Communications in
Analysis and Geometry
Volume 26, Number 3, 461–504, 2018

Logarithmically spiraling helicoids

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We construct helicoid-like embedded minimal disks with axes along self-similar curves modeled on logarithmic spirals using PDE methods. The surfaces have a self-similarity inherited from the curves and the nature of the construction. Moreover, inside of a "logarithmic cone", the surfaces are embedded.

1. Introduction

In this article we construct helicoid-like embedded minimal disks with axes modeled on a class of embedded self-similar curves called logarithmic spirals. Logarithmic spirals are solutions $\gamma(z)$ to the initial value problem

$$\kappa(z) = \kappa_0 e^{-\xi z}, \quad \tau(z) = \tau_0 e^{-\xi z}$$

where τ and κ denote the torsion and curvature of the unknown curve, respectively, and where the constant ξ controls the rate of exponential propagation. We emphasize that such surfaces can be constructed using the analytic methods of Meeks-Weber [13]. The purpose of our article is to apply the PDE based methods pioneered by Kapouleas in his work on singular perturbation to the construction of minimal laminations on curves in $\mathbb{R}^3$.

Our main theorem can be roughly stated as:

Theorem 1.1. *For each $\delta > 0$ sufficiently small, there exist minimal disks S_δ such that*

1) *Up to rigid motion, the surfaces S_δ exhibit a discrete dilation invariance (see* (1.7)*).*

2010 Mathematics Subject Classification: 53A05, 53C21.
Key words and phrases: Differential geometry, minimal surfaces, partial differential equations, perturbation methods.
C. Breiner was supported in part by NSF grant DMS-1308420 and an AMS-Simons Travel Grant. S. J. Kleene was partially supported by NSF grant DMS-1004646.

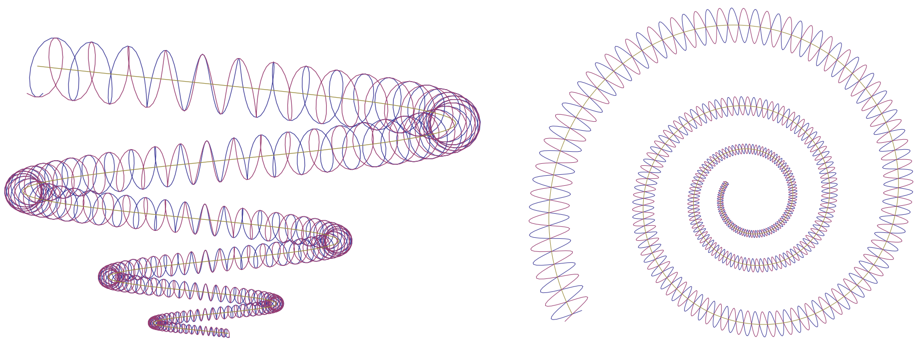

Figure 1: Two examples of logarithmic spirals with the boundaries of the corresponding surfaces S_δ.

2) *The surfaces* S_δ *are embedded inside an open set* $\hat{T}$ *containing* $\gamma(z)$ *independent of* δ*. As* $\delta \to 0$ *the surfaces* S_δ *converge smoothly away from* $\gamma(z)$ *to a foliation of* $\hat{T}$ *by planes orthogonal to* $\gamma(z)$*.*

This result will follow from a more precise statement of the theorem which we record below.

A motivation for applying PDE methods in this context is the following more general question: Given a singly periodic minimal surface Σ, a smooth curve γ, and a non-negative function $\lambda : \gamma \to \mathbb{R}^+$, can one obtain a minimal surface in a tubular neighborhood of γ by bending Σ along γ and scaling by λ? If so, what are the restrictions on the scale function λ? This and related problems arise in several contexts, including gluing constructions for minimal surfaces and the theory of Colding-Minicozzi type laminations, and in certain special cases is well understood. One of the simplest non-trivial cases is that of a constant scale function $\lambda \equiv c$ and a periodic curve γ, which arises naturally in highly symmetric gluing constructions such as [9], where Kapouleas has developed the theory extensively. In this case, the candidate surface can be constructed with the same periodicity as the underlying curve and the problem descends to a compact quotient of the periodic minimal surface Σ which simplifies the analysis considerably.

There are several constructions in the literature which are not compact in any quotient. Meeks and Weber in [13] use the Björling theorem to construct helicoidal minimal surfaces in tubes along embedded $C^{1,1}$ curves where λ scales like the curvature of the curve. In [6], Hoffman and White employ variational techniques to construct minimal laminations in tubes with singularities on prescribed compact subsets of curves. The second author, in [12], uses the Weierstrass representation to prove the same result as in [6]. For additional constructions in the same spirit, see also [3, 4, 11]. All of the above constructions fall short of answering the general question posed above.

In this paper, we consider only curves γ that satisfy a self-similarity condition which allows us to descend to compact quotients. Therefore, the PDE methods employed in this article do not shed light on the more general question. However, our expectation is that PDE methods may have the potential to do so, and this is our primary motivation for using them here.

1.1. Precise statement of the main theorem

We perform the gluing construction on initial surfaces that are self-similar bendings and rescaling of the helicoid. The axes of the surfaces are spiraling curves that lie in a three parameter family of logarithmic spirals. We give a complete description of this family, denoted by $\mathfrak{L}$, in Subsection 3.1 and note here that the parameters for the space $\mathfrak{L}$ are given by $(\kappa_0, \tau_0, \xi) \in \mathbb{R}^+ \times \mathbb{R}^2$. Because we are interested in a lamination, the curves will also depend upon $\delta > 0$ where δ is the parameter used to describe the family S_δ. Every $\gamma \in \mathfrak{L}$ has a self-similar property that implies that curvature and torsion are completely determined by the equations $\kappa(z) = \delta\kappa_0 e^{-\delta\xi z}, \tau(z) = \delta\tau_0 e^{-\delta\xi z}$. Notice that $z \in \mathbb{R}$ is the parameter for the curve γ.

It will be convenient to denote the matrix of transformation of the Frenet frame $\{T, N, B\}$ for the unscaled ($\delta = 1$) curves by T (see (3.3)) and to denote its norm by

$$|\mathsf{T}| := \sqrt{\kappa_0^2 + \tau_0^2}. \tag{1.1}$$

Note that for $\delta \neq 1$, the matrix of transformation simply has the form $\delta\mathsf{T}$.

The global isometry group of the curve γ is generated by the action $e^{\delta\xi\tilde{\theta}} R_{\delta|\mathsf{T}|\tilde{\theta}}$ where R_t corresponds to a rotation in the xy-plane by angle t. The isometry of γ thus implies a periodicity of the form

$$\gamma(\theta + 2\pi/(\delta|\mathsf{T}|)) = e^{2\pi\xi/|\mathsf{T}|}\gamma(\theta) \text{ or } \gamma(\theta + 2\pi) = e^{2\pi\delta\xi} R_{2\pi\delta|\mathsf{T}|}\gamma(\theta). \tag{1.2}$$

We use $\gamma \in \mathfrak{L}$ to define a map $M : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$M(x, y, z) := \gamma(z) + e^{\delta\xi z}\{xN(z) + yB(z)\} \tag{1.3}$$

where here $N(z), B(z)$ are the normal and binormal vectors along γ. We then construct helicoid-like minimal surfaces as graphs over a surface obtained by composing M with the conformal parameterization of the helicoid

$$F(s,\theta) = \sinh(s)\sin(\theta)\mathbf{e}_x + \sinh(s)\cos(\theta)\mathbf{e}_y + \theta\mathbf{e}_z. \tag{1.4}$$

When $\delta > 0$ is sufficiently small, the map M is a diffeomorphism of a tube along the z-axis, with radius comparable to $1/\delta$, onto its image. Properties of γ and the diffeomorphism imply that as $\kappa_0 \to 0, \tau_0 \to 0, \delta \to 0$, the embedded component of $M \circ F$ about γ converges to a rigid motion of the helicoid.

The geometry of the immersion $M \circ F$ possesses a periodicity inherited from properties of the curve γ and by (1.2),

$$M \circ F(s, \theta + 2\pi) = e^{2\pi\delta\xi} R_{2\pi\delta|\mathsf{T}|} M \circ F(s,\theta). \tag{1.5}$$

Since we solve the problem on periodic function spaces, the normal graphs over surfaces satisfying the previous property will exhibit the symmetry we denoted *discrete dilation invariance* in Theorem 1.1. A more precise statement of the theorem is the following:

Theorem 1.2. *Given $\kappa_0, \tau_0, \xi \in \mathbb{R}$ with $\kappa_0 > 0$, there exist $\epsilon_1 > 0$ and $\delta_0 > 0$ so that: For any $0 < \delta < \delta_0/|\xi|$ and $\ell > 16$ satisfying $\delta(1 + |\mathsf{T}| + |\xi|)\ell \leq \epsilon_1$, there exists a curve $\gamma \in \mathfrak{L}$ and a periodic function $u(s,\theta) : \Lambda \subset \mathbb{R}^2 \to \mathbb{R}$ where $\Lambda := \{(s,\theta) : \cosh(s) \leq \ell/4\}$ such that for M given by $\gamma \in \mathfrak{L}$ and*

$$G(s,\theta) := M \circ F(s,\theta), \tag{1.6}$$

the normal graph over G by $w(s,\theta) := e^{\delta\xi\theta}u(s,\theta)$ is an immersed minimal disk with boundary.

The surface G_w satisfies a discrete dilation invariance *of the form*

$$G_w(s, \theta + 2\pi) = e^{2\pi\delta\xi} R_{2\pi\delta|\mathsf{T}|} G_w(s,\theta). \tag{1.7}$$

Moreover, the surface is embedded if

$$\ell \leq \frac{1}{\delta\xi}\left(\frac{e^{\pi\xi/|\mathsf{T}|} - 1}{e^{\pi\xi/|\mathsf{T}|} + 1}\right)\sqrt{\frac{\tau_0^2 + \xi^2}{|\mathsf{T}|^2 + \xi^2}}.$$

1.2. Structure of the article

In Section 2, we introduce various preliminary notions that will be needed throughout the article. In Subsection 2.1, we introduce notation for norms and function spaces, and in Subsection 2.2 we formalize the process of obtaining weighted $C^{k,\alpha}$ estimate for a broad class of homogeneous quantities. This formalization is extremely useful and helps streamline the presentation since almost every geometric quantity we estimate–the mean curvature, e.g.–is such a quantity. The discussion generalizes a similar one in [2].

In Section 3 we estimate local perturbations of geometric quantities on G, where here we heavily exploit the group action on γ. We completely describe the three parameter family of curves $\mathfrak{L}$ by giving an explicit parameterization for each curve in Subsection 3.1. The curves are determined by an initial curvature $\kappa_0 > 0$, torsion τ_0, and rate of exponential parameterization ξ. We demonstrate the self-similarity properties for these curves by explicit computation. In the case of vanishing torsion, the traces of these curves constitute the family of *logarithmic spirals*, and in accordance with this terminology we refer to the whole family as *logarithmic spirals*. The maps M in (1.3) are locally, after modding out by dilations and rigid motions, the identity map plus an exponentially growing perturbation term. In this normalization the perturbation term is periodic, which allows us to consider the problem on one fundamental domain of the helicoid. Roughly speaking, if u is 2π periodic in θ and $H[u]$ is the mean curvature of a normal graph over G by the function u, then the function $Q[u] := e^{\delta\xi\theta}\cosh^2(s)H[e^{\delta\xi\theta}u]$ is also a 2π periodic function in θ. In this way, the analysis descends to cylinders of finite (but not uniformly bounded) length.

We solve the linear problem in Section 4, where the main result appears in Proposition 4.16. This result proves the invertibility of the stability operator $\mathcal{L}_F$ on flat cylinders, Λ, of length arccosh (ℓ), for appropriately modified inhomogeneous terms. The strategy is motivated by the work of Kapouleas in various gluing problems [7–9]. A novel feature of this work is the decomposition of the inhomogeneous term E into a θ-independent function and a function $\mathring{E}$, which has "zero average on meridian circles" (see Definition 4.6). The θ-independent function can be inverted via direct integration, so the main work is to invert $\mathring{E}$. We first prove, in Proposition 4.5, that the operator has a bounded inverse in exponentially weighted Hölder spaces $\mathcal{X}^k$ supported on Λ, as long as $\mathring{E}$ is "orthogonal" to a three dimensional kernel which is spanned by the translational Killing fields. The weighting allows for a rate of exponential growth of power $3/4$ though, in fact, any growth rate less than 1 also works. Here orthogonal means L^2-orthogonal with respect to the pull-back of F to the sphere under the Gauss map. The Gauss map for minimal surfaces is conformal with conformal factor $\frac{|A|^2}{2}$ and in the case of the helicoid descends to the quotient as a conformal diffeomorphism onto the sphere minus the north and south pole (corresponding to the asymptotic normal along both ends of the helicoid). In this way, the study of the stability operator $\mathcal{L}_F$ on the helicoid in θ-periodic function spaces can be understood as the study of $\Delta_{\mathbb{S}^2} + 2$ on the sphere.

To modify $\mathring{E}$, we define functions u_x and u_y, which we can control geometrically, and whose graphs over F can be used to prescribe the kernel content of the mean curvature. In this way, we orthogonalize the error terms for which we are solving and are able to apply Proposition 4.5 to the general setting. In articles by Kapouleas and his many coauthors (see [1, 5, 10] among others), the functions $\mathcal{L}_F u_x, \mathcal{L}_F u_y$ are referred to as the "substitute kernel". Notice that while the space of translational Killing fields is three dimensional, the space of modifications is only two dimensional. The θ-independence of the third translation function and the averaging property of $\mathring{E}$ immediately guarantee the projection of $\mathring{E}$ in this direction is always zero. The functions u_x, u_y grow exponentially at a rate proportional to $\cosh(s)$, which is faster than the allowable growth rate in the space $\mathcal{X}^k$ ($\cosh^{3/4}(s)$), so that a bounded inverse does not extend to the full Hölder space $\mathcal{X}^k$.

In Section 5, we define a map from an appropriate Banach space and show the estimates are sufficient to invoke Schauder's fixed point theorem. Every point in the Banach space corresponds to a triple (v, b_x, b_y) where $v \in \mathcal{X}^2$ and $b_x, b_y \in \mathbb{R}$. For a fixed point, setting $f = e^{\delta\xi\theta}(v + b_x u_x + b_y u_y)$, in Section 6 we demonstrate that G_f is an embedded minimal disk. We define the Banach space so that

$$|b_x| + |b_y| \leq \zeta\delta|\mathsf{T}|\ell^{1/4}, \quad v \approx \zeta\delta|\mathsf{T}|\ell^{1/4}\cosh^{3/4}(s)$$

where ζ is a fixed constant which we must choose sufficiently large. Since $|s| \leq \operatorname{arccosh}(\ell/4)$ on Λ, v is then bounded by a constant uniformly proportional to $\zeta\delta|\mathsf{T}|\ell$, which we can keep arbitrarily small by taking δ small. As u_x, u_y grow exponentially, the estimates for these functions are of a weaker form:

$$\sup_{\Lambda} |b_x u_x| \approx \zeta\delta|\mathsf{T}|\ell^{1/4}\cosh(s) \leq \zeta\delta|\mathsf{T}|\ell^{5/4}.$$

This bound and the previous give some indication of the constraints that fix an upper bound δ.

With these estimates in hand, we use properties of the map M and the helicoid embedding F to prove embeddedness in Section 6. We first prove that the map M in (1.3) is a diffeomorphism from a tubular neighborhood of the z axis, with radius proportional to δ^{-1}, onto its image, which we call a *logarithmic cone*. The symmetry recorded in (1.5) gives an indication of the symmetry possessed by these logarithmic cones, which is inherited from the curves γ. We next prove that graphs over F by $(v + b_x u_x + b_y u_y)\, \nu_F + \mathbf{x}$ are small on Λ, if the norms on v, b_x, b_y are sufficiently small and $\mathbf{x}$ has small C^1 norm. The smallness depends only on properties of the helicoid. We then use properties of M to prove that G_f can be locally described as such a graph over F. The diffeomorphism property for M then implies embeddedness.

2. Preliminaries

2.1. Notation and conventions

Throughout this paper we make extensive use of cut-off functions, and we adopt the following notation: Let $\psi_0 : \mathbb{R} \to [0, 1]$ be a smooth function such that

1) ψ_0 is non-decreasing
2) $\psi_0 \equiv 1$ on $[1, \infty)$ and $\psi_0 \equiv 0$ on $(-\infty, -1]$
3) $\psi_0 - 1/2$ is an odd function.

For $a, b \in \mathbb{R}$ with $a \neq b$, let $\psi[a, b] : \mathbb{R} \to [0, 1]$ be defined by $\psi[a, b] = \psi_0 \circ L_{a,b}$ where $L_{a,b} : \mathbb{R} \to \mathbb{R}$ is a linear function with $L(a) = -3, L(b) = 3$. Then $\psi[a, b]$ has the following properties:

1) $\psi[a, b]$ is weakly monotone.
2) $\psi[a, b] = 1$ on a neighborhood of b and $\psi[a, b] = 0$ on a neighborhood of a.
3) $\psi[a, b] + \psi[b, a] = 1$ on $\mathbb{R}$.

Definition 2.1. Given a function $u \in C^{j,\alpha}(D)$, where $D \subset \mathbb{R}^m$, the (j, α) *localized Hölder norm* is given by

$$\|u\|_{j,\alpha}(p) := \|u : C^{j,\alpha}(D \cap B_1(p))\|.$$

We let $C^{j,\alpha}_{loc}(D)$ denote the space of functions for which $\| - \|_{j,\alpha}$ is pointwise finite.

Definition 2.2. Given a positive function $f : D \to \mathbb{R}$, we let the space $C^{j,\alpha}(D, f)$ be the space of functions for which the *weighted norm* $\|- : C^{j,\alpha}(D, f)\|$ is finite, where we take

$$\|u : C^{j,\alpha}(D, f)\| := \sup_{p\in D} f(p)^{-1}\|u\|_{j,\alpha}(p)$$

Definition 2.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces with norms $\|- : \mathcal{X}\|$ and $\|- : \mathcal{Y}\|$, respectively. Then $\mathcal{X} \cap \mathcal{Y}$ is naturally a Banach space with norm $\|- : \mathcal{X} \cap \mathcal{Y}\|$ given by

$$\|f : \mathcal{X} \cap \mathcal{Y}\| = \|f : \mathcal{X}\| + \|f : \mathcal{Y}\|.$$

Let $\mathcal{X}$ be a Banach space with norm $\|- : \mathcal{X}\|$ and suppose $S \subset \mathcal{X}$. For convenience, throughout the paper we will sometimes write $\|- : S\|$, where for any $f \in S$ we simply let

$$\|f : S\| := \|f : \mathcal{X}\|.$$

2.2. Estimating homogeneous quantitites

Let E be the Euclidean space $E := E^{(1)} \times E^{(2)} = \mathbb{R}^{3\times 2} \times \mathbb{R}^{3\times 4}$. We denote points of E by $\underline{\nabla} = (\nabla, \nabla^2)$, where

$$\nabla = (\nabla_1, \nabla_2), \quad \nabla^2 = (\nabla^2_{11}, \nabla^2_{22}, \nabla^2_{12}, \nabla^2_{21}).$$

We then consider functions $\Phi(\underline{\nabla})$ on E with the property

$$\Phi(c\underline{\nabla}) = c^d\Phi(\underline{\nabla})$$

for real numbers c and d. We call such a function a *homogeneous function of degree d*. It is straightforward to verify that a homogeneous degree d function has the property that its j^{th} derivative $D^{(j)}\Phi$ is homogeneous degree $d - j$.

Remark 2.4. We will assume throughout this section that all functions $\Phi : E \to \mathbb{R}$ refer to smooth functions which are homogeneous of degree d. We also presume such Φ are uniformly bounded in any C^k on compact subsets of the space

$$E_0 := \{\underline{\nabla} \in E : \mathfrak{a}(\nabla) \neq 0\}.$$

Notice E is just a Euclidean space so for any $V \in E$, we make the identification $T_V E = E$. We extend this for each $k \in \mathbb{Z}^+$ and observe that $D^{(k)}\Phi(V) : E^k \to \mathbb{R}$. For clarity we provide the following definition.

Definition 2.5. Let $k \in \mathbb{Z}^+$, $V, W_1, \ldots, W_k \in E$. Then

$$D^{(k)}\Phi\Big|_V (W_1 \otimes \cdots \otimes W_k) := D^{(k)}\Phi(V)(W_1 \otimes \cdots \otimes W_k).$$

For brevity, we denote the k-th tensor product of W with itself by

$$\otimes^{(k)} W := W \otimes \cdots \otimes W.$$

Definition 2.6. Given an immersion $\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, we set $\underline{\nabla}[\phi] := (\nabla\phi, \nabla^2\phi)$. A *homogeneous quantity of degree* d on ϕ is then a function of the form $\Phi[\phi] := \Phi(\underline{\nabla}[\phi])$ for some homogeneous function Φ on E.

Examples of such functions are the mean curvature, unit normal, components of the metric and its dual, the Christoffel symbols and the coefficients of the Laplace operator for ϕ in the domain D.

We want to estimate the linear and higher order changes of homogenous quantities along ϕ due to addition of small vector fields. To do this concisely, we refer to a map $\underline{\nabla}(s, \theta) : D \subset \mathbb{R}^2 \to E$ as an *immersion* if the quantity

$$\mathfrak{a}(\underline{\nabla}) = \mathfrak{a}(\nabla) := 2\sqrt{\det \nabla^T \nabla}/|\nabla|^2 \tag{2.1}$$

is everywhere non-zero, and otherwise we refer to it simply as a *vector field.*

Lemma 2.7. *For a map* $\underline{\nabla} : D \to E$, $|\mathfrak{a}(\nabla)| \leq 1$ *with equality if and only if* $|\nabla_1| = |\nabla_2|$ *and* $\nabla_1 \cdot \nabla_2 = 0$. *In particular,* $\mathfrak{a}(\nabla\phi) = 1$ *if and only if* ϕ *is a conformal immersion.*

Proof. Since $\mathfrak{a}$ is homogeneous degree 0 it suffices to consider the case that $|\nabla_1| = 1$, $|\nabla_2| := r$ for $r \in [0, 1]$. We can then write

$$\begin{aligned}\mathfrak{a}^2(\nabla) &= 4\frac{|\nabla_1|^2|\nabla_2|^2 - (\nabla_1 \cdot \nabla_2)^2}{(|\nabla_1|^2 + |\nabla_2|^2)^2} \\ &= 4\frac{r^2 - \cos^2(\theta)r^2}{(1 + r^2)^2} = 4(1 - \cos^2(\theta))\frac{r^2}{(1 + r^2)^2}\end{aligned}$$

For each θ, the right hand side achieves a unique maximum at $r = 1$ with value $(1 - \cos^2(\theta))$, which gives the claim. $\square$

Definition 2.8. Given an immersion $\underline{\nabla}$ and a vector field $\mathcal{E}$, we set

$$R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla}) := \int_0^1 \frac{(1-\sigma)^k}{k!}\, D\Phi^{(k+1)}\Big|_{\underline{\nabla}(\sigma)} (\otimes^{(k+1)}\mathcal{E})d\sigma \tag{2.2}$$

where $\otimes^{(k)}\mathcal{E}$ denotes the k-fold tensor product of $\mathcal{E}$ with itself and where $\underline{\nabla}(\sigma) := \underline{\nabla} + \sigma\mathcal{E}$.

When $\underline{\nabla}$ and $\mathcal{E}$ are of the form $\underline{\nabla} = \underline{\nabla}\phi$ and $\mathcal{E} = \underline{\nabla}V$ we write

$$R^{(k)}_{\Phi,V}(\phi) := R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla}).$$

Note that $R_{\Phi,\mathcal{E}}(\underline{\nabla})$ is simply the order k Taylor remainder so that:

Proposition 2.9. *We have*

$$\Phi(\underline{\nabla}+\mathcal{E}) - \Phi(\underline{\nabla}) - D\Phi|_{\underline{\nabla}}(\mathcal{E}) - \cdots - \frac{1}{k!}\, D^{(k)}\Phi\Big|_{\underline{\nabla}} \left(\otimes^{(k)}\mathcal{E}\right) \tag{2.3}$$
$$= R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla})$$

Proof. Set $f(\sigma) := \Phi(\underline{\nabla}(\sigma))$. Recall the integral form of the Taylor remainder theorem implies

$$f(1) - f(0) - \cdots - \frac{1}{k!} f^{(k)}(0) = \int_0^1 \frac{(1-\sigma)^k}{k!} f^{(k+1)}(\sigma)d\sigma.$$

The claim then follows by computing explicitly the derivatives of f in terms of Φ. □

Since we are interested in immersions, we provide a quantitative statement that well controlled variations of immersions remain immersions.

Proposition 2.10. *Let $\underline{\nabla}$ and $\mathcal{E}$ be points in E, with $\underline{\nabla} \in E_0$, satisfying*

$$|\mathcal{E}| < \epsilon|\mathfrak{a}(\nabla)||\nabla|.$$

Then for ϵ sufficiently small, independent of ∇ and $\mathcal{E}$,

$$|\mathfrak{a}(\nabla+\mathcal{E}) - \mathfrak{a}(\nabla)| < C|\mathcal{E}|/|\nabla|.$$

Proof. The definition of $\mathfrak{a}$ implies there exists $C > 0$ independent of $\underline{\nabla}$ such that

$$C\mathfrak{a}(\nabla) \leq \frac{|\nabla_1|}{|\nabla_2|} \leq C\mathfrak{a}^{-1}(\nabla).$$

This then gives

$$\begin{aligned}(1 + C\mathfrak{a}^2)|\nabla_2|^2 \leq |\nabla|^2 \leq (1 + C\mathfrak{a}^{-2})|\nabla_2|^2\\(1 + C\mathfrak{a}^2)|\nabla_1|^2 \leq |\nabla|^2 \leq (1 + C\mathfrak{a}^{-2})|\nabla_1|^2,\end{aligned}$$

so that for $0 < \epsilon < (4\mathfrak{a}^2 + 4C)^{-1/2}$,

$$\begin{aligned}&|\mathcal{E}_1|^2 < \epsilon^2\mathfrak{a}^2(1 + C\mathfrak{a}^{-2})|\nabla_1|^2 < \frac{1}{4}|\nabla_1|^2\\&|\mathcal{E}_2|^2 < \frac{1}{4}|\nabla_2|^2.\end{aligned}$$

Set $\nabla(\sigma) := \nabla + \sigma\mathcal{E}$. Then

$$\begin{aligned}\frac{1}{2}|\nabla_2| \leq |\nabla_2(\sigma)| \leq \frac{3}{2}|\nabla_2|\\\frac{1}{2}|\nabla_1| \leq |\nabla_1(\sigma)| \leq \frac{3}{2}|\nabla_1|.\end{aligned}$$

It is then straightforward to check that $|\nabla|\ D\mathfrak{a}|_{\nabla(\sigma)}$ is uniformly bounded for $\sigma \in [0, 1]$, so that using (2.2), (2.3)

$$|\mathfrak{a}(\nabla + \mathcal{E}) - \mathfrak{a}(\nabla)| = \left|R^{(0)}_{\mathfrak{a},\mathcal{E}}(\nabla)\right| < C|\mathcal{E}|/|\nabla|.$$

□

Using the previous estimates and the scaling properties of homogeneous functions, we record here an estimate we use with great frequency. In particular, this estimate allows us to control appropriately weighted Hölder estimates on the remainder terms of a homogeneous function by Hölder estimates on the variation field $\mathcal{E}$.

Proposition 2.11. *There exists $\tilde{\epsilon} > 0$ such that if $\underline{\nabla} : D \to E$ is an immersion and $\mathcal{E} : D \to E$ is a vector field satisfying*

$$\begin{aligned}&\|\mathcal{E} : C^{j,\alpha}(D, \mathfrak{a}(\nabla)|\nabla|)\| \leq C(j, \alpha)\tilde{\epsilon}, \qquad \textit{and}\\&\ell_{j,\alpha}(\nabla) := \|\nabla : C^{j,\alpha}(D, |\nabla|)\| < \infty,\end{aligned}$$

then

$$\left\| R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla}) : C^{j,\alpha}(D, |\nabla|^d) \right\| \leq C(\Phi, \ell_{j,\alpha}, \mathfrak{a}, k) \left\| \mathcal{E} : C^{j,\alpha}(D, |\nabla|) \right\|^{k+1}.$$

Proof. Since $D^{(k+1)}\Phi$ is homogeneous degree $d-(k+1)$ we can write

$$R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla}) = |\nabla|^{d-(k+1)} \int_0^1 \frac{(1-\sigma)^k}{k!} \, D\Phi^{(k+1)}\Big|_{\underline{\nabla}(\sigma)/|\nabla|} (\otimes^{(k+1)}\mathcal{E}) d\sigma,$$

where as before we have set $\underline{\nabla}(\sigma) := \underline{\nabla} + \sigma\mathcal{E}$. We then have

$$\left\| R^{(k)}_{\Phi,\mathcal{E}}(\underline{\nabla}) \right\|_{j,\alpha} \leq C \, \||\nabla|\|^{d-(k+1)}_{j,\alpha} \left\| D^{(k+1)}\Phi\Big|_{\underline{\nabla}(\sigma)/|\nabla|} \right\|_{j,\alpha} \|\mathcal{E}\|^{k+1}_{j,\alpha}. \tag{2.4}$$

With $C(j,\alpha)\tilde{\epsilon} \leq \epsilon$ from Proposition 2.10, the hypotheses imply $\underline{\nabla}(\sigma)/|\nabla|$ remains in a fixed compact subset of E_0 and

$$\left\| D^{(k+1)}\Phi\Big|_{\underline{\nabla}(\sigma)/|\nabla|} \right\|_{j,\alpha} \leq C(\ell_{j,\alpha}, \mathfrak{a}, k).$$

Additionally,

$$\||\nabla|\|_{j,\alpha} / |\nabla|(s,\theta) \leq C\ell_{j,\alpha}.$$

Dividing both sides of (2.4) by $|\nabla|^d$ then gives the claim. □

3. Geometric quantities on G

In this section we record estimates of the relevant geometric data for the immersion G. In the first subsection, we give an explicit parameterization for each curve in $\mathfrak{L}$. To get good estimates, throughout this section we presume that for a given κ_0, τ_0, ξ, we choose $0 < \delta < \delta_0/|\xi|$ such that $\delta(1 + |\mathsf{T}| + |\xi|) < \tilde{\epsilon}$. We then define G by using any curve $\gamma \in \mathfrak{L}$ defined by the parameterization (3.1) for $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi]$.

3.1. A family of logarithmic spirals

We give an explicit parameterization for any $\gamma \in \mathfrak{L}$. For any triple $(\kappa_0, \tau_0, \xi) \in \mathbb{R}^+ \times \mathbb{R}^2$ we define the following two quantities

$$b(\kappa_0, \tau_0, \xi) := \frac{\tau_0}{\kappa_0}\sqrt{\kappa_0^2 + \tau_0^2 + \xi^2}; \qquad c(\kappa_0, \tau_0, \xi) := \sqrt{\kappa_0^2 + \tau_0^2}.$$

Given constants $\kappa_0 > 0$, $\xi, \tau_0 \in \mathbb{R}$, let $\gamma[\kappa_0, \tau_0, \xi](z) : \mathbb{R} \to \mathbb{R}^3$ be a smooth curve determined by

$$\gamma[\kappa_0, \tau_0, \xi](t) = \frac{e^{\xi t}}{\sqrt{\xi^2 + b^2 + c^2}} \left(\mathbf{e}_r(ct) + \frac{b}{\xi}\mathbf{e}_z \right) \tag{3.1}$$

where we have abbreviated $\mathbf{e}_r(t) := (\cos(t), \sin(t), 0)$ and $\mathbf{e}_z = (0, 0, 1)$.

We demonstrate that such a curve satisfies the properties that

(1) $ds = e^{\xi t} dt$ where s is the arclength parameter.

(2) curvature $\kappa(t)$ and torsion $\tau(t)$ are described by

$$\kappa(t) = e^{-\xi t}\kappa_0, \quad \tau(t) = e^{-\xi t}\tau_0.$$

We first calculate derivatives and note that

$$\gamma'(t) = \frac{e^{\xi t}}{\sqrt{\xi^2 + b^2 + c^2}} \left(\xi \mathbf{e}_r(ct) + b\mathbf{e}_z + c\mathbf{e}_r^{\perp}(ct) \right) \tag{3.2}$$

where of course $\mathbf{e}_r^{\perp}(t) := (-\sin t, \cos t, 0)$.

The arclength calculation follows immediately from (3.2) and thus the unit tangent has the form

$$T = \frac{\gamma'}{|\gamma'|} = \frac{\xi \mathbf{e}_r(ct) + b\mathbf{e}_z + c\mathbf{e}_r^{\perp}(ct)}{\sqrt{\xi^2 + b^2 + c^2}}.$$

It follows that

$$\frac{dT}{ds} = e^{-\xi t} \frac{\xi c \mathbf{e}_r^{\perp}(ct) - c^2 \mathbf{e}_r(ct)}{\sqrt{\xi^2 + b^2 + c^2}}$$

and

$$\kappa = \left| \frac{dT}{ds} \right| = e^{-\xi t} \frac{c\sqrt{\xi^2 + c^2}}{\sqrt{\xi^2 + b^2 + c^2}}.$$

Since one can directly verify that $\frac{c\sqrt{\xi^2+c^2}}{\sqrt{\xi^2+b^2+c^2}} = \kappa_0$, we have demonstrated the appropriate formula for κ.

To find τ, note first that

$$N = \frac{T'}{|T'|} = \frac{\xi c \mathbf{e}_r^{\perp}(ct) - c^2 \mathbf{e}_r(ct)}{c\sqrt{\xi^2 + c^2}}$$

and thus

$$B = T \wedge N = \frac{-\xi bc \mathbf{e}_r(ct) - bc^2 \mathbf{e}_r^{\perp}(ct) + (\xi^2 c + c^3)\mathbf{e}_z}{c\sqrt{\xi^2 + c^2}\sqrt{\xi^2 + b^2 + c^2}}.$$

Since

$$\frac{dB}{dt} = \frac{-\xi bc^2 \mathbf{e}_r^{\perp}(ct) + bc^3 \mathbf{e}_r}{c\sqrt{\xi^2 + c^2}\sqrt{\xi^2 + b^2 + c^2}}$$

we observe that

$$\frac{dB}{ds} = e^{-\xi t} \frac{-\xi bc^2 \mathbf{e}_r^{\perp}(ct) + bc^3 \mathbf{e}_r}{c\sqrt{\xi^2 + c^2}\sqrt{\xi^2 + b^2 + c^2}} = -e^{-\xi t} \frac{bc}{\sqrt{\xi^2 + b^2 + c^2}} N.$$

The result for τ now follows by observing that $\tau_0 = \frac{bc}{\sqrt{\xi^2 + b^2 + c^2}}$.

We point out the Frenet-Serret equations may be written

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa_0 & 0 \\ -\kappa_0 & 0 & \tau_0 \\ 0 & -\tau_0 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

and we let

$$\mathsf{T} := \begin{pmatrix} 0 & \kappa_0 & 0 \\ -\kappa_0 & 0 & \tau_0 \\ 0 & -\tau_0 & 0 \end{pmatrix}. \tag{3.3}$$

3.2. The normalized derivatives $\tilde{\nabla} G^{(k)}$

In order to conveniently estimate geometric quantities on G such as the mean curvature, we want to normalize G and its derivatives in a way that controls for rotations and dilations, whose effect on quantities such as the mean curvature and unit normal are easily extracted.

Definition 3.1. We let $R(\theta)$ be the rotation given below:

$$R(\theta) := N(\theta) \otimes \mathbf{e}_x^* + B(\theta) \otimes \mathbf{e}_y^* + T(\theta) \otimes \mathbf{e}_z^*$$

where $\{\mathbf{e}_x^*, \mathbf{e}_y^*, \mathbf{e}_z^*\}$ is the dual basis in $\mathbb{R}^3$ to the standard basis. We then define the normalized derivatives of G as:

$$\tilde{\nabla}^{(k)} G(s,\theta) := e^{-\delta\xi\theta} R(\theta) \nabla^{(k)} G(s,\theta),$$

and we set

$$\underline{\tilde{\nabla}}[G] := \left(\tilde{\nabla} G, \tilde{\nabla}^2 G\right).$$

Proposition 3.2. *The following statements hold:*

1) *The normalized derivatives $\tilde{\nabla} G(s,\theta)$ are 2π-periodic in θ.*
2) *Letting $\tilde{\nu}_G := \nu(\tilde{\nabla}[G])$,*
$$\nu_G = \nu(\nabla G) = R^{-1}(\theta)\,(\tilde{\nu}_G)\,.$$
3) *For $H_G := H(\underline{\nabla} G)$,*
$$H_G = e^{-\delta\xi\theta} H(\underline{\tilde{\nabla}}[G]).$$

Proof. Item (1) follows directly from the definition of F in (1.4). Items (2) and (3) follow from the fact that ν and H are homogeneous degree 0 and -1 quantities, respectively, and their behavior under rotations and dilations. □

3.3. Comparing immersions on spirals with straight lines

In [2], we considered immersions of the form:

$$G_{0\delta}(s,\theta) = \left(e^{\delta\theta}\sin(\theta)\sinh(s), e^{\delta\theta}\cos(\theta)\sinh(s), \frac{1}{\delta}e^{\delta\theta}\right). \tag{3.4}$$

Theorem 3.3 (Theorem 1 from [2]). *There are constants $\epsilon_0, \delta_0 > 0$ sufficiently small so that for any $0 < \delta < \delta_0$, there is a function $u_{0\delta}(s)$: $[-\epsilon_0\delta^{-1/4}, \epsilon_0\delta^{-1/4}] \to \mathbb{R}$ such that:*

1) *The normal graph over $G_{0\delta}$ by the function $w_{0\delta}(s,\theta) := e^{\delta\xi\theta}u_{0\delta}(s)$ is an embedded minimal surface with boundary.*
2) *$u_{0\delta}(s)$ is an odd function.*

3) *There is a constant $C > 0$ sufficiently large so that $u_{0\delta}$ satisfies the estimate*

$$\|u_{0\delta} : C^{j,\alpha}([0, \epsilon_0\delta^{-1/4}], s^2)\| \leq C\delta.$$

Notice that for any $\xi \neq 0$, we can apply this theorem to immersions $G_{0\,\delta\xi}$ as long as $0 < \delta|\xi| < \delta_0$ and all domain bounds and estimates then have δ replaced by $\delta|\xi|$. Moreover, as $\xi \to 0$, an appropriate translation of $G_{0\,\delta\xi}$ converges to the helicoidal embedding by F. Thus, when $\xi = 0$, it will be natural to replace estimates for $G_{0\,0}$ by estimates for F.

Throughout, we will take the liberty of suppressing the dependence of these immersions and maps on δ, ξ from the notation and instead write

$$G_0 := G_{0\,\delta\xi}, \quad u_0 := u_{0\,\delta\xi}, \quad w_0 := w_{0\,\delta\xi}.$$

As a first approximation to our solution we wish to compare the geometry of G with that of G_0. To do this we normalize the derivatives of G_0 by taking

$$\tilde{\nabla} G_0^{(k)}(s,\theta) := e^{-\delta\xi\theta}\nabla G_0(s,\theta). \tag{3.5}$$

Lemma 3.4. *For all $k \in \mathbb{Z}^+$ there exists C independent of k so that*

$$\left|\tilde{\nabla}^{(k)} G(s,\theta) - \tilde{\nabla}^{(k)} G_0(s,\theta)\right| < C\delta|\mathsf{T}|\cosh(s).$$

Proof. Notice first that $\delta|\mathsf{T}| < 1$. By definition,

$$R(\theta)\frac{\partial^k}{\partial s^k}G = \frac{\partial^k}{\partial s^k}G_0$$

so any differentiation only in s will vanish in the difference. Let $\mathbf{e}_r := \sin\theta N + \cos\theta B$ and $\mathbf{e}_r^\perp := \cos\theta N - \sin\theta B$. Then

$$\begin{aligned}(G_0)_\theta &= e^{\delta\xi\theta}R(\theta)\mathbf{e}_3 + e^{\delta\xi\theta}\sinh(s)R(\theta)\mathbf{e}_r^\perp + \delta\xi e^{\delta\xi\theta}\sinh(s)R(\theta)\mathbf{e}_r\\ G_\theta &= e^{\delta\xi\theta}\mathbf{e}_3 + e^{\delta\xi\theta}\sinh(s)\mathbf{e}_r^\perp + \delta\xi e^{\delta\xi\theta}\sinh(s)\mathbf{e}_r + \delta e^{\delta\xi\theta}\sinh(s)\mathsf{T}\mathbf{e}_r\end{aligned}$$

and thus

$$e^{-\delta\xi\theta}\left(R(\theta)G_\theta - (G_0)_\theta\right) = \delta\sinh(s)R(\theta)(\mathsf{T}\mathbf{e}_r).$$

This immediately proves the estimates for $k = 1$. The higher order estimates follow inductively since $0 < \delta|\mathsf{T}| < 1$. $\square$

From Lemma 3.4, we can obtain a good estimate for the mean curvature of G_{w_0}, the normal graph over G by the function w_0. Since $(G_0)_{w_0}$ is a minimal surface, we expect that the failure of G_{w_0} to be minimal is controlled by the geometry of the modeling curve for G and the scale δ. We first need to compare the unit normal field along along the immersions.

To demonstrate that we may use Proposition 2.11 to estimate geometric quantities of small graphs over G we record the following lemma.

Lemma 3.5. *The following statements hold:*

(1) *There exists $C > 0$ independent of G such that*

$$C^{-1}\cosh(s) \leq |\tilde{\nabla}[G]| \leq C\cosh(s)$$

(2) *Recalling the definition of $\ell_{j,\alpha}$ from Proposition 2.11,*

$$\ell_{j,\alpha}(\underline{\tilde{\nabla}}[G]) \leq C(j,\alpha)$$

(3) *There exists $C > 0$ independent of G such that*

$$1 - [\mathfrak{a}(\tilde{\nabla}[G])] \leq C\delta\left(|\mathsf{T}| + |\xi|\right)$$

Proof. Items (1) and (2) are direct consequences of Lemma 3.4 and the corresponding estimates for F. To prove item (3), first note that

$$|\tilde{\nabla}^{(k)}G_0 - \nabla^{(k)}F| \leq C\delta|\xi|\cosh(s).$$

The result now follows from the triangle inequality, Lemma 3.4, Proposition 2.10 and the fact that $\mathfrak{a}[F] = 1$. □

Lemma 3.6. *For any $j \in \mathbb{Z}^+$*

$$\left\|\tilde{\nu}_G(s,\theta) - \nu_{G_0}(s,\theta) : C^{j,\alpha}(\mathbb{R}^2)\right\| < C(j,\alpha)|\mathsf{T}|\delta.$$

Proof. First note that $\delta(1 + |\mathsf{T}| + \xi) < \tilde{\epsilon}$ of Proposition 2.11. Thus for ν, we can apply Proposition 2.11 with $k = 0$ and $d = 0$. The result then follows by Lemma 3.4 and Proposition 2.11 with $\underline{\nabla} = \underline{\tilde{\nabla}}G$ and $\mathcal{E} = \underline{\tilde{\nabla}}G - \underline{\tilde{\nabla}}G_0$. □

We will exploit the periodicity of the helicoid and consider graphs over $G = M \circ F$ that are periodic in θ. To that end, we define the appropriate quotient space of $\mathbb{R}^2$.

Definition 3.7. Let Ω be the quotient of $\mathbb{R}^2$ by the translation $(s,\theta) \mapsto (s, \theta + 2\pi)$.

Given an immersion G, we look for functions $u : \Omega \to \mathbb{R}$ such that $G + u\nu_G$ is minimal. Because of the homogeneity of H and its invariance under rotations, we consider variation fields of the following form.

Definition 3.8. Given a function $u : \Omega \to \mathbb{R}$, we set

$$\mathcal{E}_\delta[u] := e^{-\delta\xi\theta} R(\theta) \underline{\nabla}(e^{\delta\xi\theta} u \nu_G). \tag{3.6}$$

The self-similarity of γ allows us to consider the mean curvature up to the natural localized rotation and dilation of G by $e^{-\delta\xi\theta}R(\theta)$. To that end, we define the map Q.

Definition 3.9. The map $Q[u] : C^2(D) \to C^0(D)$ is given as follows:

$$\begin{aligned} Q[u] &:= e^{\delta\xi\theta} \cosh^2(s) H(\underline{\nabla}[G] + \underline{\nabla}[e^{\delta\xi\theta}(u+u_0)\nu_G]) \\ &= \cosh^2(s) H(\tilde{\underline{\nabla}}[G] + \mathcal{E}_\delta[u+u_0]). \end{aligned} \tag{3.7}$$

An important consequence of the definition is that Q preserves the periodicity property.

Lemma 3.10. *Q maps the space $C^2(\Omega)$ into $C^0(\Omega)$.*

Proof. Since we have already verified in Proposition 3.2, item (1), that $\tilde{\underline{\nabla}}G$ is periodic, the preservation of periodicity follows once we verify that $\mathcal{E}_\delta(u)$ maps periodic functions to periodic functions. Item (1) of Proposition 3.2 also implies $\tilde{\nu}_G$ is 2π periodic in θ and thus, all derivatives of $\tilde{\nu}_G$ are 2π periodic in θ. By direct calculation, it is enough to show that $R(\theta)\partial_\alpha \nu_G$ is 2π periodic for $\alpha = s, \theta, ss, s\theta, \theta\theta$.

First, note that the vector $R(\theta)\mathbf{e}_i$ is independent of θ. Since T is fixed, and

$$R'(\theta)\mathbf{e}_i = (R(\theta)\mathbf{e}_i)' - R(\theta)\delta\mathsf{T}\mathbf{e}_i = -R(\theta)\delta\mathsf{T}\mathbf{e}_i, \tag{3.8}$$

we note that $R'(\theta)\mathbf{e}_i$ is independent of θ. A similar calculation with second derivatives immediately implies $R''(\theta)\mathbf{e}_i$ is also θ independent. Since we will need an estimate on $|R''(\theta)|$ later, we record here

$$0 = (R(\theta)\mathbf{e}_i)'' = R''(\theta)\mathbf{e}_i + 2R'(\theta)\mathbf{e}_i' + R(\theta)\mathbf{e}_i'' = \left(R''(\theta) + 2R'(\theta)\delta\mathsf{T} + R(\theta)\delta^2\mathsf{T}^2\right)\mathbf{e}_i \tag{3.9}$$

Now, suppose $\nu_G = \sum_i \alpha_i \mathbf{e}_i$. Then since $\tilde{\nu}_G = R(\theta)\nu_G$, the α_i are all 2π periodic in θ. Moreover, one quickly calculates

$$\begin{aligned}
(\tilde{\nu}_G)_s - R(\theta)(\nu_G)_s &= (\tilde{\nu}_G)_{ss} - R(\theta)(\nu_G)_{ss} = 0\\
(\tilde{\nu}_G)_\theta - R(\theta)(\nu_G)_\theta &= \sum_i \alpha_i R'(\theta)\mathbf{e}_i\\
(\tilde{\nu}_G)_{s\theta} - R(\theta)(\nu_G)_{s\theta} &= \sum_i (\alpha_i)_s R'(\theta)\mathbf{e}_i\\
(\tilde{\nu}_G)_{\theta\theta} - R(\theta)(\nu_G)_{\theta\theta} &= \sum_i \left(2(\alpha_i)_\theta R'(\theta)\mathbf{e}_i + 2\alpha_i R'(\theta)\delta\mathsf{T}\mathbf{e}_i + \alpha_i R''(\theta)\delta\mathsf{T}\mathbf{e}_i\right).
\end{aligned}$$

Since all terms on the right are 2π periodic in θ and all derivatives of $\tilde{\nu}_G$ are 2π periodic in θ, Q preserves periodicity. □

We use the estimates of Lemmas 3.5, 3.6 , and 3.10 to determine estimates for the mean curvature of G_{w_0} which by definition correspond to estimates on $Q[0]$. This bound will appear again in the fixed point argument of Section 5.

Proposition 3.11. *For* $\Omega^* := \Omega \cap \{s : |s| \le \epsilon(\delta\xi)^{-1/4}\}$,

$$\left\|Q[0] : C^{j,\alpha}(\Omega^*, \cosh(s))\right\| < C(j,\alpha)\delta|\mathsf{T}|. \tag{3.10}$$

Proof. Define $\mathcal{E}_{0\delta}[u] := e^{-\delta\xi\theta}\underline{\nabla}(e^{\delta\xi\theta}u\nu_{G_0})$. To prove the norm bound, we first write

$$\begin{aligned}
Q[0] &= \cosh^2(s)H(\underline{\tilde{\nabla}}[G] + \mathcal{E}_\delta[u_0])\\
&= \cosh^2(s)\left(H(\underline{\tilde{\nabla}}[G] + \mathcal{E}_\delta[u_0]) - H(\underline{\tilde{\nabla}}G_0 + \mathcal{E}_{0\delta}[u_0])\right)\\
&= -\cosh^2(s)R^{(0)}_{H,\underline{\nabla}}(\mathcal{E}),
\end{aligned}$$

where $\underline{\nabla} := \underline{\tilde{\nabla}}[G] + \mathcal{E}_\delta[u_0]$ and $\mathcal{E} = \underline{\tilde{\nabla}}[G_0 - G] + (\mathcal{E}_{0\delta}[u_0] - \mathcal{E}_\delta[u_0]) := T_1 + T_2$. Recall that T_1 has been estimated in Lemma 3.4:

$$\|T_1\|_{j,\alpha} \le C(j,\alpha)\delta|\mathsf{T}|\cosh(s) \tag{3.11}$$

so we focus on the term

$$T_2 := \mathcal{E}_{0\delta}[u_0] - \mathcal{E}_\delta[u_0] = e^{-\delta\xi\theta}\underline{\nabla}(e^{\delta\xi\theta}u_0\nu_{G_0}) - e^{-\delta\xi\theta}R(\theta)\underline{\nabla}(e^{\delta\xi\theta}u_0\nu_G).$$

By direct computation, we determine the components of T_2. Projected onto $E^{(1)}$,

$$\big((u_0)_s\,(R(\theta)\nu_G - \nu_{G_0}) + u_0\,(R(\theta)(\nu_G)_s - (\nu_{G_0})_s)\,,\\ \delta\xi u_0\,(R(\theta)\nu_G - \nu_{G_0}) + u_0\,(R(\theta)(\nu_G)_\theta - (\nu_{G_0})_\theta)\,\big)$$

and onto $E^{(2)}$

$$\begin{aligned}&\big((u_0)_{ss}\,(R(\theta)\nu_G - \nu_{G_0}) + 2(u_0)_s\,(R(\theta)(\nu_G)_s - (\nu_{G_0})_s)\\ &\quad + u_0\,(R(\theta)(\nu_G)_{ss} - (\nu_{G_0})_{ss})\,,\\ &\delta\xi(u_0)_s\,(R(\theta)\nu_G - \nu_{G_0}) + \delta\xi u_0\,(R(\theta)(\nu_G)_s - (\nu_{G_0})_s)\\ &\quad + (u_0)_s\,(R(\theta)(\nu_G)_\theta - (\nu_{G_0})_\theta) + u_0\,(R(\theta)(\nu_G)_{s\theta} - (\nu_{G_0})_{s\theta})\,,\\ &(\delta\xi)^2u_0\,(R(\theta)\nu_G - \nu_{G_0}) + 2\delta\xi u_0\,(R(\theta)(\nu_G)_\theta - (\nu_{G_0})_\theta)\\ &\quad + u_0\,(R(\theta)(\nu_G)_{\theta\theta} - (\nu_{G_0})_{\theta\theta})\,\big).\end{aligned}$$

From Lemma 3.6, since $\tilde{\nu}_G = R(\theta)\nu_G$

$$\|R(\theta)\nu_G - \nu_{G_0} : C^0\| \leq C\delta|\mathsf{T}|.$$

For the projection onto $E^{(1)}$, we use Lemma 3.6, the triangle inequality, and (3.8) to get, for $i \in \{s,\theta\}$,

$$|R(\theta)(\nu_G)_i - (\nu_{G_0})_i| \leq C\delta|\mathsf{T}|. \tag{3.12}$$

For the projection onto $E^{(2)}$, we observe first that for $i,j \in \{s,\theta\}$,

$$(\tilde{\nu}_G)_{ij} - R(\theta)\,(\nu_G)_{ij} = (R(\theta))_{ij}\,\nu_G + (R(\theta))_i\,(\nu_G)_j + (R(\theta))_j\,(\nu_G)_i\,.$$

Appealing to Lemma 3.6, the triangle inequality, (3.8) again coupled with (3.9), we observe that, since $\delta|\mathsf{T}| < 1$,

$$|R(\theta)(\nu_G)_{ij} - (\nu_{G_0})_{ij}| \leq C\delta|\mathsf{T}|. \tag{3.13}$$

Combining (3.11), (3.12) and (3.13), and noting further that $0 \leq \delta|\xi| < 1$,

$$\begin{aligned}\|\mathcal{E}\|_{j,\alpha} &\leq C(j,\alpha)\delta|\mathsf{T}|\,(\cosh(s) + \|u_0\|_{j+2,\alpha})\\ &\leq C(j,\alpha)\delta|\mathsf{T}|\,\big(\cosh(s) + s^{j+2}\big)\,.\end{aligned}$$

Using the estimates from Lemma 3.5, we apply Proposition 2.11 with $H = \Phi$ and $d = -1$. □

4. The linear problem

The goal of this section is to prove Proposition 4.16 which shows that, modulo a two dimensional space of exponentially growing functions, the linear operator $\cosh^2(s)\mathcal{L}_F$ is invertible in appropriately weighted Hölder spaces. These weighted spaces will be defined on subsets of Ω.

Definition 4.1. Set

$$\Lambda := \Omega \cap \{|s| \leq \text{arccosh}\,(\ell)\} \tag{4.1}$$

where $\ell \in (2, \cosh(\epsilon_0(\delta|\xi|)^{-1/4}))$ is a constant to be determined and where ϵ_0 is as in the statement of Theorem 3.3. The upper bound for ℓ is the maximum scale on which the functions u_0 of [2] are defined. The main theorem does not allow such a generous upper bound for ℓ, though ℓ should be considered as a large constant. The weighted spaces on which we solve the linear problem now take the following form.

Definition 4.2. Let $\mathcal{X}^k$, $k=0,2$ be the space of functions $f(s,\theta)$ in $C^{k,3/4}_{loc}(\Lambda)$ such that

$$\|f : \mathcal{X}^k\| := \|f : C^{k,3/4}(\Lambda, \cosh^{3/4}(s))\| < \infty. \tag{4.2}$$

Note that the spaces $\mathcal{X}^k$ are Banach spaces with norm $\|- : \mathcal{X}^k\|$. Our main result will follow from the fact that linearized problem on the surface F is invertible in the spaces $\mathcal{X}^k$, and the fact that we can treat the linearized problem on G as a perturbation of the linearized problem on F.

For a locally class C^2 immersion $\Phi(s,\theta) : D \to \mathbb{R}^3$, we recall the *stability operator* of the immersion:

$$\mathcal{L}_\Phi := \Delta_\Phi + |A_\Phi|^2. \tag{4.3}$$

Here Δ_Φ and $|A_\Phi|^2$ respectively denote the Laplace operator and the squared norm of the second fundamental form of the immersion.

4.1. Obstructions to the linear problem

In the spirit of Kapouleas, we will solve the linear problem on F by modifying an inhomogeneous $E \in \mathcal{X}^0$ so that the modified function is L^2 orthogonal to the obstructions to invertibility. We first record some relevant properties and then discuss the obstructions. The modifying functions will be determined and understood in Subsection

Let g_F, ν_F and A_F be the metric, the unit normal, and the second fundamental for F, respectively. Then

$$\begin{aligned} g_F(s,\theta) &= \cosh^2(s)(ds^2+d\theta^2) \\ \nu_F(s,\theta) &= -\cosh^{-1}(s)\cos(\theta)\mathbf{e}_x + \cosh^{-1}(s)\sin(\theta)\mathbf{e}_y + \tanh(s)\mathbf{e}_z \\ A_F(s,\theta) &= -2dsd\theta \\ \cosh^2(s)\mathcal{L}_F &= \Delta_\Omega + 2\cosh^{-2}(s). \end{aligned} \tag{4.4}$$

Note that if $u:\Omega\to\mathbb{R}$ then the periodicity of F and ν_F imply that u can be extended to a graph over the full helicoid. Moreover, to understand the behavior of $\mathcal{L}_F u$, it is enough to understand its behavior on a fundamental domain of the helicoid. That is, we can consider the problem only on Ω rather than on all of $\mathbb{R}^2$. In the same spirit, we can analyze the Gauss map $\nu_F: F\to\mathbb{S}^2$ on Ω. On this subdomain, ν_F is a conformal diffeomorphism with conformal factor $|A_F|^2/2$ onto the punctured sphere $\mathbb{S}^2\setminus\{(0,0,\pm1)\}$.

Definition 4.3. Let $\mathcal{K}$ be the space of bounded functions $\kappa:\Omega\to\mathbb{R}$ such that $\mathcal{L}_F\kappa=0$. Then standard theory implies that $\mathcal{K}$ is spanned by the functions

$$\kappa_x=\cos(\theta)\cosh^{-1}(s),\quad \kappa_y=\sin(\theta)\cosh^{-1}(s),\quad \kappa_z=\tanh(s).$$

The functions κ_x, κ_y and κ_z are (up to sign) the x, y, and z components of the unit normal for F. Indeed, if we lift κ_x, κ_y and κ_z to the sphere via the Gauss map of F, then the lifts correspond respectively to the restriction to $\mathbb{S}^2$ of the ambient coordinate functions x, y and z.

4.2. Inverting the stability operator in the spaces $\mathcal{X}^k$ modulo $\mathcal{K}$

We first demonstrate that we can invert the operator $\cosh^2(s)\mathcal{L}_F$ over the space of functions orthogonal to $\mathcal{K}$.

Definition 4.4. Let

$$(\mathcal{X}^k)^\perp := \left\{ f\in\mathcal{X}^k : \int_\Lambda f\kappa\, d\mu_\Omega = 0 \text{ for all } \kappa\in\mathcal{K}\right\}.$$

We find it convenient to solve the linear problem for inhomogeneous E with $E|_{\partial\Lambda}=0$ as a few technical arguments are made easier. As a trade-off, we have to be a bit more careful in applying the linear theory to the fixed point argument in Section 5.

Proposition 4.5. *Let $(\mathcal{X}_0^0)^\perp \subset (\mathcal{X}^0)^\perp$ be the subspace of functions that vanish on $\partial\Lambda$. Then there is a bounded linear map*

$$\mathcal{R}_F^\perp : (\mathcal{X}_0^0)^\perp \to \mathcal{X}^2$$

such that for $E \in (\mathcal{X}_0^0)^\perp$,

$$\cosh^2(s)\mathcal{L}_F\mathcal{R}_F^\perp[E] = E \text{ on } \Lambda.$$

The proof of Proposition 4.5 follows in three steps which will take up the entirety of this subsection. We first show that Δ_Ω can be inverted over the space of Hölder functions that satisfy a convenient averaging property. In the second step, we show that there is a sufficiently large constant s_0 so that on $\Lambda \cap \{s > s_0\}$ the operator $\cosh^2(s)\mathcal{L}_F$ can be solved in the spaces $\mathcal{X}^k$ as a perturbation of Δ_Ω. In this way, we reduce to the case that the error term has large but fixed compact support. In the third step, we solve for the error term by lifting the problem to the sphere using the Gauss map of ν_F, where the problem reduces to an eigenvalue problem for the stability operator on the sphere.

Step 1: Inverting over the space of inhomogeneous terms with "zero average".

Definition 4.6. Let

$$\mathring{C}_{loc}^{k,\alpha}(X) := \left\{ E \in C^{k,\alpha}(X) : \int_{-\pi}^{\pi} E(s,\theta)d\theta = 0 \text{ for all } (s,\theta) \in X \subset \Omega \right\}. \tag{4.5}$$

We say that any function in the space (4.5) has **zero average along meridians**. Given a positive weight function f, we then denote

$$\mathring{C}^{k,\alpha}(X,f) := C^{k,\alpha}(X,f) \cap \mathring{C}_{loc}^{k,\alpha}(X).$$

We will prove the invertibility of Δ_Ω over $\mathring{C}^{0,\alpha}(\Lambda)$ by first considering the invertibility of the local problem.

Lemma 4.7. *Let $A_0 \subset \Lambda$ be the annulus*

$$A_0 := \Omega \cap \{|s| \le 5/8\}$$

and let $\mathring{C}_0^{0,\alpha}(A_0)$ be the space of $\mathring{C}^{0,\alpha}$ functions on Ω with support on A_0.

Given a compact set K containing A_0 and $k \geq 0$, there is a bounded linear map

$$\mathring{\mathcal{R}}_0[-] : \mathring{C}_0^{0,\alpha}(A_0) \to \mathring{C}^{2,\alpha}(\Omega) \cap \mathring{C}^k(\Omega \setminus K, \cosh^{-1}(s))$$

such that

$$\Delta_\Omega \mathring{\mathcal{R}}_0[E] = E.$$

Proof. Lemma 4.7 can be established several ways. We choose the following approach. Let Ω_L be the domain

$$\Omega_L := \Omega \cap \{|s| \leq L\}. \tag{4.6}$$

In other words Ω_L is just the flat cylinder of length $2L$ centered at the meridian $\{s = 0\}$. Standard elliptic theory gives the existence of functions $u_L \in C^{2,\alpha}_{loc}(\Omega_L)$ satisfying:

$$\Delta u_L = E, \quad u_L(\pm L, \theta) = 0. \tag{4.7}$$

We now integrate both sides of the first equality in (4.7) in θ to obtain the equality

$$\left(\int_{-\pi}^{\pi} u_L(s, \theta) d\theta \right)_{ss} = 0.$$

The boundary conditions in (4.7) then imply that

$$\int_{-\pi}^{\pi} u_L(s, \theta) d\theta = 0.$$

That is, $u_L \in \mathring{C}^{2,\alpha}_{loc}(\Omega_L)$.

Using the zero averaging property and the Poincare inequality for K we determine that

$$\sup_K |u_L| \leq C(K) \|E : \mathring{C}^{0,\alpha}(K)\| \leq C(K) \|E : \mathring{C}^{0,\alpha}(A_0)\|. \tag{4.8}$$

We then define the map

$$\mathring{\mathcal{R}}_0[E] = u_\infty(s, \theta) := \lim_{L \to \infty} u_L(s, \theta). \tag{4.9}$$

From (4.8), the limit in (4.9) exists in $C^{2,\alpha}(\Omega)$ and by continuity u_∞ solves

$$\Delta_\Omega u_\infty = E, \quad \int_{-\pi}^{\pi} u_\infty(s,\theta)d\theta = 0.$$

The exponential decay of u_∞ in both the positive and negative s directions then follows directly. $\square$

We are now ready to prove the invertibility of Δ_Ω over $\mathring{C}^{0,\alpha}_0(\Lambda)$. Following standard arguments, we will sum up the solutions to the local problem found by Lemma 4.7 and show that the estimates are sufficient to establish convergence.

Proposition 4.8. *Given $\rho \in (-1,1)\setminus\{0\}$, $k \geq 0$, $\alpha \in (0,1)$ and a compact set K containing Λ, there is a bounded linear map*

$$\mathcal{R}_{\rho\,\alpha}[-] : \mathring{C}^{0,\alpha}_0(\Lambda, \cosh^\rho(s)) \to \mathring{C}^{2,\alpha}(\Omega, \cosh^\rho(s)) \cap \mathring{C}^k(\Omega\setminus K, \ell^\rho \cosh^{-1}(s))$$

such that

$$\Delta_\Omega \mathcal{R}_{\rho\,\alpha}[E] = E.$$

Proof. Fix $\mathring{E} \in \mathring{C}^{0,\alpha}_0(\Lambda, \cosh^\rho(s))$ and set $\beta := \|\mathring{E} : \mathring{C}^{0,\alpha}_0(\Lambda, \cosh^\rho(s))\|$. For each integer i, let A_i be the annulus $A_i := A_0 + i$. Note that the set $\{A_i\}_{i\in\mathbb{Z}}$ is a locally finite covering of Ω such that $A_i \cap A_j = \emptyset$ if $|i-j| > 1$. Let $\{\psi_i\}$ be a partition of unity subordinate to $\{A_i\}$ such that $\psi_i(s+1) = \psi_{i+1}(s)$. Setting $\mathring{E}_i(s,\theta) := \psi_i(s)\mathring{E}(s-i,\theta)$, it is straightforward to verify that $\mathring{E}_i \in \mathring{C}^{0,\alpha}_0(A_0)$. The decay assumption on $\mathring{E}$ further implies that

$$\|\mathring{E}_i : \mathring{C}^{0,\alpha}_0(A_0)\| \leq C\beta\cosh^\rho(i).$$

We now define the functions

$$\mathring{u}_i(s,\theta) := \mathring{\mathcal{R}}_0[E_i](s+i,\theta).$$

From Lemma 4.7, we determine the estimates

$$\begin{aligned}\|\mathring{u}_i : C^{2,\alpha}(A_j)\| &\leq C\beta\cosh^\rho(i)\cosh^{-1}(j-i)\\ &\leq C\beta\begin{cases} e^j e^{(\rho-1)i}, & i > j\\ e^{-j}e^{(1+\rho)i}, & i \leq j.\end{cases}\end{aligned}$$

Taking finite sums, we determine that

$$\left\| \sum_{i=0}^{n} \mathring{u}_i : C^{2,\alpha}(A_j) \right\| \leq \frac{C\beta}{1-|\rho|} \cosh^\rho(j).$$

Thus, the partial sums converge to a limiting function $\mathring{u}$ with zero average along meridians satisfying

$$\Delta_\Omega \mathring{u} = \mathring{E}, \quad \|\mathring{u} : C^{2,\alpha}(A_j)\| \leq \frac{C\beta}{1-|\rho|} \cosh^\rho(j).$$

In other words $\mathring{u}$ satisfies the estimate

$$\|\mathring{u} : C^{2,\alpha}(\Omega, \cosh^\rho(s))\| \leq \frac{C}{1-|\rho|} \|\mathring{E} : C^{0,\alpha}(\Omega, \cosh^\rho(s))\|.$$

By setting $\mathcal{R}_{\rho\alpha}[\mathring{E}] := \mathring{u}$ we prove the proposition. □

Step 2: Solving $\cosh^{-2}(s)\mathcal{L}_F$ as a perturbation of Δ_Ω. We now proceed with the second step in the proof of Proposition 4.5. Notice that the following technical lemma is stated much more generally than we will need. The reader may find it helpful to reduce the statement to when $\rho = 3/4$ and $k = 3$.

Lemma 4.9. *Let $\mathcal{L} := \Delta_\Omega + P(s)$ be a second order linear operator defined on Ω, and assume that*

$$\|P(s) : C^{0,\alpha}(\Omega, 1)\| \leq \epsilon. \tag{4.10}$$

Given $\rho \in (-1, 1) \setminus \{0\}$, $k \geq 0$, $\alpha \in (0, 1)$ and a compact set K containing Λ, there exists $\epsilon_* > 0$ such that for all $0 < \epsilon < \epsilon_*$, there exists a bounded linear map

$$\mathcal{R}_{\rho\,\alpha}[\mathcal{L}, -] : \mathring{C}_0^{0,\alpha}(\Lambda, \cosh^\rho(s)) \to \mathring{C}^{2,\alpha}(\Omega, \cosh^\rho(s)) \cap \mathring{C}^k(\Omega \setminus K, \ell^\rho \cosh^{-1}(s))$$

such that

$$\mathcal{L}\mathcal{R}_{\rho\,\alpha}[\mathcal{L}, E] = E \text{ on } \Lambda.$$

Proof. Let $E \in \mathring{C}_0^{0,\alpha}(\Lambda, \cosh^\rho(s))$. We proceed by iteration. Set $u_0 := \mathcal{R}_{\rho\,\alpha}[E]$ so that $\mathcal{L}u_0 = E + P(s)u_0$. In order to apply Proposition 4.8, we introduce the cutoff function $\tilde{\psi}(s) := \psi[\operatorname{arccosh}(\ell) + 3, \operatorname{arccosh} \ell + 2](|s|)$. Set $E_1 :=$

$-\tilde{\psi}Pu_0$. Then ψE_1 is supported in a fixed neighborhood of Λ and we can apply the previous proposition on that neighborhood and get corresponding estimates. It follows that $E_1 \in C^{0,\alpha}(\Lambda)$ and satisfies the estimate

$$\|E_1 : \mathring{C}^{0,\alpha}(\Lambda, \cosh^\rho(s))\| \le C(\alpha,\rho)\epsilon \|E : \mathring{C}_0^{0,\alpha}(\Lambda, \cosh^\rho(s))\|.$$

For simplicity, let $\Lambda_1 := \{(s,\theta) \in \Omega : |s| \le \operatorname{arccosh}(\ell) + 3\}$ denote the neighborhood of interest in this proof. Taking ϵ_* sufficiently small we can achieve the estimate,

$$\|E_1 : \mathring{C}_0^{0,\alpha}(\Lambda_1, \cosh^\rho(s))\| \le \frac{1}{2}\|E : \mathring{C}_0^{0,\alpha}(\Lambda, \cosh^\rho(s))\|.$$

Inductively define the sequence (E_k, u_k) by

$$u_k = \mathcal{R}_{\rho\,\alpha}[E_k], \quad E_{k+1} = E_k - \tilde{\psi}\mathcal{L}u_k.$$

We continue to use the cutoffs so that $E_k \in \mathring{C}_0^{0,\alpha}(\Lambda_1, \cosh^\rho(s))$ for each k. Appealing to the modified version of Proposition 4.8, we get the estimates

$$\begin{aligned}&\|u_i : \mathring{C}^{2,\alpha}(\Omega, \cosh^\rho(s)) \cap \mathring{C}^k(\Omega \setminus K, \ell^\rho \cosh^{-1}(s))\| \\ &\le C\|E_i : \mathring{C}_0^{0,\alpha}(\Lambda_1, \cosh^\rho(s))\| \le C2^{-i}.\end{aligned}$$

The partial sums $v_n := \sum_{i=0}^n u_i$ satisfy $\tilde{\psi}\mathcal{L}v_n = E - E_{n+1}$ on Λ and

$$\begin{aligned}&\|v_n : \mathring{C}^{2,\alpha}(\Omega, \cosh^\rho(s)) \cap \mathring{C}^k(\Omega \setminus K, \ell^\rho \cosh^{-1}(s))\| \\ &\le C_0\|E : \mathring{C}_0^{0,\alpha}(\Lambda, \cosh^\rho(s))\|\end{aligned}$$

where C_0 is a uniform constant independent of n. The limit $u := \lim_{n\to\infty} v_n$ then exists in $C^{2,\alpha}_{loc}$ and by continuity satisfies

$$\tilde{\psi}\mathcal{L}u := E, \quad \int_{-\pi}^{\pi} \tilde{\psi}u(s,\theta)d\theta = 0 \text{ on } \Lambda.$$

Moreover, u satisfies the appropriate weighted Hölder estimates. Since $\mathcal{L}(\tilde{\psi}u) - \tilde{\psi}\mathcal{L}u$ on Λ, the conclusion therefore follows by setting $\tilde{\psi}u := \mathcal{R}_{\rho\,\alpha}[\mathcal{L}, E]$. □

As the operator $\cosh^2(s)\mathcal{L}_F$ does not satisfy the condition (4.10), to use a perturbation technique we must modify this operator by cutting off the curvature term for s near zero.

Definition 4.10. Given $a \in \mathbb{R}$, we define the operator $\hat{\mathcal{L}}_{F\,a}$ as follows:

$$\hat{\mathcal{L}}_{F\,a} := \Delta_\Omega + 2\psi[a-1,a](|s|)\cosh^{-2}(s).$$

From the definition and the properties of $\mathcal{L}_F$, it immediately follows that

Lemma 4.11. 1) *$\hat{\mathcal{L}}_{F\,a} = \cosh^2(s)\mathcal{L}_F$ on $\Omega \cap \{|s| \geq a\}$.*

2) *For any $\epsilon > 0$ there exists an a sufficiently large so that*

$$\|\hat{\mathcal{L}}_{F\,a} - \Delta_\Omega, C^{0,\alpha}(\Omega, \cosh^{-1}(s))\| \leq \epsilon.$$

3) *The operator $\hat{\mathcal{L}}_{F\,a}$ preserves the class of functions with zero average over meridian circles on Ω. In other words, given $f \in \mathring{C}^{k,\alpha}(\Omega)$, $\hat{\mathcal{L}}_{F\,a} f \in \mathring{C}^{k-2,\alpha}(\Omega)$.*

We are now ready to quantify the error induced by inverting $\hat{\mathcal{L}}_{F\,a}$ rather than $\cosh^2(s)\mathcal{L}_F$. We describe the error via the following map. Note that throughout what follows, in all applications we will use $\rho = 3/4, \alpha = 3/4$.

Definition 4.12. The map $\mathfrak{T}[-] : \mathring{\mathcal{X}}_0^k \to C^{k,3/4}(\Omega, \cosh^{3/4}(s))$ is given as follows:

$$\mathfrak{T}[E](s,\theta) := E - \hat{\psi}(s)\cosh^2(s)\mathcal{L}_F(s)\mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E](s,\theta).$$

Here the function $\hat{\psi}(s) := \psi[\operatorname{arccosh}(\ell) + 2, \operatorname{arccosh}\ell + 1](|s|)$ is a fixed cutoff function. We choose $\hat{\psi}(s)$ so that $\hat{\psi} \equiv 1$ on Λ and $\operatorname{supp}(\hat{\psi}) \subset \{(s,\theta) \in \Lambda_1 : \tilde{\psi} \equiv 1\}$. Such a choice of $\hat{\psi}$ removes from $\mathfrak{T}$ the small error induced in Lemma 4.9 that comes from the cutoff $\tilde{\psi}$. In fact $\hat{\psi}$ cuts off the second term in a neighborhood where that term is identically zero. Thus, derivatives of $\hat{\psi}$ will not contribute to the estimates below in any way.

Proposition 4.13. *The following statements hold:*

(1) *For a sufficiently large, $\mathfrak{T}[-]$ is well-defined.*

(2) *$\mathfrak{T}[E]$ is compactly supported on the set $\Lambda \cap \{|s| \leq a\}$.*

(3) *There is a constant $C(\rho)$ depending only on $\rho = 3/4$ so that*

$$\|\mathfrak{T}[E] : C^{0,3/4}(\Lambda \cap \{|s| \leq a\}, 1)\| \leq C\|E : \mathcal{X}^0\|.$$

(4) *$\mathfrak{T}$ maps $(\mathring{\mathcal{X}}_0^0)^\perp$ into $(\mathring{\mathcal{X}}_0^0)^\perp$ where $E \in (\mathring{\mathcal{X}}_0^0)^\perp$ if $E \in (\mathcal{X}_0^0)^\perp$ and E has zero radial average along meridian circles.*

Proof. Statements (1) and (2) are obvious from Definition 4.12. Statement (3) follows directly from Lemma 4.9 and the definition of the maps $\mathcal{R}_{\rho\,\alpha}[\mathcal{L}, -]$. To see (4), note first that the zero average condition on meridians is preserved by the definition of $\mathfrak{T}[-]$, Lemma 4.9 and the definition of $\hat{\mathcal{L}}_{F\,a}$. We then immediately get that

$$\int_\Lambda \mathfrak{T}[E]\kappa_z d\mu_\Omega = 0,$$

since $\kappa_z = \kappa_z(s)$ is θ-independent. Additionally, we have that

$$\int_\Lambda \mathfrak{T}[E]\kappa_x d\mu_\Omega = \int_\Lambda E\kappa_x d\mu_\Omega - \int_\Lambda \mathcal{L}_F \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]\kappa_x d\mu_F$$

Note that by assumption the first term above is zero when E is in $(\mathcal{X}^k)^\perp$. Considering the support of E and the definition of Λ, for $L \geq \operatorname{arccosh}(\ell)$, $\Lambda \subset \Omega_L$. (Recall the definition of Ω_L in (4.6).) By our choice of $\hat{\psi}$ and the properties of the function $\mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]$, we note that

$$\begin{aligned}\int_\Lambda \mathcal{L}_F \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]\kappa_x d\mu_F &= \lim_{L\to\infty} \int_{\Omega_L} \hat{\psi} \mathcal{L}_F \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]\kappa_x d\mu_F \\ &= \lim_{L\to\infty} \int_{\Omega_L} \mathcal{L}_F \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]\kappa_x d\mu_F \\ &= \lim_{L\to\infty} \int_{-\pi}^{\pi} \Big(\kappa_x(L,\theta)\nabla_s \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E](L,\theta) \\ &\qquad - \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E](L,\theta)\nabla_s\kappa_x(L,\theta)\Big)d\theta,\end{aligned}$$

where we have used that $\mathcal{L}_F\kappa_x = 0$. Considering the growth rates of $\mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, E]$ and κ_x, we see the right hand side above is equal to zero. The claim then follows immediately. □

Step 3: Solving for the error term by lifting to the sphere. We are now ready to finish the proof of Proposition 4.5. Let E be a function in $(\mathcal{X}^0_0)^\perp$ and set $\beta := \|E : (\mathcal{X}^0)^\perp\|$. Set

$$\bar{E}(s) := \frac{1}{2\pi}\int_{-\pi}^{\pi} E(s,\theta)d\theta, \quad \mathring{E}(s,\theta) := E(s,\theta) - \bar{E}(s).$$

It is then straightforward to verify that $\mathring{E}$ is in $(\mathring{\mathcal{X}}^0_0)^\perp$ and

$$\|\mathring{E} : (\mathring{\mathcal{X}}^0_0)^\perp\| \leq C\beta.$$

The equation

$$\cosh^2(s)\mathcal{L}_F u = \mathfrak{T}[\mathring{E}]$$

on Λ is equivalent to

$$(\Delta_{\mathbb{S}^2} + 2)\, u = 2\mathfrak{T}[\mathring{E}]/(|A_F|^2 \cosh^2(s)) \tag{4.11}$$

on the sphere, where we have abused notation slightly and identified functions with their lifts to the sphere under the Gauss map of F. Since $\mathfrak{T}[\mathring{E}]$ is supported on the set $\Omega \cap \{|s| \leq a\}$,

$$\|\mathfrak{T}[\mathring{E}]/(|A_F|^2 \cosh^2(s)) : L^2(\mathbb{S}^2)\| \leq C\|\mathfrak{T}[\mathring{E}]/|A_F|^2 : (\mathring{\mathcal{X}}^0)^\perp\| \leq C\beta.$$

Moreover, by Proposition 4.13 (4), $\mathfrak{T}[\mathring{E}]/(|A_F|^2 \cosh^2(s))$ is orthogonal to the kernel of $\Delta_{\mathbb{S}^2} + 2$. (Recall that the kernel is spanned by the coordinate functions restricted to $\mathbb{S}^2$.) Thus, there is a solution u to (4.11) satisfying

$$\|u : W^{2,2}(\mathbb{S}^2)\| \leq C\beta.$$

Standard elliptic theory then implies that

$$\|u\|_{2,\alpha} \leq C\|\mathring{E}\|_{0,\alpha}.$$

Define the function

$$\mathring{v} := u + \mathcal{R}_{\rho\,\alpha}[\hat{\mathcal{L}}_{F\,a}, \mathring{E}].$$

Then $\mathring{v}$ satisfies $\cosh^2(s)\mathcal{L}_F\mathring{v} = \mathring{E}$ on Λ and the estimate

$$\|\mathring{v} : \mathring{\mathcal{X}}^2\| \leq C\|\mathring{E} : (\mathring{\mathcal{X}}^0)^\perp\|.$$

We then solve for the function $\bar{E}$ by direct integration. In particular, in [2] we proved that the expression

$$\bar{v}(s) := \left(\int_0^s \tanh^{-2}(s) \int_0^{s'} \tanh(s'') \cosh^{-2}(s)\bar{E}(s'')ds''ds'\right) \tanh(s) \tag{4.12}$$

satisfies $\cosh^2(s)\mathcal{L}_F\bar{v} = \bar{E}$. Moreover, by Lemma 8 of [2], making the appropriate modifications for the norms of interest here, we establish there exists a uniform $C > 0$ such that

$$\|\bar{v} : \mathcal{X}^2\| \leq C\|\bar{E} : \mathcal{X}^0\|.$$

We conclude the proof of Proposition 4.5 by setting

$$v := \bar{v} + \mathring{v} \text{ and } \mathcal{R}_F^{\perp}[E] := v.$$

4.3. Solving the linear problem on $\mathcal{X}_0^0$

We now solve the more general linear problem by decomposing any $E \in \mathcal{X}_0^0$ into a θ independent function and a function $\mathring{E}$, with zero average along meridian circles. We invert the θ independent function directly using (4.12). To invert $\mathring{E}$, we modify the function by its projection onto the space $\mathcal{K}$. The resulting function is orthogonal to $\mathcal{K}$ and thus can be inverted via Proposition 4.5.

To begin, we define functions that, as graphs over the helicoid, will induce linear error that can be used to modify the inhomogeneous term.

Definition 4.14. Let $\psi(s)$ be the cutoff function $\psi(s) := \psi[1,2](|s|)$ and set

$$u_x(s,\theta) = \frac{1}{4\pi}\psi(s)\cos(\theta)\cosh(s),$$
$$u_y(s,\theta) = \frac{1}{4\pi}\psi(s)\sin(\theta)\cosh(s), \quad u_z(s,\theta) = \frac{1}{4\pi}\psi(s)\,|s|.$$

We denote the linear error in the mean curvature due to adding the graphs of u_x, u_y and u_z by

$$w_x := \cosh^2(s)\mathcal{L}_F u_x, \quad w_y := \cosh^2(s)\mathcal{L}_F u_y, \quad w_z := \cosh^2(s)\mathcal{L}_F u_z.$$

Note that we chose the functions w_x, w_y, w_z so that the projection of E onto $\mathcal{K}$ can be captured by linear combinations of the functions w.

Proposition 4.15. *For $i, j \in \{x, y, z\}$,*

$$\int_{\Omega} \kappa_i w_j \, d\mu_{\Omega} = \delta_{ij}.$$

Proof. Let N be a large constant and set $D := D_N = \Omega \cap \{|s| \leq N\}$ and $\partial = \partial_D := F(|s| = N)$. Then

$$\int_D \kappa_i w_j \, d\mu_\Omega = \int_D \kappa_i \mathcal{L}_F u_j \, d\mu_F = \int_{\partial D} \kappa_i \nabla^F_\eta u_j \, d\mu_\partial - u_j \nabla^F_\eta \kappa_i \, d\mu_\partial \tag{4.13}$$

where ∇^F denotes the surface gradient on F and where η is the outward pointing conormal at ∂D. We do not write in the $\cosh^2(s)$ term in the second integral since $\cosh^2(s)\, d\mu_\Omega = d\mu_F$. Notice that ∂D consists of four components. From (4.4), on the components $(\pm N, \theta)$, $\nabla^F_\eta := \pm \cosh^{-1}(N)\partial_s$ for $\pm s > 0$ and $d\mu_\partial = \cosh(N) d\theta$. On the components $(s, \pm\pi)$, $\nabla^F_\eta := \pm \cosh^{-1}(s)\partial_\theta$ for $\pm\theta > 0$ and $d\mu_\partial = \cosh(s) ds$.

We first consider the case $i = j$. Observe that for $\theta = \pm\pi$, $u_i \partial_\theta \kappa_i = \kappa_i \partial_\theta u_i = 0$ for $i \in \{x, y, z\}$. Thus, we are only concerned with the boundary components $s = \pm N$. Thus

$$\begin{aligned}
4\pi \int_D \kappa_x w_x \, d\mu_\Omega &= \int_{-\pi}^{\pi} \cos^2(\theta) \tanh(N) d\theta - \int_{-\pi}^{\pi} - \cos^2(\theta) \tanh(N) d\theta \\
&\quad + \int_{-\pi}^{\pi} \cos^2(\theta)(-\tanh(-N)) d\theta - \int_{-\pi}^{\pi} \cos^2(\theta) \tanh(-N) d\theta \\
&= 4 \tanh(N) \int_{-\pi}^{\pi} \cos^2(\theta) d\theta \to 4\pi.
\end{aligned}$$

The same estimate follows for $\int_\Omega \kappa_y w_y \, d\mu_F$. For $w = w_z$ and $\kappa = \kappa_z$, on $s = \pm N$ we have that

$$\nabla^F_\eta \kappa_z = \cosh^{-3}(N).$$

In this case, the second term on the right hand side of (4.13) converges to zero as N goes to infinity, and we note that

$$4\pi \int_D \kappa_z w_z \, d\mu_\Omega = \int_{-\pi}^{\pi} \tanh(N) d\theta - \int_{-\pi}^{\pi} \tanh(-N) d\theta \to 4\pi.$$

For the case $i \neq j$, one can easily show that the boundary curves defined by $s = \pm N$ will not contribute to the integral. Indeed $\int_{-\pi}^{\pi} u_i \partial_s \kappa_j \, d\theta = \int_{-\pi}^{\pi} \kappa_j \partial_s u_i \, d\theta = 0$ for all instances of $i \neq j$ since the integrand will contain either $\sin\theta \cos\theta$, $\sin\theta$, or $\cos\theta$. We demonstrate why the other integrals vanish for the case $i = x, j = y$ and leave further cases to the reader.

Using again (4.4), the formulas imply that

$$4\pi \int_D \kappa_y w_x \, d\mu_\Omega = \int_{-N}^{N} \left(-\sin^2\theta - \cos^2\theta\right) \psi(s)\, ds + \int_{-N}^{N} -\left(-\sin^2\theta - \cos^2\theta\right) \psi(s)\, ds = 0$$

□

Now that we understand the modifying functions, we can solve the linear problem for inhomogeneous E with non-trivial projection onto $\mathcal{K}$.

Proposition 4.16. *There is a bounded linear map*

$$\mathcal{R}_F[-] : \mathcal{X}_0^0 \to \mathcal{X}^2 \times \mathbb{R}^2$$

such that for $E \in \mathcal{X}_0^0$ *and* $(v, b_x, b_y) := \mathcal{R}_F[E]$*,*

$$\cosh^2(s)\mathcal{L}_F v = E - b_x w_x - b_y w_y \text{ on } \Lambda.$$

Proof. Given $E \in \mathcal{X}_0^0$, we set

$$\bar{E}(s) := \int_{-\pi}^{\pi} E(s,\theta) d\theta, \quad \mathring{E}(s,\theta) = E(s,\theta) - \bar{E}(s).$$

Then $\mathring{E}$ is in $\mathring{\mathcal{X}}_0^0$. Since κ_z is θ-independent and $\mathring{E}$ has zero average along meridians,

$$\int_\Lambda \mathring{E}\, \kappa_z \, d\mu_\Omega = 0.$$

Moreover, the definition of the function space $\mathcal{X}^0$ and the definition of Λ together imply

$$\left| \int_\Lambda \mathring{E}\, \kappa_x \, d\mu_\Omega \right| + \left| \int_\Lambda \mathring{E}\, \kappa_y \, d\mu_\Omega \right| \le C \| \mathring{E} : \mathcal{X}^0 \|.$$

Thus, there exist constants b_x and b_y with

$$|b_x|, \quad |b_y| \le C\|E : \mathcal{X}^0\|$$

so that $E^\perp := \mathring{E} - b_x w_x - b_y w_y$ lies in $(\mathcal{X}_0^0)^\perp$. Using Proposition 4.5 we define the function

$$v^\perp := \mathcal{R}_F^\perp[E^\perp].$$

We then use (4.12) to determine a function $\bar{v}$. We now define $v := v^\perp + \bar{v}$. By definition, v satisfies

$$\cosh^2(s)\mathcal{L}_F v = E - b_x w_x - b_y w_y \text{ on } \Lambda$$

and the estimates imply that

$$\|v : \mathcal{X}^2\| + |b_x| + |b_y| \leq C\|E : \mathcal{X}_0\|. \qquad \square$$

5. Finding exact solutions

Recall that finding a graph over G so that the resulting surface has $H = 0$ is equivalent to finding a function $u \in C^{2,\alpha}$ with $Q[u] = 0$. We will solve this problem via standard gluing methods, invoking a fixed point theorem for a given map Ψ from some Banach space we designate. Estimates for the map will require a strong understanding of $Q[u]$ and to that end we first consider a natural decomposition of Q.

Definition 5.1. Let $\mathcal{L}_Q[u]$ denote the linearization of the operator Q at 0, and set

$$R_Q[u] := Q[u] - Q[0] - \mathcal{L}_Q[u].$$

We first record linear estimates, which are controlled by properties of the immersions F and G.

Lemma 5.2. *Given $\tau_0, \xi \in \mathbb{R}$ and $\kappa_0 > 0$, choose $\delta > 0$ such that $C(1 + |\mathsf{T}| + \xi)\delta < \tilde{\epsilon}$ where $\tilde{\epsilon}$ is from Proposition 2.11 and C is a universal constant arising from the norm bounds in the linear problem. Then for any G defined according to a curve $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi] \in \mathfrak{L}$,*

$$\|Q[0] : \mathcal{X}^0\| \leq C\delta\ell^{1/4}|\mathsf{T}| \tag{5.1}$$

and for $u \in \mathcal{X}^2$,

$$\|\mathcal{L}_Q[u] - \cosh^2(s)\mathcal{L}_F u : \mathcal{X}^0\| \leq C\delta(1 + |\mathsf{T}| + |\xi|)\|u : \mathcal{X}^2\|. \tag{5.2}$$

Proof. Proposition 3.11 implies

$$\|Q[0]\|_{j,3/4}(s,\theta) < C\delta|\mathsf{T}|\cosh(s)$$

and thus

$$\cosh^{-3/4}(s)\|Q[0]\|_{j,3/4}(s,\theta) \le C\delta|\mathsf{T}|\cosh^{1-3/4}(s) \le C\delta\ell^{1/4}|\mathsf{T}|.$$

For the second estimate, recall that

$$Q[u] = \cosh^2(s)H(\tilde{\underline{\nabla}}[G] + \mathcal{E}_\delta[u_0+u])$$

Then for $\underline{\nabla}_0 := \tilde{\underline{\nabla}}[G] + \mathcal{E}_\delta[u_0]$,

$$\begin{aligned}\mathcal{L}_Q[u] &= \dot{Q}[u] = \cosh^2(s)\ DH|_{\underline{\nabla}_0}(\mathcal{E}_\delta[u])\\ \mathcal{L}_F u &= \ DH|_{\underline{\nabla}F}(\underline{\nabla}[u\,\nu_F]).\end{aligned}$$

We then write

$$\begin{aligned}\mathcal{L}_Q[u] - \cosh^2(s)\mathcal{L}_F u &= \cosh^2(s)\left\{DH|_{\underline{\nabla}_0}(\mathcal{E}_\delta[u]) - DH|_{\underline{\nabla}F}(\mathcal{E}_\delta[u])\right\}\\ &\quad - \cosh^2(s)\ DH|_{\underline{\nabla}F}(\underline{\nabla}[u\,\nu_F] - \mathcal{E}_\delta[u])\\ &:= I + II.\end{aligned}$$

Lemma 3.4, (3.4) and (3.5), and the triangle inequality imply

$$\|\tilde{\underline{\nabla}}[G] - \underline{\nabla}F : C^k(\Lambda,\cosh(s))\| \le C\delta + C\delta|\mathsf{T}|. \tag{5.3}$$

Moreover, the estimate for u_0 in Theorem 3.3 (with $\delta\xi$ replacing δ) and Definition 3.8 imply

$$\|\mathcal{E}_\delta[u_0] : C^{0,\alpha}(\Lambda,\cosh(s))\| \le C\delta|\xi|. \tag{5.4}$$

Combining (5.3) and (5.4), we note that

$$\|\underline{\nabla}_0 - \underline{\nabla}F : C^{0,\alpha}(\Lambda,\cosh(s))\| \le C\delta(1+|\mathsf{T}|+|\xi|) < \tilde{\epsilon}.$$

Thus we may apply Proposition 2.11 with $\underline{\nabla} = \underline{\nabla}F$ and $\mathcal{E} = \underline{\nabla}_0 - \underline{\nabla}F$. Notice here that $\mathfrak{a}(\underline{\nabla}) = 1$, $|\nabla| = \cosh(s)$, and $d = -2$. Thus

$$\|I\|_{0,\alpha} \le C\cosh^{-1}(s)\|\mathcal{E}\|_{0,\alpha}\|u\|_{2,\alpha} \le C\delta\cosh^{-1}(s)(1+|\mathsf{T}|+|\xi|)\|u\|_{2,\alpha}.$$

Again using Definition 3.8 and estimates on the derivatives of ν_G, we observe that

$$\|\underline{\nabla}[u\,\nu_F] - \mathcal{E}_\delta[u]\|_{0,\alpha} \leq C\delta(1+|\mathsf{T}|)\|u\|_{2,\alpha}$$

and it follows that $\|II\|_{0,\alpha} \leq C\delta(1+|\mathsf{T}|)\|u\|_{2,\alpha}$. □

We now define the Banach space on which we will solve the fixed point theorem.

Definition 5.3. For $\zeta \gg 1$, set

$$\Xi := \{(u, b_x, b_y) \in \mathcal{X}^2 \times \mathbb{R}^2 : \|u : \mathcal{X}^2\| \leq \zeta\delta\ell^{1/4}|\mathsf{T}|, |b_x|, |b_y| \leq \zeta\delta\ell^{1/4}|\mathsf{T}|\}.$$

Notice that by definition, Ξ is a compact, convex subset of $\mathcal{X}^2 \times \mathbb{R}^2$. Thus, we are in a setting where it is natural to apply Schauder's fixed point theorem.

Now we control the higher order terms of $Q[u]$ where $u = v + b_x u_x + b_y u_y$ with $(v, b_x, b_y) \in \Xi$.

Proposition 5.4. *Given $\zeta \gg 1$, $\tau_0, \xi \in \mathbb{R}$ and $\kappa_0 > 0$, choose any $0 < \delta|\xi| < \delta_0$ and $\ell > 16$ such that $C\delta(1 + |\mathsf{T}| + |\xi|)\ell^{5/4} < \min\{1/(4\zeta), \tilde{\epsilon}/\zeta\}$. Here C is a universal constant arising from the norm bounds in the linear problem and δ_0 comes from Theorem 3.3. Consider any G defined according to a curve $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi] \in \mathfrak{L}$. Given $(v, b_x, b_y) \in \Xi$, and $f := v + b_x u_x + b_y u_y$*

$$\|R_Q[f] : \mathcal{X}^0\| \leq C\zeta^2\delta^2|\mathsf{T}|^2\ell^{3/4}.$$

Proof. The definition of $R_Q[f]$ implies that

$$R_Q[f] = \cosh^2(s) R^{(1)}_{H,\mathcal{E}_\delta[f]}(\underline{\nabla}_0).$$

From Definition 3.8,

$$\begin{aligned}
\|\mathcal{E}_\delta[f]\|_{0,\alpha} &\leq C\|f\|_{2,\alpha} \leq C\|v\|_{2,\alpha} + (|b_x| + |b_y|)\cosh(s) \\
&\leq C\zeta\delta|\mathsf{T}|\ell^{1/4}\left(\cosh^{3/4}(s) + \cosh(s)\right) \\
&\leq C\zeta\delta|\mathsf{T}|\ell^{1/4}\cosh(s).
\end{aligned}$$

Applying Proposition 2.11 with $\underline{\nabla} = \underline{\nabla}_0$ and $\mathcal{E} = \mathcal{E}_\delta[f]$, $\Phi = H$ and $d = -1$ implies that

$$\cosh(s)\|R_Q[f]\|_{0,\alpha} \leq C\zeta^2\delta^2|\mathsf{T}|^2\ell^{1/2}\cosh^2(s).$$

As $|s| \leq \operatorname{arccosh}(\ell)$ on Λ, the estimate immediately follows. □

Using all of the previous estimates, we define a map Ψ that takes Ξ into Ξ. As previously mentioned, because we chose to simplify the linear problem by considering only inhomogeneous terms with zero boundary data on Λ, we now have a small amount of technical work to do. In particular, to invert $Q[u]$, we need to first cut it off by a function ψ'.

Proposition 5.5. *For* $\psi(s):=\psi[\operatorname{arccosh}(\ell/2), \operatorname{arccosh}(\ell/4)](|s|)$*, and* $\psi'(s) := \psi[\operatorname{arccosh}(\ell), \operatorname{arccosh}(\ell/2)](|s|)$*, set*

$$\Psi[v, b_1, b_2] = (\psi v, b_1, b_2) - \mathcal{R}_F[\psi' Q[\psi v + b_1 u_x + b_2 u_y]]$$

Given $\zeta \gg 1$*,* $\tau_0, \xi \in \mathbb{R}$ *and* $\kappa_0 > 0$*, for any* $0 < \delta|\xi| < \delta_0$ *and* $\ell > 16$ *such that* $C\delta(1 + |\mathsf{T}| + |\xi|)\ell^{1/2} < \min\{1/(4\zeta), \tilde{\epsilon}/\zeta\}$*,* $\Psi(\Xi) \subset \Xi$*. (Again, we consider any* G *defined according to a curve* $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi] \in \mathfrak{L}$*.)*

Proof. We begin by noting that

$$\begin{aligned} Q[\psi v + b_1 u_x + b_1 u_y] &= Q[0] + \mathcal{L}_Q[\psi v + b_1 u_x + b_2 u_y] \\ &\quad + R_Q[\psi v + b_1 u_x + b_2 u_y] \\ &= Q[0] + \cosh^2(s)\mathcal{L}_F(\psi v + b_1 u_x + b_2 u_y) \\ &\quad + R_Q[\psi v + b_1 u_x + b_2 u_y] \\ &\quad + \left(\mathcal{L}_Q - \cosh^2(s)\mathcal{L}_F\right)(\psi v + b_1 u_x + b_2 u_y). \end{aligned}$$

First define $(u^0, b_x^0, b_y^0) := \mathcal{R}_F[\psi' Q[0]]$. Proposition 4.16, the estimate (5.1), and the uniform bounds on ψ' imply that

$$\|u^0 : \mathcal{X}^2\| + |b_x^0| + |b_y^0| \leq C\|\psi' Q[0] : \mathcal{X}_0^0\| \leq C\delta|\mathsf{T}|\ell^{1/4}. \tag{5.5}$$

Set $(u', b_x', b_y') := \mathcal{R}_F[\psi' R_Q[\psi v + b_1 u_x + b_2 u_y]]$. Then, Propositions 4.16 and 5.4 imply that

$$\begin{aligned} &\|u' : \mathcal{X}^2\| + |b_x'| + |b_y'| \\ &\leq C\|\psi' R_Q[\psi v + b_1 u_x + b_2 u_y] : \mathcal{X}_0^0\| \leq C\zeta^2\delta^2|\mathsf{T}|^2\ell^{3/4}. \end{aligned} \tag{5.6}$$

Finally, set

$$R'' := \left(\mathcal{L}_Q - \cosh^2(s)\mathcal{L}_F\right)(\psi v + b_1 u_x + b_2 u_y)$$

and $(u'', b''_x, b''_y) := \mathcal{R}_F[\psi' R'']$. By Proposition 4.16, (5.2), the estimates for v, b_1, b_2, and the decay control on ψ,

$$\begin{aligned}(5.7)\quad \|u'' : \mathcal{X}^2\| + |b''_x| + |b''_y| &\le C\|R'' : \mathcal{X}^0_0\| \\ &\le C\delta^2(1 + |\mathsf{T}| + |\xi|)\zeta|\mathsf{T}|\ell^{1/2} \le C\delta\zeta|\mathsf{T}|\ell^{1/2}.\end{aligned}$$

The definitions of u_x, u_y imply that $\mathcal{L}_F(b_1 u_x + b_2 u_y) = 0$ for all $|s| \ge 2$. Moreover, as $\psi v \equiv 0$ for all s with $\cosh(s) > \ell/2$, $\psi' \mathcal{L}_F(\psi v + b_1 u_x + b_2 u_y) = \mathcal{L}_F(\psi v + b_1 u_x + b_2 u_y)$. In other words,

$$\mathcal{R}_F[\psi' \mathcal{L}_F(\psi v + b_1 u_x + b_2 u_y)] = (\psi v, b_1, b_2).$$

Therefore

$$\Psi(v, b_1, b_2) = (-u^0 - u' - u'', -b^0_x - b'_x - b''_x, -b^0_y - b'_y - b''_y).$$

Combining (5.5), (5.6) and (5.7), as long as $\zeta > 4C$, the hypothesis on δ allows us to conclude that

$$\begin{aligned}(5.8)\quad \| - u_0 - u' - u'' : \mathcal{X}^2\| &\le C\delta|\mathsf{T}|\ell^{1/4} + C\zeta^2\delta^2|\mathsf{T}|^2\ell^{3/4} + C\zeta\delta|\mathsf{T}|\ell^{1/2} \\ &\le 3\zeta\delta\ell^{1/4}|\mathsf{T}|/4\end{aligned}$$

Similar estimates hold for the coefficients of u_x, u_y:

$$(5.9)\qquad |b^0_x - b'_x - b''_x| \le 3\zeta\delta\ell^{1/4}|\mathsf{T}|/4, \quad |b^0_y - b'_y - b''_y| \le 3\zeta\delta\ell^{1/4}|\mathsf{T}|/4.$$

□

6. The Main Theorem

We prove the main theorem via a fixed point argument and all of the necessary estimates have now been recorded. We require two further technical propositions which allows us to prove embeddedness in the main theorem. The first proposition demonstrates the size of the neighborhood of γ on which M acts as a diffeomorphism. The second proposition demonstrates that we may consider graphs over G as coming from graphs over F.

Proposition 6.1. *Given $\delta > 0$ and $\gamma \in \mathfrak{L}$ given by $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi]$ as in* (3.1)*, for any α satisfying*

$$0 \le \alpha < \frac{e^{\pi\xi/|\mathsf{T}|} - 1}{e^{\pi\xi/|\mathsf{T}|} + 1},$$

recalling $|\mathsf{T}|^2 = \kappa_0^2 + \tau_0^2$, the map M defined in (1.3) *is a diffeomorphism of the tubular neighborhood*

$$T(\alpha) := \left\{ (x, y, z) : x^2 + y^2 \le \frac{\alpha^2}{\delta^2\xi^2} \frac{\tau_0^2 + \xi^2}{|\mathsf{T}|^2 + \xi^2} \right\}$$

onto its image.

Proof. Note that

$$\begin{aligned} DM|_{(x,y,z)} = e^{\delta\xi z}\big(&\mathbf{e}_x^* \otimes N + \mathbf{e}_y^* \otimes B \\ &+ \mathbf{e}_z^* \otimes T + \delta\, x\mathbf{e}_z^* \otimes N + \delta\, y\mathbf{e}_z^* \otimes B\big). \end{aligned} \tag{6.1}$$

Thus, det $DM|_{(x,y,z)} = e^{3\delta\xi z}$ and M is a local diffeomorphism at each point.

We now show M is a global diffeomorphism. First note that for each $z = c$, $M(x, y, c)$ is a disk contained in a plane through the origin with normal direction $T(c)$. Now suppose that $M(x_1, y_1, z_1) = M(x_2, y_2, z_2) = \mathbf{p}$. Then $\mathbf{p} \cdot T(z_1) = \mathbf{p} \cdot T(z_2) = 0$. The definition of γ implies that $T(z) \cdot \mathbf{e}_z$ is constant and thus $\mathbf{p} \cdot (T(z_1) - T(z_2)) = 0$ is a condition on radial values (i.e., the values in the directions $\mathbf{e}_x, \mathbf{e}_y$). We immediately conclude that $T(z_1) = \pm T(z_2)$. Thus, the periodicity for γ' coming from (3.2) implies that up to reordering z_1 and z_2,

$$z_1 - z_2 = \pi n/(\delta|\mathsf{T}|) \text{ for } n \in \mathbb{Z}^+.$$

Using the properties of the embedding γ described in (3.1), we observe that

$$|M(0, 0, z)| = |\gamma(z)| = \frac{e^{\delta\xi z}}{\delta\xi} \sqrt{\frac{\tau_0^2 + \xi^2}{|\mathsf{T}|^2 + \xi^2}}.$$

Therefore

$$\begin{aligned}|M(0,0,z+n\pi/(\delta|\mathsf{T}|)) - M(0,0,z)| &\geq \left(e^{n\pi\xi/|\mathsf{T}|}-1\right)|M(0,0,z)| \\ &\geq \left(e^{n\pi\xi/|\mathsf{T}|}-1\right)\frac{e^{\delta\xi z}}{\delta\xi}\sqrt{\frac{\tau_0^2+\xi^2}{|\mathsf{T}|^2+\xi^2}} \\ |M(x,y,z)-M(0,0,z)| &= e^{\delta\xi z}\sqrt{x^2+y^2}.\end{aligned}$$

Presuming that $\sqrt{x^2+y^2} \leq \frac{\alpha}{\delta\xi}\sqrt{\frac{\tau_0^2+\xi^2}{|\mathsf{T}|^2+\xi^2}}$ for some α to be determined, we note that

$$\begin{aligned}&|M(x_1,y_1,z+n\pi/(\delta|\mathsf{T}|)) - M(x_0,y_0,z)| \\ \geq\ &-|M(x_1,y_1,z+n\pi/(\delta|\mathsf{T}|)) - M(0,0,z+n\pi/(\delta|\mathsf{T}|))| \\ &-|M(x_0,y_0,z)-M(0,0,z)| \\ &+|M(0,0,z+n\pi/(\delta|\mathsf{T}|)) - M(0,0,z)| \\ \geq\ &\frac{e^{\delta\xi z}}{\delta\xi}\sqrt{\frac{\tau_0^2+\xi^2}{|\mathsf{T}|^2+\xi^2}}\left[\left(e^{n\pi\xi/|\mathsf{T}|}-1\right)-\alpha(1+e^{n\pi\xi/|\mathsf{T}|})\right].\end{aligned}$$

Thus, requiring that

$$\left(e^{n\pi\xi/|\mathsf{T}|}-1\right)-\alpha(1+e^{n\pi\xi/|\mathsf{T}|}) > 0$$

gives that the points $M(x_1,y_1,z+n\pi/(\delta|\mathsf{T}|))$ and $M(x_0,y_0,z)$ do not intersect. This completes the proof. □

Proposition 6.2. *There is a constant $\epsilon_F > 0$ so that: Given constants b_x and b_y and a vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^3$, satisfying*

$$|b_x|,\ |b_y| \leq \epsilon_F, \quad \|\mathbf{v} : C^1(\mathbb{R}^2)\| \leq \epsilon_F,$$

the surface $F + \mathbf{v} + (b_x u_x + b_y u_y)\nu_F$ is an embedded surface in $\mathbb{R}^3$.

Proof. In the following, we refer to the angle that the projection of a vector onto the plane $\{z=0\}$ makes with the vector $\mathbf{e}_x$ as the argument, and for a vector $\mathbf{v}$ we denote it by $Arg(\mathbf{v})$. Recall that

$$\begin{aligned}F(s,\theta) &= \sinh(s)\mathbf{e}_r(\theta) + \theta\mathbf{e}_z \\ \nu_F(s,\theta) &= -\cosh^{-1}(s)\mathbf{e}_r^{\perp}(\theta) + \tanh(s)\mathbf{e}_z\end{aligned} \tag{6.2}$$

where we have denoted $\mathbf{e}_r(\theta) := \sin(\theta)\mathbf{e}_x + \cos(\theta)\mathbf{e}_y$ and $\mathbf{e}_r^\perp(\theta) = \cos(\theta)\mathbf{e}_x - \sin(\theta)\mathbf{e}_y$. Set $\tilde{F} := F + \mathbf{v} + (b_x u_x + b_y u_y)\nu_F$. Then it follows from (6.2) that

$$|Arg(\tilde{F}(s,\theta)) - Arg(F(s,\theta))| \leq C\epsilon. \tag{6.3}$$

Let $x_i = (s_i, \theta_i)$, $i = 1, 2$ be two points such that $\tilde{F}(x_1) = \tilde{F}(x_2)$ and assume first that $s_1 > 0,\ s_2 < 0$. Then it follows from (6.3) that $|\theta_1 - \theta_2 - (2k+1)\pi| \leq C\epsilon$. We can write

$$\begin{aligned} u_x \nu_F(s,\theta) &= -\frac{1}{4\pi}\psi_0(s)\cos(\theta)\mathbf{e}_r^\perp(\theta) + \frac{1}{4\pi}\psi_0(s)\cos(\theta)\sinh(s)\mathbf{e}_z, \\ u_y \nu_F(s,\theta) &= -\frac{1}{4\pi}\psi_0(s)\sin(\theta)\mathbf{e}_r^\perp(\theta) + \frac{1}{4\pi}\psi_0(s)\sin(\theta)\sinh(s)\mathbf{e}_z. \end{aligned}$$

This then gives that

$$\left| (b_x u_x + b_y u_y)\,\nu_F|_{x_1}^{x_2} \right|^2 \leq C\epsilon^2 \left(1 + (\sinh(s_1) + \sinh(s_2))^2\right) \tag{6.4}$$

Moreover, from (6.2), we have

$$|F(x_2) - F(x_1)|^2 \geq (1 - C\epsilon)(\sinh(s_2) + \sinh(s_1))^2 + (\pi - 2\epsilon)^2 \tag{6.5}$$

Combining (6.4) and (6.5) implies

$$|\tilde{F}(x_2) - \mathbf{v}(x_2) - \tilde{F}(x_1) + \mathbf{v}(x_1)| \geq \pi - C\epsilon.$$

Here C is just a universal constant depending on F. Now suppose $|s_1 - s_2| \leq 10$. Then the uniform C^1 bounds on $\mathbf{v}$ imply

$$\begin{aligned} |\tilde{F}(x_2) - \tilde{F}(x_1)| &\geq |\tilde{F}(x_2) - \mathbf{v}(x_2) - \tilde{F}(x_1) + \mathbf{v}(x_1)| - |\mathbf{v}(x_2) - \mathbf{v}(x_1)| \\ &\geq \pi - C\epsilon. \end{aligned}$$

On the other hand, if $|s_1 - s_2| > 10$, then the C^0 estimate on $\mathbf{v}$ is enough as in this case,

$$\begin{aligned} |\tilde{F}(x_2) - \tilde{F}(x_1)| &\geq |\tilde{F}(x_2) - \mathbf{v}(x_2) - \tilde{F}(x_1) + \mathbf{v}(x_1)| - |\mathbf{v}(x_2) - \mathbf{v}(x_1)| \\ &\geq \pi - C\epsilon - |\mathbf{v}(x_1)| - |\mathbf{v}(x_2)| \geq \pi - C\epsilon. \end{aligned}$$

Since C depends only on F, we can choose $0 < \epsilon < \epsilon_F$ so that $\pi - C\epsilon > 0$ and thus no self-intersections exist. A similar argument gives the same result when both s_1 and s_2 are positive. The result then follows immediately. $\square$

We now gather all of the estimates from the previous section in combination with the previous proposition to prove the main theorem.

Theorem 6.3. *Given $\tau_0, \xi \in \mathbb{R}$, $\kappa_0 > 0$, and $\zeta \gg 1$, choose any $0 < \delta < \delta_0/|\xi|$ and $\ell > 16$ such that $C\delta(1 + |\mathsf{T}| + |\xi|)\ell < \min\{1/(4\zeta), \tilde{\epsilon}/\zeta, \epsilon_F/(4\zeta)\}$. Here C is a universal constant arising from the norm bounds in the linear problem and δ_0 comes from Theorem 3.3. Consider any G defined according to a curve $\gamma[\delta\kappa_0, \delta\tau_0, \delta\xi] \in \mathfrak{L}$. Then there exists $(v, b_x, b_y) \in \Xi$ such that the surface $G + e^{\delta\xi\theta}\left(v + b_x u_x + b_y u_y + u_0\right)\nu_G$*

1) *is an immersed smooth surface with boundary.*
2) *is minimal on the domain $\{(s,\theta)|\cosh(s) \le \ell/4\}$.*
3) *is embedded when*

$$\ell \le \frac{1}{\delta\xi}\left(\frac{e^{\pi\xi/|\mathsf{T}|} - 1}{e^{\pi\xi/|\mathsf{T}|} + 1}\right)\sqrt{\frac{\tau_0^2 + \xi^2}{|\mathsf{T}|^2 + \xi^2}}.$$

To conclude the theorem found in the introduction, simply set $\epsilon_1 \le \min\{1/(4C\zeta), \tilde{\epsilon}/(C\zeta), \epsilon_F/(4C\zeta)\}$.

Proof. To prove embeddedness, note that for $|z| \le \frac{1}{\delta}$, extending the calculation of (6.1) to consider also D^2M implies

$$\|M(x,y,z) - (x,y,z) : C^1(\{|z| \le 1/\delta\})\| = O(\delta(z^2 + xz + yz)(1 + |\mathsf{T}|)).$$

Thus for $|\theta| \le 4\pi$ and $\Lambda_{4\pi} := \{(s,\theta)|(s,\theta) \in \Lambda, |\theta| \le 4\pi\}$,

$$\|G(s,\theta) - F(s,\theta) : C^1(\Lambda_{4\pi})\| \le \delta O(1 + \cosh(s)(1 + |\mathsf{T}|)).$$

Let $\mathbf{v} := e^{\delta\xi\theta} f\nu_G$ and $\mathbf{w} = e^{\delta\xi\theta} f\left(\nu_G - \nu_F\right)$. Then

$$\begin{aligned} &\|G + e^{\delta\xi\theta} f\nu_G - \left(F + e^{\delta\xi\theta} f\nu_F\right) : C^1(\Lambda_{4\pi})\| \\ &= \|G + \mathbf{v} - (F - \mathbf{v}) + \mathbf{w} : C^1(\Lambda_{4\pi})\| \\ &\le \delta O(1 + \cosh(s)(1 + |\mathsf{T}|)) + \|\mathbf{w} : C^1(\Lambda_{4\pi})\|. \end{aligned}$$

Taking f of the form $f = v + u_x b_x + u_y b_y + u_0$ for $(v, b_x, b_y) \in \Xi$, and using the estimate of (5.3),

$$\begin{aligned} \|\mathbf{w} : C^1(\Lambda_{4\pi})\| &\le \|f : C^1(\Lambda_{4\pi})\|\|\nu_G - \nu_F : C^1(\Lambda_{4\pi})\| \\ &\le C\delta^2\zeta(|\mathsf{T}| + |\mathsf{T}|^2)\ell^{5/4}. \end{aligned}$$

Thus, the graph over G can be viewed as a graph over F by $(b_x u_x + b_y u_y)\nu_F + \tilde{\mathbf{w}}$ where $\tilde{\mathbf{w}} = (v + u_0)\nu_F + \mathbf{w} + (G - F)$. The definition of Ξ implies

$$|b_x| + |b_y| \leq \epsilon_F/2.$$

As $\delta\ell < 1$, the perturbation vector field has the bound

$$\begin{aligned}\|\tilde{\mathbf{w}} : C^1(\Lambda_{4\pi})\| &\leq \|G - F + \mathbf{w} : C^1(\Lambda_{4\pi})\| + \|(v + u_0)\nu_F : C^1(\Lambda_{4\pi})\| \\ &\leq C\left(\delta\ell(1 + |\mathsf{T}|) + \delta\zeta(1 + |\mathsf{T}|)\ell + \delta\zeta|\mathsf{T}|\ell^{1/4}\right) \\ &\leq \epsilon_F.\end{aligned}$$

The self-similarity of the curve implies that, up to some translation, the same argument can be done for any domain of Λ with $\theta \in [\tilde{\theta} - 2\pi, \tilde{\theta} + 2\pi]$. Applying Proposition 6.2, we conclude the surface is locally an embedding. We appeal to Proposition 6.1 to get global embeddedness.

The minimality will follow from a fixed point argument. Namely, since $\Phi(\Xi) \subset \Xi$ and Ξ is a compact, convex subset of a Banach space, there exists $(v, b_x, b_y) \in \Xi$ such that

$$(v, b_x, b_y) = (\psi v, b_x, b_y) - \mathcal{R}_F[\psi' Q[\psi v + b_x u_x + b_y u_y]].$$

This implies $\mathcal{R}_F[\psi' Q[\psi v + b_x u_x + b_y u_y]] = ((1 - \psi)v, 0, 0)$ and thus on the region where $\psi = 1$,

$$Q[v + b_x u_x + b_y u_y] = 0.$$

The definition of Q, see (3.7), implies

$$H(\tilde{\underline{\nabla}} G + \mathcal{E}_\delta[v + b_x u_x + b_y u_y + u_0]) = 0$$

and thus $G + e^{\delta\xi\theta}\,(v + b_x u_x + b_y u_y + u_0)\,\nu_G$ has $H = 0$. □

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Received June 10, 2014