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3-manifolds and generalized Baumslag-Solitar groups

Alberto L. Delgado, Derek J. S. Robinson, and Mathew Timm

This article studies the relationship between 3-manifolds and generalized Baumslag-Solitar groups. We classify the generalized Baumslag-Solitar groups that are fundamental groups of compact orientable 3-manifolds. More generally, we show that many generalized Baumslag-Solitar groups which are not 3-manifold groups are special types of quotients of generalized Baumslag-Solitar groups which are.

A *generalized Baumslag-Solitar group*, or *GBS-group*, is the fundamental group of a graph of groups whose vertex and edge groups are infinite cyclic. We describe these in greater detail below but one can think of them as groups built from amalgamated products and HNN-constructions.

It is natural to ask which GBS-groups are the fundamental groups of compact 3-manifolds. The works of Heil [8] and Shalen [16] give partial answers in this direction. In this paper we continue this analysis by applying the standard combinatorial description of GBS-groups in terms of weighted graphs. This point of view has been quite productive; see, among many other works [1, 3–6] and [13–15].

To this end, let $\mathbb{Z}^* = \mathbb{Z}\backslash 0$. A *generalized Baumslag-Solitar graph*, or *GBS-graph*, (Γ, ω) consists of a finite, connected, directed graph Γ in which we allow loops and multiple edges, and a *weight function* ω on the edges of Γ:

$$\omega(e) = (\omega^-(e), \omega^+(e)) \in \mathbb{Z}^* \times \mathbb{Z}^*.$$

We say the weights $\omega^-(e)$ and $\omega^+(e)$ are associated to the initial and terminal vertices e^- and e^+ of e, respectively.

It is convenient to denote an oriented weighted edge by its incident vertices: if the edge e has initial vertex x and terminal vertex y with weights $\omega(e) = (m, n)$, write $(e, \omega) = [x^m, y^n]$; if e is a loop with vertex x, write

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$(e, \omega) = [x^m, x^n]$. We use addition $\oplus$ to join weighted edges that share a vertex. For example, we denote by $[x^m, y^n] \oplus [y^r, z^s]$ a GBS-graph whose underlying graph has exactly two edges each incident with a vertex y.

In Section 1 below we describe in detail how to associate a GBS-group with the GBS-graph (Γ, ω). The associated group, $\pi_1(\Gamma, \omega)$, is called the *fundamental group* of (Γ, ω). In particular,

$$BS(m, n) = \pi_1([x^m, x^n]) = \langle x, t : t^{-1}x^m t = x^n\rangle$$

and

$$K(m, n) = \pi_1([x^m, y^n]) = \langle a, b : a^m = b^n\rangle$$

The first of these examples is an HNN-construction while the second is an amalgamated free product. They form the building blocks for the GBS-groups. Notice that $K(m, n)$ is a torus knot group whenever m and n are relatively prime.

A relation of the form $t^{-1}x^m t = x^n,\ m, n \neq 0$, among elements of a group is called a *Baumslag-Solitar relation*. These relations play a critical role in the subsequent analysis. Of course, the Baumslag-Solitar groups $BS(m, n)$ exhibit such relations, but so also does any GBS-group whose underlying graph contains a circuit. Except in some trivial cases, GBS-groups described by graphs with no circuit also contain elements satisfying $BS(m, m)$ relations while the GBS-groups based on a bouquet of loops have presentations which exhibit only Baumslag-Solitar relations.

We highlight three results. The first two, due to Heil, strongly restrict the GBS-groups which can appear as fundamental groups of 3-manifolds. In the first of our statements, we have adapted Heil's result to our present language.

[8, Proposition 1]. *Suppose $|m|, |n|, |r|, |s| \geq 2$, and x, y, z are distinct vertices. If $|n| \neq |r|$, then $\pi_1([x^m, y^n] \oplus [y^r, z^s])$ is not the fundamental group of a 3-manifold.*

[8, Proposition 2]. *$BS(m, n)$ is the fundamental group of a 3-manifold if and only if $|m| = |n|$.*

The third result, due to Shalen [16], characterizes when a Baumslag-Solitar relation can hold in the fundamental group of a compact orientable 3-manifold. See [9, 10] also.

[16, Theorem 1]. *Let g and t be elements in the fundamental group of a compact 3-manifold group. If g and t satisfy a Baumslag-Solitar relation $t^{-1}g^m t = g^n$ for nonzero integers $m,\ n$, then g has finite order or $|m| = |n|$.*

We say that a GBS-graph (Γ, ω) is $\pm$*locally weight constant*, or $\pm LWC$, if the weights adjacent to each vertex are equal up to sign, although the weight may depend on the vertex. (Γ, ω) is *reduced* if its only edges with at least one weight on them equal to ± 1 are loops. Our first result characterizes those GBS-groups that are fundamental groups of compact orientable 3-manifolds in terms of the underlying GBS-graphs.

Theorem 3.1. *Let (Γ, ω) be a reduced GBS-graph. The following are equivalent:*

1) $\pi_1(\Gamma, \omega)$ *is the fundamental group of a compact orientable 3-manifold;*
2) $\pi_1(\Gamma, \omega)$ *is the fundamental group of an orientable 3-manifold;*
3) (Γ, ω) *is* $\pm LWC$.

If $G = \pi_1(\Gamma, \omega)$ is the fundamental group of a compact 3-manifold we say that G is a GBS *3-manifold group*. It is instructive to compare Theorem 3.1 to [13, 8.1] and the remarks following it. Those results say that any GBS-group with a $\pm LWC$ GBS-graph embed in some $BS(m, \pm m)$. This helps explain why the $\pm LWC$ condition appears here.

Many GBS-groups fail to be 3-manifold groups in a manner that can be very precisely measured. To do so requires a bit more background.

First, the *elementary* GBS-groups are $\mathbb{Z}, \mathbb{Z} \times \mathbb{Z}$, and the Klein bottle group. They can be realized as the GBS-groups whose associated GBS-graphs, respectively, have a single vertex and no edges, as $BS(1,1)$, and as $BS(1,-1)$. Note also that $BS(1,-1) \simeq K(2,2)$. All other GBS-groups are *non-elementary*. Of course, the elementary GBS-groups are topological fundamental groups of compact orientable 3-manifolds: $\mathbb{Z} = \pi_1(S_1 \times D^2)$, $\mathbb{Z} \times \mathbb{Z} = \pi_1(S^1 \times S^1 \times I)$, and the Klein bottle group is the fundamental group of the twisted S^1 bundle over the Möbius band.

Kropholler [12] associates to each non-elementary GBS-group G, its *modular homomorphism* Δ into the group of multiplicative rationals. When $\Delta(G) \subseteq \{1, -1\}$, then G and any GBS-graph having G as its fundamental group are called *unimodular*.

Recall that every finitely generated group is the quotient of a finitely generated free group. Since every finitely generated free group is the fundamental group of a compact orientable 3-manifold, it follows that every finitely generated group is a quotient of the fundamental group of a compact 3-manifold. Thus, interesting results involving quotients of 3-manifold groups require additional structure. In this paper we require additional structure on both the 3-manifolds and the quotients.

To this end, we are mainly interested in *pinch maps* between GBS-groups. We call the surjective image of a pinch map a *pinch quotient*. If the groups in question are GBS 3-manifold groups and if the manifold is compact or orientable, we say the same of the pinch quotient. See Section 1 below and [3]. We now state our second main result.

Theorem 3.2 *Suppose that* (Γ, ω) *is a non-elementary GBS-graph. The following are equivalent:*

1) $\pi_1(\Gamma, \omega)$ *is a compact orientable pinch quotient of a GBS 3-manifold group;*
2) $\pi_1(\Gamma, \omega)$ *is an orientable pinch quotient of a GBS 3-manifold group;*
3) (Γ, ω) *is unimodular.*

It is well known [4, 13], that the unimodularity of a GBS-group, an *a priori* algebraic property, is equivalent to the graph properties of *tree dependence* and *skew tree dependence*. (See the definition below.) This yields an additional equivalence in Theorem 3.2.

By Levitt [13, 2.6], a non-elementary GBS-group is unimodular if and only if it contains an infinite cyclic normal subgroup. Consequently, by the Seifert Fibered Space Theorem [2, 7], see also [9, 12.8], [16], the manifolds delivered by Theorem 3.2 will necessarily be Seifert fibered.

This paper is organized as follows. Section 1 contains background on the fundamental group of a GBS-graph. Those familiar with the language and definitions may skip this material or refer to it for notation. Section 2 contains the topological lemma we need for our main results. Section 3 contains the proofs of our main results.

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1. Preliminaries

Throughout this paper, we let (Γ, ω) be a GBS-graph and choose a maximal subtree T of Γ.

The fundamental group of a GBS-graph. The fundamental group $\pi_1(\Gamma, \omega)$ has generators

$$g_x,\ t_e,\ \text{for } x \in V(\Gamma), e \in E(\Gamma)\backslash E(T),$$

and defining relations

$$\begin{cases} g_{e^+}^{\omega^+(e)} = g_{e^-}^{\omega^-(e)}, & e \in E(T), \\ t_e^{-1}\, g_{e^+}^{\omega^+(e)}\, t_e = g_{e^-}^{\omega^-(e)}, & e \in E(\Gamma)\backslash E(T). \end{cases}$$

It is well known that up to isomorphism $\pi_1(\Gamma, \omega)$ is independent of the choice of maximal subtree.

Choice of signs. By replacing generators of the fundamental group of a GBS-graph by their inverses we can change the signs of various weights without affecting the isomorphism class of the corresponding GBS-group. Consequently, we assume that the weights on the edges of any desired maximal subtree are positive and that at most the initial weight of any edge off the maximal subtree is negative.

Tree and skew tree dependence. Whenever $e = [x, y]$ is an edge of Γ off T, there exists a unique path in T from x to y. Reading along this path, we obtain a relation $g_x^{p_1(e)} = g_y^{p_2(e)}$ where $p_1(e)$ and $p_2(e)$ are the respective products of the nearest and farthest weight values of the edges in the path from x to y.

If the vector $(\omega^-(e), \omega^+(e))$ is a rational multiple of $(p_1(e), p_2(e))$, then e is said to be *T-dependent*. If the vector $(\omega^-(e), -\omega^+(e))$ is a rational multiple of $(p_1(e), p_2(e))$, then e is said to be *skew T-dependent*. Otherwise e is *T-independent*. If e is a loop, then by convention e is T-dependent precisely when $\omega^-(e) = \omega^+(e)$; e is skew T-dependent when $\omega^-(e) = -\omega^+(e)$.

If every non-tree edge of Γ is T-dependent, we say that (Γ, ω) is *T-dependent*. If every non-tree edge is T- or skew T-dependent and if there exists at least one skew T-dependent non-tree edge, we say that (Γ, ω) is *skew T-dependent*.

If there exists a maximal subtree T such that (Γ, ω) is T-dependent or skew T-dependent, we say that (Γ, ω) is, respectively, *tree* or *skew tree dependent*. Clearly, a GBS-tree is tree-dependent.

The modular homomorphism. Let $\Delta = \Delta_G$ be the modular homomorphism defined for the non-elementary GBS-group $G = \pi_1(\Gamma, \omega)$. Levitt [13, 2.5, 2.6] shows that (Γ, ω) is tree dependent if and only if $\Delta(g) = 1$ for every $g \in G$ and skew tree dependent if and only if $\Delta(G) = \{1, -1\}$. Thus the non-elementary GBS-graph (Γ, ω) is tree or skew tree dependent if and only if G is unimodular. Since tree and skew tree dependence of a non-elementary GBS-group can be defined in terms of its modular homomorphism these properties are independent of the choice of maximal subtree. For additional discussion of the modular homomorphism see Heil [8], Kropholler [11], and Section 4 in [4].

Pinch maps and geometric homomorphisms. The *geometric maps* form a class of homomorphisms induced by the underlying graphical structure of the GBS-graphs; see [3] for a further description of such maps. Here we consider only two types of geometric maps: the pinch and contraction maps. We do not need to develop the contraction maps extensively. The pinch maps, however, we describe in further detail.

Let (Γ, χ) and (Γ, ω) be GBS-graphs and let $(e, \chi) = [u^m, v^n]$ be a weighted edge of (Γ, χ). Let d be a positive common divisor of m and n. Suppose $\omega(f) = \chi(f)$ for every edge $f \neq e$ of Γ and that $\omega(e) = (\frac{m}{d}, \frac{n}{d})$. The assignment $g_v \mapsto g_v$ induces a map $\varphi : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \omega)$. Following [3, 2.1], this map is a *pinch along* e *of pinch degree* $pd_e(\varphi) = d$. It is always an epimorphism and it has a non-trivial kernel if and only if $d > 1$. A composition of pinch maps along a sequence of edges is called a *pinch map*.

Pinch degree. Let $(\Gamma, \chi), (\Gamma, \omega)$ be reduced, unimodular GBS-graphs and $\varphi : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \omega)$ a pinch map. For an ordering $\Lambda : e_1, e_2, \ldots, e_n$ of the edges of Γ, let $d_i = pd_{e_i}(\theta)$. The *pinch degree of* θ *relative to* Λ is the n-tuple $(d_1, \ldots, d_n)$.

Reduced GBS-graphs. A GBS-graph is *reduced* if only loops carry a weight of absolute value 1. Using either elementary collapses [5, 6] or, equivalently, the contraction homomorphisms of [3], any GBS-graph can be converted to a reduced GBS-graph without changing the associated GBS-group. Note that such a conversion preserves the $\pm LWC$ condition.

2. $\pm LWC$ graphs and some topology

Suppose (Γ, ω) is tree or skew tree dependent and let N be the set of edges off T whose weights differ in sign. Then (Γ, ω) is tree dependent exactly when N is empty and is skew tree dependent otherwise. Recall that when N is non-empty the convention holds that the initial weight of each edge in N negative.

Lemma 2.1. *If* (Γ, ω) *is* $\pm LWC$, *then* $\pi_1(\Gamma, \omega)$ *is the fundamental group of a compact orientable Seifert fibered 3-manifold.*

We give two proofs of this lemma. Both proofs are highly dependent on the combinatorial structure of the GBS-graph used to define the group. The first makes use of the structural results for GBS-groups from [14]. While it shows that a manifold of the desired type exists, we learn little about its structure. The second explicitly describes how to build the desired manifold as a regular 3-dimensional neighborhood of the canonical 2-complex associated to the GBS-group [1].

Proof 1. By [13, 8.1] and following remarks, $\pi_1(\Gamma, \omega)$ embeds in $BS(n, \pm n)$ for some n. By [8, 2], $BS(n, \pm n)$ is the fundamental group of a compact orientable irreducible 3-manifold. Consequently $\pi_1(\Gamma, \omega)$ is itself the fundamental group of an orientable irreducible 3-manifold. Replacing this manifold, if necessary, by its core, we may assume the manifold is compact and orientable. Since (Γ, ω) is $\pm LWC$, $\pi_1(\Gamma, \omega)$ is unimodular and, by [13] and [4], it contains a cyclic normal subgroup. By the Seifert Fibered Space Theorem [2], [7] the manifold is Seifert fibered. □

Proof 2. We may assume that (Γ, ω) is reduced. Let (Γ, T, N) be as above. Embed Γ on a compact orientable surface S. Let $N(\Gamma)$ be a small closed regular neighborhood of Γ on S with Γ in its interior. Let $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi]\}$. Put $M = N(\Gamma) \times S^1$, and identify the surface $N(\Gamma)$ with $N(\Gamma) \times 1 \subseteq M$. The Seifert fibration of M given by the product structure has base space $N(\Gamma)$. We proceed to modify successively $N(\Gamma)$ and M to create a 3-manifold with the desired properties.

We first identify each edge e in N (if there are any) with a copy of the unit inteval $e = [0, 1]_e$ and consider a neighborhood $[\frac{1}{4}, \frac{3}{4}]_e \times [\epsilon, -\epsilon] \subseteq N(\Gamma)$ using the convention that $e \cap \big([\frac{1}{4}, \frac{3}{4}]_e \times [-\epsilon, \epsilon]\big) = [\frac{1}{4}, \frac{3}{4}]_e \times 0$. Cut open M along $\frac{1}{2} \times [-\epsilon, \epsilon] \times S^1$.

Reattach the copy of $\frac{1}{2} \times [-\epsilon, \epsilon] \times 1$ contained in $[\frac{1}{4}, \frac{1}{2}]_e \times [-\epsilon, \epsilon] \times S^1$ to the copy of $\frac{1}{2} \times [-\epsilon, \epsilon] \times 1$ contained in $[\frac{1}{2}, \frac{3}{4}]_e \times [-\epsilon, \epsilon] \times S^1$ via $(\frac{1}{2}, t, 1) \mapsto (\frac{1}{2}, -t, 1)$. Observe that this creates a cross-cap in $N(\Gamma)$ for each edge e in the set N of edges with one negative and one positive weight.

Then for each edge $e \in N$, in $[\frac{1}{4}, \frac{1}{2}]_e \times [-\epsilon, \epsilon] \times S^1$ glue the fiber $\frac{1}{2} \times t \times S^1$ to the fiber $\frac{1}{2} \times -t \times S^1$ contained in $[\frac{1}{2}, \frac{3}{4}]_e \times [-\epsilon, \epsilon] \times S^1$ via $(\frac{1}{2}, t, e^{i\theta}) \to (\frac{1}{2}, -t, e^{-i\theta})$. We replace the original Seifert fibered 3-manifold M with this resulting manifold. Then M is again Seifert fibered and has base space $N(\Gamma)$ as before.

The space $N(\Gamma)$ contains a copy of Γ. For each $e \in N$ the copy of the circuit in Γ formed from the union of e and the unique path from e^- to e^+ in T is an orientation reversing curve on a non-orientable surface. Hence M is an orientable Seifert fibered 3-manifold with boundary. Note that it contains an embedded copy of its base space $N(\Gamma)$.

We continue to modify M. For each vertex v of Γ take a small closed disk D_v contained in the interior of the embedded copy of $N(\Gamma)$ in such a way that (i) $v \in int\, D_v$ and (ii) whenever v is a vertex of an edge $e \in N$, D_v lies in the complement of the small rectangle $[\frac{1}{4}, \frac{3}{4}]_e \times [-\epsilon, \epsilon]$ used to form the cross-cap in $N(\Gamma)$ associated to e. The 2-torus $\partial D_v \times S^1$ forms the boundary of the solid torus $D_v \times S^1$ in M. For each vertex v of Γ, fix a point $p_v \in \partial D_v = \partial D_v \times 1$. The homotopy classes of $\lambda_v = p_v \times S^1$ and $\mu_v = \partial D_v$ serve as generators for $\pi_1(\partial D_v \times S^1, p_v)$. We remove $\bigcup\{int\, D_v \times S^1 : v \in V(\Gamma)\}$ from M and replace the original M with this resulting space.

Next, for each vertex v, let ω_v be the absolute value of (any of the) weights adjacent to v. Change the Seifert fibration of $(D_v \times S^1) - ((int\, D_v) \times S^1)$ so that $S^1_v = v \times S^1$ is an exceptional fiber of index ω_v with "gluing parameters" $(\omega_v, \omega_v - 1)$; that is, fiber $D_v \times S^1$ by $(\omega_v, \omega_v - 1)$-torus knots and $v \times S^1$. Sew this Seifert fibered copy of $D_v \times S^1$ back into M via a homeomorphism that sends λ_v to the ordinary fiber $\lambda_v^{\omega_v} \mu_v^{\omega_v - 1}$ in the boundary of the Seifert fibered copy of $D_v \times S^1$, and that sends μ_v to the crossing curve $\lambda_v^{\omega_v - 1} \mu_v^{\omega_v - 2}$ in the boundary of the Seifert fibered torus. By replacing our present M with the results of these operations, we may suppose that the following hold:

- M is a Seifert fibered 3-manifold with boundary;
- the base space of the Seifert fibration of M is $N(\Gamma)$;
- $N(\Gamma)$ is a regular neighborhood of Γ;
- each vertex v in Γ is an exceptional point of index ω_v on the base space;
- the corresponding exceptional fiber S^1_v has index ω_v and gluing parameters $(\omega_v, \omega_v - 1)$.

In [1] we describe how to associate to the GBS-graph (Γ, ω) an aspherical 2-complex $K = K(\Gamma, \omega)$ having the property that $\pi_1(K) \simeq \pi_1(\Gamma, \omega)$. Recall that $\Gamma \subseteq N(\Gamma)$. If we denote by $\eta : M \to N(\Gamma)$ the projection of the Seifert fibration, then $\eta^{-1}(\Gamma) = K$. Since $N(\Gamma)$ is a regular neighborhood of Γ, there exists a strong deformation retract from $N(\Gamma)$ onto Γ which induces a Seifert fibration preserving strong deformation retract of M onto $\eta^{-1}(\Gamma)$. Consequently $\pi_1(\Gamma, \omega) = \pi(M)$.

By construction, when N is empty, the base $N(\Gamma)$ of M is orientable, there are no orientation reversing curves on $N(\Gamma)$, and M is orientable. On the other hand, when N is nonempty, $N(\Gamma)$ is non-orientable but contains the needed orientation reversing curves to make M orientable. □

Each of the choices in the second proof of Lemma 2.1 can influence the homeomorphism class of the resulting 3-manifold: first, varying the orientable surface into which Γ is embedded can influence the number of boundary components of M; second, instead of choosing the second gluing parameter for the exceptional fiber to be $\omega_v - 1$, we could have used any positive integer relatively prime to ω_v instead. The resulting aspherical Seifert fibered 3-manifolds, although possibly non-homeomorphic, would still have $K(\Gamma, \omega)$ as a strong deformation retract and consequently share $\pi_1(\Gamma, \omega)$ as their fundamental group.

A non-elementary unimodular GBS-group has a unique maximal cyclic normal subgroup called the *cyclic radical of* G, see [13] or [4]. The results there imply that if the GBS-graph (Γ, ω) is reduced and if $G = \pi_1(\Gamma, \omega)$ is unimodular, then for each vertex v of Γ there exists a unique positive integer c_v with the property that $g_v^{c_v}$ generates the cyclic radical of G. We call c_v the *total weight of* v *in* G and write

$$c_v = \omega^{tot}(v).$$

In Section 5 of [4] we provide an algorithm for computing total weights. Recall our convention that it is the initial weight which is negative on every non-tree edge in a skew tree dependent graph carrying a negative weight.

Lemma 2.2. *Suppose* (Γ, ω) *is non-elementary, reduced, and either tree or skew tree dependent. Define a new weight function* $\bar{\omega}$ *on* Γ *by* $\bar{\omega}(e) = (\pm\omega^{tot}(e^-), \omega^{tot}(e^+))$*. Then* $(\Gamma, \bar{\omega})$ *is* $\pm LWC$ *and there is a pinch map* $\theta : \pi_1(\Gamma, \bar{\omega}) \to \pi_1(\Gamma, \omega)$ *such that along each edge* e *of* Γ

$$pd_e(\theta) = \pm\frac{\omega^{tot}(e^-)}{\omega^-(e)} = \frac{\omega^{tot}(e^+)}{\omega^+(e)}.$$

The minus signs occur when (Γ, ω) *is skew tree dependent and* e^- *is the initial vertex of a non-tree edge on which* $\omega^-(e)$ *is negative.*

This result restates [4, Theorem 2] and the remarks on pinch maps that follow it. By applying this equation along the edges of a path which starts at a vertex of known total weight, we may easily compute the total weight at any vertex of the graph.

The GBS-graph $(\Gamma, \bar{\omega})$ is called *the total weight cover of* (Γ, ω), the pinch map $\theta : \pi_1(\Gamma, \bar{\omega}) \to \pi_1(\Gamma, \omega)$ is the *canonical pinch map for* $\pi_1(\Gamma, \omega)$ and $\pi_1(\Gamma, \bar{\omega})$ is the *canonical pinch pre-image of* $\pi_1(\Gamma, \omega)$.

Proposition 2.3. *If* (Γ, ω) *is tree or skew tree dependent, then* $\pi_1(\Gamma, \omega)$ *is a canonical pinch quotient of an orientable 3-manifold GBS-group* $\pi_1(\Gamma, \bar{\omega})$.

Proof. If $\pi_1(\Gamma, \omega)$ is elementary, the result is obvious, so assume this is not the case. Since reducing the graph does not change the isomorphism class of the associated GBS-group, we may assume that (Γ, ω) has no weight equal to 1 on any vertex of degree 1. Let $(\Gamma, \bar{\omega})$ be the total weight cover of (Γ, ω). Then by Lemma 2.2, the canonical pinch map from $\pi_1(\Gamma, \bar{\omega})$ to $\pi_1(\Gamma, \omega)$ exists. But by Lemma 2.1, $\pi_1(\Gamma, \bar{\omega})$ is the fundamental group of a compact orientable 3-manifold group. □

3. Orientable 3-manifold pinch pre-images

We can now prove our main results. Note that the assumption that (Γ, ω) is reduced is essential. For example, the non-reduced GBS handcuff graph $[u^m, u^{\pm m}] \oplus [u^1, v^1] \oplus [v^m, v^{\pm m}]$ is the GBS-graph of the fundamental group of an orientable Seifert fibered 3-manifold and yet it does not satisfy the $\pm LWC$ condition.

Theorem 3.1. *Let* (Γ, ω) *be reduced and non-elementary. Then the following are equivalent:*

1) $\pi_1(\Gamma, \omega)$ *is the fundamental group of a compact orientable 3-manifold;*

2) $\pi_1(\Gamma, \omega)$ *is the fundamental group of an orientable 3-manifold;*

3) (Γ, ω) *is* $\pm LWC$.

Proof. (1) ⇒ (2). This implication is obvious.

(2) ⇒ (3). We begin by noting that both the property of being an orientable 3-manifold group and that of being a GBS-group are inherited by at least certain subgroups of each of these types of groups. Specifically,

- a subgroup of the fundamental group of an orientable 3-manifold group is also an orientable 3-manifold group, and
- if Δ is a connected subgraph of Γ, then $\pi_1(\Delta, \omega)$ is a subgroup of $\pi_1(\Gamma, \omega)$.

Let $G = \pi_1(\Gamma, \omega)$. Assume *(2)* is true and proceed by contradiction. Among all counterexamples, take one for which Γ has, first, the fewest number of vertices and, then, the fewest number of edges. As a consequence of the bulleted items above and [8, 1] and [8, 2] it follows that (Γ, ω) is one of the following:

(i) $[u^m, v^n]_1 \oplus [v^r, u^s]_2$, with $|m|, |n|, |r|, |s| \geq 2$, $u \neq v$, and $m \neq \pm s$,

(ii) $[u^{\pm m}, u^m]_1 \oplus [u^{\pm 1}, u^1]_2$ with $m \geq 2$, or

(iii) $[u^{\pm 1}, u^1] \oplus [u^m, v^n]$, with $m, n \geq 2$, $u \neq v$

We now proceed to show that the fundamental group of none of these is an orientable 3-manifold group.

Assume (Γ, ω) is as in (i). Form the cyclic double cover of (Γ, ω) and lift the weights on Γ to this double cover in the obvious way. This double cover GBS-graph now contains a path of length 2 with a central vertex that violates the $\pm LWC$ condition and on which all weights have absolute value at least 2. Applying the hereditary properties above and [8], it follows that the fundamental group of this double cover graph is not a 3-manifold group. Since this group is the fundamental group of some double cover of any 3-manifold with fundamental group G, it follows that G is also not a 3-manifold group. This is a contradiction.

For case (ii), note that one can apply an elementary expansion as in [6, 2.1] to the weighted loop $[u^{\pm m}, u^m]$, then apply the above hereditary property for GBS-subgraphs and case (ii) reduces to case (iii).

For case (iii), suppose that G is the fundamental group of an orientable 3-manifold M. Since G is finitely presented, [9, 8.6] implies M has a compact core with the same fundamental group. We may therefore assume M is compact and orientable. Furthermore G is not cyclic and not a free product. Hence [9, 3.3] and [9, 3.13] imply M is irreducible. Since M has infinite first homology, [9, 6.6] implies M is sufficiently large. Thus, we may assume that M is compact, orientable, irreducible and sufficiently large.

In the present case, G has an infinite cyclic normal subgroup: specifically G is unimodular and the unique maximal infinite cyclic normal subgroup in this group is generated by the power $h = g_v^m$ of the vertex generator g_v. Because G has an infinite cyclic normal subgroup, the Seifert Fibered Space Theorem [2], [7] implies M is Seifert fibered.

But $F = G/\langle h \rangle = \langle u, v, t : u^m = v^n = 1, [t, u] = 1 \rangle \cong \mathbb{Z}_n * (\mathbb{Z} \ltimes \mathbb{Z}_m)$ is a member of the class of groups known as Fuchsian groups. By the Grusko Decomposition Theorem, this factorization of G as a free product is unique. Also, Fuchsian groups have the property that their nontrivial finite subgroups have finite centralizers. However, this is not true for the factor $\mathbb{Z}_m$ in the semi-direct product, this is a contradiction. Therefore, G is not the fundamental group of a 3-manifold as desired.

(3) ⇒ (1). If (Γ, ω) is $\pm LWC$, it is tree or skew tree dependent. Consequently, $\pi_1(\Gamma, \omega)$ is unimodular and the indicated implication follows from Lemma 2.1. □

Theorem 3.2 *Suppose that* (Γ, ω) *is non-elementary. Then the following are equivalent:*

1) $\pi_1(\Gamma, \omega)$ *has a compact orientable 3-manifold group pinch preimage;*
2) $\pi_1(\Gamma, \omega)$ *has an orientable 3-manifold group pinch preimage;*
3) (Γ, ω) *is tree or skew tree dependent;*
4) $\pi_1(\Gamma, \omega)$ *is unimodular.*

Proof. (3) $\Leftrightarrow$ *(4).* This was noted earlier and follows from [13] or [4].

(1) $\Rightarrow$ *(2).* This implication is obvious.

(2) $\Rightarrow$ *(3).* Assume there exists an orientable 3-manifold pinch preimage $\theta : \pi_1(\Gamma, \bar{\omega}) \to \pi_1(\Gamma, \omega)$ with a $\pm LWC$ weight function $\bar{\omega}$. It follows easily from the definitions that $(\Gamma, \bar{\omega})$ is tree or skew tree dependent. But pinch maps clearly preserve tree and skew tree dependence. Consequently (Γ, ω) must be tree or skew tree dependent.

(3) $\Rightarrow$ *(1).* This follows from Proposition 2.3. □

Corollary 3.3. *Suppose* (Γ, ω) *is reduced, unimodular, with total weight cover* $(\Gamma, \bar{\omega})$. *Let* $\theta : \pi_1(\Gamma, \bar{\omega}) \to \pi_1(\Gamma, \omega)$ *be the orientable 3-manifold group pinch preimage of* $\pi_1(\Gamma, \omega)$. *Suppose that* $\pi_1(\Gamma, \chi)$ *is a GBS 3-manifold group and that there exists a surjective pinch map* $\psi : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \omega)$. *Then there exists a pinch map* $\psi' : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \bar{\omega})$ *and a positive integer* k *such that* $\psi = \theta \circ \psi'$ *and* $pd_e(\psi') = k$ *for every* $e \in E(\Gamma)$.

$$\begin{array}{ccc} \pi_1(\Gamma, \chi) & \xrightarrow{\psi'} & \pi_1(\Gamma, \bar{\omega}) \\ & \searrow^{\psi} & \downarrow \theta \\ & & \pi_1(\Gamma, \omega) \end{array}$$

Proof. This follows immediately from Theorem 3.2 and [4]. It is simply a restatement of [4, Theorem 3] in the topological language used in the current paper. □

Thus we may reasonably argue that $\theta : \pi_1(\Gamma, \bar{\omega}) \to \pi_1(\Gamma, \omega)$ is the *minimal orientable 3-manifold pinch preimage of* $\pi_1(\Gamma, \omega)$. We have noted above that the orientable 3-manifold pinch preimage of a unimodular GBS-group is the canonical 3-manifold group associated to a given unimodular GBS-group.

Two elementary facts about pinch maps provide the background for our final observation. First, the composition of pinch maps is a pinch map whose pinch degree along each edge is the product of the pinch degrees of the individual pinch maps. It is also clear that a pinch map $\varphi : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \omega)$ along e with $pd_e(\varphi) = m$ exists if and only if m is a common divisor of $\chi^-(e)$ and $\chi^+(e)$.

Choose an ordering Λ of the edges $e_1, \ldots, e_n$ of Γ and determine the pinch degree of the canonical pinch map θ and every other pinch map $\varphi : \pi_1(\Gamma, \chi) \to \pi_1(\Gamma, \omega)$ from 3-manifold groups onto $\pi_1(\Gamma, \omega)$ relative to Λ. Order this collection of pinch degrees lexicographically. Then θ has the minimal pinch degree relative to Λ among all pinches from 3-manifold groups onto $\pi_1(\Gamma, \omega)$.

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Department of Mathematics, Illinois State University
Normal, IL 61790, USA
E-mail address: `aldelg2@ilstu.edu`

Department of Mathematics, University of Illinois at Urbana-Champaign
Urbana, IL 61801, USA
E-mail address: `dsrobins@illinois.edu`

Department of Mathematics, Bradley University
Peoria, IL 61625, USA
E-mail address: `mtimm@bradley.edu`

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