h_{*(K_1, \dots, K_m)}$ . (An important example of such an operation is Orlicz addition; see [13, (5.2)] and note that the associated function  $F$  in this case is defined implicitly. See also Section 8 below.) Then there is an  $M \in \mathcal{CC}^m$  such that  $F = h_M$ . To see

this, let  $x \in \mathbb{R}^n$  and  $K_1, \dots, K_m \in \mathcal{K}_o^n$ . Choose  $K'_1, \dots, K'_m \in \mathcal{K}_o^n$  such that  $h_{K'_j}(x) = h_{K_j}(x)$  and  $h_{K'_j}(-x) = 0$ ,  $j = 1, \dots, m$ . Then, by (33),

$$\begin{aligned} F(h_{K_1}(x), \dots, h_{K_m}(x)) &= F(h_{K'_1}(x), \dots, h_{K'_m}(x)) \\ &= (* (h_{K'_1}, \dots, h_{K'_m}))(x) = h_{*(K'_1, \dots, K'_m)}(x) \\ &= h_M(h_{-K'_1}(x), h_{K'_1}(x), \dots, h_{-K'_m}(x), h_{K'_m}(x)) \\ &= h_M(0, h_{K_1}(x), \dots, 0, h_{K_m}(x)), \end{aligned}$$

for all  $x \in \mathbb{R}^n$ . Replacing  $M$  by its projection onto the  $\{x_2, x_4, \dots, x_{2m}\}$ -plane in  $\mathbb{R}^{2m}$  and identifying the latter with  $\mathbb{R}^m$ , we see that

$$F(h_{K_1}(x), \dots, h_{K_m}(x)) = h_M(h_{K_1}(x), \dots, h_{K_m}(x)),$$

for all  $x \in \mathbb{R}^n$  and  $K_1, \dots, K_m \in \mathcal{K}_o^n$ , and hence  $F = h_M$ .

An operation  $* : (\mathcal{S}^n)^m \rightarrow \mathcal{S}^n$  is called *section covariant* if

$$(*(K_1, \dots, K_m)) \cap S = *(K_1 \cap S, \dots, K_m \cap S),$$

for any subspace  $S$  in  $\mathbb{R}^n$ . Important examples are radial addition and, more generally,  $p$ th radial addition for  $-\infty \leq p \neq 0 \leq \infty$  [12, Section 5.4] and radial Orlicz addition [14, 31]. A straightforward extension of the proof of [12, Theorem 7.17] implies that if an operation  $* : (\mathcal{S}_s^n)^m \rightarrow \mathcal{S}^n$  is positively homogeneous and rotation and section covariant, then there is a function  $F : [0, \infty)^m \rightarrow [0, \infty)$  such that

$$\rho_{*(K_1, \dots, K_m)}(x) = F(\rho_{K_1}(x), \dots, \rho_{K_m}(x)),$$

for all  $K_1, \dots, K_m \in \mathcal{S}_s^n$  and all  $x \in \mathbb{R}^n \setminus \{o\}$ . In other words, such operations between  $o$ -symmetric star sets correspond to pointwise operations between their radial functions when defined by  $*(\rho_{K_1}, \dots, \rho_{K_m}) = \rho_{*(K_1, \dots, K_m)}$ .

### 7. Pointwise operations on real-valued functions

Throughout this section,  $\Phi(A)$  will be a class of real-valued functions on a nonempty subset  $A$  of  $\mathbb{R}^n$ ,  $n \geq 2$ .

Define

$$(34) \quad D = \{s \in \mathbb{R} : f(x) = s \text{ for some } f \in \Phi(A) \text{ and } x \in A\}.$$

If  $\Phi(A)$  is a cone, then  $D$  is also a cone and hence unless it is trivial, it must be  $\mathbb{R}$ ,  $[0, \infty)$ , or  $(-\infty, 0]$ .

Let  $\alpha$  be a cardinal number such that  $\alpha \leq |D|$ , where  $|D|$  is the cardinality of the set  $D$  defined by (34). We say that  $\Phi(A)$  has the  $\alpha$ -point property if whenever  $S \subset D$  has cardinality  $\alpha$ , there is an injection  $\phi : S \rightarrow \Phi(A)$  and an  $x \in A$  such that  $(\phi(s))(x) = s$  for all  $s \in S$ .

Let  $*$  be a pointwise operation on  $\Phi(A)^m$ ,  $m \geq 2$ , with associated function  $F$ . We call the set  $D^m \subset \mathbb{R}^m$  the *proper domain* of  $F$ .

Any class  $\Phi(A)$  containing the constant functions with values in  $D$  has the  $\alpha$ -point property for any  $\alpha \leq |D|$ . The same is true for any cone  $\Phi(A)$  of nonnegative functions. For cones of arbitrary functions, this is not true; indeed, the cone generated by the functions  $f(t) = \max\{0, t\}$  and  $g(t) = \min\{0, t\}$ ,  $t \in \mathbb{R}$ , does not even have the 2-point property. It is not difficult to see that any cone that has the 2-point property also has the  $\alpha$ -point property for any  $\alpha \leq |D|$ .

**Lemma 7.1.** *Let  $*$  be a pointwise operation on  $\Phi(A)^m$  whose associated function  $F$  has proper domain  $D^m$ .*

- (a) *If  $\Phi(A)$  is a cone with the 2-point property, then  $*$  is positively homogeneous if and only if  $F$  is positively homogeneous on  $D^m$ .*
- (b) *If  $\Phi(A)$  has the  $\aleph_0$ -point property, then  $*$  is pointwise continuous if and only if  $F$  is continuous on  $D^m$ .*
- (c) *If  $m = 2$  and  $\Phi(A)$  has the 3-point property, then  $*$  is associative if and only if  $F$  is associative on  $D^2$ , that is,  $F(F(s, t), u) = F(s, F(t, u))$ , for  $s, t, u \in D$ .*

*Proof.* All three stated properties of  $F$  clearly imply the corresponding property of  $*$ , even without assuming an  $\alpha$ -point property. We therefore only consider the remaining implications.

(a) Since  $\Phi(A)$  is a cone with the 2-point property, it has the  $\alpha$ -point property when  $\alpha \leq |D|$ , and  $D$  is also a cone. Given  $s_1, \dots, s_m \in D$ , choose  $f_j \in \Phi(A)$  and  $x \in A$  such that  $f_j(x) = s_j$ ,  $j = 1, \dots, m$ . Let  $r \geq 0$ . Then

$$\begin{aligned} F(rs_1, \dots, rs_m) &= F(rf_1(x), \dots, rf_m(x)) = (*)(rf_1, \dots, rf_m)(x) \\ &= r(*)(f_1, \dots, f_m)(x) = rF(f_1(x), \dots, f_m(x)) \\ &= rF(s_1, \dots, s_m). \end{aligned}$$

(b) Given  $s_{ij} \in D$ ,  $i \in \mathbb{N} \cup \{0\}$ ,  $j = 1, \dots, m$ , such that  $s_{ij} \rightarrow s_{0j}$  as  $i \rightarrow \infty$  for  $j = 1, \dots, m$ , choose  $f_{ij} \in \Phi(A)$  and  $x \in A$  such that  $f_{ij}(x) = s_{ij}$ ,

$i \in \mathbb{N} \cup \{0\}$ ,  $j = 1, \dots, m$ . Then

$$\begin{aligned} F(s_{i1}, \dots, s_{im}) &= F(f_{i1}(x), \dots, f_{im}(x)) = (*f_{i1}, \dots, f_{im})(x) \\ &\rightarrow (*f_{01}, \dots, f_{0m})(x) = F(f_{01}(x), \dots, f_{0m}(x)) \\ &= F(s_{01}, \dots, s_{0m}), \end{aligned}$$

as  $i \rightarrow \infty$ .

(c) Given  $s, t, u \in D$ , choose  $f, g, h \in \Phi(A)$  and  $x \in A$  such that  $f(x) = s$ ,  $g(x) = t$ , and  $h(x) = u$ . Then

$$\begin{aligned} F(F(s, t), u) &= F(F(f(x), g(x)), h(x)) \\ &= ((f * g) * h)(x) = (f * (g * h))(x) = F(s, F(t, u)). \quad \square \end{aligned}$$

We say that  $\Phi(A)$  has the *point separation property* (or *strict point separation property*) if whenever  $f(x) \leq g(x)$  (or  $f(x) < g(x)$ , respectively) for some  $f, g \in \Phi(A)$  and  $x \in A$ , there are  $\bar{f}, \bar{g} \in \Phi(A)$  with  $\bar{f}(x) = f(x)$ ,  $\bar{g}(x) = g(x)$ , and  $\bar{f} \leq \bar{g}$  (or  $\bar{f} < \bar{g}$ , respectively).

Any class  $\Phi(A)$  containing the constant functions with values in  $D$  has the point separation property and strict point separation property. Also, any cone  $\Phi(A)$  of nonnegative functions (or of negative functions) has the point separation property. Another example is provided by the following lemma, which will find use later. Notation for the various classes of functions is defined in Section 2. Recall that for classes  $\text{Cvx}(A)$  or  $\text{Cvx}^+(A)$ , it is always assumed that  $A$  is a nontrivial convex set in  $\mathbb{R}^n$ .

**Lemma 7.2.** *Let  $\Phi(A)$  be  $\text{Cvx}(A)$ ,  $\text{Cvx}^+(A)$ ,  $\text{Supp}(\mathbb{R}^n)$ , or  $\text{Supp}^+(\mathbb{R}^n)$ . Then  $\Phi(A)$  has the  $\alpha$ -point property, for each  $\alpha \leq |\mathbb{R}|$ , and the point separation property.*

*Proof.* Since  $\text{Cvx}(A)$  contains all the constant functions, and  $\text{Cvx}^+(A)$  and  $\text{Supp}^+(\mathbb{R}^n)$  are cones of nonnegative functions, our remarks above show that the lemma holds for these three classes.

Let  $\Phi(A) = \text{Supp}(\mathbb{R}^n)$ . The  $\alpha$ -point property for any  $\alpha \leq |\mathbb{R}|$  follows easily by fixing  $x = e_1$  and setting  $\phi(s) = h_{\{se_1\}}$  for  $s \in S \subset \mathbb{R}$  in the definition of this property. For the point separation property, suppose that  $K, L \in \mathcal{K}^n$ ,  $x \in \mathbb{R}^n$ , and  $h_K(x) \leq h_L(x)$ . Choose  $\bar{K}, \bar{L} \in \mathcal{K}^n$  such that  $h_{\bar{K}}(x) = h_K(x)$ ,  $h_{\bar{L}}(x) = h_L(x)$ , and  $\bar{K} \subset \bar{L}$ . Then  $h_{\bar{K}} \leq h_{\bar{L}}$ , as required.  $\square$

By our earlier remarks,  $\text{Cvx}(A)$  and  $\text{Cvx}^+(A)$  have the strict point separation property, but  $\text{Supp}(\mathbb{R}^n)$  and  $\text{Supp}^+(\mathbb{R}^n)$  do not, since all such functions vanish at the origin.



**Lemma 7.3.** *Let  $*$  be a pointwise operation on  $\Phi(A)^m$ ,  $m \geq 2$ , whose associated function  $F$  has proper domain  $D^m$  and assume that  $\Phi(A)$  has the  $2m$ -point property and the point separation property (or strict point separation property). Then  $*$  is monotonic (or strictly monotonic, respectively) if and only if  $F$  is increasing (or strictly increasing, respectively) in each variable on  $D^m$ .*

*Proof.* If  $F$  is increasing (or strictly increasing) in each variable on  $D^m$ , then  $*$  is clearly monotonic (or strictly monotonic, respectively).

Suppose that  $*$  is monotonic. Given  $s_j, t_j \in D$  with  $s_j \leq t_j, j = 1, \dots, m$ , use the  $2m$ -point property to choose  $f_j, g_j \in \Phi(A)$  and  $x \in A$  such that  $f_j(x) = s_j$  and  $g_j(x) = t_j, j = 1, \dots, m$ . Using the point separation property, we may assume without loss of generality that  $f_j \leq g_j, j = 1, \dots, m$ . Then

$$\begin{aligned} F(s_1, \dots, s_m) &= F(f_1(x), \dots, f_m(x)) = (* (f_1, \dots, f_m))(x) \\ &\leq (* (g_1, \dots, g_m))(x) = F(g_1(x), \dots, g_m(x)) = F(t_1, \dots, t_m). \end{aligned}$$

The case when  $*$  is strictly monotonic follows in a similar fashion. □

**Theorem 7.4.** *Let  $\Phi(A)$  be a cone of nonnegative functions and let  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  be a pointwise operation. Suppose that  $*$  is positively homogeneous, either monotonic or pointwise continuous, and associative. Then for all  $f, g \in \Phi(A)$  and  $x \in A$ , we have  $(f * g)(x) = F(f(x), g(x))$ , where  $F$  is either one of the six types of functions listed in Corollary 4.2 or one of the three types of functions listed in Proposition 3.2, respectively.*

*Proof.* Let  $F$  be the function associated with  $*$ . The assumptions imply that  $F$  is nonnegative and has proper domain  $[0, \infty)^2$  and that  $\Phi(A)$  has the  $\aleph_0$ -point property and the point separation property. If  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  is positively homogeneous, pointwise continuous, monotonic, or associative, then by Lemma 7.1(a),(b),(c) and Lemma 7.3,  $F$  is positively homogeneous, continuous, increasing in each variable, or associative, respectively. The conclusion follows from Proposition 3.2 and Corollary 4.2. □

The following simple lemma provides circumstances in which the positive homogeneity of  $*$  is guaranteed.

**Lemma 7.5.** *Let  $\Phi(A) \subset \Psi(A)$  be classes of positively homogeneous functions on  $A$  and let  $*$  :  $\Phi(A)^m \rightarrow \Psi(A)$  be a pointwise operation. If  $\Phi(A)$  is a cone, then  $*$  is positively homogeneous.*

*Proof.* Let  $F$  be the function associated with  $*$ . If  $r \geq 0$ , then for  $f, g \in \Phi(A)$ ,  $f * g \in \Psi(A)$  is positively homogeneous and hence

$$\begin{aligned} r(f * g)(x) &= (f * g)(rx) = F(f(rx), g(rx)) \\ &= F(rf(x), rg(x)) = (rf * rg)(x), \end{aligned}$$

for all  $x \in A$ , so  $*$  is positively homogeneous.  $\square$

The assumptions on  $\Phi(A)$  in the next theorem are satisfied if  $\Phi(A)$  is a cone containing the constant functions.

**Theorem 7.6.** *Let  $\Phi(A)$  be a cone and let  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  be a pointwise operation whose associated function  $F$  has proper domain  $\mathbb{R}^2$ . Suppose that  $\Phi(A)$  has the 2-point property and the point separation property, and that  $*$  is positively homogeneous, monotonic, and associative. Then for all  $f, g \in \Phi(A)$  and  $x \in A$ , we have  $(f * g)(x) = F(f(x), g(x))$ , where  $F$  is one of the 40 types of functions listed in Theorem 4.7.*

*Proof.* Since  $\Phi(A)$  is a cone with the 2-point property, it also has the 3-point property. Let  $F$  be the function associated with  $*$ . If  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  is positively homogeneous, monotonic, or associative, then by Lemma 7.1(a),(c) and Lemma 7.3,  $F$  is positively homogeneous, increasing in each variable, or associative, respectively. The conclusion follows from Theorem 4.7.  $\square$

**Example 7.7.** In each of the following examples, we assume that  $\Phi(A)$  is a cone of functions on a nonempty set  $A$  in  $\mathbb{R}^n$  and that either functions in  $\Phi(A)$  are nonnegative or  $\Phi(A)$  contains the constant functions. We will make use of the functions  $H_1$ ,  $H_2$ , and  $H_3$  defined before Corollary 4.8.

(a) Choose  $x_0 \in A$  and define

$$(f * g)(x) = f(x_0) + g(x),$$

for all  $f, g \in \Phi(A)$  and  $x \in A$ . Then  $*$  is positively homogeneous, pointwise continuous, monotonic, and associative, but not pointwise. This shows that the pointwise property cannot be dropped in Theorems 7.4 and 7.6.

(b) Define

$$(f * g)(x) = H_1(f(x), g(x)) = 1,$$

for all  $f, g \in \Phi(A)$  and  $x \in A$ . Then  $*$  is pointwise continuous, monotonic, and associative, but not positively homogeneous. This shows that the latter property cannot be omitted in Theorems 7.4 and 7.6.

(c) Define  $(f * g)(x) = H_2(f(x), g(x))$ , for all  $f, g \in \Phi(A)$  and  $x \in A$ . By Lemmas 7.1 and 7.3, the operation  $*$  is positively homogeneous and associative, but not monotonic and not pointwise continuous. This shows that pointwise continuity and monotonicity cannot be omitted in Theorems 7.4 and 7.6.

(d) Define

$$(f * g)(x) = H_3(f(x), g(x)) = 2f(x) + g(x),$$

for all  $f, g \in \Phi(A)$  and  $x \in A$ . Then  $*$  is positively homogeneous, pointwise continuous, and monotonic, but not associative. This shows that associativity cannot be dropped in Theorems 7.4 and 7.6.

The operation  $*$  in Theorems 7.4 and 7.6 need not be commutative, even if  $*$  is pointwise continuous and monotonic, as is shown by the operation  $*$  with associated function  $F = \Pi_1$ , for example. In particular, the associated function need not be of the form (6).

**Theorem 7.8.** *Let  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  be a pointwise operation whose associated function  $F$  has proper domain  $D^2$ , where  $D \subset \mathbb{R}$  is an interval open on one side. Suppose that  $\Phi(A)$  has the  $\aleph_0$ -point property and the strict point separation property. Then  $*$  is pointwise continuous, strictly monotonic, and associative if and only if there is a continuous and strictly monotonic function  $\phi : D \rightarrow \mathbb{R}$ , such that for all  $f, g \in \Phi(A)$  and  $x \in A$ , we have*

$$(35) \quad (f * g)(x) = \phi^{-1}(\phi(f(x)) + \phi(g(x))).$$

*Proof.* By Lemmas 7.1 and 7.3, the operation  $*$  is pointwise continuous, strictly monotonic, and associative if and only if its associated function  $F$  is continuous, strictly increasing in each variable, and associative. In view of Proposition 3.5, this yields the assertion. □

Let  $m \geq 2$ . We say that  $\Phi(A)$  has the *m-linear interpolation property* with associated subclass  $\Gamma(A)$  if there is a  $\Gamma(A) \subset \Phi(A)$  such that for all  $s_j, t_j \in D, j = 1, \dots, m$ , there are  $f_j \in \Gamma(A)$  and  $x, y \in A$  such that  $f_j(x) = s_j, f_j(y) = t_j$ , and  $f_j$  is linear on  $[x, y]$ , for  $j = 1, \dots, m$ .

Note that if  $\Phi(A)$  has the *m-linear interpolation property* with associated subclass  $\Gamma(A)$ , then  $\Phi(A)$  also has the *m-point property*, and since  $m \geq 2$ , if  $\Phi(A)$  is a nontrivial cone this extends to the  *$\alpha$ -point property*, for any  $\alpha \leq |\mathbb{R}|$ .

If  $\Phi(A)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(A)$ , then  $D$  is an interval, because if  $s_1, t_1 \in D$  and  $0 \leq \lambda \leq 1$ , we can choose  $f \in \Gamma(A)$  and  $x, y \in A$  such that  $f(x) = s_1$ ,  $f(y) = t_1$ , and

$$(1 - \lambda)s_1 + \lambda t_1 = (1 - \lambda)f(x) + \lambda f(y) = f((1 - \lambda)x + \lambda y) \in D.$$

**Remark 7.9.** (a) For each  $m \geq 2$ , the class  $\text{Cvx}(A)$  has the  $m$ -linear interpolation property with the affine functions on  $A$  as the associated subclass.

(b) If  $A$  is compact and  $m \geq 2$ , then  $\text{Cvx}^+(A)$  also has the  $m$ -linear interpolation property with the nonnegative affine functions on  $A$  as the associated subclass. If  $A$  is arbitrary, then functions of the form  $f(x) = a\|x_0 - x\|$ , where  $x_0 \in \mathbb{R}^n$  and  $a > 0$ , restricted to  $A$ , can serve as the associated subclass.

(c) For each  $m \geq 2$ , the class  $\text{Supp}(\mathbb{R}^n)$  (or  $\text{Supp}^+(\mathbb{R}^n)$ ), for which  $D = \mathbb{R}$  (or  $D = [0, \infty)$ , respectively), has the  $m$ -linear interpolation property with the support functions of singletons (or line segments, respectively) as the associated subclass. Indeed, let  $s_j, t_j \in \mathbb{R}$ ,  $j = 1, \dots, m$ . Then we may take  $x = e_1$ ,  $y = e_2$ , and  $f_j = h_{\{s_j e_1 + t_j e_2\}}$  (or  $f_j = h_{[0, s_j e_1 + t_j e_2]}$ , respectively).

**Lemma 7.10.** *Let  $\Phi(A)$  be a class of functions on a nontrivial convex set  $A$  in  $\mathbb{R}^n$ , let  $m \geq 2$ , and let  $\Phi(A)$  have the  $m$ -linear interpolation property with associated subclass  $\Gamma(A)$ . Suppose that  $*$  is a pointwise operation on  $\Phi(A)^m$  whose associated function  $F$  has proper domain  $D^m$ . Consider the statements:*

- (a)  $*(f_1, \dots, f_m) \in \text{Cvx}(A)$ , for all  $f_1, \dots, f_m \in \Phi(A)$ ;
- (b)  $*(f_1, \dots, f_m) \in \text{Cvx}(A)$ , for all  $f_1, \dots, f_m \in \Gamma(A)$ ;
- (c)  $F$  is convex on  $D^m$ .

Then (a)  $\Rightarrow$  (b)  $\Rightarrow$  (c). If  $*$  is monotonic,  $\Phi(A) \subset \text{Cvx}(A)$ , and  $\Phi(A)$  has the  $2m$ -point property and the point separation property, then (a)  $\Leftrightarrow$  (b)  $\Leftrightarrow$  (c).

*Proof.* Clearly, (a) implies (b). Suppose (b) holds. Let  $s_j, t_j \in D$ ,  $j = 1, \dots, m$ , and let  $0 \leq \lambda \leq 1$ . Set  $s = (s_1, \dots, s_m)$  and  $t = (t_1, \dots, t_m)$ . Choose  $f_j \in \Gamma(A)$  and  $x, y \in A$  such that  $f_j(x) = s_j$  and  $f_j(y) = t_j$ , with  $f_j$  linear on  $[x, y]$ . Then

$$f_j((1 - \lambda)x + \lambda y) = (1 - \lambda)f_j(x) + \lambda f_j(y) = (1 - \lambda)s_j + \lambda t_j,$$

for  $j = 1, \dots, m$ . Let  $H(z) = (* (f_1, \dots, f_m))(z) = F(f_1(z), \dots, f_m(z))$ , for all  $z \in A$ . By assumption,  $H$  is convex, so

$$\begin{aligned} F((1 - \lambda)s + \lambda t) &= F((1 - \lambda)s_1 + \lambda t_1, \dots, (1 - \lambda)s_m + \lambda t_m) \\ &= F(f_1((1 - \lambda)x + \lambda y), \dots, f_m((1 - \lambda)x + \lambda y)) \\ &= H((1 - \lambda)x + \lambda y) \leq (1 - \lambda)H(x) + \lambda H(y) \\ &= (1 - \lambda)F(f_1(x), \dots, f_m(x)) + \lambda F(f_1(y), \dots, f_m(y)) \\ &= (1 - \lambda)F(s) + \lambda F(t). \end{aligned}$$

This proves that  $F$  is convex on  $D^m$ , so (c) is true.

Now suppose that  $*$  is monotonic,  $\Phi(A) \subset \text{Cvx}(A)$ ,  $\Phi(A)$  has the  $2m$ -point property and the point separation property, and (c) holds. Let  $f_j \in \Phi(A)$ ,  $j = 1, \dots, m$ , let  $x, y \in A$ , and let  $0 \leq \lambda \leq 1$ . By Lemma 7.3,  $F$  is increasing in each variable. Using this and the convexity of  $f_j$  and  $F$ , we obtain

$$\begin{aligned} &(* (f_1, \dots, f_m))((1 - \lambda)x + \lambda y) \\ &= F(f_1((1 - \lambda)x + \lambda y), \dots, f_m((1 - \lambda)x + \lambda y)) \\ &\leq F((1 - \lambda)f_1(x) + \lambda f_1(y), \dots, (1 - \lambda)f_m(x) + \lambda f_m(y)) \\ &= F((1 - \lambda)(f_1(x), \dots, f_m(x)) + \lambda(f_1(y), \dots, f_m(y))) \\ &\leq (1 - \lambda)F(f_1(x), \dots, f_m(x)) + \lambda F(f_1(y), \dots, f_m(y)) \\ &= (1 - \lambda)(* (f_1, \dots, f_m))(x) + \lambda(* (f_1, \dots, f_m))(y). \end{aligned}$$

This shows that  $* (f_1, \dots, f_m)$  is convex, so (a) holds. □

**Lemma 7.11.** *Let  $\Phi(A)$  be  $\text{Cvx}(A)$ ,  $\text{Cvx}^+(A)$ ,  $\text{Supp}(\mathbb{R}^n)$ , or  $\text{Supp}^+(\mathbb{R}^n)$ . If  $* : \Phi(A)^m \rightarrow \text{Cvx}(A)$ ,  $m \geq 2$ , is a pointwise operation, then the associated function  $F$  is increasing in each variable and hence  $*$  is monotonic.*

*Proof.* Suppose that  $\Phi(A)$  is  $\text{Cvx}(A)$  or  $\text{Cvx}^+(A)$ . Let  $s_1, \dots, s_m, t_1 \in D$  with  $s_1 < t_1$  be given. Define  $f_j \equiv s_j$  on  $A$ , for  $j = 2, \dots, m$ , and choose a convex function  $f_1$  on  $A$  and  $x, y \in A$  such that  $f_1(x) = f_1(y) = t_1$  and  $f_1((x + y)/2) = s_1$ . By assumption, the function

$$\begin{aligned} H(z) &= F(f_1(z), s_2, \dots, s_m) \\ &= F(f_1(z), \dots, f_m(z)) = (* (f_1, \dots, f_m))(z), \end{aligned}$$

for  $z \in A$ , is convex. Therefore

$$\begin{aligned} F(s_1, \dots, s_m) &= F\left(f_1\left(\frac{x+y}{2}\right), s_2, \dots, s_m\right) = H\left(\frac{x+y}{2}\right) \\ &\leq \frac{1}{2}H(x) + \frac{1}{2}H(y) = F(t_1, s_2, \dots, s_m). \end{aligned}$$

Hence  $F$  is increasing in the first variable, and similarly  $F$  is increasing in the other variables. Thus  $F$  is increasing in each variable. By Lemma 7.2, we can apply Lemma 7.3 to conclude that  $*$  is monotonic.

Now suppose that  $\Phi(A)$  is  $\text{Supp}(\mathbb{R}^n)$  or  $\text{Supp}^+(\mathbb{R}^n)$ . Let  $s_1, \dots, s_m, t_1 \in D$  with  $s_1 < t_1$  be given. Let

$$K = \begin{cases} \text{conv}\{0, (2s_1 - t_1)e_1 + t_1e_2, t_1e_1 + (2s_1 - t_1)e_2\}, & \text{if } s_1 \geq 0, \\ [(2s_1 - t_1)e_1 + t_1e_2, t_1e_1 + (2s_1 - t_1)e_2], & \text{if } s_1 < 0, \end{cases}$$

$f_1 = h_K$ ,  $x = e_1$ , and  $y = e_2$ . When  $\Phi(A)$  is  $\text{Supp}^+(\mathbb{R}^n)$ , then  $s_1 \in D = [0, \infty)$ , so  $o \in K$ . Then a routine calculation shows that we again have  $f_1(x) = f_1(y) = t_1$  and  $f_1((x+y)/2) = s_1$ . If  $f_j = h_{\{s_j(e_1+e_2)\}}$ ,  $j = 2, \dots, m$ , then  $f_j(x) = f_j((x+y)/2) = f_j(y) = s_j$ , for  $j = 2, \dots, m$ . With these expressions in hand, the argument follows that in the previous paragraph.  $\square$

**Lemma 7.12.** *Let  $F : [0, \infty)^m \rightarrow \mathbb{R}$ ,  $m \geq 2$  be a positively homogeneous and convex function. If  $F$  is increasing in each variable, then there is an  $M \in \mathcal{K}^m$  such that  $F = h_M$  on  $[0, \infty)^m$  and  $h_M$  is increasing in each variable.*

*Proof.* For  $s_1, \dots, s_m \in \mathbb{R}$ , let

$$G(s_1, \dots, s_m) = F(s_1^+, \dots, s_m^+),$$

where  $t^+ = \max\{t, 0\}$  is the positive part of  $t$ . Then  $G = F$  on  $[0, \infty)^m$  and it is easy to check, using the properties of  $F$  and the fact that the positive part function is convex, that  $G$  is positively homogeneous and convex on  $\mathbb{R}^m$ . It then follows that there is an  $M \in \mathcal{K}^m$  such that  $G = h_M$  and hence  $F = h_M$  on  $[0, \infty)^m$ . Using the fact that the positive part function is increasing, we see that  $G = h_M$  is also increasing in each variable.  $\square$

**Example 7.13.** The assumption that  $F$  is increasing in each variable in Lemma 7.12 is essential. Indeed, [17, Theorem 2.2] implies that if  $C$  is a cone with nonempty interior properly contained in  $\mathbb{R}^n$ , then there is a positively homogeneous and convex function  $F$  on  $C$  that cannot be extended to the support function of a set in  $\mathcal{K}^n$ . A specific example is the

function  $F(x_1, x_2) = x_1 + x_2 - 2\sqrt{x_1x_2}$ , for  $x_1, x_2 \geq 0$ , which is the support function of the unbounded closed convex set  $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq x_1/(x_1 - 1), x_1 < 1\}$ .

**Theorem 7.14.** *Let  $\Phi(A) \subset \text{Cvx}(A)$  be a cone and let  $*$  be a pointwise operation on  $\Phi(A)^m$ ,  $m \geq 2$ , that is positively homogeneous. Let  $D^m$  be the proper domain of the function  $F$  associated with  $*$ . Suppose that  $\Phi(A)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(A)$ , and that  $*(f_1, \dots, f_m) \in \text{Cvx}(A)$ , whenever  $f_1, \dots, f_m \in \Gamma(A)$ . If*

- (a)  $D = \{0\}$ , or  $D = \mathbb{R}$ , or
- (b)  $D = [0, \infty)$ ,  $\Phi(A)$  has the point separation property, and  $*$  is monotonic,

then there is an  $M \in \mathcal{K}^m$  such that

$$(36) \quad (*(f_1, \dots, f_m))(x) = h_M(f_1(x), \dots, f_m(x)),$$

for all  $f_1, \dots, f_m \in \Phi(A)$  and all  $x \in A$ .

If in addition  $*(f_1, \dots, f_m) \geq 0$  whenever  $f_1, \dots, f_m \in \Gamma(A)$ , then (36) holds with  $M \in \mathcal{K}_o^m$ .

*Proof.* Since the cone  $\Phi(A)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(A) \subset \Phi(A)$ , it also has the  $\alpha$ -point property for any  $\alpha \leq |D|$ . By Lemma 7.1(a), the function  $F$  associated with  $*$  is positively homogeneous on its proper domain  $D^m$ , where  $D$  is a cone in  $\mathbb{R}$ , i.e.,  $D \in \{\{0\}, (-\infty, 0], [0, \infty), \mathbb{R}\}$ . Our assumptions and Lemma 7.10 imply that  $F$  is convex on  $D^m$ . When (a) holds, the case  $D = \{0\}$  is trivial and if  $D = \mathbb{R}$ , then  $F : \mathbb{R}^m \rightarrow \mathbb{R}$  is a positively homogeneous convex function, so there is an  $M \in \mathcal{K}^m$  with  $F = h_M$ . (This follows from the continuity of  $F$  and [25, Theorem 13.2].) Thus (36) holds.

If (b) holds, Lemma 7.3 implies that  $F$  is increasing in each variable on  $[0, \infty)^m$ . Then Lemma 7.12 yields an  $M \in \mathcal{K}^m$  such that  $F = h_M$ .

To prove the last statement, suppose that  $*(f_1, \dots, f_m) \geq 0$  whenever  $f_1, \dots, f_m \in \Gamma(A)$ . Since  $\Phi(A)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(A)$ , given  $s_j \in D$ ,  $j = 1, \dots, m$ , there are  $f_j \in \Gamma(A)$  and  $x \in A$  such that  $f_j(x) = s_j$ ,  $j = 1, \dots, m$ . Then

$$h_M(s_1, \dots, s_m) = F(f_1(x), \dots, f_m(x)) = (*(f_1, \dots, f_m))(x) \geq 0.$$

This implies that the set  $M$  in (36) may be replaced by  $\text{conv}(M \cup \{o\}) \in \mathcal{K}_o^m$ . □

Examples 7.7(a) and (b) show that the pointwise and positive homogeneity assumptions cannot be dropped in Theorem 7.14. The assumption that  $*$  is monotonic in Theorem 7.14(b) also cannot be dropped. To see this, recall that by Remark 7.9(b) with  $A = B^n$ , if  $m \geq 2$ , then  $\text{Cvx}^+(B^n)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(B^n)$  consisting of the nonnegative affine functions on  $B^n$ . The operation  $*$  :  $\text{Cvx}^+(B^n)^2 \rightarrow \text{Cvx}^+(B^n)$  with associated function

$$F(s, t) = \begin{cases} 0, & \text{if } st > 0, \\ \max\{s, t\}, & \text{otherwise,} \end{cases}$$

for  $s, t \in D = [0, \infty)$ , is positively homogeneous and has the property that  $f * g \in \text{Cvx}^+(B^n)$ , for all  $f, g \in \Gamma(B^n)$ . But if (36) held, then  $h_M(s, t) = F(s, t) = 0$ , for  $s, t > 0$  would imply  $M \subset (-\infty, 0]^2$ , which contradicts  $h_M(1, 0) = F(1, 0) = 1$ .

**Theorem 7.15.** *Let  $\Phi(A)$  be  $\text{Cvx}(A)$  or  $\text{Supp}(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  :  $\Phi(A)^m \rightarrow \Phi(A)$ ,  $m \geq 2$ , is pointwise and positively homogeneous.*
- (b) *There is an  $M \in \mathcal{K}^m$  with  $M \subset [0, \infty)^m$  such that  $*$  =  $\oplus_M$ , i.e., (36) holds for all  $f_1, \dots, f_m \in \Phi(A)$  and all  $x \in A$ .*

*When  $\Phi(A) = \text{Supp}(\mathbb{R}^n)$ , the assumption of positive homogeneity can be omitted.*

*All the above statements hold when  $\text{Cvx}(A)$  and  $\text{Supp}(\mathbb{R}^n)$  are replaced by  $\text{Cvx}^+(A)$  and  $\text{Supp}^+(\mathbb{R}^n)$ , respectively, where in (b)  $M \in \mathcal{K}_o^m$  is 1-unconditional (or, equivalently, the intersection of a 1-unconditional compact convex set with  $[0, \infty)^m$ ).*

*Proof.* First we show that (a) $\Rightarrow$ (b). Clearly, all four classes  $\Phi(A)$  under consideration have the point separation property and the  $m$ -linear interpolation property with associated subclass  $\Gamma(A) = \Phi(A)$ . By Lemma 7.11,  $F$  is increasing in each variable and  $*$  is monotonic, so Theorem 7.14 implies that (36) holds, where  $h_M = F$  is increasing in each variable.

Now assume that  $\Phi(A)$  is  $\text{Cvx}(A)$  or  $\text{Supp}(\mathbb{R}^n)$ . As  $h_M$  is increasing,  $h_M(-e_j) \leq h_M(0) = 0$ , for  $1 \leq j \leq m$ . This implies  $M \subset [0, \infty)^m$ .

In the remaining cases,  $\Phi(A)$  only contains nonnegative functions, so  $M \in \mathcal{K}_o^m$  by the last statement in Theorem 7.14. Using the fact that  $h_M$  is increasing in each variable, its subadditivity is inherited by the function  $G(s_1, \dots, s_m) = h_M(|s_1|, \dots, |s_m|)$ , for  $s_1, \dots, s_m \in \mathbb{R}$ . (See [13, p. 440] for



a similar argument.) Consequently,  $G$  is the support function of some  $M' \in \mathcal{K}^m$ , which must be 1-unconditional due to the symmetry of  $G$ . As  $h_M = h_{M'}$  on  $[0, \infty)^m$ , (36) holds with  $M$  replaced by  $M'$ .

If  $\Phi(A)$  consists of support functions, then  $*$  is positively homogeneous, by Lemma 7.5, so this assumption can be omitted.

For (b) $\Rightarrow$ (a), note that the operation  $\oplus_M$  is pointwise and positively homogeneous, so it suffices to show that  $\oplus_M : \Phi(A)^m \rightarrow \Phi(A)$  when  $M$  has the stated properties. To this end, we first claim that the right-hand side of (36) is a convex function. By Lemma 7.10, it suffices to show that  $h_M$  is increasing in each variable. To see this, recall that by (5), we have

$$h_M(s_1, \dots, s_m) = \sup\{s_1x_1 + \dots + s_mx_m : (x_1, \dots, x_m) \in M\}.$$

If  $M \subset [0, \infty)^m$ , then  $x_j \geq 0$ , for  $j = 1, \dots, m$ , so  $h_M$  is increasing on  $\mathbb{R}^m$ . If, on the other hand,  $M$  is 1-unconditional and  $s_j \geq 0$ ,  $j = 1, \dots, m$ , the supremum is attained for some  $x_j \geq 0$ ,  $j = 1, \dots, m$ , so  $h_M$  is increasing on  $[0, \infty)^m$ .

Therefore  $*(f_1, \dots, f_m)$  is a convex function, for all  $f_1, \dots, f_m \in \Phi(A)$ , where  $\Phi(A)$  is any of the four classes under consideration. This completes the proof when  $\Phi(A) = \text{Cvx}(A)$ . When  $\Phi(A) = \text{Cvx}^+(A)$ , we have  $M \in \mathcal{K}_o^m$  and hence  $*(f_1, \dots, f_m)$  is also nonnegative. If  $\Phi(A)$  is  $\text{Supp}(\mathbb{R}^n)$  or  $\text{Supp}^+(\mathbb{R}^n)$ , then [12, Corollary 6.6] and  $M \subset [0, \infty)^m$  (or  $M \cap [0, \infty)^m \subset [0, \infty)^m$  for the 1-unconditional  $M \in \mathcal{K}_o^n$ , respectively) imply that  $*(f_1, \dots, f_m) \in \Phi(A)$ .  $\square$

We can now characterize Volle's operations (29).

**Theorem 7.16.** *Let  $\Phi(A)$  be  $\text{Cvx}^+(A)$  or  $\text{Supp}^+(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  is pointwise and positively homogeneous.*
- (b) *Either  $f * g = af$  or  $f * g = ag$ , for some  $a \geq 0$  and all  $f, g \in \Phi(A)$ , or there is a monotone norm  $\|\cdot\|$  such that*

$$(37) \quad (f * g)(x) = (f +_{\|\cdot\|} g)(x) = \|(f(x), g(x))\|,$$

*for all  $f, g \in \Phi(A)$  and  $x \in A$ .*

*When  $\Phi(A) = \text{Supp}^+(\mathbb{R}^n)$ , the assumption of positive homogeneity can be omitted.*

*Proof.* If (a) holds, then by Theorem 7.15 with  $m = 2$ , there is a 1-unconditional  $M \in \mathcal{K}_o^2$  such that  $(f * g)(x) = h_M(f(x), g(x))$ , for all  $f, g \in$

$\Phi(A)$  and  $x \in A$ . If  $M$  is not full dimensional, then it must be a (possibly degenerate)  $o$ -symmetric line segment parallel to one of the coordinate axes. This implies that either  $f * g = af$  or  $f * g = ag$ , for some  $a \geq 0$ . Otherwise,  $M'$  is an  $o$ -symmetric convex body and  $h_{M'} = \|\cdot\|$  is a monotone norm.

The converse is clear since all the latter operations are positively homogeneous. □

**Theorem 7.17.** *Let  $\Phi(A)$  be  $\text{Cvx}^+(A)$  or  $\text{Supp}^+(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  is pointwise, positively homogeneous, and associative.*
- (b) *Either  $f * g = 0$ , or  $f * g = f$ , or  $f * g = g$ , for all  $f, g \in \Phi(A)$ , or  $f * g = f +_p g$ , for some  $1 \leq p \leq \infty$  and all  $f, g \in \Phi(A)$ .*

*When  $\Phi(A) = \text{Supp}^+(\mathbb{R}^n)$ , the assumption of positive homogeneity can be omitted.*

*Proof.* If (a) holds, the operation  $*$  must be one of the operations listed in Theorem 7.16. It follows that its associated function  $F$  is positively homogeneous, continuous, and associative. Then  $F$  must be one of the functions listed in Proposition 3.2. However, the cases when  $0 \neq p < 1$  are excluded since these do not preserve convexity.

The converse is clear since all the listed operations are positively homogeneous and associative. □

**Theorem 7.18.** *Let  $\Phi(A)$  be  $\text{Cvx}(A)$  or  $\text{Supp}(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  :  $\Phi(A)^2 \rightarrow \Phi(A)$  is pointwise, positively homogeneous, and associative.*
- (b) *Either  $f * g = 0$ , or  $f * g = f$ , or  $f * g = g$ , or  $(f * g)(x) = \max\{f(x), 0\}$ , or  $(f * g)(x) = \max\{g(x), 0\}$ , or  $f * g = f + g$ , for all  $f, g \in \Phi(A)$ , or there are  $1 \leq p \leq \infty$  and  $-\infty \leq q \leq 0$  such that*

$$(38) \quad (f * g)(x) = \begin{cases} (f +_p g)(x), & \text{if } f(x), g(x) \geq 0, \\ f(x), & \text{if } f(x) \geq 0, g(x) < 0, \\ g(x), & \text{if } f(x) < 0, g(x) \geq 0, \\ -(|f| +_q |g|)(x), & \text{if } f(x), g(x) < 0, \end{cases}$$

*for all  $f, g \in \Phi(A)$  and  $x \in A$ .*

*When  $\Phi(A) = \text{Supp}(\mathbb{R}^n)$ , the assumption of positive homogeneity can be omitted.*

*Proof.* Suppose that (a) holds. By Lemma 7.1(c), Theorem 7.15, and the continuity of support functions, (36) holds, where the function  $h_M : \mathbb{R}^2 \rightarrow \mathbb{R}$  is positively homogeneous, continuous, associative. By Lemma 7.11,  $h_M$  is also increasing in each variable. Then  $h_M$  must be one of the 40 functions listed in Theorem 4.7. By Remark 7.9(b),(c) and Lemma 7.10,  $h_M$  must be convex. However, the only nonzero convex functions among the 40 listed are the first and second functions in (12), with  $G = M_0$ , which can be written as  $F(s, t) = \max\{s, 0\}$  and  $F(s, t) = \max\{t, 0\}$ , respectively; the first function in (13), with  $p \geq 1$  and  $G = M_q$ , where  $q \leq 0$ , which leads to (38); the two functions in (16), namely  $F(s, t) = s$  and  $F(s, t) = t$ ; and the function in (17), with  $p = 1$  and  $t_0 = -1$ , which is just  $F(s, t) = s + t$ . (To check the convexity of the first function in (13), with  $p \geq 1$  and  $G = M_q$ , where  $q \leq 0$ , one shows that the Hessian of  $F$  vanishes and  $F_{ss} \geq 0$  for  $s, t < 0$ , and notes that  $F_s = 1$  on the coordinate axes.)

The converse is clear since all the listed operations are positively homogeneous and associative. □

It is interesting to identify the compact convex sets  $M$  in  $\mathbb{R}^2$  whose support functions  $h_M$  in (36) give rise to the seven operations provided by Theorem 7.18. They are, in order:  $M = \{o\}$ ,  $M = \{e_1\}$ ,  $M = \{e_2\}$ ,  $M = [o, e_1]$ ,  $M = [o, e_2]$ ,  $M = \{e_1 + e_2\}$ , and for (38),  $M$  equal to one of a family of convex bodies satisfying  $[e_1, e_2] \subset M \subset [0, 1]^2$ . In the latter case, if  $q = 0$ , then  $M$  is the part of the unit ball in  $l_p^2$ , contained in  $[0, \infty)^2$ , where  $1/p + 1/p' = 1$ , and this corresponds to the general  $L_p$  addition defined by (31). All these sets  $M$  are contained in  $[0, \infty)^2$  and the corresponding operations in Theorem 7.18 represent a new family that play the same role for arbitrary convex functions as  $L_p$  addition does for nonnegative convex functions.

Examples 7.7(a), (b), and (d) define operations that preserve nonnegativity and convexity. Thus Examples 7.7(a) and (b) show that the pointwise and positive homogeneity assumptions cannot be omitted in Theorems 7.15, 7.16, 7.17, and 7.18, and the associativity assumption in Theorems 7.17 and 7.18 cannot be removed due to Example 7.7(d).

**Lemma 7.19.** *Let  $* : (\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  be an arbitrary operation such that*

$$(39) \quad r(pK * qK) = (rpK) * (rqK),$$

*for all  $K \in \mathcal{K}_o^n$  and  $r > 0$  and some  $p, q > 0$  with  $p \neq q$ . If  $*$  has an identity  $I \in \mathcal{K}_o^n$  such that  $I * K = K = K * I$ , for all  $K \in \mathcal{K}_o^n$ , then  $I = \{o\}$ . The same holds true for operations  $* : (\mathcal{K}^n)^2 \rightarrow \mathcal{K}^n$ .*

*Proof.* Taking  $r = 1/p$  in (39), we obtain

$$r(pI * qI) = (rpI) * (rqI) = I * (rqI) = rqI,$$

and hence  $pI * qI = qI$ . With  $r = 1/q$  instead, we get  $pI * qI = pI$ , so  $pI = qI$ . Since  $p \neq q$  and  $I \in \mathcal{K}_o^n$  (or  $I \in \mathcal{K}^n$ , as appropriate), it follows that  $I = \{o\}$ .  $\square$

The following result is in the spirit of [12, Theorem 7.9]; note that the latter deals only with operations between  $o$ -symmetric sets.

**Theorem 7.20.** *Let  $*$  :  $(\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  be defined by*

$$(40) \quad h_{K*L}(x) = F(h_K(x), h_L(x)),$$

for some  $F : [0, \infty)^2 \rightarrow [0, \infty)$  and all  $K, L \in \mathcal{K}_o^n$  and  $x \in \mathbb{R}^n$ . Then  $F = h_M$  for some 1-unconditional  $M \in \mathcal{K}_o^n$ . Furthermore,

- (a)  $*$  is associative if and only if  $F$  is given by one of the four operations in Theorem 7.17(b) with  $f = h_K$  and  $g = h_L$ , where  $h_{K*L} = h_K * h_L$ ;
- (b)  $*$  is associative and has an identity if and only if  $* = +_p$ , for some  $1 \leq p \leq \infty$ .

For operations  $*$  :  $(\mathcal{K}^n)^2 \rightarrow \mathcal{K}^n$  satisfying (40) for some  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  and all  $K, L \in \mathcal{K}^n$  and  $x \in \mathbb{R}^n$ , we have  $F = h_M$  for some  $M \in \mathcal{K}^n$  such that  $M \subset [0, \infty)^2$ . In this case,

- (c)  $*$  is associative if and only if  $F$  is given by one of the seven operations in Theorem 7.18(b) with  $f = h_K$  and  $g = h_L$ , where  $h_{K*L} = h_K * h_L$ ;
- (d)  $*$  is associative and has an identity if and only if  $* = +$ .

*Proof.* Let  $*$  :  $(\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  be defined by (40). Define a pointwise operation  $\diamond : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  by  $h_K \diamond h_L = h_{K*L}$ . In view of (40), the first statement in the theorem follows directly from Theorem 7.15 with  $*$  replaced by  $\diamond$ . It is easy to check that  $\diamond$  is associative or has the identity property if  $*$  is associative or has an identity, respectively, where in the latter case we can appeal to Lemma 7.19 to see that the identity must be  $\{o\}$ . Then (a) and (b) are consequences of Theorem 7.17.

The argument for operations  $*$  :  $(\mathcal{K}^n)^2 \rightarrow \mathcal{K}^n$  is similar, where Theorem 7.18 is used instead of Theorem 7.17; note the operation in (38) does not have the identity property.  $\square$

Of the seven operations mentioned in Theorem 7.20(c), the one corresponding to (38) seems to be new. It is easily seen to be continuous in the Hausdorff metric and  $GL(n)$  covariant. As was remarked above,  $M$  is one of

a family of convex bodies satisfying  $[e_1, e_2] \subset M \subset [0, 1]^2$ ; the new operation represents a spectrum, parameterized by  $q$ , of extensions of  $L_p$  addition to arbitrary compact convex sets. The case  $q = 0$  corresponds to the extension given in [12, Example 6.7], but this is essentially just the usual  $L_p$  addition applied to the convex hulls of sets with the origin. We suggest the case  $q = -\infty$  instead as the most natural extension. It has the advantage, for example, of preserving convexity, unlike the extension proposed by Lutwak, Yang, and Zhang in [19] (see also [12, p. 2311]).

Another continuous, associative, and  $GL(n)$ -covariant operation  $*$ :  $(\mathcal{K}^n)^2 \rightarrow \mathcal{K}^n$  different from  $L_p$  addition is given in [12, Example 9.5]. Of course, both operations must coincide with  $L_p$  addition when restricted to the  $\alpha$ -symmetric sets, by [12, Corollary 7.10] (or Theorem 7.17). Note that while Theorem 7.20 can be regarded as a contribution to the understanding of projection covariant operations between arbitrary compact convex sets, the role of associativity is still not completely clear, since the general form of such operations is (33).

**Theorem 7.21.** *Let  $*$ :  $\text{Cvx}(A)^2 \rightarrow \text{Cvx}(A)$  be a pointwise operation. The following statements are equivalent:*

- (a)  *$*$  is strictly monotonic, associative, and satisfies  $0 * 0 = 0$ ;*
- (b)  *$*$  has the identity property;*
- (c)  *$f * g = f + g$ , for all  $f, g \in \text{Cvx}(A)$ .*

*Proof.* If (a) holds, then Lemma 7.10 shows that the function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  associated with  $*$  is convex and hence continuous on  $\mathbb{R}^2$ . By Lemma 7.1(b),  $*$  is also pointwise continuous, so Theorem 7.8 implies that there is a  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  such that (35) holds. If  $\phi(0) = a$ , then since  $0 * 0 = 0$  we have  $0 = (0 * 0)(x) = \phi^{-1}(2a)$ , for any  $x \in A$ , so  $\phi(0) = 2a$  and hence  $a = 0$ . Then (b) follows directly from (35) and  $\phi(0) = 0$ .

Assume that (b) holds. Again by Lemma 7.10, the function  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  associated with  $*$  is convex, and since  $*$  has the identity property, we have  $F(s, 0) = s$  and  $F(0, t) = t$ , for all  $s, t \in \mathbb{R}$ . If  $G(s, t) = F(s, t) - s - t$ , then  $G$  is also convex on  $\mathbb{R}^2$  and vanishes on the coordinate axes. For  $s, t \in \mathbb{R}$  with  $st \neq 0$ , choose  $s_0, t_0 \in \mathbb{R}$  such that  $(s, t)$  lies on the line segment  $[(s_0, 0), (0, t_0)]$ . Since  $G(s_0, 0) = G(0, t_0) = 0$ , the convexity of  $G$  implies that  $G(s, t) \leq 0$ . If  $G(s, t) < 0$ , we can choose  $s_1, t_1 \in \mathbb{R}$  such that the relative interior of the line segment  $[(s, t), (s_1, t_1)]$  meets one of the coordinate axes, say at  $(s_2, t_2)$ . Now the convexity of  $G$  implies that  $G(s_2, t_2) < 0$ , a contradiction. Hence  $G \equiv 0$  and so  $F(s, t) = s + t$ , yielding  $f * g = f + g$  for all  $f, g \in \text{Cvx}(A)$ . Therefore (c) holds.

Since (c) obviously implies (a), the proof is complete.  $\square$

**Example 7.22.** Define

$$(f * g)(x) = \log \left( e^{f(x)} + e^{g(x)} \right),$$

for all  $f, g \in \text{Cvx}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . The function  $F(s, t) = \log(e^s + e^t)$ ,  $s, t \in \mathbb{R}$ , is convex and (strictly) increasing in each variable, since its Hessian vanishes and  $F_{ss} > 0$ ,  $F_s > 0$ , and  $F_t > 0$  on  $\mathbb{R}^2$ , and corresponds to taking  $\phi(r) = e^r$ ,  $r \in \mathbb{R}$ , in Theorem 7.8. Therefore the operation  $*$ :  $\text{Cvx}(\mathbb{R}^n)^2 \rightarrow \text{Cvx}(\mathbb{R}^n)$  is pointwise continuous, strictly monotonic, and associative, but it is not positively homogeneous, nor does it have the identity property. Note however that  $*$  has the somewhat unnatural property that  $0 * 0 = \log 2$ , showing that the condition  $0 * 0 = 0$  cannot be omitted in Theorem 7.21(a).

Suppose we define

$$(f * g)(x) = \log \left( e^{f(x)} + e^{g(x)} \right) - \log 2,$$

for all  $f, g \in \text{Cvx}(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ . The function  $F(s, t) = \log(e^s + e^t) - \log 2$ ,  $s, t \in \mathbb{R}$ , is convex and (strictly) increasing in each variable, so Lemma 7.10 shows that  $*$ :  $\text{Cvx}(\mathbb{R}^n)^2 \rightarrow \text{Cvx}(\mathbb{R}^n)$ , and  $*$  is clearly strictly monotonic and satisfies  $0 * 0 = 0$ . This shows that the associativity assumption in Theorem 7.21(a) cannot be dropped.

The operation defined by (38) shows that the assumption in Theorem 7.21 that  $*$  is strictly monotonic cannot be replaced by the weaker assumption that  $*$  is monotonic. The pointwise assumption in Theorem 7.21 is also necessary, as can be seen for part (a) by defining

$$(f * g)(x) = f(x) + f(-x) + g(x) + g(-x)$$

and for part (b) by choosing  $x_0 \in A$  and defining

$$(f * g)(x) = e^{g(x_0)} f(x) + e^{f(x_0)} g(x),$$

for all  $f, g \in \text{Cvx}(A)$  and  $x \in A$ .

**Remark 7.23.** There does not seem to be a natural version of Theorem 7.21 that applies to the class  $\text{Cvx}^+(A)$ . Of course, the properties in Theorem 7.21(a),(b) are also satisfied when  $*$  =  $+_p$ ,  $1 \leq p \leq \infty$ . However, these properties do not even characterize  $L_p$  addition. A counterexample

is the pointwise operation  $*$  with associated function  $F$  given by (6) with  $\phi(t) = t \log(t + 1)$ ,  $t \geq 0$ . A routine calculation shows that  $\phi$  is continuous, strictly increasing, and convex on  $[0, \infty)$ , and that  $\phi/\phi'$  is convex on  $(0, \infty)$ . Mulholland [21, Theorem 2], improving an earlier result of Bosanquet [8] (see also [16, 106(ii), p. 88]) shows that these properties imply the midpoint convexity [21, Inequality (SM)] of  $F$ . As  $F$  is continuous, it follows that  $F$  is convex on  $[0, \infty)^2$ , so Lemma 7.10 implies that  $*$ :  $\text{Cvx}^+(A)^2 \rightarrow \text{Cvx}^+(A)$ . That  $*$  also satisfies all properties in Theorem 7.21(a),(b) is straightforward.

On the other hand, the class of pointwise operations  $*$ :  $\text{Cvx}^+(A)^2 \rightarrow \text{Cvx}^+(A)$  satisfying Theorem 7.21(a) is quite restricted. It is necessary that the function  $F$  associated with  $*$  is of the form (6) with a continuous and strictly monotonic function  $\phi: [0, \infty) \rightarrow [0, \infty)$ , and if  $\phi$  is differentiable, both  $\phi$  and  $-\log \phi'$  must be convex. To see this, note that Remark 7.9(b), Lemmas 7.2, 7.3, and 7.10, and Theorem 7.21(a) imply that  $F$  is convex and strictly increasing in each variable on its proper domain  $[0, \infty)^2$ . Lemma 7.12 shows that  $F$  is the restriction to  $[0, \infty)^2$  of the support function of a convex body, and hence continuous. By Lemmas 7.1(b) and 7.2,  $*$  is pointwise continuous. Therefore, by Theorem 7.8,  $*$  must take the form (35) and then  $F$  must be given by (6), where  $\phi: [0, \infty) \rightarrow [0, \infty)$  is continuous and strictly monotonic. When  $\phi$  is differentiable, it is not difficult to prove that any suitable function  $\phi$  must be convex on  $[0, \infty)$ ; see [21, Section 8] for further remarks on necessary conditions for [21, Inequality (SM)] to hold for  $\phi$ . The convexity of  $-\log \phi'$  is equivalent to  $\partial^2 F(s, t)/\partial s^2 \geq 0$ ,  $s, t \geq 0$ , which is satisfied as  $F$  is convex.

### 8. Orlicz addition of functions

Let  $\Phi(A)$  be a class of Borel functions on a nonempty subset  $A$  of  $\mathbb{R}^n$ ,  $n \geq 2$ . Define  $D$  by (34). Suppose that  $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}$  exists such that for  $d_1, \dots, d_m \in D$ , not all zero, there is a unique solution  $\lambda$  of

$$(41) \quad \varphi \left( \frac{d_1}{\lambda}, \dots, \frac{d_m}{\lambda} \right) = 1.$$

Then for  $f_1, \dots, f_m \in \Phi(A)$  and  $x \in A$ , we define  $(+_\varphi(f_1, \dots, f_m))(x)$  by

$$(42) \quad \varphi \left( \frac{f_1(x)}{(+_\varphi(f_1, \dots, f_m))(x)}, \dots, \frac{f_m(x)}{(+_\varphi(f_1, \dots, f_m))(x)} \right) = 1,$$

if  $f_1(x) \cdots f_m(x) \neq 0$  and by  $(+_\varphi(f_1, \dots, f_m))(x) = 0$  otherwise. We call the pointwise operation  $+_\varphi$  *Orlicz addition*.

Let  $\text{Cvx}_0^+([0, \infty)^m)$ ,  $m \in \mathbb{N}$ , be the set of convex functions  $\varphi : [0, \infty)^m \rightarrow [0, \infty)$  that are increasing in each variable and satisfy  $\varphi(o) = 0$  and  $\varphi(e_j) = 1$ ,  $j = 1, \dots, m$ . (The normalization is a matter of convenience and other choices are possible.) For  $\varphi \in \text{Cvx}_0^+([0, \infty)^m)$ , Orlicz addition of sets in  $\mathcal{K}_o^n$  was introduced in [13] (where the notation  $\Phi_m$  was used for  $\text{Cvx}_0^+([0, \infty)^m)$ ), and this operation corresponds to Orlicz addition of functions in  $\text{Supp}^+(\mathbb{R}^n)$  when defined by (42). It follows from the results in [13] that  $+_\varphi : \text{Supp}^+(\mathbb{R}^n)^m \rightarrow \text{Supp}^+(\mathbb{R}^n)$  and  $+_\varphi$  is positively homogeneous, pointwise continuous, monotonic, and has the identity property, but it is not in general commutative or associative.

More generally, when  $\varphi \in \text{Cvx}_0^+([0, \infty)^m)$  and  $\Phi(A)$  is a class of *non-negative* Borel functions on  $A$ , the Orlicz addition defined by (42) fits into a general framework also introduced in [13] and which we now briefly describe. In [13, Section 4], take  $Z = \Phi(A)^m$  and suppose that  $\mu$  is a Borel measure in  $\Phi(A)^m$ . With the notation of [13],  $L_\varphi(\Phi(A)^m, \mu)$  is a vector space and (4.2) there defines a norm  $\|\cdot\|_\varphi$  on this space. (The triangle inequality is proved explicitly in [13, Proposition 4.1] and the positive homogeneity is an easy consequence of the definition (4.2).) Following [13], define  $h_x : \Phi(A)^m \rightarrow [0, \infty)^m$  by

$$h_x(f_1, \dots, f_m) = (f_1(x), \dots, f_m(x)),$$

for all  $x \in A$ . We define

$$\begin{aligned} F_\mu(x) &= \|h_x\|_\varphi \\ &= \inf \left\{ \lambda > 0 : \int_{\Phi(A)^m} \varphi \left( \frac{f_1(x)}{\lambda}, \dots, \frac{f_m(x)}{\lambda} \right) d\mu(f_1, \dots, f_m) \leq 1 \right\}, \end{aligned}$$

for all  $x \in A$ . Some conditions on  $\mu$  are required to ensure that the previous integral is well defined and that  $h_x \in L_\varphi(\Phi(A)^m, \mu)$ , and for these we can follow [14]. We assume that  $\mu$  is a nonzero finite Borel measure in  $\Phi(A)^m$  with support  $C$  contained in a bounded separable subset of  $\Phi(A)^m$ , where the topology is the one generated by the supremum pseudonorm on  $\Phi(A)$ . The function on  $C \times A$  that maps  $(f_1, \dots, f_m, x)$  to the previous integrand is continuous in each of the first  $m$  variables and Borel measurable in  $x$ . Then the fact that  $C$  is separable ensures that this function is jointly Borel measurable, and  $F_\mu(x) \in \mathbb{R}$  as  $C$  is bounded; see the second paragraph of [14, Section 3].

Using the properties of  $\varphi$  and the fact that  $\|\cdot\|_\varphi$  is a norm, it is easy to show that if  $\Phi(A) = \text{Cvx}^+(A)$ , then  $F_\mu \in \text{Cvx}^+(A)$ .



The special case when  $\mu = \delta_{\{(f_1, \dots, f_m)\}}$ , for some fixed  $f_1, \dots, f_m \in \Phi(A)$ , leads, as in [13], to the formula

$$(+_\varphi(f_1, \dots, f_m))(x) = F_\mu(x) = \inf \left\{ \lambda > 0 : \varphi \left( \frac{f_1(x)}{\lambda}, \dots, \frac{f_m(x)}{\lambda} \right) \leq 1 \right\},$$

for  $x \in A$ . This formula can be used to define Orlicz addition  $+_\varphi$  on  $\Phi(A)^m$  and is equivalent to (42). Moreover, in some cases, we have  $+_\varphi : \Phi(A)^m \rightarrow \Phi(A)$ ; for example, when  $\Phi(A)$  is the class of nonnegative Borel or nonnegative continuous functions on  $A$ , or, as we have seen,  $\text{Cvx}^+(A)$  or  $\text{Supp}^+(\mathbb{R}^n)$ .

In [13], it was shown that Orlicz addition and  $M$ -addition of sets in  $\mathcal{K}_o^n$  are intimately related, and the same is true for addition of functions. Let  $\Phi(A)$  be a class of *arbitrary* Borel functions on a nonempty subset  $A$  of  $\mathbb{R}^n$ . Recall that for  $M \in \mathcal{K}^n$ , the  $M$ -sum  $\oplus_M(f_1, \dots, f_m)$  of  $f_1, \dots, f_m \in \Phi(A)$  is defined by (30). Our first simple observation is that if (30) holds, then (42) holds with  $\varphi = h_M$  and  $+_\varphi = \oplus_M$ . This is an immediate consequence of the homogeneity of  $h_M$ .

In particular, the operation  $\oplus_M : \Phi(A)^m \rightarrow \Phi(A)$  resulting from (38), where  $\Phi(A)$  is  $\text{Cvx}(A)$  or  $\text{Supp}(\mathbb{R}^n)$ , is also an Orlicz addition, since in (42) one need only take  $m = 2$  and  $\varphi = h_M$ . This shows that the definition (42) can be valid even for classes of functions that are not necessarily nonnegative. Now suppose that (42) defines an Orlicz addition on  $\Phi(A)^m$ , where  $\Phi(A)$  is  $\text{Cvx}(A)$  or  $\text{Supp}(\mathbb{R}^n)$ . By multiplying numerators and denominators of the components in (42) and using the uniqueness of the solution to (41), we see that  $+_\varphi$  is positively homogeneous. Since  $+_\varphi$  is clearly pointwise, when  $+_\varphi : \Phi(A)^m \rightarrow \Phi(A)$  and  $\Phi(A)$  is one of the four classes in Theorem 7.15, we conclude from this theorem that  $+_\varphi = \oplus_M$ , for some suitable  $M$ .

### 9. Pointwise operations on extended real-valued convex functions

Recall from Section 2 that  $\overline{\text{Cvx}}(A)$  is the class of extended-real-valued convex functions on a nontrivial convex set  $A$  in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $\overline{\text{Supp}}(\mathbb{R}^n)$  is the class of support functions of sets in  $\mathcal{CC}^n$ . In this section we consider pointwise operations  $* : \Upsilon(A)^m \rightarrow \Upsilon(A)$ ,  $m \geq 2$ , where  $\Upsilon(A) \subset \overline{\text{Cvx}}(A)$ , with associated function  $F$  defined on its proper domain  $D^m \subset (-\infty, \infty]^m$ , where

$$D = \{s \in (-\infty, \infty] : f(x) = s \text{ for some } f \in \Upsilon(A) \text{ and } x \in A\}.$$

For such operations, we define the Properties 1–10 listed in Section 5 in exactly the same way. We adopt the convention  $\infty \cdot 0 = 0 \cdot \infty = 0$ , so that a

positively homogeneous operation  $*$  satisfies  $0 * \dots * 0 = 0$ . If  $*$  is a pointwise operation with associated function  $F$ , then  $F : E \subset (-\infty, \infty]^m \rightarrow (-\infty, \infty]$ . The  $\alpha$ -point property and the point separation property are defined as in Section 7, and then Lemmas 7.1 and 7.3 hold with almost the same proofs, when the 3-point property instead of the 2-point property is assumed in Lemma 7.1(a).

Let  $m \geq 2$ . We say that  $\Upsilon(A)$  has the  $m$ -linear interpolation property with associated subclass  $\Gamma(A) \subset \Upsilon(A)$  if for all  $s_j, t_j \in D \cap \mathbb{R}$ ,  $j = 1, \dots, m$ , there are  $f_j \in \Gamma(A)$  and  $x, y \in A$  such that  $f_j(x) = s_j$ ,  $f_j(y) = t_j$ , and  $f_j$  is linear on  $[x, y]$ , for  $j = 1, \dots, m$ . Furthermore, we say that  $\Upsilon(A)$  has the  $m$ - $V$ -property if for all  $s_j, t_j \in D$  with  $s_j \leq t_j$ ,  $j = 1, \dots, m$ , there are  $f_j \in \Upsilon(A)$  and  $x, y \in A$  such that  $f_j(x) = f_j(y) = t_j$  and  $f_j((x + y)/2) = s_j$ , for  $j = 1, \dots, m$ .

For each  $m \geq 2$ , the classes  $\overline{\text{Cvx}}(A)$ ,  $\overline{\text{Cvx}}^+(A)$ ,  $\overline{\text{Supp}}(\mathbb{R}^n)$ , and  $\overline{\text{Supp}}^+(\mathbb{R}^n)$  all have the  $m$ -linear interpolation property for suitable subclasses (compare Remark 7.9) and the  $m$ - $V$ -property.

**Lemma 9.1.** *Let  $\Upsilon(A) \subset \overline{\text{Cvx}}(A)$  and suppose that  $* : \Upsilon(A)^m \rightarrow \Upsilon(A)$ ,  $m \geq 2$ , is a pointwise operation whose associated function  $F$  has proper domain  $D^m$ .*

- (a) *If  $\Upsilon(A)$  has the  $m$ -linear interpolation property, then  $F$  is convex on  $(D \cap \mathbb{R})^m$ .*
- (b) *If  $\Upsilon(A)$  has the  $m$ - $V$ -property, then  $F$  is increasing in each variable on  $D^m$ .*

*Proof.* For (a), just replace  $D$  by  $D \cap \mathbb{R}$  in the first part of the proof of Lemma 7.10. Part (b) follows from the argument in the first part of the proof of Lemma 7.11, using  $* : \Upsilon(A)^m \rightarrow \Upsilon(A)$  and the  $m$ - $V$ -property.  $\square$

For the next result, we need some notation. Let

$$t^\ulcorner = \begin{cases} 0, & \text{if } -\infty < t \leq 0, \\ \infty, & \text{if } 0 < t \leq \infty \end{cases} \quad \text{and} \quad t^\lrcorner = \begin{cases} 0, & \text{if } t \in \mathbb{R}, \\ \infty, & \text{if } t = \infty. \end{cases}$$

As both functions are increasing and convex, the functions  $f^\ulcorner(x) = f(x)^\ulcorner$  and  $f^\lrcorner(x) = f(x)^\lrcorner$  are convex whenever  $f : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is convex. If  $h_K \in \overline{\text{Supp}}^+(\mathbb{R}^n)$ , then  $h_K \geq 0$ , so

$$(h_K)^\ulcorner(x) = \begin{cases} 0, & \text{if } h_K(x) = 0, \\ \infty, & \text{otherwise} \end{cases} = h_{N(K,o)^\circ}(x),$$

for  $x \in \mathbb{R}^n$ , where  $N(K, o)$  is the normal cone of  $K$  at  $o$ , and

$$N(K, o)^\circ = \{y \in \mathbb{R}^n : x \cdot y \leq 0 \text{ for all } x \in N(K, o)\}$$

is its polar cone; see, for example, [25, Section 14] for details. In a similar way,  $h_K^\perp = h_{S(K^\circ, o)^\circ}$ , where  $K^\circ = \{y \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } x \in K\}$  is the polar set of  $K$ , and  $S(K^\circ, o) = \text{cl}\{\lambda x : x \in K^\circ, \lambda \geq 0\}$  is the support cone of  $K^\circ$  at  $o$ . This shows in particular that  $\lrcorner$  and  $\llcorner$  map functions in  $\overline{\text{Supp}}^+(\mathbb{R}^n)$  to support functions of nonempty closed convex cones.

The following result corresponds to Theorem 7.16. As in that theorem, (46) can also be formulated in terms of monotone norms.

**Theorem 9.2.** *Let  $\Upsilon(A)$  be  $\overline{\text{Cvx}}^+(A)$  or  $\overline{\text{Supp}}^+(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  :  $\Upsilon(A)^2 \rightarrow \Upsilon(A)$  is pointwise and positively homogeneous.*
- (b) *The operation  $*$  is one of the following:*

$$(43) \quad \begin{aligned} f * g &= f^\perp g^\perp, & f * g &= f^\perp, & f * g &= g^\perp, & f * g &= f^\perp g^\lrcorner, \\ f * g &= f^\lrcorner g^\perp, & f * g &= f^\perp g^\lrcorner + g^\perp, & f * g &= f^\lrcorner g^\perp + f^\perp, \\ f * g &= f^\perp + g^\perp, & \text{or } f * g &= f^\lrcorner g^\perp + f^\perp g^\lrcorner, \end{aligned}$$

or

$$(44) \quad f * g = f^\lrcorner + g^\perp, \quad \text{or } f * g = f^\perp + g^\lrcorner,$$

or there is an  $a > 0$  such that

$$(45) \quad \begin{aligned} f * g &= af + f^\lrcorner g^\perp, & f * g &= af + g^\perp, \\ f * g &= ag + f^\perp g^\lrcorner, & \text{or } f * g &= ag + f^\lrcorner, \end{aligned}$$

or there is a 1-unconditional set  $M \in \mathcal{CC}_o^2$  such that

$$(46) \quad (f * g)(x) = h_M(f(x), g(x)),$$

for all  $x \in \mathbb{R}^n$  and  $f, g \in \Upsilon(A)$ .

When  $\Upsilon(A) = \overline{\text{Supp}}^+(\mathbb{R}^n)$ , the assumption of positive homogeneity can be dropped.

*Proof.* For (a)  $\Rightarrow$  (b), note that  $\Upsilon(A)$  is a cone that has the  $|\mathbb{R}|$ -point property, the 2-linear interpolation property with associated subclass  $\Upsilon(A)$ , and the 2- $V$ -property. If  $\Upsilon(A) = \overline{\text{Cvx}}^+(A)$ , then  $*$  is positively homogeneous by assumption, while if  $\Upsilon(A) = \overline{\text{Supp}}^+(\mathbb{R}^n)$ , this follows from the proof of Lemma 7.5. In either case, Lemma 7.1(a) implies that the function  $F : [0, \infty]^2 \rightarrow [0, \infty]$  associated with  $*$  is also positively homogeneous. Lemma 9.1 implies that  $F$  is convex on  $(D \cap \mathbb{R})^2 = [0, \infty)^2$  and increasing in each variable on  $D^2 = [0, \infty]^2$ . We claim that  $F$  is lower semicontinuous on  $[0, \infty)^2$ . Indeed, let  $s^{(k)} = (s_1^{(k)}, s_2^{(k)}) \in [0, \infty)^2$ ,  $k \in \mathbb{N}$ , converge to  $s = (s_1, s_2) \in [0, \infty)^2$ . If  $s_i > 0$ , we may assume that  $s_i^{(k)} > 0$ , so

$$\alpha_k = \max \left\{ s_i / s_i^{(k)} : i \in \{1, 2\}, s_i > 0 \right\} > 0.$$

Since  $\alpha_k s_i^{(k)} \geq s_i$ ,  $i = 1, 2$ , we obtain

$$(47) \quad F(s^{(k)}) = \alpha_k^{-1} F(\alpha_k s^{(k)}) \geq \alpha_k^{-1} F(s) \rightarrow F(s),$$

as  $k \rightarrow \infty$ , proving the claim. Let  $\overline{F} : \mathbb{R}^2 \rightarrow [0, \infty]$  be defined by  $\overline{F} = F$  on  $[0, \infty)^2$  and  $\overline{F} = \infty$ , otherwise. Then  $\overline{F}$  is positively homogeneous, convex, proper (since  $F(0) = 0$ ), and lower semicontinuous. The same is true for the function  $H(s, t) = \overline{F}(|s|, |t|)$ ,  $s, t \in \mathbb{R}$ , as follows from the fact that on  $[0, \infty)^2$ ,  $\overline{F} = F$  is increasing in each variable. By [25, Theorem 13.2], there is an  $M \in \mathcal{CC}^2$  such that  $H = h_M$ . Due to the symmetry of  $H$ , the set  $M$  is 1-unconditional and hence  $M \in \mathcal{CC}_o^2$ . The fact that  $H = \overline{F} = F$  on  $[0, \infty)^2$  yields (46) whenever  $f(x), g(x) < \infty$ .

It remains to determine the values of  $F(s, \infty)$  and  $F(\infty, t)$ , for  $s, t \in [0, \infty)$ , and  $F(\infty, \infty)$ . Suppose that  $F(r, \infty) < \infty$  for  $0 < r < s$ , which implies that  $F(r, \infty) < \infty$  and  $F(r, 0) < \infty$  for  $0 < r < \infty$  due to the positive homogeneity and increasing property of  $F$ . Set  $f = h_{sB^n}$  and  $g = h_{(-\infty, 0]^n}$ , and note that  $f, g \in \Upsilon(A)$ . Then

$$(f * g)(x) = F(f(x), g(x)) = \begin{cases} F(s\|x\|, 0), & \text{if } x \in [0, \infty)^n, \\ F(s\|x\|, \infty), & \text{otherwise,} \end{cases}$$

is convex and finite on  $\mathbb{R}^n$ , and hence continuous, so  $F(s, \infty) = F(s, 0)$ . If, on the other hand,  $F(r, \infty) = \infty$ , for some  $r \in (0, s)$ , positive homogeneity yields  $F(s, \infty) = \infty$ . This and the same argument for  $F(\infty, t)$  gives

$$(48) \quad F(\cdot, \infty) \in \{F(\cdot, 0), \infty\} \quad \text{and} \quad F(\infty, \cdot) \in \{F(0, \cdot), \infty\},$$

on  $(0, \infty)$ . The positive homogeneity of  $F$  also implies that

$$(49) \quad F(0, \infty) \in \{0, \infty\}, \quad F(\infty, 0) \in \{0, \infty\}, \quad \text{and} \quad F(\infty, \infty) \in \{0, \infty\}.$$

We now distinguish several cases depending on the dimension of  $M$ . Assume first that  $M = \{0\}$ . The increasing property of  $F$ , (48), and (49) allow for the functions  $F(s, t) \equiv 0 \equiv h_M(s, t)$ ,  $F(s, t) = s^\perp t^\perp$ ,  $F(s, t) = s^\perp$ ,  $F(s, t) = t^\perp$ ,  $F(s, t) = s^\perp t^\lceil$ ,  $F(s, t) = s^\lceil t^\perp$ ,  $F(s, t) = s^\lceil t^\lceil$ ,  $F(s, t) = s^\perp t^\lceil + t^\perp$ ,  $F(s, t) = s^\lceil t^\perp + s^\perp$ ,  $F(s, t) = s^\perp + t^\perp$ , or  $F(s, t) = s^\lceil t^\perp + s^\perp t^\lceil$ . This yields the operations in (43).

When the 1-unconditional set  $M$  is 1-dimensional, it must be an  $\alpha$ -symmetric line segment or line parallel to a coordinate axis. In the latter case, the increasing property of  $F$ , (48), and (49) provide the possibilities  $F(s, t) = h_M(s, t)$ ,  $F(s, t) = s^\lceil + t^\perp$ , or  $F(s, t) = s^\perp + t^\lceil$ . This yields the operations in (44). When  $M$  is a line segment of length  $2a > 0$ , say, we obtain  $F(s, t) = h_M(s, t)$ ,  $F(s, t) = as + s^\lceil t^\perp$ ,  $F(s, t) = as + t^\perp$ ,  $F(s, t) = at + s^\perp t^\lceil$ , or  $F(s, t) = at + s^\perp$ . This yields the operations in (45).

Finally, when  $M$  is full-dimensional, there is an  $x \in M \cap (0, \infty)^2$ , so

$$F(s, t) = h_M(s, t) \geq (s, t) \cdot x \rightarrow \infty,$$

when  $(s, t) \in [0, \infty)^2$  is such that at least one component tends to  $\infty$ . The increasing property of  $F$  then implies  $F = h_M$  on  $[0, \infty]^2$ , yielding the operation (46) also obtained in the other cases for lower-dimensional  $M$ .

For (b)  $\Rightarrow$  (a), note that all the operations in (b) are clearly pointwise and positively homogeneous. Those listed in (43), (44), and (45) map  $\Upsilon(A)^2$  to  $\Upsilon(A)$  because  $\Upsilon(A)$  is closed under addition, the product of increasing nonnegative convex functions is convex, and the product of support functions of two closed convex cones is the support function of their intersection. Finally, the argument used in the proof of Theorem 7.15 for  $\text{Cvx}^+(A)$  and  $\text{Supp}^+(\mathbb{R}^n)$  shows that the operation in (46) maps  $\Upsilon(A)^2$  to  $\Upsilon(A)$ .  $\square$

In the statement of Theorem 9.2, one can replace  $M \in \mathcal{CC}_o^2$  with an  $M \in \mathcal{K}_o^2$  if the operations

$$(50) \quad f * g = f^\lceil + g^\lceil, \quad f * g = af + g^\lceil, \quad \text{and} \quad f * g = ag + f^\lceil,$$

for  $a \geq 0$ , are added to those already listed. Indeed, the only unbounded 1-unconditional sets in  $\mathcal{CC}_o^2$  are  $\mathbb{R}^2$  and the slabs  $\{x \in \mathbb{R}^2 : |x \cdot e_i| \leq a\}$ ,  $a \geq 0$ ,  $i = 1, 2$ , corresponding to the three functions in (50).

A counterpart of Theorem 7.17 can now be proved.

**Theorem 9.3.** *Let  $\Upsilon(A)$  be  $\overline{\text{Cvx}}^+(A)$  or  $\overline{\text{Supp}}^+(\mathbb{R}^n)$ . The following are equivalent.*

- (a) *The operation  $*$  is pointwise, positively homogeneous, and associative.*
- (b) *The operation  $*$  is one of the following:*

$$(51) \quad f * g = f^\perp g^\perp, \quad f * g = f^\perp + g^\perp, \quad f * g = f^\lrcorner + g^\lrcorner,$$

or

$$(52) \quad \begin{aligned} f * g &= af + g^\perp, & f * g &= ag + f^\perp, \\ f * g &= af + g^\lrcorner, & f * g &= ag + f^\lrcorner, \end{aligned}$$

with  $a \in \{0, 1\}$ , or  $f * g = 0$ ,  $f * g = f$ ,  $f * g = g$ , or there is a  $1 \leq p \leq \infty$  with

$$f * g = f +_p g,$$

for all  $f, g \in \Upsilon(A)$ .

When  $\Upsilon(A) = \overline{\text{Supp}}^+(\mathbb{R}^n)$ , the assumption of positive homogeneity can be dropped.

*Proof.* We have  $(s + t)^\perp = s^\perp + t^\perp$ ,  $(st)^\perp = s^\perp t^\perp$ ,  $(s^\perp)^\perp = s$ , and  $(s^\perp)^\lrcorner = s^\perp$ , for all  $0 \leq s, t \leq \infty$ , and the same relations with  $\perp$  and  $\lrcorner$  interchanged. Using this, one can show that the 11 operations in (b) are the only associative operations among those in (43), (44), (45), and (50). For the case where  $*$  is given by (46) with  $M \in \mathcal{K}^2$ , Theorem 7.17 yields the remaining operations. □

Theorems 9.2 and 9.3 yield positively homogeneous, projection covariant operations  $*$ :  $(\mathcal{CC}_o^n)^2 \rightarrow \mathcal{CC}_o^n$  different from  $M$ -addition. One example deriving from the first operation in (45) is

$$K * L = aK + (N(K, o)^\circ \cap S(L^\circ, o)^\circ),$$

where  $a > 0$  is fixed and  $K, L \in \mathcal{CC}_o^n$ . As  $*$  is pointwise when considered as operation on support functions,  $*$  is projection covariant. The operation

$$K * L = K + S(L^\circ, o)^\circ,$$

for  $K, L \in \mathcal{CC}_o^n$ , is positively homogeneous, projection covariant, and associative. Indeed, the corresponding operation on support functions is the first one in (52) with  $a = 1$ .

We transfer these results to operations on closed convex sets.

**Theorem 9.4.** *Let  $*$  :  $(\mathcal{CC}_o^n)^2 \rightarrow \mathcal{CC}_o^n$  be defined by*

$$h_{K*L}(x) = F(h_K(x), h_L(x)),$$

for some  $F : [0, \infty]^2 \rightarrow [0, \infty]$  and all  $K, L \in \mathcal{CC}_o^n$  and  $x \in \mathbb{R}^n$ . Then  $F$  is given by one of the 16 operations in Theorem 9.2 with  $f = h_K$  and  $g = h_L$ , where  $h_{K*L} = h_K * h_L$ . Furthermore,

- (a) *The operation  $*$  is associative if and only if  $F$  is given by the one of the 11 operations in Theorem 9.3 with  $f = h_K$  and  $g = h_L$ , where  $h_{K*L} = h_K * h_L$ .*
- (b) *The operation  $*$  is associative and satisfies the identity property (with identity  $\{0\}$ ) if and only if  $* = +_p$ , for some  $1 \leq p \leq \infty$ .*

*Proof.* Let  $*$  :  $(\mathcal{CC}_o^n)^2 \rightarrow \mathcal{CC}_o^n$  be defined as in the statement of the theorem. As in the proof of Theorem 7.20, it is easy to show that the pointwise operation  $\diamond : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$  defined by  $h_K \diamond h_L = h_{K*L}$  is associative or has the identity property if  $*$  is associative or has the identity property (with identity  $\{0\}$ ), respectively. The first claim in the statement of the theorem follows directly from Theorem 9.2 and (a) follows from Theorem 9.3. Of the operations listed in Theorem 9.3, only  $+_p$  has the identity property, yielding (b). □

It seems that results corresponding to Theorems 9.2 and 9.3 for operations  $*$  :  $\overline{\text{Supp}}(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}(\mathbb{R}^n)$  cannot be obtained in a similar fashion, because the increasing property of  $F$  is insufficient to establish the lower semicontinuity needed, as in the proof of Theorem 9.2, to identify  $\overline{F}$  (or  $H$ ) as  $h_M$  for a suitable set  $M$ . However, we can state the following result, corresponding to the implication (b) $\Rightarrow$ (c) in Theorem 7.21.

**Theorem 9.5.** *Let  $*$  :  $\overline{\text{Cvx}}(A)^2 \rightarrow \overline{\text{Cvx}}(A)$  be a pointwise operation. Then  $*$  has the identity property if and only if  $f * g = f + g$ , for all  $f, g \in \overline{\text{Cvx}}(A)$ .*

*Proof.* Suppose that  $*$  :  $\overline{\text{Cvx}}(A)^2 \rightarrow \overline{\text{Cvx}}(A)$  is pointwise, with associated function  $F : (-\infty, \infty]^2 \rightarrow (-\infty, \infty]$ , and that  $*$  has the identity property. Since  $\overline{\text{Cvx}}(A)$  has the 2-linear interpolation property,  $F$  is convex on  $\mathbb{R}^2$  by Lemma 9.1(a). The identity property for  $*$  implies that the convex set

$\{(s, t) \in \mathbb{R}^2 : F(s, t) < \infty\}$  contains the coordinate axes and therefore must be  $\mathbb{R}^2$ . Consequently,  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $* : \text{Cvx}(A)^2 \rightarrow \text{Cvx}(A)$ , so by Theorem 7.21,  $F(s, t) = s + t$ , for all  $s, t \in \mathbb{R}$ . Since  $\overline{\text{Cvx}}(A)$  has the 2- $V$ -property, Lemma 9.1(b) implies that  $F$  is increasing in each variable on  $(-\infty, \infty]^2$ . Therefore  $F = \infty$  on  $(-\infty, \infty]^2 \setminus \mathbb{R}^2$  and it follows that  $F(s, t) = s + t$ , for all  $s, t \in (-\infty, \infty]$ , and hence  $f * g = f + g$ , for all  $f, g \in \overline{\text{Cvx}}(A)$ . The converse is obvious.  $\square$

### 10. Arbitrary operations

In this section, we consider operations between functions that are not necessarily pointwise. It is assumed throughout that  $n \geq 2$ . The following proposition follows from results of Milman and Rotem [20, Theorems 2.2 and 6.1], the ideas behind which we employ often. See Section 5 for the definitions of the homothety and pointwise-continuous-from-below properties.

**Proposition 10.1.** *Let  $* : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$  be monotonic, associative, and have the homothety and identity properties. Then  $* = +_p$ , for some  $1 \leq p \leq \infty$ . The same holds for operations  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  that are in addition pointwise continuous from below.*

The statement of the relevant part of [20, Theorems 2.2 and 6.1] is as follows, where properties assumed are the natural analogues for sets of those defined in Section 5 for functions, continuity is respect to the Hausdorff metric, and continuity from below makes use of the ordering of sets by inclusion. Then Proposition 10.1 follows from Proposition 10.2 by defining  $h_K * h_L = h_{K*L}$ . Note that the identity in Proposition 10.2 is  $\{o\}$  (and, by Lemma 7.19, must be for operations  $* : (\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$ ), but other results in [20] allow for different identity elements.

**Proposition 10.2.** *Let  $* : (\mathcal{CC}_o^n)^2 \rightarrow \mathcal{CC}_o^n$  be monotonic, associative, and have the homothety and identity properties (with identity  $\{o\}$ ). Then  $* = +_p$ , for some  $1 \leq p \leq \infty$ . The same holds for operations  $* : (\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  that are in addition continuous from below.*

The second statement of Proposition 10.1 overlaps with Theorem 7.17, trading the pointwise assumption for monotonicity, homothety, identity, and continuity-from-below properties. With Theorem 7.15 in hand, we know that a pointwise operation  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  must be  $\oplus_M$  for some 1-unconditional compact convex set  $M$  in  $\mathbb{R}^2$ . Then  $* = \oplus_M$  is monotonic and



continuous, and if it is associative, then it also has the homothety property, as can be seen in view of (24) by defining  $\xi : \mathbb{N} \rightarrow [0, \infty)$  by  $\xi(1) = 1$  and  $\xi(k) = h_M(\xi(k - 1), 1)$ , for  $k = 2, \dots$ . However, the remaining assumption in Proposition 10.1, the identity property, is only satisfied if  $M$  contains  $e_1$  and  $e_2$  in its boundary.

The methods used in Proposition 10.1 do not appear to produce a result for associative operations  $*$  :  $\text{Supp}(\mathbb{R}^n)^2 \rightarrow \text{Supp}(\mathbb{R}^n)$  that overlaps with Theorem 7.18. Indeed, the homothety property is not satisfied when  $*$  is the operation corresponding to (38).

Our main goal is to strengthen and extend Propositions 10.1 and 10.2 via the weak homogeneity property defined in Section 5. See Theorems 10.5 and 10.8 and Corollary 10.9. Though the following result is not stated explicitly in [20], its proof is essentially given there.

**Theorem 10.3.** *Let  $\Upsilon(A)$  be a cone of nonnegative extended-real-valued functions on a nonempty set  $A$  in  $\mathbb{R}^n$  and let  $*$  :  $\Upsilon(A)^2 \rightarrow \Upsilon(A)$  be monotonic and have the homothety and identity properties. Then  $*$  is weakly homogeneous.*

*Proof.* Since  $*$  has the homothety property, there is a function  $\xi(m)$ ,  $m \in \mathbb{N}$ , such that (24) holds. The monotonicity and identity properties imply that for a suitable function  $f$ , we have  $f * f \geq f * 0 = f$  and hence  $\xi(2) \geq 1$ . If  $\xi(2) = 1$ , these properties imply  $f * g = \max\{f, g\}$ , as in [20, Proposition 4.2], and the claim follows. Hence we may assume that  $\xi(2) > 1$ .

In the proof of [20, Claim 3.2],  $\xi(m)$  is shown to be increasing and multiplicative, which forces it to be of the form  $\xi(m) = m^q$  for some  $q > 0$ . Let  $Q = \{(m/n)^q : m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}\}$ . As in the proof of [20, Claim 3.3], for  $r \geq 0$ ,  $s = (m/n)^q$ ,  $t = (m'/n')^q$ , and a suitable function  $f$ , we can use the homothety property to obtain

$$\begin{aligned} (rsf) * (rtf) &= \left( r \left( \frac{m}{n} \right)^q f \right) * \left( r \left( \frac{m'}{n'} \right)^q f \right) \\ &= \left( (mn')^q \left( \frac{r^{1/q}}{nn'} \right)^q f \right) * \left( (m'n)^q \left( \frac{r^{1/q}}{nn'} \right)^q f \right) \\ &= (mn' + m'n) \odot \left( \left( \frac{r^{1/q}}{nn'} \right)^q f \right) = r \left( \frac{m}{n} + \frac{m'}{n'} \right)^q f, \end{aligned}$$

where, as in (24), the number to the left of the symbol  $\odot$  indicates the number of times that the operation  $*$  is taken. Combining this with the

same equations for  $r = 1$  yields

$$(rsf) * (rtf) = r((sf) * (tf)),$$

as required. □

For  $x \in A$  and  $s \in (-\infty, \infty]$ , define

$$\delta_{s,x}(y) = \begin{cases} s, & \text{if } y = x, \\ \infty, & \text{otherwise,} \end{cases}$$

for  $y \in A$ . Then  $\delta_{s,x} \in \overline{\text{Cvx}}^+(A)$ , for  $s \in [0, \infty]$ . Note also that if  $x \neq y$ , then  $\delta_{s,x} + \delta_{t,y} = \infty$ , so it is natural to allow the identically infinite function in our considerations.

The restriction to  $r, s, t > 0$  in the next result is needed because  $0 \cdot \delta_{s,x} = 0 \neq \delta_{0,x}$ .

**Lemma 10.4.** *Let  $x \in A \subset \mathbb{R}^n$  and let  $\Upsilon(A)$  be a cone of nonnegative extended-real-valued functions such that  $\delta_{s,x} \in \Upsilon(A)$ , for  $s \geq 0$ . If  $*$  :  $\Upsilon(A)^2 \rightarrow \Upsilon(A)$  is monotonic and has the identity property, then there is an  $F_x : [0, \infty]^2 \rightarrow [0, \infty]$  such that*

$$(53) \quad \delta_{s,x} * \delta_{t,x} = \delta_{F_x(s,t),x},$$

for  $s, t \geq 0$ , where  $F_x(s, t) \geq \max\{s, t\}$  and  $F_x$  is increasing in each variable. If in addition  $*$  is associative or weakly homogeneous, then  $F_x$  satisfies the associativity equation or (vi) of Section 3 (weak homogeneity) for  $r, s, t > 0$ , respectively.

*Proof.* If  $x \in A$  and  $s, t \geq 0$ , then

$$\delta_{s,x} * \delta_{t,x} \geq \max\{\delta_{s,x} * 0, 0 * \delta_{t,x}\} = \max\{\delta_{s,x}, \delta_{t,x}\}.$$

Then (53) must hold for some  $F_x(s, t) \geq \max\{s, t\}$ . The monotonicity of  $*$  and (53) show that  $F_x$  is increasing in each variable.

Suppose that  $*$  is associative and  $s, t, u \geq 0$ . Then by (53),

$$\begin{aligned} \delta_{F_x(s, (F_x(t,u)),x)} &= \delta_{s,x} * \delta_{F_x(t,u),x} = \delta_{s,x} * (\delta_{t,x} * \delta_{u,x}) \\ &= (\delta_{s,x} * \delta_{t,x}) * \delta_{u,x} = \delta_{F_x(F_x(s,t),u),x}, \end{aligned}$$

so  $F_x$  satisfies the associativity equation.

Let  $r, s, t > 0$ . Then  $F_x(s, t) \geq \max\{s, t\} > 0$ . In view of (53), if  $*$  has the weak homogeneity property, there is a set  $Q$ , dense in  $[0, \infty]$  and containing 0, such that

$$(54) \quad \begin{aligned} \delta_{rF_x(s,t),x} &= r\delta_{F_x(s,t),x} = r(\delta_{s,x} * \delta_{t,x}) \\ &= (r\delta_{s,x}) * (r\delta_{t,x}) = \delta_{rs,x} * \delta_{rt,x} = \delta_{F_x(rs,rt),x}, \end{aligned}$$

for all positive  $s, t \in Q$ . This shows that  $F_x$  satisfies (vi) of Section 3 for  $r, s, t > 0$ . □

Henceforth in this section the set  $A$  is assumed to be a nontrivial convex set.

We now show that in Proposition 10.1 the homothety property can be replaced by weak homogeneity and an additional weaker assumption, defined as follows. We say that  $* : \overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$  (or  $* : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$ ) has the  $\delta$ -finite property if whenever  $x \in A$  (or  $x \in S^{n-1}$ , respectively), there is an  $s_x > 0$  such that

$$(55) \quad (\delta_{s_x,x} * \delta_{s_x,x})(x) < \infty$$

(or

$$(56) \quad (h_{H(s_x,x)} * h_{H(s_x,x)})(x) < \infty,$$

respectively, where  $H(s, x) = \{y \in \mathbb{R}^n : x \cdot y \leq s\}$ ). Note that when  $x \in S^{n-1}$ ,  $h_{H(s,x)}(y) = \delta_{s,x}(y)$ , for all  $y \in S^{n-1}$ . In either situation, if  $*$  has the homothety property then  $*$  has the  $\delta$ -finite property, since if  $x \in A$ , then by (24), we have

$$(\delta_{1,x} * \delta_{1,x})(x) = \xi(2)\delta_{1,x}(x) = \xi(2) < \infty$$

and similarly if  $x \in S^{n-1}$ , then (24) yields  $(h_{H(1,x)} * h_{H(1,x)})(x) = \xi(2) < \infty$ .

**Theorem 10.5.** *Let  $* : \overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$  (or  $* : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$ ) be monotonic, associative, weakly homogeneous, and have the identity and  $\delta$ -finite properties. Then  $* = +_p$ , for some  $1 \leq p \leq \infty$ .*

*Proof.* Suppose that  $* : \overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$  has all the listed properties. Fix  $x \in A$ . By Lemma 10.4, (53) holds for a nontrivial function  $F_x : [0, \infty)^2 \rightarrow [0, \infty]$  that is increasing in each variable and satisfies the associativity equation and (vi) of Section 3 (weak homogeneity) for  $r, s, t > 0$ . The  $\delta$ -finite property allows us to conclude that there is an  $s_x > 0$  such that

$F_x(s_x, s_x) < \infty$ , from which the increasing and weak homogeneity properties of  $F_x$  easily imply that  $F_x : [0, \infty)^2 \rightarrow [0, \infty)$ . By Proposition 3.3, the restriction of  $F_x$  to  $(0, \infty)^2$  satisfies the hypotheses of Proposition 3.4, which then implies that on  $(0, \infty)^2$ , we have  $F_x = 0$ , or  $F_x = \Pi_1$ , or  $F_x = \Pi_2$ , or  $F_x = M_{p(x)}$ , for some  $-\infty \leq p(x) \neq 0 \leq \infty$ . By Lemma 10.4, we have  $F_x(s, t) \geq \max\{s, t\}$ , for  $s, t \geq 0$ , and it follows that on  $(0, \infty)^2$ , we have  $F_x = M_{p(x)}$ , for some  $0 < p(x) \leq \infty$ . Since  $F_x$  is increasing in each variable on  $[0, \infty)^2$ , this yields  $F_x(s, 0) = F_x(0, s) = s$ , so by (53), we obtain

$$(57) \quad \delta_{s,x} * \delta_{0,x} = \delta_{0,x} * \delta_{s,x} = \delta_{s,x},$$

for  $s \geq 0$ .

We claim that

$$(58) \quad f * \delta_{0,x} = \delta_{0,x} * f = \delta_{f(x),x},$$

for each  $f \in \overline{\text{Cvx}}^+(A)$ . Indeed, arguing as in [20, Claim 3.4], we have

$$f * \delta_{0,x} \geq f * 0 = f \quad \text{and} \quad f * \delta_{0,x} \geq 0 * \delta_{0,x} = \delta_{0,x},$$

which shows that  $f * \delta_{0,x} \geq \delta_{f(x),x}$ . On the other hand, (57) implies

$$f * \delta_{0,x} \leq \delta_{f(x),x} * \delta_{0,x} = \delta_{f(x),x}.$$

Thus  $f * \delta_{0,x} = \delta_{f(x),x}$  and  $\delta_{0,x} * f = \delta_{f(x),x}$  follows similarly, proving the claim.

Now let  $f, g \in \overline{\text{Cvx}}^+(A)$ . By (58), we have

$$(59) \quad (f * g) * \delta_{0,x} = \delta_{(f*g)(x),x}.$$

Also, from (57), we have  $\delta_{0,x} * \delta_{0,x} = \delta_{0,x}$ , so following [20, Claim 3.5], and using the associativity of  $*$ , (58), and  $F_x = M_{p(x)}$ , we obtain

$$\begin{aligned} (f * g) * \delta_{0,x} &= (f * g) * (\delta_{0,x} * \delta_{0,x}) = f * (g * \delta_{0,x}) * \delta_{0,x} \\ &= f * (\delta_{0,x} * g) * \delta_{0,x} = (f * \delta_{0,x}) * (g * \delta_{0,x}) \\ &= \delta_{f(x),x} * \delta_{g(x),x} = \delta_{M_{p(x)}(f(x),g(x)),x}. \end{aligned}$$

Comparing this and (59), we see that

$$(60) \quad (f * g)(x) = M_{p(x)}(f(x), g(x)).$$

Next, we prove that  $p(x)$  is independent of  $x$ . The choice  $f = g \equiv 1$  in (60) implies that  $q(x) = 2^{1/p(x)}$  is convex on  $A$ . By (60) with  $f = g$ , the function  $q$  has the property that  $f q \in \overline{\text{Cvx}}^+(A)$  is convex for all  $f \in \overline{\text{Cvx}}^+(A)$ . Suppose that  $q$  is not constant on  $A$ . Then there is a non-degenerate line segment  $L = \{x_0 + ty_0 : 0 \leq t \leq 1\} \subset A$  such that on  $L$ ,  $q$  strictly decreases with  $t$ . Let  $f \in \overline{\text{Cvx}}^+(A)$  be such that  $f(x_0 + ty_0) = t$ , for  $0 \leq t \leq 1$ . Since  $h(t) = (f q)(x_0 + ty_0)$  is convex on  $[0, 1]$ , we have

$$h((1 - \lambda)0 + \lambda 1) \leq (1 - \lambda)h(0) + \lambda h(1),$$

for  $0 < \lambda < 1$ , which reduces to  $q(\lambda) \leq q(1)$ . This contradiction means that  $q$ , and therefore  $p$ , must be constant on  $A$ . The restriction  $p \geq 1$  is necessary to preserve convexity.

For operations  $* : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$ , we can argue as above, replacing  $\delta_{s,x}$  by the support function of the half-space  $H(s, x) = \{y \in \mathbb{R}^n : x \cdot y \leq s\}$ ,  $x \in S^{n-1}$ , to obtain (60). It then suffices to observe that the operation  $* : (\mathcal{K}_s^n)^2 \rightarrow \mathcal{K}^n$  defined by

$$h_{K*L}(x) = (h_K * h_L)(x) = M_{p(x)}(h_K(x), h_L(x))$$

is associative and projection covariant, from which [12, Theorem 7.9] implies that  $p(x) = p \geq 1$ , for all  $x \in S^{n-1}$ . □

Before describing examples showing that none of the assumptions in Theorem 10.5 can be omitted, we make the following general observation. Suppose that  $\diamond$  is an associative operation between functions in a given class of nonnegative functions. Define  $*$  by

$$(61) \quad f * g = \begin{cases} f \diamond g, & \text{if } f \neq 0 \text{ and } g \neq 0, \\ \max\{f, g\}, & \text{if } f = 0 \text{ or } g = 0, \end{cases}$$

for all  $f$  and  $g$  in the class. Then  $*$  is also associative and has the identity property. Moreover, if  $\diamond$  is weakly homogeneous or has the  $\delta$ -finite property, then  $*$  also has these properties, respectively.

**Example 10.6.** Consider the following five operations  $* : \overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$ .

(a) For all  $f, g \in \overline{\text{Cvx}}^+(A)$ , let  $f \diamond g = f$  and define  $f * g$  by (61). Of the properties listed in Theorem 10.5,  $*$  has all except that it is not monotonic.

(b) Let  $*$  be defined by (29), where the monotone norm is generated by a 1-unconditional planar convex body, containing  $e_1$  and  $e_2$  in its boundary,

different from the unit ball in  $l_p^2$ ,  $1 \leq p \leq \infty$ . Of the properties listed in Theorem 10.5,  $*$  has all except that it is not associative.

(c) For all  $f, g \in \overline{\text{Cvx}}^+(A)$ , let  $f \diamond g = f + g + 1$  and define  $f * g$  by (61). Of the properties listed in Theorem 10.5,  $*$  has all except that it is not weakly homogeneous. A more exotic example with the same properties is as follows. Let  $F : [0, \infty)^2 \rightarrow [0, \infty)$  be as in Remark 7.23 and define  $(f * g)(x) = F(f(x), g(x))$ , for all  $f, g \in \overline{\text{Cvx}}^+(A)$  and  $x \in A$  such that  $f(x), g(x) < \infty$ , and  $(f * g)(x) = \infty$ , otherwise. That  $*$  :  $\overline{\text{Cvx}}^+(A)^2 \rightarrow \overline{\text{Cvx}}^+(A)$  is an easy consequence of the fact that  $F$  is convex and increasing in each variable.

(d) Let  $f * g = f$ , for all  $f, g \in \overline{\text{Cvx}}^+(A)$ . Of the properties listed in Theorem 10.5,  $*$  has all except the identity property.

(e) If  $f \in \overline{\text{Cvx}}^+(A)$ , let  $M_f = \sup\{f(x) : x \in A\}$  and define

$$f \diamond g = M_f + M_g,$$

for all  $f, g \in \overline{\text{Cvx}}^+(A)$ . Of the properties listed in Theorem 10.5, the operation  $*$  defined in (61) has all except that it is not  $\delta$ -finite.

For operations  $*$  :  $\overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$ , the definitions in (a), (b), and (d) serve the same purpose. Examples for operations  $*$  :  $\overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$  corresponding to (c) and (e) above may be obtained as follows.

If  $f, g \in \overline{\text{Supp}}^+(\mathbb{R}^n)$ , let  $(f \diamond g)(x) = f + g + \|x\|$ , for  $x \in \mathbb{R}^n$ , and then define  $f * g$  by (61). Then  $*$  has all the properties listed in Theorem 10.5 except weak homogeneity. Note that the second example under (c) above does not work when  $f$  and  $g$  are support functions.

For  $f \in \overline{\text{Supp}}^+(\mathbb{R}^n)$ , let  $R_f = \sup\{f(u) : u \in S^{n-1}\}$ . If  $f, g \in \overline{\text{Supp}}^+(\mathbb{R}^n)$ , let  $(f \diamond g)(x) = (R_f + R_g)\|x\|$ , for  $x \in \mathbb{R}^n$ , and then define  $f * g$  by (61). Then  $*$  has all the properties listed in Theorem 10.5 except the  $\delta$ -finite property.

**Example 10.7.** For  $f, g \in \overline{\text{Cvx}}(\mathbb{R}^n)$ , the function

$$(f \square g)(x) = \inf_{y+z=x} \{f(y) + g(z)\},$$

for  $x \in \mathbb{R}^n$ , is called the *infimal convolution* of  $f$  and  $g$ . See, for example, [25, p. 34] or [27, p. 39]. The operation  $\square$  preserves convexity, but the infimum may be  $-\infty$ . However, we may consider  $\square$  as an operation  $\square : \overline{\text{Cvx}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Cvx}}^+(\mathbb{R}^n)$ . It is then positively homogeneous, monotonic, commutative, and associative. Since  $f \square \delta_{0,o} = f$ ,  $\square$  has an identity different from the function 0. Lemma 10.4 still holds with  $\Upsilon(A) = \overline{\text{Cvx}}^+(\mathbb{R}^n)$ , because the proof is valid when the function 0 is replaced by  $\delta_{0,o}$ ; in this case  $F_x(s, t) = \infty$ , if  $x \neq o$ ,

and  $F_o(s, t) = s + t$ . But Theorem 10.5 does not apply, as  $\square$  does not satisfy the  $\delta$ -finite property.

We say that  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  has the *point-bounded property* if whenever  $x \in S^{n-1}$ , there are reals  $c_x, s_x > 0$  such that for all  $f, g \in \text{Supp}^+(\mathbb{R}^n)$  with  $f(x), g(x) < s_x$ , we have  $(f * g)(x) < c_x$ .

If  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  is monotonic and has the homothety property, then  $*$  is point-bounded. Indeed, suppose that  $x \in S^{n-1}$  and  $f, g \in \text{Supp}^+(\mathbb{R}^n)$  satisfy  $f(x), g(x) < 1$ . If  $h = \max\{f, g\}$ , then  $h \in \text{Supp}^+(\mathbb{R}^n)$  and  $h(x) < 1$ , so by the homothety property,

$$(f * g)(x) \leq (h * h)(x) = \xi(2)h(x) < \xi(2),$$

so  $*$  has the point-bounded property with  $s_x = 1$  and  $c_x = \xi(2)$ .

By Proposition 10.3 and our remarks concerning the  $\delta$ -finite and point-bounded properties, Theorem 10.5 and the next result extend and strengthen Proposition 10.1. The following argument follows that of [20, Proposition 6.1] and makes use of the continuity-from-below assumption.

**Theorem 10.8.** *Let  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  be monotonic, associative, weakly homogeneous, pointwise continuous from below, and have the identity and point-bounded properties. Then  $* = +_p$ , for some  $1 \leq p \leq \infty$ .*

*Proof.* It is more convenient to work with sets than with functions, so we begin by noting that if  $* : \text{Supp}^+(\mathbb{R}^n)^2 \rightarrow \text{Supp}^+(\mathbb{R}^n)$  has the stated properties, we can define an operation  $* : (\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  by  $h_{K*L} = h_K * h_L$  and this operation is also monotonic, associative, and has  $\{o\}$  as identity. By [27, Theorem 1.8.15], this operation is continuous from below in the sense that if  $(K_i)$  and  $(L_i)$  are increasing sequences in  $\mathcal{K}_o^n$  converging to  $K$  and  $L$  in  $\mathcal{K}_o^n$ , respectively, then  $K_i * L_i \rightarrow K * L$ , as  $i \rightarrow \infty$ .

For  $\bar{K}, \bar{L} \in \mathcal{CC}_o^n$ , define

$$(62) \quad \bar{K} \diamond \bar{L} = \text{cl} \cup \{K * L : K \subset \text{relint } \bar{K}, L \subset \text{relint } \bar{L} \text{ and } K, L \in \mathcal{K}_o^n\}.$$

It is not hard to show (see the proof of [20, Proposition 6.1]) that  $\bar{K} \diamond \bar{L} \in \mathcal{CC}_o^n$ . As  $*$  is monotonic and continuous from below,  $K \diamond L = K * L$  for  $K, L \in \mathcal{K}_o^n$ , showing that  $\diamond$  is an extension of  $*$ .

The monotonicity of  $\diamond$  follows easily from that of  $*$ . If  $\bar{K} \in \mathcal{CC}_o^n$ , then

$$\begin{aligned} \bar{K} \diamond \{o\} &= \text{cl} \cup \{K * \{o\} : K \subset \text{relint } \bar{K} \text{ and } K \in \mathcal{K}_o^n\} \\ &= \text{cl} \cup \{K : K \subset \text{relint } \bar{K} \text{ and } K \in \mathcal{K}_o^n\} = \bar{K}, \end{aligned}$$

so  $\{o\}$  is an identity for  $\diamond$ . With this in hand, the weak homogeneity of  $*$  :  $(\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  (meaning that Property 5 of Section 5 holds for the support functions of the sets concerned) implies that  $\diamond$  is weakly homogeneous.

The associativity of  $\diamond$  will also follow easily from that of  $*$  if we can prove that

$$(63) \quad (\overline{K} \diamond \overline{L}) \diamond \overline{M} = \text{cl} \cup \{(K * L) * M : K \subset \text{relint } \overline{K}, L \subset \text{relint } \overline{L}, \\ M \subset \text{relint } \overline{M}, \text{ and } K, L, M \in \mathcal{K}_o^n\}.$$

To this end, note first that for  $K, L, M \in \mathcal{K}_o^n$  with  $K \subset \text{relint } \overline{K}, L \subset \text{relint } \overline{L}$ , and  $M \subset \text{relint } \overline{M}$ , we have

$$(K * L) * M = (K \diamond L) \diamond M \subset (\overline{K} \diamond \overline{L}) \diamond \overline{M},$$

where we used that  $\diamond$  is a monotonic extension of  $*$ . Therefore in (63), the right-hand side is contained in the left-hand side. For the reverse inclusion, let  $J, M \in \mathcal{K}_o^n$  be such that  $J \subset \text{relint } (\overline{K} \diamond \overline{L})$  and  $M \subset \text{relint } \overline{M}$ . Then there is a convex polytope  $P = \text{conv} \{p_1, \dots, p_m\}$  such that  $J \subset P \subset \text{relint } (\overline{K} \diamond \overline{L})$ . (Without loss of generality, we may assume that  $\dim(\overline{K} \diamond \overline{L}) = n$ . Then  $J \subset \text{int } (\overline{K} \diamond \overline{L})$ , so  $\lambda J \subset \text{int } (\overline{K} \diamond \overline{L})$  for some  $\lambda > 1$ . The existence of  $P$  then follows from [27, Theorem 1.8.19].) Now

$$p_i \in P \subset \text{relint } (\overline{K} \diamond \overline{L}) \\ = \text{relint} \cup \{K * L : K \subset \text{relint } \overline{K}, L \subset \text{relint } \overline{L} \text{ and } K, L \in \mathcal{K}_o^n\},$$

so there are  $K_i, L_i \in \mathcal{K}_o^n$  with  $K_i \subset \text{relint } \overline{K}, L_i \subset \text{relint } \overline{L}$ , and  $p_i \in K_i * L_i$ , for  $i = 1, \dots, m$ . Let  $K = \text{conv} \cup \{K_i : 1 \leq i \leq m\}$  and  $L = \text{conv} \cup \{L_i : 1 \leq i \leq m\}$ . Then the increasing property of  $*$  implies that  $P = \text{conv} \{p_1, \dots, p_m\} \subset K * L$  and therefore  $J \subset K * L$ . The increasing property of  $*$  yields  $J * M \subset (K * L) * M$ , completing the proof of the reverse inclusion. Thus  $\diamond$  is associative.

Define  $\diamond : \overline{\text{Supp}}^+(\mathbb{R}^n)^2 \rightarrow \overline{\text{Supp}}^+(\mathbb{R}^n)$  by  $h_{\overline{K}} \diamond h_{\overline{L}} = h_{\overline{K \diamond L}}$ . Then  $\diamond$  is monotonic, associative, weakly homogeneous, and has the identity property. We claim that  $\diamond$  has the  $\delta$ -finite property. Indeed, let  $x \in S^{n-1}$  and let  $c_x, s_x > 0$  witness the point-bounded property of  $*$ . Then

$$H_{s_x, x} \diamond H_{s_x, x} = \text{cl} \cup \{K * L : K, L \subset \text{relint } H_{s_x, x} \text{ and } K, L \in \mathcal{K}_o^n\}.$$

Since  $h_K(x), h_L(x) < s_x$  for all  $K, L$  in the right-hand side of the previous equation, we have  $h_{K * L}(x) < c_x$  and hence  $h_{H_{s_x, x} \diamond H_{s_x, x}}(x) < c_x < \infty$ , as required.



Since  $\diamond$  satisfies all the properties listed in Theorem 10.5, we have  $\diamond = +_p$ , for some  $1 \leq p \leq \infty$ . Let  $\varepsilon > 0$  and let  $K_\varepsilon = K + \varepsilon B^n$ , for each  $K \in \mathcal{K}_o^n$ . If  $K, L \in \mathcal{K}_o^n$ , then

$$K +_p L = K \diamond L \subset K * L \subset K_\varepsilon \diamond L_\varepsilon = K_\varepsilon +_p L_\varepsilon,$$

where the two inclusions follow directly from (62). Since  $+_p$  is continuous, on letting  $\varepsilon \rightarrow 0+$ , we obtain  $K * L = K +_p L$ , as required.  $\square$

The following corollary strengthens Proposition 10.2. Note that by Lemma 7.19, any identity for a weakly homogeneous operation  $*$  :  $(\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  must be  $\{o\}$ .

**Corollary 10.9.** *Let  $*$  :  $(\mathcal{CC}_o^n)^2 \rightarrow \mathcal{CC}_o^n$  be monotonic, associative, weakly homogeneous, and have the identity property (with identity  $\{o\}$ ) and the  $\delta$ -finite property. Then  $* = +_p$ , for some  $1 \leq p \leq \infty$ .*

*The same holds for operations  $*$  :  $(\mathcal{K}_o^n)^2 \rightarrow \mathcal{K}_o^n$  that are in addition continuous from below, if the  $\delta$ -finite property is replaced by the point-bounded property.*

We do not know if the hypothesis of pointwise continuity from below is necessary in Theorem 10.8 and Corollary 10.9. However, the operations defined in Example 10.6 for support functions show that none of the other assumptions in these results can be dropped.

### 11. The Asplund sum

Background for the following material can be found in [26] or [27, Sections 1.6 and 9.5], for example.

Denote by  $\text{CV}(\mathbb{R}^n)$ ,  $n \geq 2$ , the class of lower semicontinuous functions in  $\overline{\text{Cvx}}(\mathbb{R}^n) \setminus \{\infty\}$ . (We warn the reader that some authors use different notation for this class.)

The class of *log-concave functions* on  $\mathbb{R}^n$ , assumed upper semicontinuous and with the zero function excluded, is

$$\text{LC}(\mathbb{R}^n) = \{\exp(-f) : f \in \text{CV}(\mathbb{R}^n)\}.$$

If  $K \in \mathcal{CC}^n$ , then  $h_K \in \text{CV}(\mathbb{R}^n)$  and  $1_K \in \text{LC}(\mathbb{R}^n)$ .

For  $f, g \in \text{LC}(\mathbb{R}^n)$ , define the *Asplund sum* (or *sup-convolution*)  $f \star g$  by

$$f \star g(x) = \sup_{x=y+z} f(y)g(z),$$

for all  $x \in \mathbb{R}^n$ . Then  $1_K \star 1_L = 1_{K+L}$  for  $K, L \in \mathcal{CC}^n$ .

For  $f \in \text{CV}(\mathbb{R}^n)$ , let  $\mathcal{L}f$  be the *Legendre transform* of  $f$ , defined by

$$(\mathcal{L}f)(x) = \sup_{y \in \mathbb{R}^n} (x \cdot y - f(y)).$$

Define  $\mathcal{S} : \text{LC}(\mathbb{R}^n) \rightarrow \text{CV}(\mathbb{R}^n)$  by  $(\mathcal{S}f) = \mathcal{L}(-\log f)$ . Then  $\mathcal{S}(1_K) = h_K$ , for all  $K \in \mathcal{CC}^n$ . The map  $\mathcal{S} : \text{LC}(\mathbb{R}^n) \rightarrow \text{CV}(\mathbb{R}^n)$  is a bijection, is order preserving (i.e.,  $f \leq g \Rightarrow \mathcal{S}f \leq \mathcal{S}g$ ), and satisfies  $\mathcal{S}(f \star g) = \mathcal{S}f + \mathcal{S}g$ .

Rotem [26, Theorem 3] proves that if  $\mathcal{T} : \text{LC}(\mathbb{R}^n) \rightarrow \text{CV}(\mathbb{R}^n)$  is order preserving and satisfies  $\mathcal{T}(1_K) = h_K$  for all  $K \in \mathcal{CC}^n$ , and  $\oplus : \text{LC}(\mathbb{R}^n)^2 \rightarrow \text{LC}(\mathbb{R}^n)$  is an operation satisfying  $\mathcal{T}(f \oplus g) = \mathcal{T}f + \mathcal{T}g$ , then

$$(\mathcal{T}f)(x) = \frac{1}{c}(\mathcal{S}f)(cx),$$

for some  $c > 0$ , and  $f \oplus g = f \star g$ .

**Theorem 11.1.** *Let  $*$  :  $\text{LC}(\mathbb{R}^n)^2 \rightarrow \text{LC}(\mathbb{R}^n)$  have the identity property (with identity  $1_{\{o\}}$ ) and be such that the operation  $\diamond : \text{CV}(\mathbb{R}^n)^2 \rightarrow \text{CV}(\mathbb{R}^n)$  defined by  $f \diamond g = \mathcal{S}(\mathcal{S}^{-1}f * \mathcal{S}^{-1}g)$ , for  $f, g \in \text{CV}(\mathbb{R}^n)$ , is pointwise. Then  $* = \star$  is the Asplund sum.*

*Proof.* If  $f \in \text{CV}(\mathbb{R}^n)$ , then

$$f \diamond 0 = \mathcal{S}(\mathcal{S}^{-1}f * \mathcal{S}^{-1}0) = \mathcal{S}(\mathcal{S}^{-1}f * 1_{\{o\}}) = \mathcal{S}(\mathcal{S}^{-1}f) = f,$$

and similarly  $0 \diamond f = f$ , so  $\diamond$  has the identity property.

It is easy to see that for each  $m \geq 2$ ,  $\text{CV}(\mathbb{R}^n)$  has the  $m$ -linear interpolation property (with associated subclass the affine functions on  $\mathbb{R}^n$ , for example) and the  $m$ - $V$ -property, so Lemma 9.1 holds when  $\Upsilon(\mathbb{R}^n) = \text{CV}(\mathbb{R}^n)$ . Since  $\text{Cvx}(\mathbb{R}^n) \subset \text{CV}(\mathbb{R}^n)$ , Theorem 9.5 also holds when  $\Upsilon(\mathbb{R}^n) = \text{CV}(\mathbb{R}^n)$ , with the same proof. By Theorem 9.5,  $\diamond = +$ , so

$$f * g = \mathcal{S}^{-1}(\mathcal{S}f \diamond \mathcal{S}g) = \mathcal{S}^{-1}(\mathcal{S}f + \mathcal{S}g) = f \star g,$$

for all  $f, g \in \text{LC}(\mathbb{R}^n)$ , as required. □

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