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Pseudo-locality for a coupled Ricci flow

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Let (M, g, ϕ) be a solution to the Ricci flow coupled with the heat equation for a scalar field ϕ. We show that a complete, κ-noncollapsed solution (M, g, ϕ) to this coupled Ricci flow with a Type I singularity at time $T < \infty$ will converge to a non-trivial Ricci soliton after parabolic rescaling, if the base point is Type I singular. A key ingredient is a version of Perelman pseudo-locality for the coupled Ricci flow.

1. Introduction

The Ricci flow [6] can be viewed as the parabolic and Euclidean version of Einstein's equation in the vacuum. In presence of matter fields, Einstein's equation becomes a coupled system. Thus we should also consider the Ricci flow coupled with other flows. The simplest is the Ricci flow coupled with the heat equation for a scalar field. This is a special case of the Ricci flow coupled with the harmonic map flow, a version of which had been instrumental in the proof of the short-time existence of solutions to the Ricci flow [3]. Coupled Ricci flows also arise as dimensional reductions of the Ricci flow in higher

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dimensions, and the coupling to a scalar field arises in particular in the Ricci flow on warped products [10].

More specifically, let M be a compact manifold. The Ricci flow coupled to the heat equation is the following system of equations for a metric $g_{ij}(t)$ and scalar field $\phi(t)$,

$$\begin{cases} \frac{\partial g}{\partial t} = -2Ric_g + 2d\phi \otimes d\phi, \qquad \phi_t = \Delta_g \phi \\ g(0) = g_0, \qquad \phi(0) = \phi_0 \end{cases} \tag{1.1}$$

where g_0 and ϕ_0 are given smooth initial data, and the coupling constant to the scalar field has been normalized to be 1. Its stationary points are solutions to Einstein's equation with ϕ the matter field and stress tensor $T_{ij} = \partial_i\phi\partial_j\phi$. This flow has been first studied extensively by List [9], who established criteria for its long-time existence, and obtained extensions to this case of Perelman's monotonicity formula and non-collapse results, as well as an extension of Hamilton's compactness theorem. Similar results for the more general case of the Ricci flow coupled with the harmonic map flow were subsequently obtained by Müller [11].

The goal of this paper is to establish a Perelman pseudo-locality theorem for the Ricci flow coupled with a scalar field. Let

$$Sic_{g,\phi} = Ric_g - d\phi \otimes d\phi \tag{1.2}$$

and denote its components by $S_{ij} = R_{ij} - \partial_i\phi\partial_j\phi$ and its trace by $S = \mathrm{tr}_g Sic_{g,\phi} = g^{ij}S_{ij} = R_g - |\nabla\phi|^2_g$. From now on, we shall omit the sub-script g and ϕ, if it is clear from the context. We prove the following pseudo-locality theorem:

Theorem 1. [Pseudo-locality] *Given $\alpha \in (0, \frac{1}{100n})$ and some const $C > 0$, there exist $\varepsilon = \varepsilon(n, \alpha, C), \delta = \delta(n, \alpha, C)$ with the following property. For any solution to the Ricci flow coupled with a scalar field $(M^n, g(t), \phi(t), p)$, $t \in [0, (\varepsilon r_0)^2]$, which has complete time slices and satisfies*

(1) *$S(g(0)) \geq -r_0^2$ on the ball $B_{g(0)}(p, r_0)$;*

(2) *$Area_{g(0)}(\partial\Omega)^n \geq (1-\delta)c_n Vol_{g(0)}(\Omega)^{n-1}$, for any $\Omega \subset B_{g(0)}(p, r_0)$, where c_n is the isoperimetric constant in $\mathbf{R}^n$,*

(3) *$|\phi_0| \leq C$ on the ball $B_{g(0)}(p, r_0)$,*

we have

$$|Rm|(x,t) \leq \alpha t^{-1} + (\varepsilon r_0)^{-2}, \tag{1.3}$$

for any (x,t) *such that* $d_{g(t)}(x,p) \leq \varepsilon r_0$ *and* $t \in (0,(\varepsilon r_0)^2]$.

A well-known conjecture of Hamilton is that blow-ups of Type I singularities in the Ricci flow should converge to a non-trivial gradient soliton. This conjecture was proved in dimension $n = 3$ by Perelman [14]. The convergence to a gradient soliton for a Type I singularity was proved in all dimensions by Naber [12], and the non-triviality of the soliton as t tends to the maximum existence time was subsequently proved by Enders, Müller, and Topping [4]. In a different direction, it was also shown by Cao and Zhang [2] that the blow-down limit of Type I κ-noncollapsed ancient solutions was a non-trivial soliton. A key ingredient in the arguments of Enders, Müller, and Topping was Perelman's pseudo-locality theorem. In this paper, we extend their arguments to the coupled Ricci flow as follows. First, List [9] has shown that the maximum existence time T for the coupled Ricci flow must satisfy

$$\lim_{t\to T} \sup_{x\in M} |Rm|^2(t,x) = \infty. \tag{1.4}$$

Thus, in analogy with the Ricci flow, a solution (M,g,ϕ) of the coupled Ricci flow with maximal time T is called of Type I if there exists some constant C_0 such that

$$\sup_{x\in M} |Rm_{g(t)}|(x,t) \leq \frac{C_0}{T-t}, \quad t \in [0,T). \tag{1.5}$$

A point p is said to be a Type I singular point if there exists a sequence (p_i,t_i), $p_i \to p, t_i \to T$, with $|Rm_{g(t_i)}|(p_i) \geq c(T-t_i)^{-1}$ for some constant $c > 0$. Next, the notion of gradient soliton for the coupled flow can be extended as a triple (g_{ij},ϕ,f) satisfying

$$Sic + \nabla^2 f - \frac{g}{2(T-t)} = 0, \quad \Delta\phi - \langle\nabla f, \nabla\phi\rangle = 0. \tag{1.6}$$

The soliton is said to be trivial if the metric g_{ij} is flat. As a consequence of the above pseudo-locality theorem, we have then

Theorem 2. *Let* $(M,g(t),\phi(t))$ *be a solution of the coupled Ricci flow (1.1) with* $|\phi_0| \leq C$*, and assume that it has a Type I singularity at time* $T < \infty$*,* *with p a Type I singularity point. Let $\lambda_i \to \infty$ be any sequence of numbers, and define a sequence of coupled Ricci flows by*

$$g_i(t) = \lambda_i g(\lambda_i^{-1} t + T), \quad \phi_i(t) = \phi(\lambda_i^{-1} t + T), \quad \forall t \in [-\lambda_i T, 0) \tag{1.7}$$

Then there exists a subsequence of (M, g_i, ϕ_i, p) which converges to a non-trivial gradient shrinking Ricci soliton $(M_\infty, g_\infty(t), \phi_\infty, p_\infty)$. The function ϕ_∞ is actually constant, so this soliton for the coupled flow actually reduces to a soliton for the usual Ricci flow.

The present paper is a revised version of a longer paper which was posted as `arXiv:1510.04332`. Some background material in `arXiv:1510.04332v3` has been replaced now by references. Otherwise the paper and main results remain the same.

2. Preliminary

For the convenience of the reader, we quote here several formulas, estimates and theorems already established by List [9] and Müller [11].

2.1. The evolution equation for the curvature

Let $(M, g_{ij}(t), \phi(t))$, $t \in [0, T)$, be a solution of the coupled Ricci flow (1.1). Then [9]

$$\frac{\partial}{\partial t} S_{jk} = \Delta S_{jk} + 2R_{ijk}{}^{l} S_{l}{}^{i} - S_{j}{}^{l} R_{lk} - S_{k}{}^{l} R_{lj} + 2\Delta\phi \cdot \phi_{jk}. \tag{2.1}$$

Taking traces with respect to g^{jk} gives

$$\frac{\partial}{\partial t} S = \Delta S + 2|S_{ij}|^2 + 2(\Delta\phi)^2 \tag{2.2}$$

List [9] showed $\phi(x,t)$ is uniformly bounded along the flow

Lemma 1. *(Lemma 5.10 in [9]) Let $(g(t), \phi(t))$ be a solution to the coupled Ricci flow (1.1) on $M \times [0, T)$ with initial data (g_0, ϕ_0), assume* $\sup|\phi_0| \leq C$*, then we have for any $t > 0$,*

$$\inf_{x\in M} \phi_0(x) \leq \phi(x,t) \leq \sup_{x\in M} \phi_0(x) \tag{2.3}$$

$$\sup_{x\in M} |\nabla\phi|^2(x,t) \leq C^2 t^{-1}, \tag{2.4}$$

The following derivative estimate holds

Proposition 1. *(Theorem 5.12 in [9], see also Theorem 3.10 in [11]) Let* $(g(t), \phi(t))$ *be a solution to the coupled Ricci flow (1.1). Fix* $x_0 \in M$ *and* $r > 0$*, if*

$$\sup_{B(T,x_0,r)} r^2|Rm| \le \tilde{C} \tag{2.5}$$

where $B(T, x_0, r)$ *is the union of metric balls* $B_t(x_0, r)$ *for all* $t \in [0, T]$*. Denote* $\Phi = (Rm, \nabla^2\phi)$*, then for any integer* $m \ge 0$ *and any* $t \in (0, T]$ *the following estimates for derivatives of* Φ *hold*

$$\sup_{B(t,x_0,r/2)} |\nabla^m \Phi|^2 \le C(n,m)\tilde{C}^{m+2}\left(r^{-2} + t^{-1}\right)^{m+2} \tag{2.6}$$

where $C = C(n, m)$ *is a constant depending only on* n *and* m*.*

2.2. Reduced distance, reduced volume and monotonicity formula

In this subsection we provide some background material on the reduced distance and volume for the coupled Ricci system. The material can be found in List [9], Müller [11] and Vulcanov [15].

Let $(g_{ij}(t), \phi(t))$ be a solution to the coupled Ricci flow (1.1) on $t \in (0, T)$. For some fixed $t_0 \in [0, T)$, set $\tau = t_0 - t$. In terms of τ, the flow becomes

$$(g_{ij})_\tau = 2S_{ij}, \qquad \phi_\tau = -\Delta\phi. \tag{2.1}$$

We can now define the $\mathcal{L}$-length of a path for coupled Ricci flow as follows. Let $\gamma : [\tau_1, \bar{\tau}] \to M$ be a path, where $0 \le \tau_1 < \bar{\tau} \le t_0$. We define the $\mathcal{L}$-length of the path γ by

$$\mathcal{L}(\gamma) = \int_{\tau_1}^{\bar{\tau}} \sqrt{\tau}\Big(S(\gamma(\tau)) + |\gamma'(\tau)|^2\Big)d\tau \tag{2.2}$$

where $S(\gamma(\tau))$ and the norm $|\gamma'(\tau)|$ are evaluated using the metric $g_{ij}(t)$ at time $t = t_0 - \tau$.

We are ready to state the definition of reduced distance $\ell_{p,t_0}(q, \tau)$ and reduced volume $\tilde{V}_{t_0}(\tau)$. Given $p \in M$, and $t_0 \in (0, T)$, for any $q \in M$ and

$0 < \bar{\tau} \le t_0$, define $L_{p,t_0}(q, \bar{\tau})$ to be the infimal $\mathcal{L}$-length of all curves γ with $\gamma(0) = p$ and $\gamma(\bar{\tau}) = q$, i.e.,

$$\begin{aligned} L_{p,t_0}(q,\bar{\tau}) &= \inf_{\gamma} \mathcal{L}(\gamma) \\ &= \inf_{\gamma} \left\{ \int_0^{\bar{\tau}} \sqrt{\tau}\Big(S(\gamma(\tau)) + |\gamma'(\tau)|^2\Big) d\tau : \gamma(0) = p, \gamma(\bar{\tau}) = q \right\} \end{aligned} \tag{2.3}$$

Define the reduced distance based at (p, t_0) by

$$\ell_{p,t_0}(q,\tau) = \frac{L_{p,t_0}(q,\tau)}{2\sqrt{\tau}}, \tag{2.4}$$

and the reduced volume by

$$\tilde{V}_{t_0}(\tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\ell_{p,t_0}(q,\tau)} dV_{g(\tau)}(q) \tag{2.5}$$

From now on we will omit the sub-scripts p, t_0 if it is clear from the context. We have the following lemma about derivatives of L and $\tilde{V}$.

Lemma 2. *(Lemma 6.9 in [11]) The first derivatives of L are given by*

$$L_{\bar{\tau}}(q,\bar{\tau}) = 2\sqrt{\bar{\tau}}S(q) - \frac{1}{2\bar{\tau}}L(q,\bar{\tau}) + \frac{1}{\bar{\tau}}K \tag{2.6}$$

and

$$|\nabla L|^2(q,\bar{\tau}) = -4\bar{\tau}S(q) + \frac{2}{\sqrt{\bar{\tau}}}L(q,\bar{\tau}) - \frac{4}{\sqrt{\bar{\tau}}}K \tag{2.7}$$

where

$$K = \int_0^{\bar{\tau}} \tau^{\frac{3}{2}} H(X(\tau)) d\tau. \tag{2.8}$$

and $X(\tau) = \gamma'(\tau)$, γ is the unique $\mathcal{L}$ geodesic joint p, q. $H(X)$ is defined by

$$H(X) = -S_\tau - \frac{1}{\tau}S - 2\langle \nabla S, X\rangle + 2Sic(X,X) \tag{2.9}$$

Lemma 3. *(Lemma 6.10 in [11]) The following inequality holds in the barrier sense*

$$L_{\bar{\tau}}(q,\bar{\tau}) + \Delta L(q,\bar{\tau}) \le \frac{n}{\sqrt{\bar{\tau}}} - \frac{1}{2\bar{\tau}}L(q,\bar{\tau}), \tag{2.10}$$

As a simple consequence, the function $\bar{L}(q,\tau) = 2\sqrt{\tau}L(q,\tau)$ satisfies the following inequality in the barrier sense

$$\bar{L}_\tau + \Delta\bar{L} \leq 2n, \tag{2.11}$$

Recall that the reduced distance $\ell(q,\tau)$ was defined in (2.4). It follows immediately from Lemmas 2 and 3 that

Lemma 4. *The reduced distance satisfies the following differential inequalities in the barrier sense,*

$$\ell_\tau = S - \frac{1}{\tau}\ell + \frac{1}{2\tau^{3/2}}K \tag{2.12}$$

$$|\nabla\ell|^2 = -S + \frac{1}{\tau}\ell - \frac{1}{\tau^{3/2}}K \tag{2.13}$$

$$\Delta\ell \leq \frac{n}{2\tau} - S - \frac{1}{2\tau^{3/2}}K \tag{2.14}$$

which imply the following inequality in the barrier sense,

$$\ell_\tau - \Delta\ell + |\nabla\ell|^2 - S + \frac{n}{2\tau} \geq 0 \tag{2.15}$$

Note also that, by the maximum principle applied to (2.11), we have $\min\bar{L}(\cdot,\tau) \leq 2n\tau$, and hence

$$\min\ell(\cdot,\tau) = \min\frac{\bar{L}(\cdot,\tau)}{4\tau} \leq \frac{n}{2}. \tag{2.16}$$

Theorem 3. *(Theorem 6.4 in [11]) Along a solution of the coupled Ricci flow, the reduced volume $\tilde{V}(\tau)$ is monotonically nonincreasing in $\tau = t_0 - t$.*

Proof. If we let $v = (4\pi\tau)^{-n/2}e^{-\ell}$, then we have

$$\frac{v_\tau}{v} = -\frac{n}{2\tau} - \ell_\tau, \qquad \frac{\Delta v}{v} = -\Delta\ell + |\nabla\ell|^2 \tag{2.17}$$

Hence by (2.15), we have $v_\tau - \Delta v + Sv \leq 0$. Since

$$\tilde{V}(\tau) = \int_M (4\pi\tau)^{-n/2}e^{-\ell(q,\tau)}dV = \int_M vdV,$$

we obtain that

$$\frac{d}{d\tau}\tilde{V}(\tau) = \int_M (v_\tau + Sv)dV \leq \int_M \Delta v dV = 0 \tag{2.18}$$

i.e., $\tilde{V}(\tau)$ is monotonically nonincreasing in τ. Q.E.D.

2.3. A κ-noncollapsing theorem

The κ-noncollapsing theorem of Perelman [14] has been generalized to the coupled Ricci flow by List [9] and Müller [11]. First, we recall Perelman's definition of κ-noncollapse:

Definition 1 (κ-noncollapsing). We say a solution (M, g, ϕ) of the coupled Ricci flow (1.1) on an interval $[0, T)$ to be κ-noncollapsing at the scale ρ, if for each $r < \rho$, and all $(x_0, t_0) \in M \times [0, T)$, the following holds: if $|Rm|(x, t) < r^{-2}$ for all (x, t) in the parabolic neighborhood of (x_0, t_0), $(x, t) \in P(x_0, t_0, r) = B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$, then $vol(B_{t_0}(x_0, r)) \geq \kappa r^n$.

Using the formulas about reduced distance and reduced volume we have stated above, one can establish the following κ-noncollapsing theorem.

Theorem 4. *(Theorem 7.2 in [9], see also Theorem 6.13 in [11]) Given n, K, ρ, c, there exists $\kappa = \kappa(n, K, \rho, c)$ such that for any solution $(M, g(t), \phi(t))$ to Ricci flow coupled with heat equation on $M \times [0, T)$, $T < \infty$ with$(M, g(0))$ complete, $|Rm| + |\nabla^2 \phi|^2 \leq K$ and $inj(M, g(0)) \geq c > 0$, then the solution is κ-noncollapsing at the scale ρ.*

3. Proof of the Pseudo-Locality Theorem

Our proof of the pseudo-locality theorem for the coupled Ricci flow follows the lines of Perelman ([14], see also [8]). We begin with the analogues for the coupled Ricci flow of several lemmas of Perelman (Lemma 8.3 in [14]) on the time evolution of the distance function and the fundamental solution of the conjugate heat equation.

3.1. The time-derivative of the distance function

Lemma 5. *Let $(M, g(t), \phi(t))$ be a complete Ricci flow coupled with the scalar heat equation. If $Ric(g(t_0)) \leq (n-1)K$ in $B_{g(t_0)}(x_0, r_0)$, then for any $x \notin B_{g(t_0)}(x_0, r_0)$, we have for $d(x, t) = d_{g(t)}(x, x_0)$*

$$\frac{\partial}{\partial t} d(x, t)\Big|_{t=t_0} - \Delta_{g(t_0)} d(x, t_0) \geq -(n-1)\left(\frac{2}{3} K r_0 + r_0^{-1}\right) \tag{3.1}$$

Proof. Let $\gamma(s), s \in [0, d(x,t_0)]$ be a normal minimal geodesic with respect to $g(t_0)$ joining x_0 and x. Then

$$\frac{\partial}{\partial t}\Big|_{t=t_0} d(x,t) = -\int_\gamma Sic(\gamma',\gamma')ds \geq -\int_\gamma Ric(\gamma',\gamma')ds \tag{3.2}$$

where $Sic = Ric - d\phi \otimes d\phi$. Note that Sic is bounded above by Ric.

At x_0, set $E_1 = \gamma'(0)$, and extend E_1 to an orthonormal basis E_i of $T_{x_0}M$. Parallelly transporting this basis along γ gives us an orthonormal basis of $T_{\gamma(s)}M$.

Recall the second variation formula of the distance function. Assume $w \in T_xM$, and let W be the Jacobi field given by $W(x_0) = 0$, $W(x) = w$. Then

$$\nabla^2 d(w,w) = I(W,W) \leq I(Y,Y) \tag{3.3}$$

where Y is any vector field with $Y(x_0) = 0$, $Y(x) = w$ and $I(W,W)$ is the index form defined by

$$I(V,W) = \int_\gamma \Big(\langle V',W'\rangle - \langle V',\gamma'\rangle\langle W',\gamma'\rangle - R(\gamma',V,\gamma',W)\Big)ds. \tag{3.4}$$

Thus

$$\begin{aligned}
\Delta_{g(t_0)} d(x,t_0) &= \sum_{i=1}^n \nabla^2 d(e_i,e_i) \\
&\leq \sum_{i=1}^n \int_\gamma |\nabla_{\gamma'}F_i|^2 - \big\langle \nabla_{\gamma'}F_i,\gamma'\big\rangle^2 - R(\gamma',F_i,\gamma',F_i)ds
\end{aligned}$$

for any vector field F_i with $F_i(x_0) = 0$ and $F_i(x) = E_i(x)$. Choosing F_i along γ as

$$F_i(\gamma(s)) = \begin{cases} \frac{s}{r_0}E_i(s) & s \in [0,r_0] \\ E_i(s) & s \in [r_0, d(x,t_0)]. \end{cases} \tag{3.5}$$

we obtain

$$\langle \nabla_{\gamma'}F_i,\gamma'\rangle = 0 \tag{3.6}$$

for all $i = 2, \cdots, n$ and $s \in [0, d(x, t_0)]$ and $|\nabla_{\gamma'} F_1|^2 - \langle \nabla_{\gamma'} F_1, \gamma' \rangle^2 = 0$. Hence

$$\begin{aligned}
\Delta_{t_0} d(x, t_0) &\leq \int_\gamma \sum_{i=2}^n |\nabla_{\gamma'} F_i|^2 - R(\gamma', F_i, \gamma', F_i) ds \\
&= \int_0^{r_0} \sum_{i=2}^n \left(\frac{1}{r_0^2} - \frac{s^2}{r_0^2} R(\gamma', E_i, \gamma', E_i) \right) ds \\
&\quad + \int_{r_0}^{d(x,t_0)} \sum_{i=2}^n -R(\gamma', E_i, \gamma', E_i) ds \\
&= \frac{n-1}{r_0} + \int_0^{r_0} (1 - \frac{s^2}{r_0^2}) Ric(\gamma', \gamma') ds - \int_0^{d(x,t_0)} Ric(\gamma', \gamma') ds \\
&\leq \frac{n-1}{r_0} + (n-1)K(r_0 - \frac{1}{3} r_0) + \frac{\partial}{\partial t} d(x, t)\Big|_{t=t_0} \qquad (3.7)
\end{aligned}$$

This completes the proof.

By similar argument, we have the following lemma, for which we omit the proof.

Lemma 6. *Let $(M, g(t), \phi(t))$ be a complete Ricci flow coupled with the scalar heat equation. Assume*

$$Ric(g(t_0), x) \leq (n-1)K$$

for any $x \in B_{g(t_0)}(x_1, r_0) \cup B_{g(t_0)}(x_2, r_0)$, and $d_{t_0}(x_1, x_2) \geq 2r_0$. Then we have

$$\frac{\partial}{\partial t} d_t(x_1, x_2)\Big|_{t=t_0} \geq -2(n-1) \left(\frac{2}{3} K r_0 + r_0^{-1} \right) \qquad (3.8)$$

where $d_t(x_1, x_2) = d_{g(t)}(x_1, x_2)$.

3.2. The localized $\mathcal{W}$-functional and conjugate heat equation

Let $\Box = \partial_t - \Delta$. The formal adjoint $\Box^*$ of $\Box$ is defined by the relation

$$0 = \int_a^b \int_M \Box\varphi \cdot \psi dV dt - \int_a^b \int_M \varphi \cdot \Box^* \psi dV dt \qquad (3.9)$$

for any $a, b \in [0, T]$, $\varphi, \psi \in C_0^\infty(M \times (a, b))$. It is readily recognized that $\Box^* = -\partial_t - \Delta + S$.

Let $u = (4\pi(T-t))^{-\frac{n}{2}}e^{-f}$ be a solution of the conjugate heat equation $\Box^* u = 0$. Set

$$v = \Big((T-t)(2\Delta f - |\nabla f|^2 + S) + f - n\Big)u. \tag{3.10}$$

By direct calculations, we have

$$\Box^* v = -2(T-t)\left(\left|S_{ij}+\nabla_i\nabla_j f - \frac{g_{ij}}{2(T-t)}\right|^2 + (\Delta\phi - \langle\nabla\phi, \nabla f\rangle)^2\right) u \tag{3.11}$$

and in particular we have $\Box^* v \le 0$. We remark that the integral of v over M is the $\mathcal{W}$-functional introduced by Perelman (see e.g. [2, 13, 14])

$$\mathcal{W}(g_{ij}, f, T-t) = (4\pi(T-t))^{-n/2}\int_M \Big((T-t)(|\nabla f|^2 + S) + f - n\Big)e^{-f}dV,$$

in case the integration by parts holds.

Similar to the Ricci flow case (see [13, 14]), we have the following differential Harnack estimate for the fundamental solutions to the conjugate heat equation.

Lemma 7. *Let u be the fundamental solution based at (p, T), namely $\Box^* u = 0$ and $u(x,t) \to \delta_p(x)$ as $t \to T$. Then we have $v \le 0$ for all $t < T$, where v is defined as in (3.10).*

Proof. For any fixed $t_0 \in (0, T)$ and any positive function h_0, solve the equation $\Box h = 0$ with initial condition $h(t_0) = h_0$. Then we have

$$\frac{d}{dt}\int_M hvdV = \int_M \Big((\Box h)v - h(\Box^* v)\Big)dV = -\int_M h(\Box^* v)dV \ge 0.$$

Hence, $\int_M hvdV$ is increasing in $t \in (t_0, T)$. Moreover, as stated by Perelman ([14], see also [13])

$$\lim_{t\to T^-}\int_M h(x,t)v(x,t)dV_{g(t)} = 0, \tag{3.12}$$

hence we have

$$\int_M h(x,t_0)v(x,t_0)dV_{g(t_0)} \le 0. \tag{3.13}$$

Since $h(x, t_0)$ and t_0 are arbitrary, we have $v(x,t) \le 0$ for any $t < T$.

3.3. A point selection lemma

We return now to proof of the pseudo-locality theorem. Without loss of generality, we can assume $r_0 = 1$. Assume that the theorem is not true, that is, there exists $\varepsilon_k, \delta_k \to 0$ such that for each k, there exists a complete solution (M_k, g_k, ϕ_k, p_k) to coupled Ricci flow (1.1), such that $S(g_k(0)) \geq -1$ on ball $B_{g_k(0)}(p_k, 1)$, $Area_{g_k(0)}(\partial\Omega)^n \geq (1-\delta_k)c_n Vol_{g_k(0)}(\Omega)^{n-1}$ for any $\Omega \subset B_{g_k(0)}(p_k, 1)$, and $|\phi_{k,0}| \leq C$, but there exists (x_k, t_k) such that $d_{g(t_k)}(x_k, p_k) \leq \varepsilon_k$ and

$$|Rm|(x_k, t_k) \geq \alpha t_k^{-1} + \varepsilon_k^{-2}. \tag{3.14}$$

Moreover, we can choose a smaller ε_k such that

$$|Rm|(x, t) \leq \alpha t^{-1} + 2\varepsilon_k^{-2} \tag{3.15}$$

for all $t \in (0, \varepsilon_k^2)$ and $d_{g_k(t)}(x, p_k) \leq \varepsilon_k$. To see this, observe that the continuous function

$$F(\varepsilon) = \varepsilon^2 \sup_{t\in(0,\varepsilon^2]} \left(\sup_{B_{g_k(t)}(p_k,\varepsilon)} |Rm|_{g_k}(t) - \alpha t^{-1} \right)$$

satisfies $F(\varepsilon_k) \geq 1$ by (3.14), and $\limsup_{\varepsilon\to 0} F(\varepsilon) \leq 0$ since $(M_k, g_k(t))$ is a smooth flow. Therefore we can choose an $\hat{\varepsilon}$ smaller than ε_k such that $1 \leq F(\hat{\varepsilon}) \leq 2$, and denote this $\hat{\varepsilon}$ as the new ε_k. It follows from the definition of $F(\varepsilon)$ that (3.14) and (3.15) both hold by slightly changing the x_k in (3.14) if necessary.

We divide the argument into several steps.

The first step is to choose some other point $(\bar{x}, \bar{t})$, such that the Riemannian curvature tensor can be controlled by $|Rm|(\bar{x}, \bar{t})$ in a parabolic neighborhood of $(\bar{x}, \bar{t})$, provided there exists an (x, t) satisfying the above hypotheses.

The following point-selection lemmas can be proved in the same way as the Ricci flow case, so we omit the proof and refer to that of claim 1 and claim 2 in section 10 of [14] (see also [8]).

Lemma 8. *For any large $A > 0$ and any solution $(M, g(t), \phi(t), p)$ to coupled Ricci flow, if there is a point $(x_0, t_0) \in M \times (0, \varepsilon^2]$ such that $|Rm|(x_0, t_0) \geq \alpha t_0^{-1} + \varepsilon^{-2}$ and $d_{t_0}(x_0, p) \leq \varepsilon$, then there is a point $(\bar{x}, \bar{t}) \in M_\alpha$ such that*

$d_{\bar t}(\bar x, p) \le (1+2A)\varepsilon$ and

$$|Rm|(x,t) \le 4|Rm|(\bar x, \bar t); \tag{3.16}$$

for any $(x,t) \in M_\alpha$, $0 < t \le \bar t$ such that

$$d_t(x,p) \le d_{\bar t}(\bar x, p) + A|Rm|^{-1/2}(\bar x, \bar t),$$

where

$$M_\alpha := \{(x,t) \in M \times (0, \varepsilon^2] : |Rm|(x,t) > \alpha t^{-1}\}.$$

Lemma 9. *Under the same assumption as previous lemma, the point $(\bar x, \bar t)$ selected above satisfies*

$$|Rm|(x,t) \le 4Q =: 4|Rm|(\bar x, \bar t) \tag{3.17}$$

for any $x \in B_{g(\bar t)}(\bar x, \frac{1}{10}AQ^{-1/2}) \times [\bar t - \frac{1}{2}\alpha Q^{-1}, \bar t]$.

Applying Lemma 9, we can find

$$(\bar x_k, \bar t_k) \in M_k \times (0, \varepsilon_k^2] \quad \text{with} \quad d_{g_k(\bar t_k)}(\bar x_k, p_k) \le (1+2A_k)\varepsilon_k,$$

satisfying the above properties, i.e.,

$$|Rm|_{g_k(t)}(x,t) \le 4Q_k = 4|Rm|(\bar x_k, \bar t_k) \tag{3.18}$$

for any $(x,t) \in \Omega_k$, where Ω_k is defined by

$$\Omega_k = \left\{(x,t);\ d_{\bar t_k}(x, \bar x_k) \le \frac{1}{10}A_k Q_k^{-1},\ \bar t_k - \frac{1}{2}\alpha Q_k^{-1} \le t \le \bar t_k\right\} \tag{3.19}$$

3.4. A gap lemma

For each k, let u_k be the fundamental solution at $(\bar x_k, \bar t_k)$ of the conjugate heat equation, i.e., u_k is the solution of

$$\Box_t^* u_k = 0$$

with initial condition Dirac function $\delta_{\bar x_k}(x)$ at time $\bar t_k$. Define as in (3.10)

$$v_k = \Big((\bar t_k - t)(S + 2\Delta f_k - |\nabla f_k|^2) + f_k - n\Big)u_k$$

where we have set $u_k = (4\pi(\bar t_k - t))^{-n/2}\, e^{-f_k}$.

Lemma 10. *There exists a uniform constant $b > 0$(independent of k) and a $\tilde{t}_k \in (t_k - \frac{1}{2}\alpha Q_k^{-1}, \bar{t}_k)$ for each k, such that*

$$\int_{B_k} v_k dV_{g_k(\tilde{t}_k)} \leq -b < 0 \tag{3.20}$$

where $B_k = B_{g_k(\tilde{t}_k)}(\bar{x}_k, \sqrt{\bar{t}_k - \tilde{t}_k})$.

Proof. The proof is by contradiction. Assume that it is not true, which means that for any $\tilde{t}_k \in (t_k - \frac{1}{2}\alpha Q_k^{-1}, \bar{t}_k)$, there exists a subsequence, still denoted by k, such that

$$\liminf_{k\to\infty} \int_{B_k} v_k dV_{g_k(\tilde{t}_k)} \geq 0. \tag{3.21}$$

Consider the following rescaling,

$$\hat{g}_k(t) = Q_k g_k(Q_k^{-1}t + \bar{t}_k), \hat{\phi}_k(t) = \phi_k(Q_k^{-1}t + \bar{t}_k) \tag{3.22}$$

for $t \in [-Q_k\bar{t}_k, 0]$. Then $(\hat{g}_k(t), \hat{\phi}_k(t))$ also satisfies the coupled Ricci flow (1.1). Note that under the parabolic rescaling Ω_k becomes

$$\hat{\Omega}_k = B_{\hat{g}(0)}\left(\bar{x}_k, \frac{1}{10}A_k\right) \times \left[-\frac{1}{2}\alpha, 0\right] \tag{3.23}$$

Now, we consider the following two cases: either along a subsequence the injectivity radius of $\hat{g}_k(0)$ at $\bar{x}_k$ has positive lower bound, or there is no such a lower bound along any subsequence.

Case 1: By List's compactness theorem (Theorem 7.5 in [9]), we can find a subsequence, again denoted by k, such that $(M_k, \hat{g}_k(t), \hat{\phi}_k(t), p_k)$ converges in the pointed C^∞-CG (Cheeger-Gromov) sense to a new complete coupled Ricci flow $(M_\infty, g_\infty, \phi_\infty, p_\infty)$ for $t \in [-\frac{1}{2}\alpha, 0]$, and $|Rm|_{g_\infty}(x,t) \leq 4$ for all $(x,t) \in M_\infty \times [-\frac{1}{2}\alpha, 0]$ and $|Rm|(x_\infty, 0) = 1$. For each M_k, we have the fundamental solution $\hat{u}_k$ based at $(\bar{x}_k, \bar{t}_k)$, $\hat{u}_k$ converge to u_∞ in the same sense, where u_∞ is a fundamental solution to the conjugate heat equation on $(M_\infty, g_\infty, \phi_\infty, x_\infty)$ based at $(x_\infty, 0)$. So v_k converge to v_∞ in the same sense, and $v_\infty \leq 0$ by Lemma 7. By the assumption (3.21), for any fixed

$t_0 \in [-\frac{1}{2}\alpha, 0]$, we have

$$\int_{B_{g_\infty(t_0)}(x_\infty,\sqrt{-t_0})} v_\infty(\cdot, t_0) dV_{g_\infty(t_0)} \geq 0, \tag{3.24}$$

Thus $v_\infty(\cdot, t_0) = 0$ on $B_{g_\infty(t_0)}(x_\infty, \sqrt{-t_0})$. Next we show that

$$v_\infty \equiv 0 \quad \text{on } M_\infty \times (t_0, 0] \tag{3.25}$$

For this, take a non-negative, non-trivial, function h_0 with compact support in $B_{g_\infty(t_0)}(x_\infty, \sqrt{-t_0})$, and solve the following heat equation

$$\partial_t h = \Delta_{g_\infty(t)} h, \quad h(\cdot, t_0) = h_0(\cdot) \tag{3.26}$$

By the maximum principle, we have $h(x, t) > 0$ for any $t > t_0$ and

$$\frac{d}{dt} \int_{M_\infty} h(x,t) v_\infty(x,t) dV_{g_\infty(t)} = \int_{M_\infty} \Box_t h v_\infty - h \Box_t^* v_\infty \geq 0. \tag{3.27}$$

Since the integral $\int hvdV$ is zero at both $t = t_0$ and $t = 0$ (see [13, 14]), it is zero for any $t \in (t_0, 0)$. Since h is strictly positive, and v_∞ non-positive, our assertion follows.

The formula (3.11) for $\Box_t^* v_\infty$ shows that $(g_\infty(t), \phi_\infty(t))$ is a complete extended Ricci soliton, i.e.,

$$Sic_\infty(g_\infty(t)) + \nabla^2 f_\infty(t) - \frac{g_\infty(t)}{-2t} = 0 \tag{3.28}$$

$$\Delta\phi_\infty - \langle \nabla\phi_\infty, \nabla f_\infty \rangle = 0 \tag{3.29}$$

for all $t \in (t_0, 0)$. We require the following version for the coupled Ricci flow of a result of Zhang [16],

Theorem 5. *Let $(M_\infty, g_\infty(t), \phi_\infty, f_\infty)$ be as in (3.28) and (3.29), then $S_\infty(t) \geq 0$ and the gradient vector $-\nabla_{g_\infty} f_\infty$ is a complete vector field, i.e. it generates a one-parameter family of diffeomorphisms $\varphi(t) : M_\infty \to M_\infty$ for all $t \in (-\infty, 0)$.*

The proof of Theorem 5 will be given in the Appendix.

We can now apply Theorem 5. The completeness of $g_\infty(t)$ implies that of ∇f_∞. Thus the vector field $-\nabla f_\infty$ can be integrated to give a family of diffeomorphisms $\varphi(t) : M_\infty \to M_\infty$, $t_0 \le t < 0$, such that

$$\frac{\partial}{\partial t}\varphi(t) = -\nabla f_\infty(t) \circ \varphi(t)$$

Consider $\tilde{g}(t) = -t^{-1}\varphi(t)^* g_\infty(t)$. We have

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{g}(t) &= t^{-2}\varphi(t)^* g_\infty(t) - t^{-1}\varphi(t)^*(L_{-\nabla f_\infty} g_\infty(t) - 2Sic(g_\infty(t)) \\ &= -2t^{-1}\varphi(t)^* \left(-Sic(g_\infty(t)) - \nabla^2 f_\infty(t) - \frac{g_\infty(t)}{2t}\right) = 0. \qquad (3.30)\end{aligned}$$

Hence $\tilde{g}(t)$ is independent of $t \in (t_0, 0)$, the Riemannian curvature of $\tilde{g}(t)$ is bounded and $|Rm_{\tilde{g}}|(x_\infty) \neq 0$. On the other hand, $|Rm|_{\tilde{g}(t)} = (-t)|Rm|_{g_\infty(t)}$, therefore $|Rm|_{g_\infty(t)}(x_\infty) = \frac{|Rm|_{\tilde{g}(t)}(x_\infty)}{-t} \gg 1$ when t is close to 0. This contradicts $|Rm|_{g_\infty(0)}(x_\infty) = 1$.

Case 2: Suppose now that there is a subsequence so that the injectivity radii of the metrics at $\bar{x}_k$ tend to zero, w.l.o.g., assume $r_k = inj(\bar{x}_k, \hat{g}_k(0)) \to 0$. Rescale $(M_k, \hat{g}_k, \hat{\phi}_k, p_k)$ by

$$\tilde{g}_k(t) = r_k^{-2}\hat{g}_k(r_k^2 t), \quad \tilde{\phi}_k(t) = \hat{\phi}_k(r_k^2 t). \qquad (3.31)$$

where $t \in [-\frac{1}{2}\alpha r_k^{-2}, 0]$. The region Ω_k becomes

$$d_{\tilde{g}_k(0)}(x, \bar{x}_k) \le \frac{1}{10}A_k r_k^{-1} \to \infty, \quad t \in \left[-\frac{1}{2}\alpha r_k^{-2}, 0\right] \qquad (3.32)$$

The injectivity radius of $\tilde{g}_k(0)$ at $\bar{x}_k$ is 1. On Ω_k, the Riemannian curvature tensor is bounded above by $4r_k^2$, hence we get a subsequence converging in C^∞-CG sense to a complete coupled Ricci flow $(M_\infty, g_\infty, \phi_\infty, p_\infty)$ for $t \in (-\infty, 0]$. Moreover the uniform curvature bound on Ω_k implies that the solution (M_∞, g_∞) is a flat metric. Hence, by similar arguments as in the first case, we get a family of solitons $(M_\infty, g_\infty, \phi_\infty, f_\infty)$ satisfying (3.28) and (3.29). Since $Rm(g_\infty(t)) = 0$, by Theorem 5, $S(g_\infty) = -|\nabla_{g_\infty}\phi_\infty|^2 \ge 0$, we have ϕ_∞ =const. Therefore

$$\nabla^2 f_\infty = \frac{g_\infty(t)}{-2t}. \qquad (3.33)$$

By the uniformization theorem in Riemannian geometry, the universal cover of (M_∞, g_∞) is isometric to $(\mathbf{R}^n, g_{can})$, $\pi : \mathbf{R}^n \to M_\infty$. Pulling back to the universal cover, we get

$$\nabla^2 \pi^* f_\infty = \frac{g_{can}}{-2t} > 0$$

Hence, $\pi^* f_\infty$ is a strictly convex function in $\mathbf{R}^n$. Since a convex function on $\mathbf{R}^n$ can never be periodic, π has to be trivial. Therefore, we have $(M_\infty, g_\infty(t)) = (\mathbf{R}^n, c(t)g_{can})$, but this contradicts the fact that $inj_{g_\infty}(x_\infty, 0)$ is finite. The proof of Lemma 10 is complete.

We establish next another version of the preceding gap lemma, but with the volume form $dV_{g_k(\tilde{t}_k)}$ replaced by $dV_{g_k(0)}$. For this, we need the assumption on the initial metrics, which we had not used as yet.

Let φ be a smooth function on $\mathbf{R}$ which is one on $(-\infty, 1]$, decreases to zero on $[1, 2]$, and is zero on $[1, \infty)$, with $(\varphi')^2 \leq 10\varphi$ and $-\varphi'' \leq 10\varphi$.

The following construction is done for each individual (M_k, g_k, ϕ_k, p_k). Let $\tilde{d}_k(x,t) = d_k(x,t) + 200n\sqrt{t}$ where $d_k(x,t) = d_{g_k(t)}(p_k, x)$, and define a function $h_k(x,t)$ by

$$h_k(x,t) = \varphi\left(\frac{\tilde{d}_k(x,t)}{10A_k\varepsilon_k}\right)$$

Lemma 11. *If the constants A_k are chosen to be large enough, then we have for all k*

$$\int_M h_k v_k dV_{g_k}\Big|_{t=0} \leq -\frac{1}{2}b < 0. \tag{3.34}$$

where b is the constant in Lemma 10.

Proof. We suppress the subindex k for notational simplicity. It is easy to show that

(3.35)

$$\Box_t h(x,t) = \frac{1}{10A\varepsilon}\left(d_t - \Delta d + \frac{100n}{\sqrt{t}}\right)\varphi'\left(\frac{\tilde{d}(x,t)}{10A\varepsilon}\right) - \frac{1}{(10A\varepsilon)^2}\varphi''\left(\frac{\tilde{d}(x,t)}{10A\varepsilon}\right)$$

We claim that, if A is large enough, then

$$\Box_t h(x,t) \leq -\frac{1}{(10A\varepsilon)^2}\varphi''\left(\frac{\tilde{d}(x,t)}{10A\varepsilon}\right). \tag{3.36}$$

For this, it suffices to show that if A is large enough, then on the support of $\varphi'(\tilde{d}(x,t)(10A\varepsilon)^{-1})$, i.e., when $10A\varepsilon \le \tilde{d}(x,t) \le 20A\varepsilon$, we have

$$d_t - \Delta d + \frac{100n}{\sqrt{t}} \ge 0. \tag{3.37}$$

Indeed, for $t \in [0,\varepsilon^2]$, we have $200n\sqrt{t} \le A\varepsilon$ if A is large enough, and hence

$$9A\varepsilon \le d(x,t) \le 21A\varepsilon$$

Let $r_0 = \sqrt{t}$. Since $r_0 \le \varepsilon$, we obviously have $x \notin B(p,r_0)$. Moreover, from (3.15), we know that $|Rm|(x,t) \le \alpha t^{-1} + 2\varepsilon^{-2}$ for $x \in B(p,r_0)$. Thus we can apply Lemma 5 to get

$$\begin{aligned} d_t - \Delta d &\ge -(n-1)\left(\frac{2}{3}(\alpha t^{-1} + 2\varepsilon^{-2})t^{1/2} + t^{-1/2}\right) \\ &= -(n-1)\left(\frac{2}{3}\alpha + \frac{4}{3}\varepsilon^{-2}t + 1\right)t^{-1/2} \ge -100nt^{-1/2}. \end{aligned} \tag{3.38}$$

This establishes the claim.

Recall that $\Box_t^* v \le 0$. Thus

$$\begin{aligned} \frac{d}{dt}\left(\int_M h(-v)dV\right) &= \int_M \left(\Box h(-v) + h\Box^* v\right)dV \le \int_M \Box h(-v)dV \\ &\le \frac{1}{(10A\varepsilon)^2}\int_M -\varphi''(-v)dV \le \frac{1}{(10A\varepsilon)^2}\int_M 10\varphi(-v)dV \\ &\le \frac{10}{10(A\varepsilon)^2}\int_M h(-v)dV \end{aligned} \tag{3.39}$$

and hence

$$\frac{d}{dt}\log\int_M h(-v)dV \le \frac{1}{10(A\varepsilon)^2}. \tag{3.40}$$

Integrating from 0 to $\tilde{t}$ yields (here $\tilde{t}$ is from Lemma 10)

$$\frac{\int_M h(-v)dV\Big|_{t=\tilde{t}}}{\int_M h(-v)dV\Big|_{t=0}} \le e^{\frac{\tilde{t}}{10A^2\varepsilon^2}} \le e^{\frac{1}{10A^2}} \tag{3.41}$$

where we used $\tilde{t} \le \varepsilon^2$. Hence, we get

$$\int_M h(-v)dV\Big|_{t=0} \ge e^{-\frac{1}{10A^2}}\int_M h(-v)dV|_{t=\tilde{t}}. \tag{3.42}$$

It suffices to show that $h \equiv 1$ on the ball $B = B_{g(\tilde{t})}(\bar{x}, \sqrt{\bar{t}-\tilde{t}})$, and then the desired inequality follows from Lemma 10.

To see this, it suffices to show for any $x \in B_{g(\tilde{t})}(\bar{x}, \sqrt{\bar{t}-\tilde{t}})$, it holds that $d_{g(\tilde{t})}(x,p) \leq 3A\varepsilon$. Recall by Lemma 9, $|Rm|(y,t) \leq 4Q = 4|Rm|(\bar{x},\bar{t})$, for all

$$(y,t) \in B_{g(\bar{t})}\left(\bar{x}, \frac{1}{10}AQ^{-1/2}\right) \times \left[\bar{t} - \frac{1}{2}\alpha Q^{-1}, \bar{t}\right].$$

And by distance comparison argument we have

$$d_{g(\tilde{t})}(y,\bar{x}) \leq e^{4Q(\bar{t}-\tilde{t})} d_{g(\bar{t})}(y,\bar{x}) \leq e^{2\alpha} d_{g(\bar{t})}(y,\bar{x}).$$

For $x \in B_{g(\tilde{t})}(\bar{x}, \sqrt{\bar{t}-\tilde{t}})$, we have

$$\begin{aligned} d_{g(\tilde{t})}(x,p) &\leq d_{g(\tilde{t})}(x,\bar{x}) + d_{g(\tilde{t})}(p,\bar{x}) \\ &\leq \sqrt{\bar{t}-\tilde{t}} + e^{2\alpha} d_{g(\bar{t})}(p,\bar{x}) \\ &\leq \sqrt{\frac{\alpha Q^{-1}}{2}} + e^{2\alpha}(1+2A)\varepsilon \\ &\leq \varepsilon + (1+2A)\epsilon \leq 3A\varepsilon, \end{aligned}$$

if A is large enough and we use the estimate $Q \geq \alpha \bar{t}^{-1} \geq \alpha\epsilon^{-2}$. Hence the desired statement follows.

We show next that the gap inequality in Lemma 11 can be expressed in a form closer to that appears in log-Sobolev inequalities.

We continue to suppress the subindex k for notational simplicity. Set $u = (4\pi(\bar{t}-t))^{-n/2}e^{-f}$, and $v = \Big((\bar{t}-t)(S + 2\Delta f - |\nabla f|^2) + f - n\Big)u$. Then we can write

$$\begin{aligned} &\int_M h(-v)dV\Big|_{t=0} \\ &= \int_M -\Big(\bar{t}(S + 2\Delta f - |\nabla f|^2) + f - n\Big)(4\pi\bar{t})^{-n/2} h e^{-f} dV \end{aligned} \tag{3.43}$$

In terms of $\tilde{u} \equiv uh$, $\tilde{f} \equiv f - \log h$, this can be rewritten as

$$\begin{aligned}
&\int_M h(-v)dV\Big|_{t=0} \\
&= \int_M -\Big(\bar{t}(S + 2\Delta\tilde{f} + 2\Delta\log h - |\nabla\tilde{f} + \nabla\log h|^2) \\
&\qquad + \tilde{f} + \log h - n\Big)(4\pi\bar{t})^{-n/2}e^{-\tilde{f}}dV \\
&= \int_M \Big(\bar{t}(-2\Delta\tilde{f} + |\nabla\tilde{f}|^2) - \tilde{f} + n\Big)(4\pi\bar{t})^{-n/2}e^{\tilde{f}}dV \\
&\quad + \int_M \Big(- \bar{t}S + \bar{t}|\nabla\log h|^2 - \log h\Big)(4\pi\bar{t})^{-n/2}e^{-\tilde{f}}dV \qquad (3.44)
\end{aligned}$$

after an integration by parts.

The second term in the above right hand side can be estimated as follows. At time $t = 0$, we have $S(g, \phi)(0) = R(g) - |\nabla\phi|^2 \geq -1$. Thus

$$\int_M -\bar{t}S\tilde{u}dV \leq \int_M \bar{t}\tilde{u}dV \leq \int_M \bar{t}udV \leq \bar{t} \leq \varepsilon^2. \qquad (3.45)$$

From the definition of h, it follows immediately that

$$|\nabla\log h|^2 = \frac{1}{(10A\varepsilon)^2}\frac{\varphi'^2}{\varphi^2} \leq \frac{1}{(10A\varepsilon)^2}\frac{10}{\varphi}$$

Since $1 \leq \varphi \leq 2$ on the support of φ', we have then

$$\int_M \bar{t}|\nabla\log h|^2 hudV \leq \frac{\bar{t}}{10A\varepsilon^2}\int_M udV \leq \frac{1}{10A^2} \qquad (3.46)$$

For the last term $-\int_M h\log hudV$, by the definition of h, it's non-zero only on the region $B(p, 20A\varepsilon)\backslash B(p, 10A\varepsilon)$. Since $-x\log x \leq \frac{1}{e} \leq 1$ on $(0, 1]$, we obtain

$$\begin{aligned}
\int_M -(h\log h)\, udV &\leq \int_{B(p,20A\varepsilon)\backslash B(p,10A\varepsilon)} udV \\
&\leq \int_M udV - \int_{B(x_0,10A\varepsilon)} udV \qquad (3.47)
\end{aligned}$$

Finally, a similar argument as above shows that

$$\int_{B(p,10A\varepsilon)} udV \geq \int_M \varphi\Big(\frac{d(x,t)}{5A\varepsilon}\Big)\, udV \geq 1 - cA^{-2},$$

for some constant c. Putting all these together, and applying Lemma 11, we obtain

$$\int_M (-\bar{t}|\nabla\tilde{f}|^2 - \tilde{f} + n)(4\pi\bar{t})^{-n/2}e^{-\tilde{f}}dV \tag{3.48}$$
$$\geq (1 - A^{-2})b - (1+c)A^{-2} - \varepsilon^2 \geq \frac{b}{2}.$$

if A is large and ε is small enough.

Define $\tilde{g} = \frac{1}{2\bar{t}}g$, then $d\tilde{V} = (2\bar{t})^{-n/2}dV$, $\bar{t}|\nabla\tilde{f}|_g^2 = \frac{1}{2}|\nabla\tilde{f}|_{\tilde{g}}^2$. Now we restore the subscript k, and normalize the function $\tilde{u}_k$ as follows

$$U_k = \frac{\tilde{u}_k}{\int_{M_k}\tilde{u}_k d\tilde{V}_{\tilde{g}_k}} = \frac{(2\bar{t}_k)^{n/2}\tilde{u}_k}{\int_{M_k}\tilde{u}dV_{g_k}}. \tag{3.49}$$

Define then the function F_k by

$$U_k = (2\pi)^{-n/2}e^{-F_k}. \tag{3.50}$$

Lemma 12. *Let F_k and U_k be defined as above. Then we have for k large*

$$\int_M \left(-\frac{1}{2}|\nabla F_k|^2 - F_k + n\right)U_k d\tilde{V}_{\tilde{g}_k} \geq \frac{1}{2}b > 0. \tag{3.51}$$

Proof. Write $\frac{\tilde{u}_k}{\sigma_k} = (4\pi\bar{t}_k)^{-n/2}e^{-F_k}$, where $\sigma_k \equiv \int_{M_k}\tilde{u}_k dV_{g_k} \to 1,$ as $k \to \infty$. From the definition of $\tilde{f}_k$, we have $F_k = \tilde{f}_k + \log\sigma_k$. Plugging this to the inequality (3.48), we get

$$\int_{M_k}\left(-\frac{1}{2}|\nabla F_k|^2 - F_k + \log\sigma_k + n\right)U_k\sigma_k d\tilde{V}_{\tilde{g}_k} \tag{3.52}$$
$$\geq (1 - A_k^{-2})b - (1+c)A_k^{-2} - \varepsilon_k^2$$

Since $A_k \to \infty$, $\varepsilon_k \to 0$ and $\sigma_k \to 1$ as $k \to \infty$, the lemma follows.

3.5. Logarithmic Sobolev inequalities

We now derive the desired contradiction. Once the estimate in Lemma 12 is established, this part of the argument is the same as for the Ricci flow. Nevertheless, we give it for the sake of completeness.

Logarithmic Sobolev inequalities have played an import role in geometric flows to derive noncollapsing properties (e.g. [5, 14]). We begin by recalling the log Sobolev inequality on $\mathbf{R}^n$. One version of it says that

$$\int_{\mathbf{R}^n}\left(-\frac{1}{2}|\nabla F|^2 - F + n\right)(2\pi)^{-n/2}e^{-F}dV \le 0 \tag{3.53}$$

for compactly supported functions $U = (2\pi)^{-n/2}e^{-F}$ satisfying $\int_{\mathbf{R}^n} UdV = 1$. It would have been easy to compare this inequality with the inequality established in Lemma 12, had both been on $\mathbf{R}^n$, but we need a version of the log Sobolev inequality on Riemannian manifolds. For this, it is convenient to make use of another version for $\mathbf{R}^n$, which can be obtained as follows.

Let $U_c(x) = c^n U(cx)$, and define $F_c(x)$ by $U_c(x) = (2\pi)^{-n/2}e^{-F_c(x)}$. Then $F_c(x) = F(cx) - n\log c$. Applying the logarithmic Sobolev inequality to $U_c(x)$ (since $\int_{\mathbf{R}^n} U_c(x)dx = 1$) gives

$$\int_{\mathbf{R}^n}\left(-\frac{1}{2}c^2|\nabla F(cx)|^2 - F(cx) + n\log c + n\right)(2\pi)^{-n/2}e^{-F(cx)}c^n dx \le 0 \tag{3.54}$$

Change of variable by $y = cx$, and maximize the right hand side term for $c \in (0,\infty)$. A simple calculation shows that the maximum is achieved at

$$c^2 = n\left(\int_{\mathbf{R}^n}|\nabla F(x)|^2(2\pi)^{-n/2}e^{-F(x)}dx\right)^{-1}$$

Substituting c back into the inequality above, we obtain the inequality

$$\int_{\mathbf{R}^n}|\nabla F|^2 U dx \ge n\exp\left(1 - \frac{2}{n}\int_{\mathbf{R}^n} FUdx\right) \tag{3.55}$$

for all $U = (2\pi)^{-n/2}e^{-F}$ and $\int_{\mathbf{R}^n} Udx = 1$.

Using symmetrization arguments, we show now that the same inequality holds on a Riemannian manifold, provided the isoperimetric inequality holds for domains on that Riemannian manifold. More precisely,

Lemma 13. *Let (M, g_{ij}) be a compact Riemannian manifold. Assume that $Area(\partial\Omega)^n \ge (1-\delta)c_n Vol(\Omega)^{n-1}$ for any $\Omega \subset B_g(p,r)$, then for any U with compact support on $B_g(p,r)$, with $\int_M UdV_g = 1$, we have*

$$\int_M |\nabla F|^2 UdV_g \ge (1-\delta)^{2/n} n\exp\left(1 - \frac{2}{n}\int_M FUdV_g\right) \tag{3.56}$$

Proof. We use another equivalent formulation of the log Sobolev inequality. On $\mathbf{R}^n$, it says that

$$\int_{\mathbf{R}^n} 4|\nabla\varphi|^2 dx \geq n \exp\left(1 + \log 2\pi + \frac{2}{n}\int 2(\log\varphi)\,\varphi^2 dx\right) \tag{3.57}$$

and the Riemannian version, equivalent to (3.56), is the following

$$\int_{M_k} 4|\nabla\varphi|^2 dV_g \geq (1-\delta)^{2/n} n \exp\left(1 + \log 2\pi + \frac{2}{n}\int_{M_k} 2(\log\varphi)\varphi^2\right) \tag{3.58}$$

where we have set $\varphi^2 = U = (2\pi)^{-n/2}e^{-F}$, $\int_M \varphi^2 dV_g = 1$ and $F = -2\log\varphi - \frac{n}{2}\log 2\pi$.

To prove (3.58), define φ^* on $(\mathbf{R}^n, g_{can})$ as the symmetrization of φ as follows. More specifically, let $M(s) = \{x \in M : \varphi(x) \geq s\}$, and definee φ^* to be the non-increasing, radially symmetric function on $\mathbf{R}^n$ satisfying $\varphi^*(0) = \sup_M \varphi$, and for any s,

$$Vol_{g_{can}}(\{\varphi^*(x) \geq s\}) = Vol_g(M(s)).$$

Recall the co-area formula states that for any function g, we have

$$\int_\Omega g(x)|\nabla u|dx = \int_{-\infty}^{\infty}\int_{u^{-1}(s)} g(x)d\sigma_s ds \tag{3.59}$$

Applying this to $M(s)$, with $g(x) = \frac{1}{|\nabla\varphi|}$, and differentiating with respect to s, we obtain

$$Vol(M(s))' = \int_{\{\varphi=s\}} \frac{1}{|\nabla\varphi|}d\sigma_g = \int_{\{\varphi^*=s\}} \frac{1}{|\nabla\varphi^*(s)|}d\sigma_{g_{can}}$$

On the other hand, since φ^* is symmetric, we have $|\nabla\varphi^*|$ is constant of $\{\varphi^* = s\}$. Hence,

$$\begin{aligned}
Area(\partial\{\varphi^* = s\})^2 &= \int_{\{\varphi^*=s\}} |\nabla\varphi^*|d\sigma_{g_{can}} \int_{\{\varphi^*=s\}} \frac{1}{|\nabla\varphi^*|}d\sigma_{g_{can}} \\
&= \left(Area(\partial\{\varphi^* = s\})^n\right)^{2/n} = \left(c_n Vol(\{\varphi^* \geq s\})^{n-1}\right)^{2/n} \\
&= \left(c_n Vol(M(s))^{n-1}\right)^{2/n} \\
&\leq (1-\delta)^{-2/n} Area^2(\partial M(s)) \\
&\leq (1-\delta)^{-2/n} \int_{\{\varphi=s\}} |\nabla\varphi|d\sigma_g \int_{\{\varphi=s\}} \frac{1}{|\nabla\varphi|}d\sigma_g. \qquad (3.60)
\end{aligned}$$

Hence, we get

$$\int_{\{\varphi=s\}} |\nabla\varphi| d\sigma_g \geq (1-\delta)^{2/n} \int_{\{\varphi^*=s\}} |\nabla\varphi^*| d\sigma_{g_{can}} \tag{3.61}$$

Applying the co-area formula again, we get

$$\int_M |\nabla\varphi|^2 dV_g \geq (1-\delta)^{2/n} \int_{\mathbf{R}^n} |\nabla\varphi^*|^2 dx, \tag{3.62}$$

and

$$\begin{aligned}
\int_M \varphi^2 \log\varphi dV_g &= \int_0^\infty \int_{\{\varphi=s\}} \frac{\varphi^2\log\varphi}{|\nabla\varphi|} d\sigma_g ds \\
&= \int_0^\infty s^2 \log s \int_{\{\varphi=s\}} \frac{1}{|\nabla\varphi|} d\sigma_g ds \\
&= \int_0^\infty s^2 \log s \int_{\{\varphi^*=s\}} \frac{1}{|\nabla\varphi^*|} d\sigma_{g_{can}} ds \\
&= \int_{\mathbf{R}^n} (\varphi^*)^2 \log\varphi^* dx
\end{aligned} \tag{3.63}$$

Putting all these together gives the logarithmic inequality (3.58) on Riemannian manifolds.

3.6. Completion of the proof of the pseudo-locality theorem

We return now to the setting of $\tilde{g} = \frac{1}{2\bar{t}} g$, then $d\tilde{V} = (2\bar{t})^{-n/2} dV$, with the functions U_k and F_k defined as in (3.49) and (3.50). The equations (3.51) and (3.56) imply that

$$(1-\delta_k)^{2/n} \frac{n}{2} \exp\left(1 - \frac{2}{n}\int_{M_k} F_k U_k d\tilde{V}_k\right) + \int_{M_k} F_k U_k d\tilde{V}_k - n \leq -\frac{b}{2} \tag{3.64}$$

Let $\eta_k = 1 - \frac{2}{n}\int_{M_k} F_k U_k d\tilde{V}_k$, the left hand side term can be written as

$$LHS = \frac{n}{2}\left((1-\delta_k)^{2/n} e^{\eta_k} - 1 - \eta_k\right) \leq -\frac{b}{2}$$

However, consider the function $(1-\delta_k)^{2/n} e^x - 1 - x$. Its minimum occurs at x_0 given by $(1-\delta_k)^{2/n} e^{x_0} = 1$, and hence its minimum is $-x_0 = \frac{2}{n}\log(1 -$

δ_k). Thus, we have

$$LHS \geq \frac{n}{2} \cdot \frac{2}{n} \log(1 - \delta_k) = \log(1 - \delta_k) \to 0, \text{ as } k \to \infty. \tag{3.65}$$

This is a contradiction, and the proof of the pseudo-locality theorem is complete.

4. Convergence of parabolic rescaling to a soliton

The goal of this section is to establish Theorem 2. It is shown in the first subsections that, in presence of a Type I singularity, the parabolic rescalings of the coupled Ricci flow will converge to a soliton. The non-triviality of the soliton will be established in the last subsection by applying the pseudo-locality theorem.

4.1. Estimates for the reduced distance

Suppose the flow satisfies the Type I condition (1.5), for any $T' < T$, we have $\sup_M |Rm|(g(t)) \leq \frac{C_0}{T'-t}$ for any $t \in [0, T')$. For any fixed $t_0 \in [0, T')$, take $r^2 = T' - t_0$, and by the Type I assumption

$$\sup_{t\in[0,t_0], x\in B_{g(t)}(p,r)} r^2 |Rm|(x, g(t)) \leq \sup_{t\in[0,t_0]} (T' - t_0)\frac{C_0}{T' - t} = C_0,$$

thus by the derivative estimates (2.6), it follows that

$$\sup_{B_{g(t)}(p,r/2)} |\nabla^m Rm|(g(t)) \leq C(n,m) C_0^{1+\frac{m}{2}} \left(\frac{1}{T' - t_0} + \frac{1}{t}\right)^{1+\frac{m}{2}}, \quad \forall t \in (0, t_0].$$

In particular, at (p, t_0), we have

$$\begin{aligned} &|\nabla Rm|(p, t_0) \leq \frac{C(n, C_0)}{(T' - t_0)^{3/2}}, \quad |\nabla^2 Rm|(p, t_0) \leq \frac{C(n, C_0)}{(T' - t_0)^2}, \\ &|\partial_t Rm|(p, t_0) \leq \frac{C(n, C_0)}{(T' - t_0)^2}, \end{aligned} \tag{4.1}$$

and since $p \in M, t_0 \in [0, T')$ are arbitrary, the above estimates hold on $M \times [0, T')$.

Set $\tau = \tau(t) = T' - t$. We will denote $\bar{g}(\tau) = g(T' - \tau)$. By the higher order estimates (4.1), we have

$$\begin{aligned} \max(|S|_{\bar{g}(\tau)}, |Sic|_{\bar{g}(\tau)}) &\leq \frac{C(n, C_0)}{\tau}, \\ \max(|\nabla S|^2_{\bar{g}(\tau)}, |\partial_\tau S|_{\bar{g}(\tau)}) &\leq \frac{C(n, C_0)}{\tau^2}. \end{aligned} \tag{4.2}$$

We are now ready to establish the following estimates for the reduced distance, which are obtained by Naber ([12]) in the Ricci flow case.

Lemma 14. *There exists a constant $C(n, C_0)$ depending only on the Type I constant C_0 such that*

1) $C(n, C_0)^{-1} \frac{d^2_{\bar{g}(\bar{\tau})}(p,q)}{\bar{\tau}} - C(n, C_0) \leq \ell(q, \bar{\tau}) \leq C(n, C_0) \frac{d^2_{\bar{g}(\bar{\tau})}(p,q)}{\bar{\tau}} + C(n, C_0)$.
2) $|\nabla \ell|^2 \leq \frac{C(n,C_0)}{\bar{\tau}} \left(1 + \frac{d^2_{\bar{g}(\bar{\tau})}(p,q)}{\bar{\tau}}\right)$.
3) $\left|\frac{\partial}{\partial \tau} \ell\right| \leq \frac{C(n,C_0)}{\bar{\tau}} \left(\frac{d^2_{\bar{g}(\bar{\tau})}(p,q)}{\bar{\tau}} + 1\right)$,

where ℓ is the reduced distance with base space-time (p, T').

Proof. We apply the estimates in (2.13). For this, we estimate the quantity K in (2.8) as follows,

$$\begin{aligned} |K| &\leq C(n, C_0) \int_0^{\bar{\tau}} \tau^{3/2} \left(\frac{1}{\tau^2} + \frac{|X|}{\tau^{3/2}} + \frac{|X|^2}{\tau}\right) d\tau \\ &\leq C(n, C_0) \int_0^{\bar{\tau}} \tau^{3/2} \left(\frac{1}{\tau^2} + \frac{|X|^2}{\tau}\right) d\tau \\ &\leq C(n, C_0) \int_0^{\bar{\tau}} \sqrt{\tau}(S + |X|^2) d\tau + C(n, C_0)\sqrt{\bar{\tau}} \\ &= C(n, C_0)\sqrt{\bar{\tau}}\ell + C(n, C_0)\sqrt{\bar{\tau}}, \end{aligned}$$

Substituting the above estimates into (2.13), we get the estimate

$$|\nabla \ell|^2 \leq \frac{C(n, C_0)}{\bar{\tau}} \ell + C(n, C_0). \tag{4.3}$$

Next, we claim that there exists a constant $C_2 = C_2(n, C_0)$ such that

$$|\ell(p, \bar{\tau})| \leq C_2, \quad \forall \bar{\tau} \in (0, T'].$$

Indeed, consider the constant map $\gamma : [0, \bar\tau] \to M$ defined by $\gamma(\tau) \equiv p$. The $\mathcal{L}$-length of this curve is bounded above by

$$\mathcal{L}(\gamma) \le \int_0^{\bar\tau} \sqrt{\tau}\frac{C(n, C_0)}{\tau} d\tau \le C(n, C_0)\sqrt{\bar\tau},$$

then the claim follows at once from the definition of $\ell(p, \bar\tau)$.

Thus from (4.3) we have

$$|\nabla\sqrt{\ell + C_1}|_{\bar g(\bar\tau)} \le \frac{C(n, C_0)}{\sqrt{\bar\tau}}, \tag{4.4}$$

The mean value theorem implies

$$\sqrt{\ell(q, \bar\tau) + C_1} - \sqrt{\ell(p, \bar\tau) + C_1} \le C(n, C_0)\frac{d_{\bar g(\bar\tau)}(p, q)}{\sqrt{\bar\tau}},$$

which is

$$\ell(q, \bar\tau) \le C(n, C_0)\frac{d^2_{\bar g(\bar\tau)}(p, q)}{\bar\tau} + C(n, C_0). \tag{4.5}$$

This is the desired upper bound for $\ell(q, \bar\tau)$. To derive the lower bound, we begin by showing that, for any minimal geodesic σ with respect to the metric $\bar g(\bar\tau)$, we have

$$\int_\sigma Sic(\sigma'(s), \sigma'(s))ds \le \frac{C(n, C_0)}{\sqrt{\bar\tau}}.$$

If $L_{\bar g(\bar\tau)}(\sigma) \le 2\sqrt{\bar\tau}$, this integral inequality follows directly from the curvature assumption $Sic \le Ric \le \frac{C_0}{\bar\tau}$.

If $d \equiv L_{\bar g(\bar\tau)}(\sigma) > 2\sqrt{\bar\tau}$, take an orthonormal basis $\{E_i\}_{i=1}^n$ of vector fields which are parallel with respect to $\bar g(\bar\tau)$ along the geodesic σ, and $E_n = \sigma'(s)$. Choose a function $\phi(s)$ satisfying $\phi(s) = \frac{s}{\sqrt{\bar\tau}}$ for $s \in [0, \sqrt{\bar\tau}]$, $\phi(s) = 1$ for $s \in [\sqrt{\bar\tau}, d - \sqrt{\bar\tau}]$, and $\phi(s) = \frac{d-s}{\sqrt{\bar\tau}}$ for $s \in [d - \sqrt{\bar\tau}, d]$. By the second variational formula of the length function (applied to each variational vector field ϕE_i for $1 \le i \le n - 1$), we have

$$0 \le \int_0^d \sum_{i=1}^{n-1} |\nabla_{E_n}(\phi E_i)|^2 - \int_0^d \phi^2(s)Ric(E_n, E_n)ds.$$

Consequently,

$$\begin{aligned}\int_\sigma Ric(\sigma',\sigma') &\le \frac{2(n-1)}{\sqrt{\bar\tau}} + \frac{n-1}{\bar\tau}C_0\int_0^d (1-\phi^2)ds\\ &= \frac{2(n-1)}{\sqrt{\bar\tau}} + \frac{4(n-1)C_0}{3\sqrt{\bar\tau}}.\end{aligned}$$

The claim follows now from $Sic \le Ric$.

Choose now $\mathcal{L}$-geodesics γ_1 and $\gamma_2 : [0,\bar\tau] \to M$ connecting the space-time points $(p,0)$ with $(p,\bar\tau)$ and $(q,\bar\tau)$, respectively. Let

$$\sigma_\tau : [0, d_{\bar g(\tau)}(\gamma_1(\tau),\gamma_2(\tau))] \to M$$

be the minimal geodesic connecting the points $\gamma_1(\tau)$ and $\gamma_2(\tau)$ under the metric $\bar g(\tau)$. Then

$$\begin{aligned}\frac{d}{d\tau} d_{\bar g(\tau)}(\gamma_1(\tau),\gamma_2(\tau)) &= \langle \nabla d, \dot\gamma_1\rangle_{\bar g(\tau)} + \langle \nabla d, \dot\gamma_2\rangle_{\bar g(\tau)} + \int_\sigma Sic(\sigma'(s),\sigma'(s))ds\\ &\le |\dot\gamma_1|_{\bar g(\tau)} + |\dot\gamma_2(\tau)|_{\bar g(\tau)} + \frac{C(n,C_0)}{\sqrt{\tau}}.\end{aligned}$$

Integrating the above inequality from $\tau \in [0,\bar\tau]$ yields

$$d_{\bar g(\bar\tau)}(p,q) \le \int_0^{\bar\tau} (|\dot\gamma_1| + |\dot\gamma_2|)\, d\tau + C(n,C_0)\sqrt{\bar\tau}.$$

The first term on the right hand side can be estimated by

$$\begin{aligned}\int_0^{\bar\tau} |\dot\gamma_1| &\le \int_0^{\bar\tau} \sqrt{\tau}|\dot\gamma_1|^2 + C(n)\sqrt{\bar\tau}\\ &\le 2\sqrt{\bar\tau}\ell(p,\bar\tau) + C(n,C_0)\sqrt{\bar\tau} \le C(n,C_0)\sqrt{\bar\tau},\end{aligned}$$

while the second term can be estimated by

$$\begin{aligned}\int_0^{\bar\tau} |\dot\gamma_2| d\tau &\le \left(\int_0^{\bar\tau} \sqrt{\tau}|\dot\gamma_1|^2\right)^{1/2} \left(\int_0^{\bar\tau} \frac{1}{\sqrt{\tau}}\right)^{1/2}\\ &\le C(n,C_0)\sqrt{\bar\tau}\,(\ell(q,\bar\tau) + C(n,C_0))^{1/2},\end{aligned}$$

Combining with these estimates we get

$$d_{\bar g(\bar\tau)}(p,q) \le \sqrt{\bar\tau}\Big(C(n,C_0) + C(n,C_0)\,(\ell(q,\bar\tau) + C(n,C_0))^{1/2}\Big),$$

that is

$$C(n,C_0)^{-1}\frac{d^2_{\bar{g}(\bar{\tau})}(p,q)}{\bar{\tau}} - C(n,C_0) \le \ell(q,\bar{\tau}).$$

This is the desired lower bound for $\ell(q,\bar{\tau})$. The remaining estimates in Lemma 14 follow from the equations (2.12) and (2.13), so the proof is completed.

We restate the lemma above in terms of the original extended Ricci flow. For any $T' \in (0,T)$, there exists a constant $C = C(n,C_0)$ such that (denote $\ell_{(p,T')}(q,t) = \ell(q,T'-t)$ as in the previous notation)

$$(4.6)\quad C(n,C_0)^{-1}\frac{d^2_{g(t)}(p,q)}{T'-t} - C(n,C_0) \le \ell_{(p,T')}(q,t) \le C(n,C_0)\frac{d^2_{g(t)}(p,q)}{T'-t} + C(n,C_0),$$

$$(4.7)\quad |\nabla_{g(t)}\ell_{(p,T')}(q,t)|^2 \le \frac{C(n,C_0)}{T'-t}\left(\frac{d^2_{g(t)}(p,q)}{T'-t}+1\right),$$

$$(4.8)\quad \left|\frac{\partial}{\partial t}\ell_{(p,T')}(q,t)\right| \le \frac{C(n,C_0)}{T'-t}\left(\frac{d^2_{g(t)}(p,q)}{T'-t}+1\right),$$

and the following inequality hold in the distributional sense,

$$(4.9)\quad \frac{\partial \ell}{\partial t} + \Delta_{g(t)}\ell - |\nabla \ell|^2 + S - \frac{n}{2(T'-t)} \le 0.$$

Take a sequence of times $\{T_i\}$ such that $T_i \nearrow T$, and we have the corresponding reduced distance functions

$$\ell_i(q,t) = \ell_{(p,T_i)}(q,t) : M \times [0,T_i] \to \mathbf{R},$$

and each ℓ_i satisfies the estimates (4.6), (4.7), (4.8) and (4.9) with the same constant $C(n,C_0)$. In particular on any compact subset $\Omega \subset M \times [0,T)$, the functions ℓ_i satisfy uniform C^0, C^1_M, and C^1_t estimates. In addition,

$$\|\ell_i\|_{W^{1,2}(\Omega)} \le C(n,C_0,\Omega).$$

Thus up to a subsequence and a diagonal argument we may assume that

$$\ell_i \rightharpoonup \ell_\infty \in W^{1,2}_{loc}(M \times [0,T))$$

weakly in $W^{1,2}_{loc}(M \times [0,T))$, and on any compact subset $\Omega \subset M \times [0,T)$, the convergence is uniform in $C^0(\Omega)$ norm. Moreover, it is not hard to see the limit function is locally Lipschitz, so the limit $\ell_\infty \in W^{1,2}_{loc}$, and ℓ_∞ satisfies similar estimate as ℓ_i in (4.6).

We define

$$V_\infty(t) = (4\pi(T-t))^{-n/2} \int_M e^{-\ell_\infty(q,t)} dV_{g(t)}. \tag{4.10}$$

Lemma 15. *The function $V_\infty(t)$ satisfies the following properties:*

(1) *$V_\infty(t) \le 1$, $\forall t \in [0,T)$.*

(2) *$V_\infty(t_1) \le V_\infty(t_2)$, for $t_1 < t_2 \in (0,T)$.*

Proof. In view of Lemma 14, we have for each ℓ_i and $t \in [0,T)$,

$$e^{-\ell_i(q,t)} \le C(n,C_0) e^{-\frac{d^2_{g(t)}(p,q)}{C(n,C_0)(T_i-t)}} \le C(n,C_0) e^{-\frac{d^2_{g(t)}(p,q)}{C(n,C_0)(T-t)}}$$

Thus the function on the right hand side in integrable with respect to $dV_{g(t)}$, so by the Lebesgue dominated convergence theorem

$$\lim_{i\to\infty} V_i(t) = \lim_{i\to\infty} (4\pi(T_i-t))^{-n/2} \int_M e^{-\ell_i(q,t)} dV_{g(t)} = V_\infty(t) \tag{4.11}$$

It follows from [14] that $\lim_{t\to T_i} V_i(t) = 1$ and $V_i(t)$ is nondecreasing in t, so for each $t \in (0,T)$ when i is large enough $V_i(t) \le \lim_{s\to T_i} V_i(s) = 1$. Part (1) of the lemma follows. Similarly, Part (2) follows from the monotonicity of $V_i(t)$ for each i, so that $V_i(t_1) \le V_i(t_2)$ for $t_1 < t_2 \in (0,T)$. Q.E.D.

Lemma 16. *The function ℓ_∞ satisfies the following inequality in the distribution sense,*

$$\partial_t \ell_\infty + \Delta_{g(t)} \ell_\infty - |\nabla \ell_\infty|^2 + S - \frac{n}{(T-t)} \ge 0. \tag{4.12}$$

Proof. Since ℓ_i converge weakly to ℓ_∞ in $W^{1,2}_{loc}(M \times [0,T))$, for any vector field V on M with compact support and $\psi \in C_c^\infty((0,T))$ we have

$$\begin{aligned} &\int_0^T \int_M \left(\psi \langle V, \nabla \ell_i \rangle_{g(t)} + \frac{d\psi}{dt} \ell_i \right) dV_{g(t)} dt \\ &\to \int_0^T \int_M \left(\psi \langle V, \nabla \ell_\infty \rangle_{g(t)} + \frac{d\psi}{dt} \ell_\infty \right) dV_{g(t)} dt. \end{aligned} \tag{4.13}$$

We aim to prove that for any nonnegative $\varphi \in C_c^\infty(M \times (0,T))$, it holds that

$$\int_0^T\int_M\left(-\frac{\partial\varphi}{\partial t}\ell_\infty+\Delta_{g(t)}\varphi\ell_\infty-\varphi|\nabla\ell_\infty|^2+S\varphi-\frac{n\varphi}{2(T-t)}\right)dV_{g(t)}dt\leq 0. \tag{4.14}$$

First note that by (4.9), the following holds for ℓ_i for i large enough,

$$\int_0^T\int_M\left(-\frac{\partial\varphi}{\partial t}\ell_i+\Delta_{g(t)}\varphi\ell_i-\varphi|\nabla\ell_i|^2+S\varphi-\frac{n\varphi}{2(T_i-t)}\right)dV_{g(t)}dt\leq 0. \tag{4.15}$$

By locally C^0 uniform convergence of ℓ_i to ℓ_∞, it follows that the first, second, fourth and last terms in (4.15) converge to those in (4.14) as $i \to \infty$, respectively. So to show (4.14), it suffices to show the third terms also converge, i.e.

$$\int_0^T\int_M \varphi|\nabla\ell_i|^2 dV_{g(t)}dt \to \int_0^T\int_M \varphi|\nabla\ell_\infty|^2 dV_{g(t)}dt. \tag{4.16}$$

Since ℓ_∞ is the weak limit of ℓ_i in $W^{1,2}_{loc}(M \times (0,T))$, it follows that

$$\int_0^T\int_M \varphi|\nabla\ell_\infty|^2 dV_{g(t)}dt \leq \liminf_{i\to\infty}\int_M \varphi|\nabla\ell_i|^2 dV_{g(t)}dt.$$

So to show (4.16) it suffices to prove that

$$\limsup_{i\to\infty}\int_0^T\int_M \varphi|\nabla\ell_i|^2 dV_{g(t)}dt \leq \int_0^T\int_M \varphi|\nabla\ell_\infty|^2 dV_{g(t)}dt, \tag{4.17}$$

noting that

$$\begin{aligned}
&\limsup_{i\to\infty}\int_0^T\int_M \varphi|\nabla\ell_i|^2 dV_{g(t)}dt \\
&= \limsup_{i\to\infty}\int_0^T\int_M \varphi\langle\nabla(\ell_i-\ell_\infty-\epsilon_i),\nabla\ell_i\rangle dV_{g(t)}dt \\
&\quad + \limsup_{i\to\infty}\int_M \langle\varphi\nabla\ell_\infty,\nabla\ell_i\rangle dV_{g(t)}dt,
\end{aligned} \tag{4.18}$$

using (4.13), we see the second term on RHS of (4.18) converges to

$$\int_0^T\int_M \varphi|\nabla\ell_\infty|^2 dV_{g(t)}dt.$$

Since ℓ_i converges uniformly to ℓ_∞ on supp$\varphi \subset M \times (0,T)$, there exists a sequence of numbers $\epsilon_i > 0$ which tend to 0 such that

$$\ell_\infty - \ell_i + \epsilon_i \geq 0, \qquad \text{on supp}\, \varphi.$$

To deal with the first term on RHS of (4.18), by (2.14) it follows that

$$\Delta_{g(t)} \ell_i \leq \frac{C(n, C_0)}{T_i - t} + C(n, C_0)\frac{\ell_i}{T_i - t},$$

in the distribution sense. Multiplying both sides by $\varphi(\ell_\infty - \ell_i + \epsilon_i)$ and integrating it follows that

$$\begin{aligned} &\int_0^T \int_M \langle \nabla(\varphi(\ell_i - \ell_\infty - \epsilon_i)), \nabla \ell_i \rangle \\ \leq &\int_0^T \int_M \frac{C(n, C_0)}{T_i - t}\varphi(\ell_\infty - \ell_i + \epsilon_i)(C(n, C_0) + \ell_i), \end{aligned}$$

the RHS tends to 0 as $i \to \infty$, because both ℓ_∞ and ℓ_i are bounded on supp φ and $\ell_\infty - \ell_i + \epsilon_i \to 0$. The LHS is equal to

$$\int_0^T \int_M \varphi\langle \nabla(\ell_i - \ell_\infty - \epsilon_i), \nabla \ell_i \rangle + \int_0^T \int_M (\ell_i - \ell_\infty - \epsilon_i)\langle \nabla\varphi, \nabla \ell_i \rangle,$$

and the second term tends to 0, so

$$\limsup_{i \to \infty} \int_0^T \int_M \varphi\langle \nabla(\ell_i - \ell_\infty - \epsilon_i), \nabla \ell_i \rangle \leq 0.$$

Thus we finish the proof of (4.16), and also that of (4.14).

Define function u_∞ by

$$u_\infty = (4\pi(T - t))^{-n/2} e^{-\ell_\infty}. \tag{4.19}$$

Then the inequality in Lemma 16 is equivalent to the inequality

$$\Box^* u_\infty = (-\partial_t - \Delta_{g(t)} + S)u_\infty \leq 0 \tag{4.20}$$

in the distribution sense. So it follows that

$$\frac{d}{dt}V_\infty = \frac{d}{dt}\int_M u_\infty dV_{g(t)} = \int_M (\partial_t + \Delta_{g(t)} - S)u_\infty \geq 0. \tag{4.21}$$

Lemma 17. *If $V_\infty(t_1) = V_\infty(t_2)$ for some $t_1 < t_2 \in (0,T)$, then $g(t)$ is a coupled gradient soliton, that is,*

$$Sic + \nabla^2 \ell_\infty - \frac{g}{2(T-t)} = 0, \quad \Delta\phi - \langle \nabla \ell_\infty, \nabla\phi\rangle = 0.$$

Proof. The existence of two such values t_1 and t_2 implies that the integrand on the right hand side of (4.21), which is known to be ≥ 0 by Lemma 16, must vanish identically. By parabolic regularity, the function u_∞ is actually C^∞ in $M \times [0,T)$. Thus the function v_∞ defined by

$$v_\infty = \big((T-t)(S + 2\Delta\ell_\infty - |\nabla\ell_\infty|^2) + \ell_\infty - n\big)\, u_\infty$$

as well as $\Box^* v_\infty$ must vanish identically. Since we have, by a direct calculation,

$$\Box^* v_\infty = \left(-2(T-t)\left|Sic + \nabla^2 \ell_\infty - \frac{g}{2(T-t)}\right|^2 - 2|\Delta\phi - \langle\nabla\phi, \nabla\ell_\infty\rangle|^2\right) u_\infty.$$

The vanishing of $\Box^* v_\infty$ implies immediately that $g(t)$ is an extended soliton, as claimed.

4.2. Blow-ups of Type I κ-noncollapsed solutions

Let $(M, g(t), \phi(t))$ be an coupled Ricci flow solution to (1.1) with $|\phi_0| \leq C$, and assume that it is of Type I, i.e.,

$$\sup_M |Rm|(g(t)) \leq \frac{C_0}{T-t}, \quad t \in [0,T),$$

and $T < \infty$ is the maximal existence time of the flow.

Take any sequence of numbers $\lambda_i \to \infty$, and define a sequence of coupled Ricci flows

$$g_i(t) = \lambda_i g(\lambda_i^{-1} t + T), \quad \phi_i(t) = \phi(\lambda_i^{-1} t + T), \quad \forall t \in [-\lambda_i T, 0).$$

By the Type I condition we have

$$\sup_M |Rm|(g_i(t)) = \lambda_i^{-1} \sup_M |Rm|(g(\lambda_i^{-1} t + T)) \leq \frac{C_0 \lambda_i^{-1}}{T - (T + \lambda_i^{-1} t)} = \frac{C_0}{-t}.$$

By the compactness Theorem (Theorem 7.5 [9]) for coupled Ricci flows, combined with the κ-noncollapsed condition, there exists a subsequence

$$(M, g_i(t), \phi_i(t), p) \to (M_\infty, g_\infty(t), \phi_\infty, p_\infty),$$

converging in the Cheeger-Gromov sense, where $(g_\infty(t), \phi_\infty(t))$ still satisfies the coupled Ricci flow equation (1.1).

By the gradient estimate (2.4) of $\phi(t)$,

$$\sup_M |\nabla\phi(t)|^2_{g(t)} \le \frac{\sup|\phi(0)|^2}{t} = \frac{C^2}{t}.$$

In terms of the rescaled solutions $(g_i(t), \phi_i(t))$,

$$\begin{aligned}\sup_M |\nabla\phi_i(t)|^2_{g_i(t)} &= \lambda_i^{-1}\sup_M |\nabla\phi(\lambda_i^{-1}t+T)|^2_{g(\lambda_i^{-1}t+T)}\\ &\le \lambda_i^{-1}\frac{C^2}{\lambda_i^{-1}t+T} = \frac{C^2}{t+\lambda_i T}.\end{aligned}$$

By the C^∞ convergence of (g_i, ϕ_i) it follows that

$$\sup_{M_\infty} |\nabla\phi_\infty(t)|^2 = 0,$$

hence $\phi_\infty = \mathrm{const}$, and the coupled Ricci flow (g_∞, ϕ_∞) becomes the standard Ricci flow.

From now on, we will denote the "reduced" function ℓ_∞ constructed in the previous section by $\ell_p : M \times [0,T) \to \mathbf{R}$. Denote

$$\ell_p^i : (M \times [-\lambda_i T, 0), g_i, \phi_i, p) \to \mathbf{R}$$

by $\ell_p^i(q,t) = \ell_p(q, \lambda_i^{-1}t + T)$. By Lemma 14, ℓ_p^i satisfy the following estimates

$$\begin{gathered}C(n,C_0)^{-1}\frac{d^2_{g_i(t)}(p,q)}{-t} - C(n,C_0) \le \ell_p^i(q,t) \le C(n,C_0)\frac{d^2_{g_i(t)}(p,q)}{-t} + C(n,C_0),\\ |\nabla\ell_p^i|^2(q,t) \le \frac{C(n,C_0)}{-t}\left(\frac{d^2_{g_i(t)}(p,q)}{-t}+1\right)\end{gathered}$$

and

$$\left|\frac{\partial}{\partial t}\ell_p^i(q,t)\right| \le \frac{C(n,C_0)}{-t}\left(\frac{d^2_{g_i(t)}(p,q)}{-t}+1\right),$$

for any $(q,t) \in M \times [-\lambda_i T, 0)$, and the last two inequalities are understood as the derivatives of locally Lipschitz functions. Note that the constant $C(n, C_0)$ above is independent of i.

So we can extract a subsequence of ℓ_p^i which converges in the Cheeger-Gromov-$C^0_{loc}(M_\infty \times (-\infty, 0))$ sense to a function $\ell^\infty : M_\infty \times (-\infty, 0) \to \mathbf{R}$, and the convergence is uniform on any compact subset of $M_\infty \times (-\infty, 0)$. Define

$$V^\infty(t) = (4\pi(-t))^{-n/2} \int_{M_\infty} e^{-\ell^\infty(q,t)} dV_{g_\infty(t)}. \tag{4.22}$$

The reduced volume functions

$$V^i(t) = (4\pi(-t))^{-n/2} \int_M e^{-\ell_p^i(q,t)} dV_{g_i(t)}(q) = V_\infty(\lambda_i^{-1} t + T) \tag{4.23}$$

are nondecreasing for each fixed t since $V_\infty(s)$ is nondecreasing in $s \in [0, T)$ and bounded above by 1. So for any fixed $t < 0$, we have

$$\lim_{i\to\infty} V^i(t) = \lim_{t'\to T} V_\infty(t'), \tag{4.24}$$

noting that the RHS is independent of t. On the other hand, by the dominated convergence theorem and smooth convergence of $g_i(t)$ to $g_\infty(t)$, we have

$$\lim_{i\to\infty} V^i(t) = \lim_{i\to\infty} (4\pi(-t))^{-n/2} \int_M e^{-\ell_p^i(q,t)} dV_{g_i(t)} = V^\infty(t), \tag{4.25}$$

Hence $V^\infty(t)$ is constant and independent of t. By Lemma 17, it follows that the limit metric $g_\infty(t)$ is a gradient shrinking Ricci soliton.

4.3. Non-triviality of the soliton

It remains only to show the Ricci soliton g_∞ is not flat, when the base point p is a Type I singularity of the flow $(g(t), \phi(t))$. We will use Perelman's pseudo locality and a contradiction argument. First recall that the pseudolocality states that there exists a uniform constant $\varepsilon_0 = \varepsilon_0(n)$ such that if the extended Ricci flow solution $(M, g(t), \phi(t))$ satisfies $\|\phi(0)\|_{L^\infty(B_{g(0)}(p,1))} \le C$ and the geometry of $(B_{g(0)}(p,1), g(0))$ is sufficiently close to that of the

Euclidean unit ball $(B(0,1), g_{can})$ in the C^2 sense, then we have

$$|Rm|(x, g(t)) \leq 10\varepsilon_0^{-2}, \quad \forall t \in (\frac{\varepsilon_0^2}{2}, \varepsilon_0^2),\ \forall x \in B_{g(\varepsilon_0^2/2)}(p, \varepsilon_0).$$

Suppose the limit metric $g_\infty(t)$ is flat, and by assumption it is also κ-noncollapsed, so by the uniformization theorem $(M_\infty, g_\infty) \cong (\mathbf{R}^n, g_{can})$. By the smooth convergence of $(M, g_i(t), \phi_i(t), p)$ to $(M_\infty, g_\infty(t))$, it follows that when i is large enough, the three conditions in the pseudolocality theorem are satisfied on $(B_{g_i(-\varepsilon_0^2)}(p, 1), g_i(-\varepsilon_0^2))$. So it follows that

$$|Rm|(x, g_i(-\varepsilon_0^2 + t)) \leq 10\varepsilon_0^{-2}, \quad \forall t \in (\frac{\varepsilon_0^2}{2}, \varepsilon_0^2),\ \forall x \in B_{g_i(-\varepsilon_0^2/2)}(p, \varepsilon_0).$$

Fix a large i, and in terms of the original flow, we have

$$|Rm|(x, g(T + \lambda_i^{-1}(t - \varepsilon_0^2))) \leq 10\lambda_i\varepsilon_0^{-2}, \tag{4.26}$$

for all $t \in (\frac{\varepsilon_0^2}{2}, \varepsilon_0^2)$ and $x \in B_{g(T - \lambda_i^{-1}\varepsilon_0^2/2)}(p, \frac{\varepsilon_0}{\sqrt{\lambda_i}})$.

However, since p is a Type I singularity point, which means there exists a sequence of space-time points (p_α, t_α) such that $t_\alpha \to T$, $p_\alpha \to p$ and

$$|Rm|(p_\alpha, g(t_\alpha)) \geq \frac{c}{T - t_\alpha}, \quad \text{for some } c > 0. \tag{4.27}$$

However, by (4.26) when α is large enough

$$|Rm|(p_\alpha, g(t_\alpha)) \leq 10\lambda_i\varepsilon_0^{-2},$$

and the RHS is uniformly bounded above, and this contradicts (4.27).

Appendix

In this appendix, we provide a proof of Theorem 5, which extends Zhang's theorem ([16]) to the case of extended Ricci flow. We may assume $t = -1$ in (3.28) and (3.29) and for notational convenience we suppress the subindex ∞ in $(M_\infty, g_\infty, \phi_\infty, f_\infty)$. We fix a point $p \in M$.

Lemma 18. *We have*

$$S + |\nabla f|^2 - f \equiv \text{const}.$$

Proof. For a function f, we will denote $f_i = \nabla_i f$ and $f_{ij} = \nabla_i \nabla_j f$, etc. We will also use a comma to denote the covariant derivative, e.g. $R_{ij,k} = \nabla_k R_{ij}$. Taking covariant derivative for (3.28), we have

$$R_{ij,j} - \phi_{ij}\phi_j - \phi_i \Delta\phi + f_{ijj} = 0,$$

by the Ricci identity

$$f_{ijj} = f_{jij} = f_{jji} + f_j R_{ij} = (\Delta f)_i + R_{ij} f_j,$$

and the contracted second Bianchi identity $R_{ij,j} = \frac{1}{2}R_{,i}$, we get

$$\frac{1}{2}R_{,i} - \phi_i \Delta\phi - \frac{1}{2}(|\nabla\phi|^2)_i + (\Delta f)_i + R_{ij} f_j = 0,$$

combining with the soliton equation (3.28), $R_{ij} = \frac{1}{2}g_{ij} - f_{ij} + \phi_i\phi_j$ and $\Delta\phi = \phi_i f_i$, we have

$$\left(\frac{1}{2}R - \frac{1}{2}|\nabla\phi|^2 + \Delta f + \frac{1}{2}f - \frac{1}{2}|\nabla f|^2\right)_i = 0$$

so we conclude that

$$\frac{1}{2}R - \frac{1}{2}|\nabla\phi|^2 + \Delta f + \frac{1}{2}f - \frac{1}{2}|\nabla f|^2$$

is a constant. Combining with $S + \Delta f = \frac{n}{2}$, we finish the proof of the lemma.

Without loss of generality, we normalize f such that

$$S + |\nabla f|^2 - f = 0.$$

By direct calculations we have

$$\Delta S = \langle \nabla f, \nabla S\rangle + S - 2|Sic|^2 - 2(\Delta\phi)^2. \tag{7.1}$$

For completeness, we give the proof of (7.1).

Noting that by Lemma 18 and Bochner formula,

$$\Delta S = \Delta f - 2|\nabla^2 f|^2 - 2\langle \nabla\Delta f, \nabla f\rangle - 2Ric(\nabla f, \nabla f),$$

and

$$-\langle \nabla f, \nabla S\rangle = -|\nabla f|^2 + \langle \nabla f, \nabla|\nabla f|^2\rangle,$$

moreover, by the soliton equation (3.28) we have

$$|\nabla^2 f|^2 = |Sic|^2 + \frac{n}{4} - S,$$

combining with the equations above, it follows that

$$\Delta S - \langle \nabla f, \nabla S\rangle = S - 2|Sic|^2 - 2Ric(\nabla f, \nabla f) + |\nabla f|^2 - \langle \nabla f, \nabla |\nabla f|^2\rangle,$$

and the last three terms on RHS are equal to

$$\begin{aligned}-2(R_{ij} + f_{ij})f_i f_j + |\nabla f|^2 &= -2\left(\frac{1}{2}g_{ij} + \phi_i\phi_j\right) f_i f_j + |\nabla f|^2 \\ &= -2(\langle \nabla\phi, \nabla f\rangle)^2\end{aligned}$$

now the formula (7.1) follows from $\Delta\phi = \langle \nabla f, \nabla \phi\rangle$.

Now we go back to the proof of nonnegativity of S for the complete soliton (3.28). We rewrite the estimate (3.7) as follows: if $Ric \le (n-1)K$ in the ball $B(p, r_0)$, then

$$\Delta d(p,\cdot) \le (n-1)(r_0^{-1} + \frac{2}{3}Kr_0) - \int_\gamma Ric(\gamma', \gamma')ds,$$

where $\gamma(s)$ is a normal geodesic initiating at p. Recall the equation $Sic + \nabla^2 f = 1/2g$, $Sic(\gamma', \gamma') + \nabla^2 f(\gamma', \gamma') = 1/2$, and

$$\begin{aligned}\frac{d}{ds} f(\gamma(s)) &= \langle \nabla f, \gamma'\rangle, \\ \frac{d^2}{ds^2} f(\gamma(s)) &= \langle \nabla_{\gamma'}\nabla f, \gamma'\rangle = \nabla^2 f(\gamma', \gamma'),\end{aligned}$$

thus

$$\begin{aligned}\int_\gamma Ric(\gamma', \gamma')ds &= \frac{1}{2}d(p,x) - \int_0^{d(p,x)} \frac{d^2}{ds^2} f(\gamma(s))ds \\ &\quad + \int_0^{d(p,x)} d\phi \otimes d\phi(\gamma', \gamma')ds \\ &\ge \frac{1}{2}d(p,x) - \langle \nabla f(\gamma(s)), \gamma'(s)\rangle|_{s=0}^{s=d(p,x)} \\ &\ge \frac{1}{2}d(p,x) - \langle \nabla f(x), \nabla_x d(p,x)\rangle - |\nabla f(p)|.\end{aligned}$$

Hence, we have for $d(x) = d(p, x)$

$$\Delta d - \langle \nabla f, \nabla d \rangle \leq (n-1)\left(\frac{2}{3}Kr_0 + r_0^{-1}\right) - \frac{1}{2}d + |\nabla f|(p).$$

For any fixed $p \in M$, note that (M, g) is complete noncompact. There exists an $r_0 = r_0(p) > 0$, such that $Ric(g) \leq (n-1)r_0^{-2}$ on the ball $B(p, r_0)$. Choose a cut-off function $\eta(x)$ such that it is 1 when x is less than equal to 1, vanishes when x is greater than 2, and $|\eta'|^2/\eta \leq 4$, $\eta' \leq 0$ and $|\eta''|, |\eta'| \leq 2$ in $\mathbf{R}$. Then define a new function on M by (by abusing notation we still use η to denote the new function)

$$\eta(x) := \eta\left(\frac{d(p, x)}{Ar_0}\right),$$

where $A > 0$ is a large constant. Define $u = S\eta$, and we calculate

$$\Delta u = \eta \Delta S + \frac{S\eta'}{Ar_0}\Delta d + \frac{S\eta''}{(Ar_0)^2} + 2\langle \nabla S, \nabla \eta \rangle, \tag{7.2}$$

We consider two cases $u_{\min} \geq 0$ or $u_{\min} < 0$. For the first case, we are already done, since that means $S(p) = u(p) \geq 0$; in the latter case, we know the minimum must be inside the support of η, which is a point in $B(p, Ar_0)$ denoted by $p_{\min}$. At the minimum point $p_{\min}$, we have $\nabla u = 0$ and $\Delta u \geq 0$, hence at $p_{\min}$, we see $\nabla S = \frac{-S\nabla\eta}{\eta}$ and $u\eta' \geq 0$, and

$$\begin{aligned}
\Delta u &= \eta\Big(\langle \nabla S, \nabla f \rangle + S - 2|Sic|^2 - 2(\Delta\phi)^2\Big) \\
&\quad + \frac{u\eta'}{Ar_0\eta}\Delta d + \frac{u\eta''}{(Ar_0)^2\eta} + 2\langle \nabla S, \nabla \eta \rangle \\
&\leq \eta\Big(\langle \nabla S, \nabla f \rangle + S - 2|Sic|^2\Big) \\
&\quad + \frac{u\eta'}{Ar_0\eta}\Delta d + \frac{u\eta''}{(Ar_0)^2\eta} - 2S\frac{|\nabla\eta|^2}{\eta} \\
&= \eta\Big(\langle \nabla S, \nabla f \rangle + S - 2|Sic|^2\Big) \\
&\quad + \frac{u\eta'}{Ar_0\eta}\Delta d + \frac{u\eta''}{(Ar_0)^2\eta} - 2u\frac{|\eta'|^2}{\eta^2}\frac{1}{(Ar_0)^2} \\
&\leq \eta\Big(\langle \nabla S, \nabla f \rangle + S - 2|Sic|^2\Big) \\
&\quad + \frac{u\eta'}{Ar_0\eta}(\langle \nabla f, \nabla d \rangle + 2nr_0^{-1} + |\nabla f|(p)) - 2u\frac{|\eta'|^2}{\eta^2}\frac{1}{(Ar_0)^2}.
\end{aligned}$$

Observing that at $p_{\min}$, the first term in RHS is

$$-\eta S\frac{\langle\nabla\eta,\nabla f\rangle}{\eta}=-\frac{u\eta'}{Ar_0\eta}\langle\nabla f,\nabla d\rangle,$$

which is cancelled with one term in RHS. Hence we have at the minimum point $p_{\min}$,

$$u-2\eta|Sic|^2+\frac{u\eta'}{Ar_0^2\eta}+\frac{2nu\eta'}{Ar_0\eta}|\nabla f|(p)-2u\frac{|\eta'|^2}{\eta^2}\frac{1}{(Ar_0)^2}\geq 0. \tag{7.3}$$

On the other hand, by Cauchy-Schwarz $-2\eta|Sic|^2\leq -2\frac{\eta S^2}{n}=-\frac{2u^2}{n\eta}$, multiplying by η in (7.3), we see that

$$\eta u-\frac{2u^2}{n}+\frac{2nu\eta'}{Ar_0^2}+\frac{u\eta'}{Ar_0}|\nabla f|(p)-2u\frac{|\eta'|^2/\eta}{(Ar_0)^2}\geq 0,$$

that is

$$0\leq -\frac{2u^2}{n}-|u|+\frac{4n|u|}{Ar_0^2}+\frac{2|u|}{Ar_0}|\nabla f|(p)+4\frac{|u|}{(Ar_0)^2},$$

thus we conclude that at the minimum point $p_{\min}$,

$$|u|\leq\frac{C(|\nabla f|(p),n)}{Ar_0^2},$$

for any constant $A>0$, and this implies

$$S(p)=u(p)\geq u_{\min}\geq -\frac{C(n,|\nabla f|(p))}{Ar_0^2}$$

Note that in this estimate, the constants $C(n,|\nabla f(p)|)$ and r_0 only depend on the chosen point p and $|\nabla f(p)|$, and are independent of A, so letting $A\to\infty$, we have $S(p)=u(p)\geq 0$. Since p is an arbitrary point in M, we conclude that $S\geq 0$ on M.

From $S+|\nabla f|^2=f$, we conclude that $|\nabla\sqrt{f}|\leq 2$, thus $|\nabla f|\leq Cd(p,x)$ outside a compact set. Hence, the completeness of the metric g implies the completeness of the vector field ∇f, which means it generates a one parameter family of diffeomorphisms.

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