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On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus

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Given a sequence of properly embedded minimal surfaces in a 3-manifold with local bounds on area and genus, we prove subsequential convergence, smooth away from a discrete set, to a smooth embedded limit surface, possibly with multiplicity, and we analyze what happens when one blows up the surfaces near a point where the convergence is not smooth.

1. Introduction

In this paper, we prove several results, most of which can be summarized as follows:

Theorem 1.1. *Let Ω be an open subset of a Riemannian 3-manifold. Let g_i be a sequence of smooth Riemannian metrics on Ω converging smoothly to a Riemannian metric g. Let $M_i \subset \Omega$ be a sequence of properly embedded surfaces such that M_i is minimal with respect to g_i. Suppose also that the area and the genus of M_i are uniformly bounded on compact subsets of Ω. Then (after passing to a subsequence) the M_i converge to a smooth, properly embedded g-minimal surface M. For each connected component Σ of M, either*

1) *the convergence to Σ is smooth with multiplicity one, or*
2) *the convergence is smooth (with some multiplicity > 1) away from a discrete set S.*

In the second case, if Σ is two-sided, then it must be stable.

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Now suppose that Ω is an open subset of $\mathbf{R}^3$. (The metric g need not be flat.) If $p_i \in M_i$ converges to $p \in M$, then (after passing to a further subsequence) either

$$\mathrm{Tan}(M_i, p_i) \to \mathrm{Tan}(M, p),$$

or there exists constants $\lambda_i > 0$ tending to ∞ such that the surfaces

$$\lambda_i(M_i - p_i)$$

converge smoothly and with multiplicity 1 *to a non-flat, complete, properly embedded minimal surface $M' \subset \mathbf{R}^3$ of finite total curvature with ends parallel to* $\mathrm{Tan}(M, p)$.

A compactness theorem very similar to Theorem 1.1 was proved by Ros [Ros95], building on earlier compactness theorems in [CS85] and [Whi87b]. (See also [And85] for some related results.)

Indeed, Ros essentially proves the above conclusions provided one also assumes

1) local bounds on total curvature, and
2) that the ambient space is $\mathbf{R}^3$ with the Euclidean metric.

Ros points out in his paper that most of the proofs would work for general Riemannian 3-manifolds. However, his proof of the last statement (that the ends of M' are parallel to $\mathrm{Tan}(M, p)$) is based on the conformality of the Gauss map of a minimal surface in $\mathbf{R}^3$, and thus does not apply to non-flat metrics.

Thus what is new in this paper is:

1) we point out that, by a result of Ilmanen, it suffices to assume local bounds on genus and area instead of local bounds on total curvature and area, and
2) we prove that the ends of the blown-up surface M' are parallel to $\mathrm{Tan}(M, p)$ for general Riemannian metrics g_i and g.

These new features are used in an essential way in the recent proof [HTW16] of existence of helicoidal surfaces of arbitrary genus in $\mathbf{R}^3$, and also in recent work of Ferrer, Martin, Mazzeo, and Rodriguez on minimal annuli in $\mathbf{H}^2 \times \mathbf{R}$ [FMMR17].

The bulk of this paper (Sections 3 and 4) is devoted to proving that the ends of M' are parallel to $\mathrm{Tan}(M, p)$.

If one drops the assumption that the M_i have locally bounded areas, the behavior becomes considerably more complicated. For example, even for simply connected M_i in an open subset of $\mathbf{R}^3$, the curvatures of the M_i can blow up on arbitrary $C^{1,1}$ curves [MW07] or on arbitrary closed subsets (such as Cantor sets) of a line [HW11], [Kle12]. In [CM04a], [CM04b], [CM04c], and[CM04d], Colding and Minicozzi prove powerful theorems analyzing the behavior of such sequences. Some extensions of their work are proved in [Mee04] and [Whi15]. (See also [BT16].) Based on those works, [Whi15, Corollary 3 and Theorem 4] formulates a compactness theorem somewhat analogous to the Compactness Theorem 1.1 in this paper.

It would be very interesting to analyze what happens if one assumes local bounds on area but not on genus. By passing to a subsequence, one can get weak convergence to a stationary integral varifold V. The limit varifold has associated to it a flat chain mod 2, and that flat chain has no boundary in the open set [Whi09]. Thus, for example, the varifold cannot have soapfilm-like triple junctions. In fact, [Whi09] also proves the slightly stronger statement that if the original surfaces are orientable, then there is an integral current T with no boundary in the open set such that T and V determine the same flat chain mod 2. (The results in [Whi09] hold for arbitrary dimension and codimension.) Nothing else seems to be known about the class of stationary integral varifolds V that arise as such a limit.

2. The Main Theorems

If M is a surface in a Riemannian 3-manifold, we let $\mathrm{TC}(M)$ denote the total curvature of M:

$$\mathrm{TC}(M) = \frac{1}{2}\int_M (\kappa_1^2 + \kappa_2^2)\, dA,$$

where κ_1 and κ_2 are the principal curvatures of M.

Theorem 2.1 (Compactness Theorem). *Let Ω be an open subset of smooth 3-manifold. Let g_i be a sequence of smooth Riemannian metrics on Ω converging smoothly to a Riemannian metric g. Let $M_i \subset \Omega$ be a sequence of properly embedded surfaces such that M_i is minimal with respect to g_i. Suppose also that the area and the genus of M_i are bounded independently of i on compact subsets of Ω.*

Then the total curvatures of the M_i are also uniformly bounded on compact subsets of Ω. After passing to a subsequence, the M_i converge to a smooth, properly embedded, g-minimal surface M, and the convergence is *smooth away from a discrete set S. For each connected component Σ of M, either*

1) *the convergence to Σ is smooth everywhere with multiplicity 1, or*
2) *the convergence to Σ is smooth with some multiplicity > 1 away from $\Sigma \cap S$. In this case, if Σ is two-sided, then it must be stable.*

If the total curvatures of the M_i are bounded by β, then S has at most $\beta/(4\pi)$ points.

Proof. See [Ilm95, Theorem 3] for a proof that the total curvatures of the M_i are uniformly bounded on compact subsets of Ω.

If the supremum of the total curvatures of the M_i is less than 4π, then we get smooth, subsequential convergence (possibly with multiplicity) everywhere. (This follows from the curvature estimate [Whi16, Theorem 24] or [Whi87b, pp. 247–248].)

It follows (after passing to a subsequence) that there is a discrete set S such that the M_i converge smoothly to M on compact subsets of $\Omega \setminus S$, where $M \setminus S$ is a smooth minimal surface properly immersed in $\Omega \setminus S$. Furthermore, if W is an open subset of Ω, then the number of points in $S \cap W$ is at most

$$\frac{1}{4\pi} \limsup_i \mathrm{TC}(M_i \cap W).$$

The points of S are precisely those points where 4π or more of total curvature concentrates. See [Whi87b] or [Whi16, Theorem 25] for details.

Since the M_i are embedded, $M \setminus S$ has no transverse self-intersections. Hence $M \setminus S$ is smooth and embedded, possibly with multiplicity. It also follows that the points in S are removable singularities of M. (This removal of singularities theorem can be proved in a variety of ways. See the appendix for one proof.) In other words, M is a smooth, embedded surface, possibly with multiplicity > 1.

For simplicity, let us assume M has just one connected component.

If M has multiplicity 1, then the convergence of M_i to M is smooth everywhere by Allard's Regularity Theorem [All72, §8] or [Sim83, §23–§24], or by the easy version of the Allard Regularity Theorem in [Whi05, Theorem 1.1] or [Whi16]. (The proof in [Whi05] is for compact surfaces, but that proof can easily be modified to handle proper, non-compact surfaces.)

Thus suppose that the M_i converge to M with multiplicity $k > 1$, and suppose that M is two-sided. Since the convergence is smooth on compact subsets of $\Omega \setminus S$, we can (away from $S \cup \partial\Omega$) express M_i as the union of k

disjoint, normal graphs over Σ. Since M_i is embedded, the functions can be ordered. Let ϕ_i be the difference of the largest and the smallest functions. Let p be a point in $\Sigma \setminus S$. By standard PDE, ϕ_i satisfies a second-order linear elliptic equation. By the Harnack inequality and the Schauder estimates, the functions $\phi_n/|\phi_n(p)|$ converge smoothly (after passing to a subsequence) to a positive Jacobi field ϕ on $\Sigma \setminus S$. (The construction described in the preceding two sentences is fairly standard. For more details, see the paragraph containing equation (7) on page 333 of [Sim87].) By [FCS80, Theorem 1], existence of such a ϕ implies that $\Sigma \setminus S$ is stable. A standard cut-off argument (cf. Corollary 5.5 in the appendix) shows that Σ and $\Sigma \setminus S$ have the same Jacobi eigenvalues. Thus Σ is stable.

This completes the proof of Theorem 2.1. □

The remaining results are local, so we can assume that Ω is an open subset of $\mathbf{R}^3$. (Of course the metrics g_i and g need not be flat.) We let $\lambda(M-p)$ denote the result of translating M by $-p$ and then dilating by λ.

Theorem 2.2 (Blow-up Theorem). *Suppose in the Compactness Theorem 2.1 that Ω is an open subset of $\mathbf{R}^3$. Suppose also that $p_i \in M_i$ converges to $p \in M$, and that $\lambda_i \to \infty$.*

Then, after passing to a subsequence, the surfaces

$$M_i' := \lambda_i(M_i - p_i)$$

converge smoothly away from a finite set Q to a complete, properly embedded, $g(p)$-minimal surface M'. Furthermore, M' must be one of the following:

1) *a multiplicity* 1 *plane,*
2) *a complete, non-flat, properly embedded, multiplicity* 1 *surface of finite total curvature, or*
3) *the union of two or more (counting multiplicity) parallel planes.*

In cases (1) and (2), the convergence of M_n' to M' is smooth everywhere.

The Blow-up Theorem 2.2 describes what can happen at various scales near a point where the curvature blows up. For example, let M_i be the catenoid that is centered at the origin, that has horizontal ends, and that has neck radius r_i, where $r_i \to 0$. The M_i converge to the horizontal plane $z = 0$ with multiplicity 2, and the convergence is smooth except at the origin, where the curvature blows up. Let $p_i = (r_i, 0, 0)$. Then $\lambda_i(M_i - p_i)$ converges to a vertical, multiplicity 1 plane if $\lambda_i r_i \to \infty$, to a catenoid if $\lambda_i r_i$ tends to a finite, nonzero limit, and to a horizontal, multiplicity 2 plane if $\lambda_i r_i \to 0$.

Proof of Theorem 2.2. The monotonicity formula implies that the areas of the M_i' are uniformly bounded on compact sets. Thus the subsequential convergence to a complete, smooth, properly embedded $g(p)$-minimal surface M' of finite total curvature and the finiteness of the set Q follow immediately from the Compactness Theorem 2.1.

Suppose that M' is not the union of one or more parallel planes. By the Strong Halfspace Theorem [HM90, Theorem 2], M' is connected. Since M' is not a plane, it is unstable (by [FCS80] or [dCP79]). Thus by the Compactness Theorem 2.1, M' has multiplicity 1.

The smooth convergence everywhere in cases (1) and (2) follows from the Compactness Theorem 2.1. □

Theorem 2.3 (No-Tilt Theorem). *In cases (2) and (3) of the Blow-up Theorem 2.2, the ends of M' are parallel to* $\mathrm{Tan}(M, p)$.

The proof will be given in Sections 3 and 4.

Theorem 2.4. *Suppose, in the Compactness Theorem 2.1, that Ω is an open subset of $\mathbf{R}^3$. (The metrics need not be flat.) Suppose that $p_i \in M_i$ converges to $p \in M$ and that $\mathrm{Tan}(M_i, p_i)$ does not converge to $\mathrm{Tan}(M, p)$. Then there exist $\lambda_i \to \infty$ such that, after a passing to a subsequence, the surfaces*

$$M_i' := \lambda_i (M_i - p_i)$$

converge smoothly and with multiplicity 1 to a complete, smooth, properly embedded, non-flat, $g(p)$-minimal surface M' of finite total curvature. Furthermore, the ends of M' must be parallel to $\mathrm{Tan}(M, p)$.

Proof. By passing to a subsequence, we can assume that $\mathrm{Tan}(M_i, p_i)$ converges to a plane P not equal to $\mathrm{Tan}(M, p)$:

$$\mathrm{Tan}(M_i, p_i) \to P \neq \mathrm{Tan}(M, p). \tag{1}$$

Let r_i be the infimum of the numbers $r > 0$ such that

$$M_i \cap \mathbf{B}(p_i, r)$$

contains a point at which the principal curvatures are $\geq 1/r$. By (1),

$$\lim r_i = 0.$$

Now let $\lambda_i = 1/r_i$.

The Blow-up Theorem 2.2 implies that, after passing to a further subsequence, the surfaces

$$M_i' := \lambda_i(M_i - p_i)$$

converge to a limit surface M'.

By the choice of λ_i, the surface M_i' converges to M' smoothly in the open ball of radius 1 about 0. Hence

$$\mathrm{Tan}(M', 0) = \lim_i \mathrm{Tan}(M_i', 0) = \lim_i \mathrm{Tan}(M_i, p_i) = P \neq \mathrm{Tan}(M, p),$$

so

$$\mathrm{Tan}(M', 0) \neq \mathrm{Tan}(M, p). \tag{2}$$

It follow that M' cannot consist of planes, since by the No-Tilt Theorem 2.3, those planes would have to be parallel to $\mathrm{Tan}(M, p)$.

Thus we are in case (2) of the Blow-up Theorem 2.2: M' is a smooth, nonflat, properly embedded, multiplicity 1 surface of finite total curvature. By the No-Tilt Theorem 2.3, the ends of M' are parallel to $\mathrm{Tan}(M, p)$. □

3. The Annulus Lemma

The proof of the No-Tilt Theorem relies heavily on the following lemma, which describes the behavior of nearly flat minimal annuli as the inner radius tends to 0:

Lemma 3.1 (Annulus Lemma). *Let g_i be a sequence of Riemannian metrics on the cylinder*

$$C(R, a) := \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 \leq R^2,\ |z| \leq a\}$$

that converge smoothly to a Riemannian metric g. For $i = 1, 2, \dots,$ suppose that $M_i \subset C(R, a)$ is a g_i-minimal surface that is the graph of a function

$$u_i : A(r_i, R) \to (-a, a),$$

where

$$A(r_i, R) = \{p \in \mathbf{R}^2 : r_i \leq |p| \leq R\}$$

and where the radii r_i are positive numbers that converge to 0. Suppose also that

$$L := \sup_i \sup |Du_i| < \infty,$$

and that

$$u_i \to 0 \text{ smoothly on } A(\eta, R) \text{ for every } \eta > 0.$$

Let λ_i be a sequence of numbers tending to infinity such that

(3) $$r' = \lim \lambda_i r_i < \infty.$$

Let $\mathbf{v}_i = (0, 0, v_i)$ be a point on the z-axis such that

(4) $$c := \sup_i \frac{\operatorname{dist}(\mathbf{v}_i, M_i)}{r_i} < \infty.$$

Let $M_i' = \lambda_i(M_i - \mathbf{v}_i)$, so that M_i' is the graph of the function

$$u_i' : A(\lambda_i r_i, \lambda_i R) \to \mathbf{R},$$
$$u_i'(q) = \lambda_i(u_i(q/\lambda_i) - v_i).$$

Then, after passing to a subsequence, the u_i' converge uniformly on compact subsets of $\mathbf{R}^2$ to a function

$$u' : A(r', \infty) \to \mathbf{R}.$$

The convergence is smooth on compact subsets of $\sqrt{x^2 + y^2} > r'$, and

(5) $$\lim_{|q| \to \infty} |Du'(q)| = 0.$$

Proof. Except for the last assertion (5), the lemma is straightforward, as we now explain. Note that the graphs of the u_i' have slopes bounded by L. Also,

$$\limsup_i \operatorname{dist}(0, M_i') \le cr' < \infty$$

by (3) and (4). By the Arzela-Ascoli Theorem, after passing to a subsequence, the u_i' converge uniformly on compact subsets of $\mathbf{R}^2$ to a limit function

$$u' : A(r', \infty) \to \mathbf{R}.$$

By standard PDE or by minimal surface regularity theory, the convergence is smooth on compact subsets of $\sqrt{x^2 + y^2} > r'$.

It remains to prove the last assertion (5). We remark that if (5) holds for one choice of $\mathbf{v}_i$, then it also holds for any other choice $\mathbf{v}_i^*$ (subject to the condition (4)). This is because if $\lambda_i(M_i - \mathbf{v}_i)$ converges to M', then, after passing to a further subsequence, the surfaces $\lambda_i(M_i - \mathbf{v}_i^*)$ converge to a limit M^*, and clearly M^* is a vertical translate of M'.

Let D be the horizontal disk of radius R centered at the origin. The smooth convergence $M_i \to D$ away from the origin implies that the mean curvature with respect to g of D vanishes everywhere except possibly at the origin. By continuity, it must also vanish at the origin. That is, D is a g-minimal surface.

By replacing R by a sufficiently small $\hat{R} > 0$ and M_i by $M_i \cap C(\hat{R}, a)$, we can assume that the disk D is strictly stable.

Claim 1. *It suffices to prove the lemma under the assumptions that the outer boundary of M_i lies in the plane $z = 0$, (i.e., that $u_i(p) \equiv 0$ when $|p| = R$), that horizontal disks (i.e., disks of the form $z =$ constant) are g_i-minimal for every i, and that the metric $g(0)$ coincides with the Euclidean metric at the origin:*

$$g(0)(\mathbf{e}_i, \mathbf{e}_j) = \delta_{ij}. \tag{6}$$

Proof of Claim 1. By the implicit function theorem[1] and the strict stability of D, there exist $\epsilon > 0$ and $\delta > 0$ with the following property: for all $|t| \leq \delta$, there is a unique smooth function

$$f^t : \{\sqrt{x^2 + y^2} \leq R\} \to \mathbf{R}$$

such that $\|f^t\|_{2,\alpha} < \epsilon$, such that

$$f^t = t \text{ on the circle } \sqrt{x^2 + y^2} = R,$$

and such that the graph of f^t is a strictly stable, g-minimal disk. Note that f^t depends smoothly on t. Note also that $\frac{\partial}{\partial t} f^t$, which may be regarded as a Jacobi field on the graph of f^t, is equal to 1 on the boundary and therefore is everywhere positive by stability. Thus the map

$$F : (x, y, z) \mapsto (x, y, f^t(x, y))$$

is a smooth diffeomorphism from the cylinder $C(R, \delta)$ onto its image.

[1] See the appendix of [Whi87b] for details of this application of the implicit function theorem.

Similarly (and also by the implicit function theorem), for all sufficiently large i and for all $|t| \le \delta$, there is a smooth function

$$f_i^t : \{\sqrt{x^2+y^2} \le R\} \to \mathbf{R}$$

such that $\|f_i^t\|_{2,\alpha} < \epsilon$, such that

$$f_i^t = u_i + t \text{ on the circle } \sqrt{x^2+y^2} = R,$$

and such that the graph of f_i^t is a strictly stable, g_i-minimal surface. Furthermore,

$$F_i^t : (x,y,z) \mapsto (x,y,f_i^t(x,y))$$

defines a smooth diffeomorphism from $C(R,a)$ to its image, and F_i converges smoothly to F as $i \to \infty$. (All these statements are consequences of the implicit function theorem.)

Now let M_i' and g_i' be the pull-back of M_i and g_i under the diffeomorphism F_i. Then M_i' and g_i' satisfy all the hypotheses of the lemma, and, in addition, horizontal disks are g_i' minimal and the outer boundary of the annulus M_i' is a horizontal circle centered at the origin. This completes the proof of Claim 1, except for (6), which can be achieved by making a further diffeomorphic change of coordinates of the form

$$(x,y,z) \mapsto (\tilde{x}(x,y,z), \tilde{y}(x,y,z), \tilde{z}(z)).$$

□

We now prove (5), under the additional assumptions indicated by Claim 1.

If the u_i are identically zero, there is nothing to prove. Thus by passing to a subsequence, we may assume, without loss of generality, that

$$z_i := \max u_i > 0.$$

(The case $\min u_i < 0$ is proved in exactly the same way.) By the maximum principle (and by the assumptions described in Claim 1), the maximum is attained on the inner boundary circle of $A(r_i, R)$. (Recall that $u_i \equiv 0$ on the outer boundary circle.) Thus

$$u_i \le z_i = \max_{|p|=r_i} u_i(p). \tag{7}$$

As explained earlier, the validity of the lemma does not depend on the choice of $\mathbf{v}_i = (0,0,v_i)$, so we may choose $\mathbf{v}_i = (0,0,z_i)$. It follows that M_i'

lies in the halfspace $z \leq 0$ for all i, and thus so does M':

$$u' \leq 0.$$

By (6), the surface M' is minimal with respect to the standard Euclidean metric.

There are now many ways to see that M' is horizontal at infinity. For example, the tangent cone at infinity to M' is a multiplicity-one Lipschitz graph and therefore is a plane (because its intersection with the unit 2-sphere must be a geodesic). Since it lies in the halfspace $\{z \leq 0\}$, the plane must be horizontal. □

4. Proof of the No-Tilt Theorem

We now prove the No-Tilt Theorem 2.3. We may assume that $p_i = p = 0$. (Otherwise replace M_i and M by $M_i - p_i$ and $M - p$, and similarly for the metrics g_i and g.) By rotation, we may assume that $\mathrm{Tan}(M, 0)$ is horizontal. Thus it suffices to prove the following special case of the No-Tilt Theorem:

Theorem 4.1. *Let Ω be an open subset of $\mathbf{R}^3$ and let g_i be a sequence of smooth Riemannian metrics on Ω that converge smoothly to a Riemannian metric g. Suppose that M_i and M are smooth, properly embedded surfaces in Ω such that M_i is g_i-minimal, M is g-minimal, and such that M_i converges smoothly, with some finite multiplicity, to M away from a discrete set of points. Suppose also that*

$$\sup_i \mathrm{TC}(M_i) < \infty.$$

Suppose that the origin is contained in each of the M_i, and suppose that $\mathrm{Tan}(M, 0)$ *is horizontal. Let λ_i be a sequence of numbers tending to ∞, and suppose that the dilated surfaces $\lambda_i M_i$ converge smoothly away from a finite set of points to a limit surface M^*.*

Then either M^* is a multiplicity one plane, or the ends of M^* are all horizontal:

$$\lim_{|q| \to \infty} \mathrm{slope}(\mathrm{Tan}(M^*, q)) = 0.$$

Proof. Let m be the multiplicity of the convergence $M_i \to M$. If M is not connected, the multiplicity could be different on different components of M. In that case, we let m be the multiplicity on the connected component of M containing the origin.

Let N be an integer such that $\sup_i \mathrm{TC}(M_i \cap U) < N$ for some open set U containing the origin.

We will prove Theorem 4.1 by double induction on the multiplicity m and on N. Thus we may assume that the theorem is true for surfaces $M_i' \to M'$ (satisfying the hypotheses of the theorem) provided

1) $m' < m$, or
2) $m' = m$ and $\sup_i \mathrm{TC}(M_i' \cap U') < N - 1$ for some neighborhood U' of 0,

where m' is the multiplicity of convergence of M_i' to M' on the connected component M' containing the origin.

Since the result is local, we can assume that M is topologically a disk. By composing with a diffeomorphism, we can assume that M is a horizontal disk centered at the origin. As in the proof of the Annulus Lemma 3.1 (see Claim 1 there), we may assume that the level sets of the height function $(x, y, z) \mapsto z$ on Ω are g_i-minimal for each i, and that the metric g agrees with the Euclidean metric at the origin. By replacing Ω by a small open set of the form

$$\{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 < R^2,\ |z| < a\},$$

we can assume that the $\overline{M_i}$ are smooth manifolds-with-boundary that converge smoothly to $\overline{M}$ away from the origin.

If the convergence of M_i to M is smooth everywhere, then the result is trivially true: in that case, every subsequence of $\lambda_i M_i$ has a further subsequence that converges smoothly to the union of one or more horizontal planes.

Thus we may assume that the convergence is not smooth. It follows that there is a sequence of points $p_i \in M_i$ converging to the origin such that $\mathrm{Tan}(M_i, p_i)$ does not converge to a horizontal plane. By passing to a subsequence, we may assume that

$$\mathrm{slope}(\mathrm{Tan}(M_i, p_i)) > L \geq 0 \tag{8}$$

for some finite $L > 0$ and for all i.

Consider the set S_i of points $q = (x, y, z)$ in M_i such that

$$\begin{aligned}&\mathrm{slope}(\mathrm{Tan}(M_i, q)) \geq L, \text{ or}\\ &|z| \geq L\sqrt{x^2 + y^2}.\end{aligned}$$

Let

$$r_i = \max\{\sqrt{x^2+y^2} : (x,y,z) \in S_i\}.$$

Note that $r_i > 0$ by (8) and that $r_i \to 0$ since $\overline{M_i} \to \overline{M}$ smoothly away from the origin. Note also that the surface

$$M_i \cap \{\sqrt{x^2+y^2} \geq r_i\}$$

is the union of m graphs of functions defined on

$$A(r_i, R) = \{p \in \mathbf{R}^2 : r_i \leq |p| \leq R\},$$

and that the tangent planes to those graphs all have slopes $\leq L$. Furthermore, $M_i \cap \{\sqrt{x^2+y^2} \geq r_i\}$ is contained in

$$\{|z| \leq L\sqrt{x^2+y^2}\},$$

so

$$M_i \subset \{|z| \leq L\sqrt{x^2+y^2}\} \cup \{|z| \leq Lr_i\} \tag{9}$$

by the maximum principle. (Recall that level sets of the height function are g_i-minimal.)

By passing to a subsequence, we may assume that $\lambda_i r_i$ converges to a limit r' in $[0,\infty]$.

If $r' < \infty$, the result follows immediately from the Annulus Lemma 3.1. (Apply the lemma to each of the annular components of $M_i \cap \{\sqrt{x^2+y^2} \geq r_i\}$.)

Thus we may assume that $r' = \infty$:

$$\lambda_i r_i \to \infty. \tag{10}$$

Let M_i' be the result of dilating M_i by $1/r_i$ about the origin. By the Compactness Theorem 2.1, the M_i' converge smoothly (after passing to a subsequence, and away from a discrete set) to a limit surface

$$M' \subset \{|z| \leq L\sqrt{x^2+y^2}\} \cup \{|z| \leq L\}.$$

Since the slopes of the M_i' are bounded by L in the region $x^2+y^2 \geq 1$, the convergence $M_i' \to M'$ is smooth in the region $x^2+y^2 > 1$.

Applying the Annulus Lemma to each of the m-components of

$$M_i \cap \{\sqrt{x^2+y^2} \geq r_i\},$$

we see that

$$\text{slope}(\text{Tan}(M', p)) \to 0 \text{ as } |p| \to \infty. \tag{11}$$

That is, the ends of M' are horizontal. Note that the number of ends of M', counting multiplicities, is m.

The Blow-up Theorem 2.2 asserts that one of the following must hold:

1) M' is non-flat, complete, with finite total curvature, and the convergence $M'_i \to M'$ is smooth and multiplicity 1 everywhere.
2) M' is the union of one or more parallel planes, possibly with multiplicity. The convergence is smooth except at isolated points.

In case (1), the surface $\lambda'_i M'_i$ converges smoothly to $\text{Tan}(M', 0)$ with multiplicity 1 for every sequence $\lambda'_i \to \infty$; in particular, this holds for the sequence $\lambda'_i := \lambda_i r_i$ (which tends to ∞ by (10)). But

$$\lambda'_i M'_i = (\lambda_i r_i)(1/r_i) M_i = \lambda_i M_i,$$

so the $\lambda_i M_i$ converge with multiplicity 1 to a plane (namely, $\text{Tan}(M', 0)$)), as desired. This completes the proof in case (1).

Thus we may assume (2): that M' is a union of parallel planes. In this case, we know that the planes are horizontal since the ends are horizontal (by (11)), and that the number of planes (counting multiplicity) is m.

Case 2(a): M' contains a plane not passing through the origin. Then the plane that does pass through the origin has multiplicity $< m$.

Let $\lambda'_i = \lambda_i r_i$, which tends to infinity by (10). Then

$$\lambda'_i M'_i = \lambda_i r_i (1/r_i) M_i = \lambda_i M_i \to M^*.$$

By the inductive hypothesis (applied to $M'_i \to M'$ and λ'_i), either M^* consists of a single multiplicity 1 plane or its ends are all horizontal. Thus we are done in this case.

Case 2(b): M' is the horizontal plane through 0 with multiplicity m.

By the choice of r_i, there is a point $q_i = (x_i, y_i, z_i)$ in M_i' such that $x_i^2 + y_i^2 = 1$ and such that

$$\text{slope}(\text{Tan}(M_i', q_i)) \geq L. \tag{12}$$

In particular, the convergence $M_i' \to M'$ fails to be smooth at one or more points the circle $\{x^2 + y^2 = 1,\, z = 0\}$.

Let U be the open ball of radius $1/2$ center at the origin. Since (as we have just shown) the convergence of $M_i' \setminus U$ to $M' \setminus U$ is not smooth near the circle $\{x^2 + y^2 = 1,\, z = 0\}$, it follows that

$$\limsup_i \text{TC}(M_i' \setminus U) \geq 4\pi.$$

In particular, we can assume (by passing to a subsequence) that

$$\text{TC}(M_i' \setminus U) > 12$$

for all i. It follows that

$$\begin{aligned} \text{TC}(M_i' \cap U) &= \text{TC}(M_i') - \text{TC}(M_i' \setminus U) \\ &< \text{TC}(M_i') - 12 \\ &< N - 12. \end{aligned}$$

Thus the proposition holds for the surfaces M_i' and M' by the inductive hypothesis. Letting $\lambda_i' = \lambda_i r_i$ (which tends to ∞ by (10)), we have

$$\lambda_i' M_i' = (\lambda_i r_i)((1/r_i)M_i) = \lambda_i M_i \to M^*.$$

By the inductive hypothesis, M^* satisfies the conclusions of the theorem. □

5. Appendix

Theorem 5.1. *Let $\mathbf{B}(p, r)$ be the open ball of radius r centered at $p \in \mathbf{R}^3$. Suppose that g is a smooth Riemannian metric on $\mathbf{B}(p, r)$, and that M is a properly embedded, g-minimal surface in $\mathbf{B}(p, r) \setminus \{p\}$ with finite total curvature and finite area, and that $p \in \overline{M}$. Then $M \cup \{p\}$ is a smoothly embedded, g-minimal surface.*

Proof. We may assume that $p = 0$. It follows from the first variation formula that $M \cap \mathbf{B}(0, \rho)$ has finite area for every $\rho < r$. (This is true for any bounded mean curvature variety in $\mathbf{B}(0,r) \setminus \{p\}$ by the first variation formula. See, for example, [Gul76, Lemma 1].) Thus by replacing $\mathbf{B}(0,r)$ by a smaller ball, we can assume that the the area of M is finite and that the total curvature of M is less than 4π.

Let $\lambda_i \to \infty$. By the Compactness Theorem 2.1, after passing to a subsequence, $\lambda_i M_i$ converges smoothly on compact subsets of $\mathbf{R}^3 \setminus \{0\}$ to a $g(0)$-minimal surface M'. Note that M' is a $g(0)$-minimal cone (it is a tangent cone to M at 0) and is smooth without transverse self-intersections, so it is a plane.

It follows that there is an $\epsilon > 0$ such that the function

$$x \in M \cap \mathbf{B}(0,\epsilon) \mapsto |x|$$

has no critical points, which implies that $M \cap \mathbf{B}(0,\epsilon)$ is a union of surfaces $D_1, \dots, D_k$, each of which is topologically a punctured disk.

By a theorem of Gulliver [Gul76], each $D_i \cap \{0\}$ is a (possibly branched) minimal disk. However, since D_i has no transverse self-intersections, $D_i \cap \{0\}$ is smoothly embedded. By the strong maximum principle, there is only one such disk. □

Remark 5.2. A different proof (not using Gulliver's Theorem) is given in [Whi87b, Theorem 2].

Remark 5.3. Whether the finite total curvature assumption is necessary is a very interesting open problem in minimal surface theory. The theorem remains true if that assumption is replaced by the assumption that M is stable [GL86], or by the assumption that M has finite Euler characteristic [CS85, Proposition 1]. It also remains true if that assumption is replaced by the assumption that M has finite genus. (Using monotonicity and lower bounds on density, one can show that for sufficiently small $\epsilon > 0$, the surface $M \cap \mathbf{B}(p,\epsilon)$ is a union of finitely many surfaces homeomorphic to punctured disks, to which one can then apply Gulliver's Theorem [Gul76].)

Theorem 5.4. *Let M be a smooth, two-dimensional Riemannian manifold without boundary, f be a smooth function on M, and p be a point in the interior of M. Then $C_c^\infty(M \setminus \{p\})$ is dense with respect to the H^1 norm in $C_c(M)$.*

Equivalently, $H_0^1(M) = H_0^1(M \setminus \{p\})$.

Corollary 5.5. *If there is a* $u \in C_c^\infty(M)$ *such that*

$$\int (|Du|^2 + f|u|^2)\, dA < 0,$$

then there is a $u \in C_c^\infty(M \setminus \{p\})$ *satisfying the same inequality.*

Proof of Theorem 5.4. The theorem is essentially local, and independent of the choice of metric. Thus we can assume that M is the open unit disk D in $\mathbf{R}^2$ with the Euclidean metric and that $p = 0$. Given $u \in C_c^\infty(D)$ and $0 < \epsilon < 1$, define $u_\epsilon : D \to \mathbf{R}$ by

$$u_\epsilon(z) = \begin{cases} u(z) & \text{if } |z| \geq \epsilon, \\ \frac{\ln|z| - \ln(\epsilon^2)}{\ln(\epsilon) - \ln(\epsilon^2)} u(z) & \text{if } \epsilon^2 \leq |z| \leq \epsilon, \\ 0 & \text{if } |z| \leq \epsilon^2. \end{cases}$$

One readily checks that u_ϵ converges to u in H^1 as $\epsilon \to 0$. Of course u_ϵ is not smooth, but it is Lipschitz and compactly supported in $C_c^\infty(D \setminus \{0\})$, so we can mollify to approximate it arbitrary well in H^1 by a function in $C_c^\infty(D \setminus \{0\}.)$ □

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