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A class of complete minimal submanifolds and their associated families of genuine deformations

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Concerning the problem of classifying complete submanifolds of Euclidean space with codimension two admitting genuine isometric deformations, until now the only known examples with the maximal possible rank four are the real Kaehler minimal submanifolds classified by Dajczer-Gromoll [11] in parametric form. These submanifolds behave like minimal surfaces, namely, if simple connected either they admit a nontrivial one-parameter associated family of isometric deformations or are holomorphic.

In this paper, we characterize a new class of complete minimal genuinely deformable Euclidean submanifolds of rank four but now the structure of their second fundamental and the way it gets modified while deforming is quite more involved than in the Kaehler case. This can be seen as a strong indication that the above classification problem is quite challenging. Being minimal, the submanifolds we introduced are also interesting by themselves. In particular, because associated to any complete holomorphic curve in $\mathbb{C}^N$ there is such a submanifold and, beside, the manifold in general is not Kaehler.

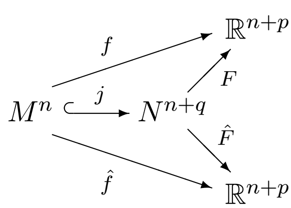
Some of the very basic question in the local and global theory of isometric immersions of Riemannian manifolds into Euclidean space remain in good part unanswered. For instance, outside some special cases it is not known which is the lowest codimension for which a given Riemannian manifold admits an isometric immersion. On one hand, there are several results that assure that a submanifold must be unique, that is, isometrically rigid, when lying in its lowest possible codimension. On the other hand, there are few theorems classifying isometrically deformable submanifolds and their deformations. This is due to the fact that rigidity is a "generic" property while being deformable is certainly not, and hence a situation harder to deal with.

The exception for the deformation problem is the case of hypersurfaces. In fact, in the local case the problem was mostly solved by Sbrana [21] and

Cartan [1] about a century ago; see [7] for details and a modern presentation. A solution to the problem for compact hypersurfaces was given by Sackstseder [20] and by Dajczer-Gromoll [10] in the complete case. But solving the deformation problem in codimension two turns out to be very challenging even in the more restrictive case of complete manifolds.

In dealing with the isometric deformation problem in higher codimension, it has to be taken into account that any submanifold of a deformable submanifold has the isometric deformations induced by the latter. In order to obtain classifications, it is natural to exclude this type of deformations and only study the remaining ones that were called genuine deformations in [5].

An isometric immersion $\hat{f}\colon M^n \to \mathbb{R}^{n+p}$ is a *genuine deformation* of a given isometric immersion $f\colon M^n \to \mathbb{R}^{n+p}$, $p \geq 2$, if there is no open subset $U \subset M^n$ along which $f|_U$ and $\hat{f}|_U$ extend isometrically. That $f\colon M^n \to \mathbb{R}^{n+p}$ and $\hat{f}\colon M^n \to \mathbb{R}^{n+p}$ *extend isometrically* means that there is an isometric embedding $j\colon M^n \hookrightarrow N^{n+q}$, $1 \leq q < p$, into a Riemannian manifold N^{n+q} and there are isometric immersions $F\colon N^m \to \mathbb{R}^{n+q}$ and $\hat{F}\colon N^m \to \mathbb{R}^{n+q}$ such that $f = F \circ j$ and $\hat{f} = \hat{F} \circ j$, i.e., the following diagram commutes:



The only general result for submanifolds that admit genuine deformations known at this time is the local result due to Dajczer-Florit [5]. In low codimension, they showed that genuine deformations are only possible for certain class of ruled submanifolds and gave a lower bound for the dimension of the rulings. In the special case of codimension two, in order to admit genuine deformations a submanifold without flat points must have rank ρ at most four at any point. By ρ we denote the rank of the Gauss map, that is, $\rho = n - \nu$, where ν stands for the standard index of relative nullity, namely, the dimension of the kernel of the second fundamental form.

In this paper, we are interested in the global problem of genuine deformations of isometric immersions with codimension two. In fact, we deal with the noncompact case since for compact submanifolds the deformation problem was already solved by Dajczer-Gromoll [12]. We point out that there exist several local results on genuine deformations in the special case of submanifold of rank $\rho = 2$ but these manifolds are never complete; see [5], [8] and [17]. In particular, there are the minimal ones that were parametrically classified in [4]. They admit a one-parameter associated family of isometric deformations whose geometric nature was recently described in [13].

At this time, there is only one classification result on deformations for complete noncompact submanifolds in Euclidean space with codimension two, namely, the one given in [11] of minimal but non-holomorphic isometric immersions of Kaehler manifolds. If simply connected such a submanifold admits a nontrivial one-parameter associated family of isometric deformations; see [9]. These submanifolds are ruled (i.e., foliated by complete Euclidean spaces) with rulings of codimension two and have rank $\rho = 4$ almost everywhere. As in the case of minimal surfaces, the associated family is obtained by composing its second fundamental form with an orthogonal parallel tensor in the tangent bundle given in terms of the complex structure of the manifold. The tensor amounts to a rotation of constant angle while keeping the the normal bundle and the induced connection unchanged. Basically, this is also the situation of the local case discussed to in the preceding paragraph.

In this paper, we parametrically construct and characterize a new class of complete minimal ruled submanifolds that also admit a one-parameter associated family of isometric deformations. As before, the rulings have codimension two and the rank is $\rho = 4$ almost everywhere. Moreover, the deformations are obtained while keeping unchanged the normal bundle and connection. But now, the second fundamental form of the deformed submanifold relates to the initial one in a much more complex form, in particular, no orthogonal tensor is involved.

It is an interesting question if the above two families of complete ruled minimal submanifolds exhaust all examples in the same class that admit genuine deformations. For instance, they may be examples such that the integral leaf exists but it is not totally geodesic. Of course, a much more challenging classification problem of complete submanifolds of rank four would be to drop one of the conditions, for instance being minimal or ruled. In the Kaehler case, it follows from [11] that there are a lot more examples without complete rulings. From the recent results in [14] it follows that this is also the situation in our case.

Finally, we observe that some arguments in this paper involve some unexpected long but straightforward computations that will be only sketched.

1. The 1-isotropic surfaces

In this section, we discuss some properties of the 1-isotropic surfaces in Euclidean space that are the basic tool for the construction of the minimal submanifolds that are the object of this paper.

Let $g\colon L^2 \to \mathbb{R}^{n+2}$ denote an isometric immersion of a two-dimensional oriented Riemannian manifold into Euclidean space. The k^{th}*-normal space* of g at $p \in L^2$ for $k \geq 1$ is given by

$$N_k^g(p) = \operatorname{span}\{\alpha_g^{k+1}(X_1, \ldots, X_{k+1}) : X_1, \ldots, X_{k+1} \in T_pL\}$$

where $\alpha_g^2 = \alpha_g\colon TL \times TL \to N_gL$ is the standard second fundamental form with values in the normal bundle and

$$\alpha_g^s\colon TL \times \cdots \times TL \to N_gL, \quad s \geq 3,$$

is the symmetric tensor called the s^{th}*-fundamental form* defined inductively by

$$\alpha_g^s(X_1, \ldots, X_s) = \left(\nabla^\perp_{X_s} \ldots \nabla^\perp_{X_3}\alpha_g(X_2, X_1)\right)^\perp.$$

Here $\nabla^\perp$ is the induced connection in the normal bundle N_gL and $(\ )^\perp$ means taking the projection onto the normal complement of $N_1^g \oplus \cdots \oplus N_{s-2}^g$ in N_gL.

Assume further that $g\colon L^2 \to \mathbb{R}^{n+2}$ is minimal and substantial. The latter means that the codimension cannot be reduced, in fact, not even locally since minimal surfaces are real analytic. Then, on an open dense subset of L^2 the normal bundle of g splits as

$$N_gL = N_1^g \oplus N_2^g \oplus \cdots \oplus N_m^g, \quad m = [(n-1)/2],$$

since all higher normal bundles have rank two except possible the last one that has rank one if n is odd; see [2], [4] or [22] for details. Moreover, the orientation of L^2 induces an orientation on each plane vector bundle N_s^g given by the ordered pair

$$\xi_1^s = \alpha_g^{s+1}(X, \ldots, X), \quad \xi_2^s = \alpha_g^{s+1}(JX, \ldots, X)$$

where $0 \neq X \in TL$ and J is the complex structure of L^2 determined by the metric and orientation.

If L^2 is simply connected, the generalized Weierstrass parametrization implies that there exists a one-parameter *associated family* of minimal immersions; see [18]. An alternative way to see this goes as follows: for each constant $\theta \in \mathbb{S}^1 = [0, \pi)$ consider the orthogonal parallel tensor field

$$J_\theta = \cos\theta I + \sin\theta J$$

where I is the identity map. Then, the symmetric section $\alpha_g(J_\theta \cdot, \cdot)$ of the bundle $\mathrm{Hom}(TL \times TL, N_g L)$ satisfies the Gauss, Codazzi and Ricci equations with respect to the normal bundle and normal connection of g; see [9]. Therefore, there exists an isometric minimal immersion $g_\theta \colon L^2 \to \mathbb{R}^{n+2}$ whose second fundamental form is

$$\alpha_{g_\theta}(X, Y) = \phi_\theta \alpha_g(J_\theta X, Y)$$

where $\phi_\theta \colon N_g L \to N_{g_\theta} L$ is the parallel vector bundle isometry that identifies the normal bundles. Explicitly, the immersion is given by the line integral

$$g_\theta(x) = \int_{p_0}^{x} g_* \circ J_\theta$$

where p_0 is any fixed point in L^2. In particular, we have that $g_{\theta *} = g_* \circ J_\theta$. Thus ϕ_θ is nothing else than parallel identification in $\mathbb{R}^{n+2}$ that identifies all normal subbundles N_j^g with $N_j^{g_\theta}$, $j \geq 1$, and for simplicity will be dropped from now on. It turns out that the associated family is *trivial* (i.e., each g_θ is congruent to g) if and only if g is a holomorphic curve with respect to some complex structure of the ambient space; cf. [4].

Remark 1. The case when L^2 above is non-simply-connected was considered in [14].

Now assume that $g \colon L^2 \to \mathbb{R}^{n+2}$, $n \geq 2$, is substantial and 1-*isotropic*. The latter means that the surface is minimal and that the ellipse of curvature at all points is a circle. Recall that the *ellipse of curvature* $\mathcal{E}^g(p) \subset N_1^g(p)$ of g at $p \in L^2$ is defined as

$$\mathcal{E}^g(p) = \{\alpha_g(X_\psi, X_\psi) : X_\psi = \cos\psi X + \sin\psi JX \quad \text{and} \quad \psi \in [0, 2\pi)\}$$

where $X \in T_pL$ has unit length.

The argument for the following result is basically due to Chern [3].

Proposition 2. *Let L_0 be the open subset of L^2 where* $\dim N_1^g(p) = 2$. *Then, $L^2 \setminus L_0$ consists at most of isolated points and the vector bundle $N_1^g|_{L_0}$ extends smoothly to a plane bundle over L^2 still denoted by N_1^g.*

Proof. The complexified tangent bundle $TL \otimes \mathbb{C}$ decomposes into the eigenspaces of the complex structure J corresponding to the eigenvalues i and $-i$ denoted by $T'L$ and $T''L$, respectively. The second fundamental form can be complex linearly extended to $TL \otimes \mathbb{C}$ with values in the complexified vector bundle $N_gL \otimes \mathbb{C}$ and then decomposed into its (p,q)-components, $p+q=2$, which are tensor products of p many 1-forms vanishing on $T''L$ and q many 1-forms vanishing on $T'L$. Since the surface is minimal the $(1,1)$-part of α_g vanishes, i.e., $\alpha_g(\partial_z, \bar{\partial}_z) = 0$ where z is a complex coordinate. We thus have the splitting

$$\alpha_g = \alpha^{(2,0)} + \alpha^{(0,2)} \quad \text{where} \quad \alpha^{(0,2)} = \overline{\alpha^{(2,0)}}. \tag{1}$$

The Codazzi equation implies that

$$\nabla^\perp_{\bar{\partial}_z} \alpha_g(\partial_z, \partial_z) = 0$$

which means that $\alpha^{(2,0)}$ is holomorphic as a $N_gL \otimes \mathbb{C}$-valued tensor field.

Since g is 1-isotropic, then $\dim N_1^g(p_0) < 2$ if and only if $\alpha_g(p_0) = 0$. Moreover, in (1) the summands are perpendicular with respect to the hermitian inner product. Hence, the zeros of α_g are precisely the zeros of $\alpha^{(2,0)}$. Since $\alpha^{(2,0)}$ is holomorphic, we conclude that its zeros are isolated, and hence $L^2 \setminus L_0$ consists at most of isolated points.

Let (U, z) be a complex chart around a point $p_0 \in L^2 \setminus L_0$ with $z(p_0) = 0$. Since $\alpha^{(2,0)}$ it is not identically zero and p_0 is a zero of it, around p_0 we may write

$$\alpha^{(2,0)} = z^m \alpha^{*(2,0)}$$

for a positive integer m, where $\alpha^{*(2,0)}$ is a tensor field of type $(2,0)$ with $\alpha^{*(2,0)}(p_0) \neq 0$. Since $\alpha^{(2,0)}(\partial_z, \partial_z) = \alpha_g(\partial_z, \partial_z)$ is isotropic, we have that $\alpha^{*(2,0)}(\partial_z, \partial_z)$ is also isotropic. Define an N_gL-valued tensor field on U by

$$\alpha^* = \alpha^{*(2,0)} + \overline{\alpha^{*(2,0)}}.$$

By definition, the (1,1)-part of α^* vanishes, hence it maps the unit tangent circle at each tangent plane into an ellipse which, in fact, is a circle of positive radius since $\alpha^{*(2,0)}(\partial_z, \partial_z)$ is isotropic. Now we may extend $N_1^g|_{L_0}$ to a plane bundle N_1^g defined over all L^2 by defining

$$N_1^g(p_0) = \text{span}\{\text{image}\,\alpha^*(p_0)\},$$

and this concludes the proof. □

To conclude this section, we show how to construct any 1-isotropic simply connected surface in parametric form. This procedure can easily be used to construct complete examples as was done in a quite similar situation in [11].

On a simply connected domain $U \subset \mathbb{C}$, a minimal surface $g\colon U \to \mathbb{R}^N$ has the generalized Weierstrass representation

$$g = \text{Re} \int^z \gamma dz$$

where the Gauss map $\gamma\colon U \to \mathbb{C}^N$ of g has the expression

$$\gamma = \frac{\beta}{2}\left(1 - \phi^2, i(1 + \phi^2), 2\phi\right)$$

being β holomorphic and $\phi\colon U \to \mathbb{C}^{N-2}$ meromorphic; see [18] for details. From [2] we have that g is 1-isotropic if and only if $(\phi', \phi') = 0$, where $(\ ,\ )$ stands for the standard symmetric inner product in $\mathbb{C}^{N-2}$. Hence, to construct any 1-isotropic surface start with a nonzero holomorphic map $\alpha_0\colon U \to \mathbb{C}^{N-4}$. Assuming that $\alpha_1\colon U \to \mathbb{C}^{N-2}$ has been defined already, set

$$\alpha_2 = \beta_2\left(1 - \phi_1^2, i(1 + \phi_1^2), 2\phi_1\right)$$

where $\phi_1 = \int^z \alpha_1 dz$ and $\beta_2 \neq 0$ is any holomorphic function. Then, the surface with Gauss map $\gamma = \alpha_1$, i.e., $g = \text{Re}\,\alpha_2$, is 1-isotropic.

2. The results

In this section, we state the results of this paper and leave the proofs for the following one.

Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $n \geq 3$, be a substantial 1-isotropic surface and let $\pi\colon \Lambda_g \to L^2$ denote the vector bundle of rank $n-2$ whose fibers are the orthogonal complement in the normal bundle N_gL of g of the extended first normal bundle N_1^g of g. Associated to g we consider the immersion

$F_g\colon \Lambda_g \to \mathbb{R}^{n+2}$ given by

$$F_g(p,v) = g(p) + v, \tag{2}$$

and denote by M^n the manifold Λ_g when it is endowed with the metric induced by F_g. By construction $F_g\colon M^n \to \mathbb{R}^{n+2}$ is an $(n-2)$-ruled submanifold with complete rulings, that is, there is an integrable tangent distribution of dimension $n-2$ whose leaves are mapped diffeomorphically by F onto complete affine subspaces of the ambient space.

In the sequel, we denote by $\mathcal{H}$ the tangent distribution orthogonal to the rulings. An embedded surface $j\colon L^2 \to M^n$ is called an *integral surface* of $\mathcal{H}$ if $j_*T_pL = \mathcal{H}(j(p))$ at every point $p \in L^2$.

Theorem 3. *Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $n \geq 4$, be a 1-isotropic substantial surface. Then the associated immersion $F_g\colon M^n \to \mathbb{R}^{n+2}$ is an $(n-2)$-ruled minimal submanifold with rank $\rho = 4$ on an open dense subset of M^n. Moreover, the rulings of F_g are complete and the integral surface L^2 of $\mathcal{H}$ is unique and totally geodesic. Furthermore, the metric of M^n is complete if L^2 is complete.*

Conversely, let $F\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, be an $(n-2)$-ruled minimal immersion with rank $\rho = 4$ on an open dense subset of M^n. Assume that $\mathcal{H}$ admits a totally geodesic integral surface $j\colon L^2 \to M^n$ which is a global cross section to the rulings. Then, the surface $g = F \circ j\colon L^2 \to \mathbb{R}^{n+2}$ is 1-isotropic and F can be parametrized by (2).

The vertical bundle $\mathcal{V} = \ker \pi_*$ of the submersion π decomposes orthogonally as

$$\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$$

on an open dense subset of L^2, where $\mathcal{V}^1$ denotes the plane bundle determined by N_2^g. We assume without loss of generality that this decomposition holds globally. In the sequel, we consider the orthogonal decomposition of the tangent bundle of M^n given by $TM = \mathcal{H} \oplus \mathcal{V}$ where we identify isometrically (and use the same notation) the subbundle $\mathcal{V}$ tangent to the rulings with the corresponding normal subbundle to g. Then, it follows from the proof that the relative nullity leaves of F are identified with the fibers of $\mathcal{V}^0$.

Let $\mathcal{J}$ be the endomorphism of TM such that $\mathcal{J}|_{\mathcal{H}}\colon \mathcal{H} \to \mathcal{H}$ is the almost complex structure in $\mathcal{H}$ determined by the orientation and restricted to $\mathcal{V}$ is the identity, and set

$$\mathcal{J}_\theta = \cos\theta I + \sin\theta \mathcal{J}.$$

Theorem 4. *Let $g\colon L^2 \to \mathbb{R}^{n+2}, n \geq 4$, be a simply connected 1-isotropic substantial surface. Then $F_g\colon M^n \to \mathbb{R}^{n+2}$ allows a smooth one-parameter family of minimal genuine isometric deformations $F_\theta\colon M^n \to \mathbb{R}^{n+2}$, $\theta \in \mathbb{S}^1$, such that $F_0 = F_g$ and each F_θ carries the same ruling and relative nullity leaves as F_g.*

Moreover, there is a parallel vector bundle isometry $\Psi_\theta\colon N_{F_g}M \to N_{F_\theta}M$ such that the relation between the second fundamental forms of F_θ and F_g is given by

$$\alpha_{F_\theta}(X,Y) = \Psi_\theta\big(R_{-\theta}\alpha_{F_g}(X,Y) + 2\kappa\sin(\theta/2)\beta(\mathcal{J}_{-\theta/2}X,Y)\big) \tag{3}$$

where R_θ is the rotation of angle θ on $N_{F_g}M$ that preserves orientation, κ is the radius of the ellipse of curvature of g and β is the traceless bilinear form defined by (19).

Remark 5. Quite similar arguments give that the above two results hold for dimension $n = 3$ and rank $\rho = 3$.

If g is holomorphic with respect to some parallel complex structure in $\mathbb{R}^{n+2}$, then taking a rotation of angle θ that preserves orientation in each N_s^g, $s \geq 2$, induces an intrinsic isometry S_θ on M^n.

Theorem 6. *If g is holomorphic then $F_g \circ S_{-\theta}$ is congruent to F_θ for any $\theta \in \mathbb{S}^1$.*

3. The proofs

Let $g\colon L^2 \to \mathbb{R}^{n+2}, n \geq 4$, be a substantial oriented minimal surface. We choose local positively oriented orthonormal frames $\{e_1, e_2\}$ in TL and $\{e_3, e_4\}$ in N_1^g such that

$$\alpha_g(e_1,e_1) = \kappa e_3 \quad \text{and} \quad \alpha_g(e_1,e_2) = \mu e_4$$

where κ, μ are the semi-axes of the ellipse of curvature. We also take a local orthonormal normal frame $\{e_5, \ldots, e_{n+2}\}$ such that $\{e_{2r+1}, e_{2r+2}\}$ is a positively oriented frame field spanning N_r^g for every even r. When $n = 2m + 1$ is odd, then e_{2m+1} spans the last normal bundle. We refer to $\{e_1, \ldots, e_{n+2}\}$

as an *adapted frame* of g and consider the one-forms

$$\omega_{ij} = \langle \tilde{\nabla} e_i, e_j \rangle \quad \text{for } 1 \leq i, j \leq n+2.$$

Then, we have from

$$\alpha_g^3(e_1, e_1, e_1) + \alpha_g^3(e_1, e_2, e_2) = 0 \quad \text{and} \quad \alpha_g^3(e_1, e_1, e_2) = \alpha_g^3(e_2, e_1, e_1)$$

that

$$\omega_{45} = -\frac{1}{\lambda} * \omega_{35} \quad \text{and} \quad \omega_{46} = -\frac{1}{\lambda} * \omega_{36} \tag{4}$$

where $\lambda = \mu/\kappa$, $*$ denotes the Hodge operator, i.e., $*\omega(e) = -\omega(Je)$, and J is the complex structure of L^2 induced by the metric and the orientation. We denote by

$$\begin{aligned} V &= a_1 e_1 + a_2 e_2, \quad W = b_1 e_1 + b_2 e_2, \\ Y &= c_1 e_1 + c_2 e_2, \quad \text{and} \quad Z = d_1 e_1 + d_2 e_2 \end{aligned}$$

the dual vector fields of $\omega_{35}, \omega_{36}, \omega_{45}$ and ω_{46}, respectively. Then (4) is equivalent to

$$Y = -\frac{1}{\lambda} JV \quad \text{and} \quad Z = -\frac{1}{\lambda} JW,$$

and hence

$$\lambda c_1 = a_2, \quad \lambda c_2 = -a_1, \quad \lambda d_1 = b_2, \quad \text{and} \quad \lambda d_2 = -b_1.$$

Clearly $F = F_g$ is an immersion and the horizontal bundle $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$ in the tangent bundle of M^n, i.e., we have at $(p, v) \in M^n$ that

$$T_{(p,v)} M = \mathcal{H}(p, v) \oplus \mathcal{V}(p, v).$$

Fixed $(p, v) \in M^n$, define a normal vector field δ_v in a neighborhood U of p by

$$\delta_v(q) = \sum_{j \geq 5} \langle v, e_j(p) \rangle e_j(q). \tag{5}$$

Let β_i, $1 \leq i \leq 2$, be the curves in M^n satisfying $\beta_i(0) = (p, v)$ given by

$$\beta_i(s) = (c_i(s), \delta_v(c_i(s)))$$

where each $c_i(s)$ is a smooth curve in L^2 such that $c_i'(0) = e_i(p)$. Then $Y_1, Y_2 \in T_{(p,v)}M$ where

$$Y_i = \beta_i'(0), \quad 1 \le i \le 2. \tag{6}$$

Let $G_i, H_i \in C^\infty(M)$, $1 \le i \le 2$, be the functions

$$G_i = t_2\omega_{56}^i + t_3\omega_{57}^i + t_4\omega_{58}^i, \quad H_i = -t_1\omega_{56}^i + t_3\omega_{67}^i + t_4\omega_{68}^i$$

where $\omega_{ij}^k = \omega_{ij}(e_k)$ and $t_j \in C^\infty(M)$ is defined by

$$t_j(q, w) = \langle w, e_{j+4}(q)\rangle, \quad 1 \le j \le 4.$$

The vertical bundle $\mathcal{V}$ can be orthogonally decomposed as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ where $\mathcal{V}^1$ denotes the plane bundle determined by N_2^g. Let $\{E_3, E_4\}$ and $\{E_5, \ldots, E_n\}$ be the local orthonormal frames of $\mathcal{V}^1$ and $\mathcal{V}^0$, respectively, such that

$$F_*E_j = e_{j+2}, \quad 3 \le j \le n.$$

Lemma 7. *The vectors $X_1, X_2 \in T_{(p,v)}M$ defined as*

$$X_i = Y_i + G_iE_3 + H_iE_4 - \sum_{j\ge 7} \langle \nabla_{e_i}^\perp \delta_v, e_j\rangle E_{j-2} \tag{7}$$

satisfy $X_1, X_2 \in \mathcal{H}(p, v)$ and

$$F_*X_1 = g_*e_1 - \varphi_1 e_3 - \frac{1}{\lambda}\varphi_2 e_4, \quad F_*X_2 = g_*e_2 - \varphi_2 e_3 + \frac{1}{\lambda}\varphi_1 e_4$$

where

$$\varphi_j = t_1^0 a_j + t_2^0 b_j, \quad j = 1, 2,$$

and $t_j^0 = t_j(p, v)$. Moreover, the normal space $N_F M(p, v)$ is spanned by

$$\xi = g_*(t_1^0 V(p) + t_2^0 W(p)) + e_3(p), \quad \eta = g_*(t_1^0 Y(p) + t_2^0 Z(p)) + e_4(p).$$

In particular, if g is 1-isotropic we have

$$\|X_1\|^2 = \|X_2\|^2 = \Omega^2 = 1 + \|t_1^0 V(p) + t_2^0 W(p)\|^2, \quad \langle X_1, X_2\rangle = 0$$

and

$$\|\xi\| = \Omega = \|\eta\|, \quad \langle \xi, \eta\rangle = 0.$$

Proof. We obtain from

$$F_*Y_i = g_{*_p}e_i(p) + \sum_{j\geq 3}\langle\nabla^\perp_{e_i}\delta_v, e_j\rangle(p)e_j(p)$$

that

$$F_*Y_i - \sum_{j\geq 5}\langle\nabla^\perp_{e_i}\delta_v, e_j\rangle(p)F_*E_{j-2} = g_{*_p}e_i(p) - \sum_{3\leq k\leq 4}\langle\nabla^\perp_{e_i}e_k, \delta_v\rangle(p)e_k(p).$$

On the other hand,

$$\begin{aligned}\langle\nabla^\perp_{e_i}\delta_v, e_5\rangle(p) &= -t^0_2\omega^i_{56}(p) - t^0_3\omega^i_{57}(p) - t^0_4\omega^i_{58}(p) = -G_i(p,v),\\ \langle\nabla^\perp_{e_i}\delta_v, e_6\rangle(p) &= t^0_1\omega^i_{56}(p) - t^0_3\omega^i_{67}(p) - t^0_4\omega^i_{68}(p) = -H_i(p,v)\end{aligned}$$

and

$$\begin{aligned}\langle\nabla^\perp_{e_i}e_3, \delta_v\rangle(p) &= t^0_1\omega^i_{35}(p) + t^0_2\omega^i_{36}(p) = t^0_1a_i(p) + t^0_2b_i(p),\\ \langle\nabla^\perp_{e_i}e_4, \delta_v\rangle(p) &= t^0_1\omega^i_{45}(p) + t^0_2\omega^i_{46}(p) = t^0_1c_i(p) + t^0_2d_i(p)\end{aligned}$$

where also $t^0_j = t_j(p,v)$, $3\leq j\leq 4$. Hence,

$$F_*X_i = g_*e_i - (t^0_1a_i + t^0_2b_i)e_3 - (t^0_1c_i + t^0_2d_i)e_4, \quad i = 1,2.$$

The remaining of the proof is immediate. □

Lemma 8. *The following equations hold:*

$$\xi_*E_3 = g_*V, \quad \xi_*E_4 = g_*W, \quad \textit{and} \quad \xi_* = 0 \quad \textit{on } \mathcal{V}^0, \tag{8}$$

$$\eta_*E_3 = g_*Y, \quad \eta_*E_4 = g_*Z, \quad \textit{and} \quad \eta_* = 0 \quad \textit{on } \mathcal{V}^0, \tag{9}$$

$$\begin{aligned}\xi_*X_1 = g_*\big(&(e_1(\varphi_1) - \kappa)e_1 + e_1(\varphi_2)e_2\\ &+ \omega^1_{12}J(t_1V + t_2W) + G_1V + H_1W\big)\\ &+ \kappa\varphi_1e_3 + (\omega^1_{34} + \lambda\kappa\varphi_2)e_4 + a_1e_5 + b_1e_6,\end{aligned}\tag{10}$$

$$\begin{aligned}\xi_*X_2 = g_*\big(&e_2(\varphi_1)e_1 + (e_2(\varphi_2) + \kappa)e_2\\ &+ \omega^2_{12}J(t_1V + t_2W) + G_2V + H_2W\big)\\ &- \kappa\varphi_2e_3 + (\omega^2_{34} + \lambda\kappa\varphi_1)e_4 + a_2e_5 + b_2e_6,\end{aligned}\tag{11}$$

$$\begin{aligned}\eta_*X_1 = g_*\big(&e_1(\psi_1)e_1 + (e_1(\psi_2) - \lambda\kappa)e_2\\ &+ \sigma\omega^1_{12}(t_1V + t_2W) - \sigma G_1JV - \sigma H_1JW\big)\\ &- (\omega^1_{34} - \kappa\psi_1)e_3 + \lambda\kappa\psi_2e_4 + \sigma a_2e_5 + \sigma b_2e_6,\end{aligned}\tag{12}$$

(13)
$$\begin{aligned}\eta_* X_2 = g_*\big((e_2(\psi_1) - \lambda\kappa)e_1 + e_2(\psi_2)e_2 \\ + \sigma\omega_{12}^2(t_1 V + t_2 W) - \sigma G_2 JV - \sigma H_2 JW\big) \\ - (\omega_{34}^2 + \kappa\psi_2)e_3 + \lambda\kappa\psi_1 e_4 - \sigma a_1 e_5 - \sigma b_1 e_6\end{aligned}$$

where $\sigma = 1/\lambda$ *and* $\psi_j = t_1^0 c_j + t_2^0 d_j$, $j = 1, 2$.

Proof. Let $\gamma(s) = (c(s), v(s))$ be a curve in M^n so that $\gamma(0) = (p, v)$ and $\gamma'(0) \in \mathcal{V}(p, v)$, that is, $c'(0) = 0$. We have that

$$\xi_*\gamma'(0) = \langle Dv/ds(0), e_5(p)\rangle g_* V(p) + \langle Dv/ds(0), e_6(p)\rangle g_* W(p),$$

or equivalently, that

$$\xi_*\gamma'(0) = \langle F_*\gamma'(0), e_5(p)\rangle g_* V(p) + \langle F_*\gamma'(0), e_6(p)\rangle g_* W(p).$$

From this we obtain (8). Similarly, we have

$$\eta_*\gamma'(0) = \langle F_*\gamma'(0), e_5(p)\rangle g_* Y(p) + \langle F_*\gamma'(0), e_6(p)\rangle g_* Z(p)$$

from which we obtain (9).

Making use of Lemma 7 and the Gauss and Weingarten formulas for g we compute equations (10) to (13). We only argue for (10) since the proof of the other equations is completely similar. We have from (7) and (8) that

$$\xi_* X_i = \xi_* Y_i + G_i g_* V + H_i g_* W, \quad 1 \le i \le 2.$$

In view of (6) and since

$$(\xi \circ \beta_i)(s) = t_1^0 g_* V(c_i(s)) + t_2^0 g_* W(c_i(s)) + e_3(c_i(s))$$

we obtain

$$\begin{aligned}\xi_* Y_1 = t_1^0\big(g_*\nabla_{e_1} V + \alpha_g(e_1, V)\big)(p) + t_2^0\big(g_*\nabla_{e_1} W + \alpha_g(e_1, W)\big)(p) \\ - \kappa(p) g_* e_1(p) + \nabla_{e_1}^{\perp} e_3(p),\end{aligned}$$

and the desired formula for $\xi_* X_1$ follows by direct computations. □

Lemma 9. *If g is a 1-isotropic surface, then the shape operators of F_g with respect to the orthonormal tangent frame*

$$E_i = X_i/\Omega, \quad i=1,2, \quad \textit{and} \quad F_*E_j = e_{j+2}, \quad 3 \le j \le n,$$

vanish along $\mathcal{V}^0$ and restricted to $\mathcal{H}\oplus\mathcal{V}^1$ are given by

(14)
$$A_\xi = \begin{bmatrix} \kappa+h_1 & h_2 & r_1 & s_1 \\ h_2 & -\kappa-h_1 & r_2 & s_2 \\ r_1 & r_2 & 0 & 0 \\ s_1 & s_2 & 0 & 0 \end{bmatrix}, \quad A_\eta = \begin{bmatrix} h_2 & \kappa-h_1 & r_2 & s_2 \\ \kappa-h_1 & -h_2 & -r_1 & -s_1 \\ r_2 & -r_1 & 0 & 0 \\ s_2 & -s_1 & 0 & 0 \end{bmatrix}$$

where $r_i\Omega=-a_i$, $s_i\Omega=-b_i$,

$$h_i = -\frac{1}{\Omega^2}\big(t_1(e_i(a_1)-a_2B_i-b_1\omega_{56}^i)+t_2(e_i(b_1)-b_2B_i+a_1\omega_{56}^i) + t_3(a_1\omega_{57}^i+b_1\omega_{67}^i)+t_4(a_1\omega_{58}^i+b_1\omega_{68}^i)\big)$$

and $B_i=\omega_{12}^i+\omega_{34}^i$, $i=1,2$.

Proof. Since g is 1-isotropic, then (10) to (13) hold for $\psi_1=\varphi_2$ and $\psi_2=-\varphi_1$. On the other hand, a straightforward computation shows that the Ricci equations

$$\langle R^\perp(e_1,e_2)e_\alpha,e_\beta\rangle=0$$

for $\alpha=3,4$ and $\beta=5,6$ are equivalent to

$$\begin{aligned} e_1(a_2)-e_2(a_1)+a_1B_1+a_2B_2-b_2\omega_{56}^1+b_1\omega_{56}^2&=0,\\ e_1(b_2)-e_2(b_1)+b_1B_1+b_2B_2+a_2\omega_{56}^1-a_1\omega_{56}^2&=0,\\ e_1(a_1)+e_2(a_2)-a_2B_1+a_1B_2-b_1\omega_{56}^1-b_2\omega_{56}^2&=0,\\ e_1(b_1)+e_2(b_2)-b_2B_1+b_1B_2+a_1\omega_{56}^1+a_2\omega_{56}^2&=0, \end{aligned}$$

and for $\alpha=3,4$ and $\beta=7,8$ are equivalent to

$$\begin{aligned} a_2\omega_{57}^1-a_1\omega_{57}^2+b_2\omega_{67}^1-b_1\omega_{67}^2&=0,\\ a_2\omega_{58}^1-a_1\omega_{58}^2+b_2\omega_{68}^1-b_1\omega_{68}^2&=0,\\ a_1\omega_{57}^1+a_2\omega_{57}^2+b_1\omega_{67}^1+b_2\omega_{67}^2&=0,\\ a_1\omega_{58}^1+a_2\omega_{58}^2+b_1\omega_{68}^1+b_2\omega_{68}^2&=0. \end{aligned}$$

We thus have that

$$\langle A_\xi E_i,E_j\rangle=-\langle F_*E_i,\xi_*E_j\rangle \text{ and } \langle A_\eta E_i,E_j\rangle=-\langle F_*E_i,\eta_*E_j\rangle,\ 1\le i,j\le n,$$

and the result follows by a straightforward computation. □

Proof of Theorem 3. We first prove the converse. Let $F\colon M^n \to \mathbb{R}^{n+2}$, $n \geq 4$, be an $(n-2)$-ruled minimal immersion with rank $\rho = 4$ on an open dense subset. Then the tangent bundle splits as $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{H}$ is orthogonal to the rulings and $\mathcal{V}$ splits as $\mathcal{V} = \mathcal{V}^1 \oplus \mathcal{V}^0$ with the fibers of $\mathcal{V}^0$ being the relative nullity leaves.

The normal space of the surface $g = F \circ j$ at any point $x \in L^2$ is given by

$$N_g L(x) = F_*(j(x))\mathcal{V} \oplus N_F M(j(x)).$$

Let Λ_g be the subbundle of the normal bundle of g whose fiber at $x \in L^2$ is $F_*(j(x))\mathcal{V}$. Observe that

$$F(p) - g \circ \pi(p) = F(p) - F(j(x)) \in F_*(j(x))\mathcal{V}$$

for any $p \in M^n$, where $x = \pi(p)$, since p and $j(x)$ belong to the same leaf of $\mathcal{V}$. Since F maps diffeomorphically the leaves of $\mathcal{V}$ onto complete affine subspaces, it follows that the map $T\colon M^n \to \Lambda_g$ given by

$$T(p) = (\pi(p), F(p) - g \circ \pi(p))$$

is a global diffeomorphism. Clearly the immersion $\tilde{F} = F \circ T^{-1}$ satisfies

$$\tilde{F}(x, v) = g(x) + v,$$

i.e., $\tilde{F} = F_g$ is of the form (2). Identifying M^n with Λ_g via T, we have that $F = F_g$ and j is the zero section of Λ_g.

It remains to show that g is 1-isotropic. Being j totally geodesic, we have that

$$\alpha_g(X, Y) = \alpha_F(j_* X, j_* Y) \tag{15}$$

for all $X, Y \in TL$. This and our assumptions imply that g is minimal. The horizontal and the vertical bundles satisfy

$$F_*(p, v)\mathcal{V} = (N_1^g(p))^\perp \subset N_g L(p),$$
$$F_*(p, v)\mathcal{H} \subseteq g_* T_p L \oplus (\Lambda_g(p))^\perp,$$
$$N_F M(p, v) \subseteq g_* T_p L \oplus (\Lambda_g(p))^\perp$$

and now (15) yields $N_1^g = \Lambda_g^\perp$.

Let $\{e_1, \dots, e_{n+2}\}$ be an adapted frame of g. Setting

$$g_{ij} = \langle X_i, X_j \rangle_F, \quad b^\xi_{ij} = \langle \xi_* X_i, F_* X_j \rangle$$
$$\text{and} \quad b^\eta_{ij} = \langle \eta_* X_i, F_* X_j \rangle, \quad i, j = 1, 2,$$

and using Lemma 7 and Lemma 8, we find that

$$g_{11} = 1 + \varphi_1^2 + \sigma^2\varphi_2^2, \quad g_{12} = (1 - \sigma^2)\varphi_1\varphi_2, \quad g_{22} = 1 + \varphi_2^2 + \sigma^2\varphi_1^2,$$

and

$$\begin{aligned}
b^\xi_{11} &= e_1(\varphi_1) - \kappa - \omega^1_{12}\varphi_2 + G_1a_1 + H_1b_1 - \kappa\varphi_1^2 - \sigma\varphi_2(\omega^1_{34} + \mu\varphi_2),\\
b^\xi_{12} &= e_1(\varphi_2) + \omega^1_{12}\varphi_1 + G_1a_2 + H_1b_2 - \kappa\varphi_1\varphi_2 + \sigma\varphi_1(\omega^1_{34} + \mu\varphi_2),\\
b^\xi_{21} &= e_2(\varphi_1) - \omega^2_{12}\varphi_2 + G_2a_1 + H_2b_1 + \kappa\varphi_1\varphi_2 - \sigma\varphi_2(\omega^2_{34} + \mu\varphi_1),\\
b^\xi_{22} &= e_2(\varphi_2) + \kappa + \omega^2_{12}\varphi_1 + G_2a_2 + H_2b_2 + \kappa\varphi_2^2 + \sigma\varphi_1(\omega^2_{34} + \mu\varphi_1)
\end{aligned}$$

and

$$\begin{aligned}
b^\eta_{11} &= e_1(\psi_1) - \omega^1_{12}\psi_2 + \sigma G_1a_2 + \sigma H_1b_2 + \omega^1_{34}\varphi_1 - \kappa(\varphi_1\psi_1 + \varphi_2\psi_2),\\
b^\eta_{12} &= e_1(\psi_2) - \mu + \omega^1_{12}\psi_1 - \sigma G_1a_1 - \sigma H_1b_1 + \omega^1_{34}\varphi_2 + \kappa(\varphi_1\psi_2 - \varphi_2\psi_1),\\
b^\eta_{21} &= e_2(\psi_1) - \mu - \omega^2_{12}\psi_2 + \sigma G_2a_2 + \sigma H_2b_2 + \omega^2_{34}\varphi_1 + \kappa(\varphi_1\psi_2 - \varphi_2\psi_1),\\
b^\eta_{22} &= e_2(\psi_2) + \omega^2_{12}\psi_1 - \sigma G_2a_1 - \sigma H_2b_1 + \omega^2_{34}\varphi_2 + \kappa(\varphi_1\psi_1 + \varphi_2\psi_2).
\end{aligned}$$

From our assumptions, we have

$$g_{11}b^\xi_{22} - g_{12}(b^\xi_{12} + b^\xi_{21}) + g_{22}b^\xi_{11} = 0 \tag{16}$$

and

$$g_{11}b^\eta_{22} - g_{12}(b^\eta_{12} + b^\eta_{21}) + g_{22}b^\eta_{11} = 0. \tag{17}$$

Viewing (16) and (17) as polynomials were the coefficients of $t_1^4, t_2^4, t_1^2t_2^2$ must vanish gives

$$(\lambda^2 - 1)(a_1^2 + a_2^2)(a_1^2 - a_2^2) = 0 = (\lambda^2 - 1)(b_1^2 + b_2^2)(b_1^2 - b_2^2)$$

and

$$(\lambda^2 - 1)a_1a_2(a_1^2 + a_2^2) = 0 = (\lambda^2 - 1)b_1b_2(b_1^2 + b_2^2).$$

Hence $\lambda = 1$ since otherwise, we have from the above that $\omega_{35} = \omega_{36} = \omega_{45} = \omega_{46} = 0$, which is a contradiction.

We now prove the direct statement. Observe that $g = F_g \circ j$, where j is the zero section of M^n. Clearly, we have that j is an integral surface of the distribution orthogonal to the rulings which is also totally geodesic and a global cross section to the rulings. Up to the uniqueness of the integral surface and completeness of M^n the proof now follows from Lemma 9. In fact, it is very easy to see that the metric of M^n is complete if the metric of L^2 is complete.

Assume that there exists a second integral surface $\tilde{j}\colon L^2 \to M^n$. Set $\tilde{g} = F_g \circ \tilde{j}$ and let $\tilde{T}\colon M^n \to \Lambda_{\tilde{g}}$ be the diffeomorphism given by

$$\tilde{T}(p) = (\pi(p), F(p) - \tilde{g}(\pi(p)).$$

Then $\tilde{T} \circ T^{-1}\colon \Lambda_g \to \Lambda_{\tilde{g}}$ is

$$\tilde{T} \circ T^{-1}(x, v) = (x, v + g(x) - \tilde{g}(x)).$$

Hence Λ_g and $\Lambda_{\tilde{g}}$ can be identified by parallel translation, thus there exists a section δ of Λ_g such that $\tilde{g} = g + \delta$. It follows from

$$\tilde{g}_* X = g_* X + \nabla^{\perp}_X \delta \tag{18}$$

that $\nabla^{\perp}_X \delta \in N_1^g$ for any $X \in TL$. If δ is constant, then g lies in an affine subspace $\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+2}$ perpendicular to δ which has been excluded. Thus, there is $\mu = \nabla^{\perp}_{X_0} \delta \neq 0$ for some $X_0 \in TL$. From (18) we have that $\nabla^{\perp}_Y \mu \in N_1^g$ for any $Y \in TL$. This easily implies that N_1^g is parallel in the normal bundle and thus g lies in $\mathbb{R}^4$, a contradiction. □

Proof of Theorem 4. For each $\theta \in \mathbb{S}^1$, we define $F_\theta\colon \Lambda_g \to \mathbb{R}^{n+2}$ by

$$F_\theta(p, v) = g_\theta(p) + v.$$

In the sequel, corresponding quantities of F_θ are denoted by the same symbol used for F_g marked with θ. That F_θ is isometric to F_g is immediate. Since the tangent frame $\{e_1, e_2\}$ has been fixed, we have for the adapted frames of g_θ that

$$e_3^\theta = R_\theta^1 e_3 \quad \text{and} \quad e_4^\theta = R_\theta^1 e_4$$

where R_θ^1 is the rotation of angle θ on N_1^g. We complete the adapted frame choosing

$$e_j^\theta = e_j, \quad 5 \leq j \leq n+2.$$

Clearly, it holds that $\omega^\theta_{34} = \omega_{34}$ and $\omega^\theta_{ij} = \omega_{ij}$ for $i, j \geq 5$. Moreover,

$$\omega^\theta_{35} = \cos\theta\omega_{35} - \sin\theta * \omega_{35} \quad \text{and} \quad \omega^\theta_{36} = \cos\theta\omega_{36} - \sin\theta * \omega_{36}.$$

Hence, the dual vector fields of ω^θ_{36} and ω^θ_{36} are given, respectively, by

$$V_\theta = J_{-\theta}V \quad \text{and} \quad W_\theta = J_{-\theta}W.$$

Thus,

$$a^\theta_1 = a_1\cos\theta + a_2\sin\theta, \quad a^\theta_2 = a_2\cos\theta - a_1\sin\theta$$

and

$$b^\theta_1 = b_1\cos\theta + b_2\sin\theta, \quad b^\theta_2 = b_2\cos\theta - b_1\sin\theta.$$

It follows from (5), (6) and (7) that

$$X^\theta_i = X_i, \quad i = 1, 2.$$

By Lemma 7, the normal bundle of F_θ is spanned by

$$\xi_\theta = g_{\theta_*}J_{-\theta}(t_1V + t_2W) + R^1_\theta e_3, \quad \eta_\theta = -g_{\theta_*}J_{\pi/2-\theta}(t_1V + t_2W) + R^1_\theta e_4.$$

A straightforward computation yields that the map $\Psi_\theta \colon N_{F_g}M \to N_{F_\theta}M$ given by

$$\Psi_\theta\xi = \xi_\theta \ \text{ and } \ \Psi_\theta\eta = \eta_\theta$$

is a parallel vector bundle isometry. The shape operators of F_θ vanish on $\mathcal{V}^0$ and restricted to $\mathcal{H} \oplus \mathcal{V}^1$ are given with respect to the frame $\{E_1, \ldots, E_n\}$ by

$$A^\theta_{\xi_\theta} = \begin{bmatrix} \kappa + h^\theta_1 & h^\theta_2 & r^\theta_1 & s^\theta_1 \\ h^\theta_2 & -\kappa - h^\theta_1 & r^\theta_2 & s^\theta_2 \\ r^\theta_1 & r^\theta_2 & 0 & 0 \\ s^\theta_1 & s^\theta_2 & 0 & 0 \end{bmatrix}, \quad A^\theta_{\eta_\theta} = \begin{bmatrix} h^\theta_2 & \kappa - h^\theta_1 & r^\theta_2 & s^\theta_2 \\ \kappa - h^\theta_1 & -h^\theta_2 & -r^\theta_1 & -s^\theta_1 \\ r^\theta_2 & -r^\theta_1 & 0 & 0 \\ s^\theta_2 & -s^\theta_1 & 0 & 0 \end{bmatrix}$$

where $r^\theta_i\Omega = -a^\theta_i$, $s^\theta_i\Omega = -b^\theta_i$ and

$$h^\theta_1 = h_1\cos\theta + h_2\sin\theta, \quad h^\theta_2 = -h_1\sin\theta + h_2\cos\theta.$$

Let L_θ denote the endomorphism of TM such that $L_\theta|_{\mathcal{V}} = 0$ and $L_\theta|_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}$ is the reflection given by

$$L_\theta|_{\mathcal{H}} = \begin{bmatrix} -\sin(\theta/2) & \cos(\theta/2) \\ \cos(\theta/2) & \sin(\theta/2) \end{bmatrix}$$

with respect to the tangent frame $\{E_1, E_2\}$. It follows easily that

$$A^{\theta}_{\Psi_\theta \xi} = A_{\mathsf{R}_\theta \xi} - 2\kappa \sin(\theta/2) L_\theta \quad \text{and} \quad A^{\theta}_{\Psi_\theta \eta} = A_{\mathsf{R}_\theta \eta} - 2\kappa \sin(\theta/2) \mathcal{J} L_\theta .$$

By a direct computation we obtain

$$\alpha_{F_\theta}(X,Y) = \Psi_\theta \left(\mathsf{R}_{-\theta} \alpha_{F_g}(X,Y) - \frac{2\kappa}{\Omega^2} \sin(\theta/2) (\langle L_\theta X, Y\rangle \xi + \langle L_\theta \mathcal{J} X, Y\rangle \eta) \right).$$

Define β as the symmetric section of $Hom(TM \times TM, N_{F_g} M)$ with nullity $\mathcal{V}$ such that

$$\beta(E_1,E_1) = \frac{1}{\Omega^2}\xi = -\beta(E_2,E_2) \quad \text{and} \quad \beta(E_1,E_2) = -\frac{1}{\Omega^2}\eta, \tag{19}$$

and the proof of (3) follows easily.

Finally, that the isometric deformation F_θ of F_g is genuine is immediate from (14) since the shape operators of F_g have rank four for any normal direction along an open dense subset. □

Proof of Theorem 6. Being g holomorphic, there exists an isometry τ of $\mathbb{R}^{n+2}$ such that $g_\theta = \tau \circ g$. The higher fundamental forms satisfy

$$\alpha^{s+1}_{g_\theta} = \tau_* \circ \alpha^{s+1}_g \ \text{ for any } \ s \geq 1.$$

It was shown in [4] that the almost complex structure J induces an almost complex structure J_s on each N^g_s defined by

$$J_s \alpha^{s+1}_g (X_1, \ldots, X_{s+1}) = \alpha^{s+1}_g (JX_1, \ldots, X_{s+1}).$$

In the present case each $J_s \colon N^g_s \to N^g_s$ is an isometry. Thus, we have

$$\alpha^{s+1}_{g_\theta} = R^s_\theta \circ \alpha^{s+1}_g,$$

where $R^s_\theta = \cos\theta I + \sin\theta J_s$. Hence $R^s_\theta = \tau_*|_{N^g_s}$. It is now easy to see that $F_\theta = \tau \circ F_g \circ S_{-\theta}$, and this concludes the proof. □

4. The case of holomorphic curves

Let the substantial surface $g \colon L^2 \to \mathbb{R}^{n+2}$, $n \geq 6$, be a holomorphic curve with respect to some parallel complex structure in $\mathbb{R}^{n+2}$. Let $\{e_1, e_2\}$ be an orthonormal tangent frame such that

$$\alpha_g^{s+1}(e_1,\dots,e_1) = \kappa_s e_{2s+1},$$
$$\alpha_g^{s+1}(e_1,\dots,e_1,e_2) = \kappa_s e_{2s+2}, \quad 1 \le s \le n/2.$$

Then, set $\tau_s = \kappa_s/\kappa_{s-1}$, $1 \le s \le n/2$, with $\kappa_0 = 1$. It is well-known that κ_s can be defined as the radius of the s^{th}-curvature ellipse (cf. [13]) and that the functions τ_s are completely determined by the metric of L^2 in an explicit form by a result of Calabi (cf. [19]).

We see next that in this case of a holomorphic curve g the second fundamental form of the associated minimal ruled submanifold $F_g\colon M^n \to \mathbb{R}^{n+2}$ is substantially simpler than in the general case and completely determined by the metric of the surface.

Proposition 10. *Let $g\colon L^2 \to \mathbb{R}^{n+2}$, $n \ge 6$, be holomorphic. Then the shape operators of $F_g\colon M^n \to \mathbb{R}^{n+2}$ with respect to the orthonormal tangent frame*

$$E_i = X_i/\Omega, \quad i = 1,2, \quad \text{and} \quad F_* E_j = e_{j+2}, \quad 3 \le j \le n.$$

vanish along $\mathcal{V}^0$ and restricted to $\mathcal{H} \oplus \mathcal{V}^1$ are given by
(20)
$$A_\xi = \begin{bmatrix} \tau_1 + h_1 & h_2 & r & 0 \\ h_2 & -\tau_1 - h_1 & 0 & r \\ r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \end{bmatrix}, \quad A_\eta = \begin{bmatrix} h_2 & \tau_1 - h_1 & 0 & r \\ \tau_1 - h_1 & -h_2 & -r & 0 \\ 0 & -r & 0 & 0 \\ r & 0 & 0 & 0 \end{bmatrix}$$

where

$$h_1 = -\frac{1}{1+(t_1^2+t_2^2)\tau_2^2}\big(t_1 e_1(\tau_2) - t_2 e_2(\tau_2) + t_3\tau_2\tau_3\big),$$
$$h_2 = -\frac{1}{1+(t_1^2+t_2^2)\tau_2^2}\big(t_1 e_2(\tau_2) + t_2 e_1(\tau_2) + t_4\tau_2\tau_3\big),$$
$$r = -\frac{\tau_2}{\sqrt{1+(t_1^2+t_2^2)\tau_2^2}}.$$

Moreover, the second fundamental form of F_g depends only on the metric of L^2.

Proof. From the choice of the normal frame and the definition of higher fundamental forms, we find that the normal connection forms

$$\omega^j_{\alpha\beta} = \langle \nabla^\perp_{e_j} e_\alpha, e_\beta\rangle, \quad 1 \le j \le 2, \quad 3 \le \alpha,\beta \le n+2,$$

satisfy

$$\begin{aligned}\alpha_g^{s+1}(e_1,\ldots,e_1) &= (\nabla_{e_1}^{\perp}\alpha_g^s(e_1,\ldots,e_1))_{N_s^g}\\ &= \kappa_{s-1}(\nabla_{e_1}^{\perp}e_{2s-1})_{N_s^g}\\ &= \kappa_{s-1}\big(\omega_{2s-1,2s+1}^1 e_{2s+1} + \omega_{2s-1,2s+2}^1 e_{2s+2}\big).\end{aligned}$$

Similarly, we find

$$\begin{aligned}\alpha_g^{s+1}(e_1,\ldots,e_1,e_2) &= \kappa_{s-1}\big(\omega_{2s,2s+1}^1 e_{2s+1} + \omega_{2s,2s+2}^1 e_{2s+2}\big),\\ \alpha_g^{s+1}(e_2,e_2,e_1\ldots,e_1) &= \kappa_{s-1}\big(\omega_{2s,2s+1}^2 e_{2s+1} + \omega_{2s,2s+2}^2 e_{2s+2}\big),\\ \alpha_g^{s+1}(e_2,e_1\ldots,e_1) &= \kappa_{s-1}\big(\omega_{2s-1,2s+1}^2 e_{2s+1} + \omega_{2s-1,2s+2}^2 e_{2s+2}\big).\end{aligned}$$

Thus, we obtain

$$\omega_{2s-1,2s+1} = \omega_{2s,2s+2} = \tau_s\omega_1, \quad \omega_{2s-1,2s+2} = -\omega_{2s,2s+1} = \tau_s\omega_2. \tag{21}$$

Moreover, from part (ii) of Lemma 6 in [23] it follows that

$$\omega_{2s+1,2s+2} = (s+1)\omega_{12} + *d\log\kappa_s, \quad 1 \le s \le n/2. \tag{22}$$

Then, using (21), (22) we have from Lemma 9 that the second fundamental form of F_g is given by (20). □

Remark 11. Notice that in order to obtain the expressions of the shape operators in the above result we only used that the first three ellipses of curvature are circles. In [16] we will discuss when M^n is Kaehler.

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