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A heat flow problem from Ericksen's model for nematic liquid crystals with variable degree of orientation

Chi-Cheung Poon

We study a heat flow problem for nematic liquid crystals with variable degree of orientation which is basically the harmonic heat flow into the round cone with a lower order term.

1. Introduction

In Ericksen's model for liquid crystal, the local molecular orientation is described by a pair (s, n), where n is a unit vector in $\mathbb{R}^3$, called the director, and s is a scalar, called the order parameter. The value of the order parameter s is usually restricted to the interval $(-\frac{1}{2}, 1)$. After a choice of material constants, the Helmholtz free energy is given by

$$\mathcal{W}_1 = \frac{1}{2}k_1|\nabla s|^2 + \frac{1}{2}k_2 s^2|\nabla n|^2 + W(s), \tag{1.1}$$

where $W(s)$ is a smooth double-well potential function satisfying

$$\lim_{s\to -\frac{1}{2}^+} W(s) = \infty, \quad \text{and} \quad \lim_{s\to 1^-} W(s) = \infty.$$

In this paper, we assume that $0 < k_2 < k_1$. An important observation is that, if we let

$$u = \sqrt{\frac{k_1 - k_2}{k_2}}sn,$$

then (s, u) is a point on the cone

$$\mathcal{C}(\mathcal{K}) = \{(s, u) \in \mathbb{R}^\times\mathbb{R}^3 : \mathcal{K}s^2 = |u|^2\}, \quad \text{with} \quad \mathcal{K} = \frac{k_1 - k_2}{k_2}.$$

Moreover, the energy function can be written as

$$\mathcal{W}_1 = \frac{k_1 - k_2}{2}\left(|\nabla s|^2 + |\nabla u|^2\right) + W(s).$$

It has been shown that for any $\mathcal{K} > 0$, any energy minimizer, (s, u), of the functional

$$\int_\Omega \left(\frac{1}{2}(|\nabla s|^2 + |\nabla u|^2) + W(s)\right) dx$$

among maps into $\mathcal{C}(\mathcal{K})$ with Dirichlet boundary data is of C^α and the dimension of the singular set $\{x : s(x) = 0\}$ is at most $m - 2$, see [4]. The purpose of the present paper is to study the corresponding heat flow problem. In the absence of a flow for the liquid crystal molecules, the evolution equations for s and n are

$$\begin{aligned} \beta(s)s_t &= \operatorname{div}\left\{\frac{\partial \mathcal{W}_1}{\partial \nabla s}\right\} - \frac{\partial \mathcal{W}_1}{\partial s} - W'(s), \\ \gamma(s)n_t \times n &= \left(\operatorname{div}\left\{\frac{\partial \mathcal{W}_1}{\partial \nabla n}\right\} - \frac{\partial \mathcal{W}_1}{\partial n}\right) \times n, \end{aligned} \tag{1.2}$$

see [1] p 1035. As in [1], we set

$$\beta(s) = 2 \qquad \text{and} \qquad \gamma(s) = \frac{k_2}{k_3}s^2.$$

Using (1.1), we may rewrite the system (1.2) as

$$\begin{aligned} 2s_t &= k_1\Delta s - k_2|\nabla n|^2 s - W'(s) \\ s^2 n_t &= k_3 \operatorname{div}(s^2\nabla n) + k_3 s^2|\nabla n|^2 n. \end{aligned} \tag{1.3}$$

In [1], it was proved that large time solutions for (1.3) exist, if the director n is planar, i.e., n always takes the form $n = (\cos\theta, \sin\theta, 0)$. Here, we would like to prove the existence of large time solutions without the planar assumption. In order to simplify the computations, we let

$$k_1 - k_2 = k_2. \tag{1.4}$$

After a scaling in time, we may assume that $k_1 = 2$. Also, we let

$$k_1 = 2k_3 \tag{1.5}$$

so that the system (1.3) is more or less the harmonic map equations into the cone $\mathcal{C}$, where

$$\mathcal{C} = \mathcal{C}(1) = \{(s, u) \in \mathbb{R}^\times\mathbb{R}^3 : s^2 = |u|^2\}.$$

This implies that $k_2 = 1$, by (1.4). As in the above, let $u = sn$. We note that $s^2|\nabla n|^2 = |\nabla u|^2 - |\nabla s|^2$. From (1.3), the equations for s and u can be written as

$$(1.6)\qquad \begin{aligned} s_t &= \Delta s - \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2} s - W'(s) \\ u_t &= \Delta u + \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2} u - \frac{W'(s)}{s} u. \end{aligned}$$

We wish to find a solution of the system (1.6) with initial-boundary conditions

$$(1.7)\qquad (s(x,t), u(x,t)) = (g(x), h(x)), \quad x \in \partial\Omega,\ t > 0,$$

$$(1.8)\qquad (s(x,0), u(x,0)) = (g(x), h(x)), \quad x \in \Omega,$$

where (g, h) is a Lipschitz map from Ω to $\mathcal{C}$, i.e., $g^2(x) = |h(x)|^2$ for all $x \in \Omega$. Due to technical reasons, we need to assume that $W(s) = F(s^2)$ for $s \in (-1, 1)$, and

$$\lim_{\tau \to 1^-} F(\tau) = \infty.$$

For detailed assumptions, see section 2 below. Our existence result is:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary. Let (g, h) be a Lipschitz map from Ω into the cone $\mathcal{C}$ and $\max_x |g(x)| < 1$ for $x \in \bar{\Omega}$. There is a continuous map $(s, u) : \Omega \times [0, \infty) \to \mathcal{C}$ such that at any point (x_0, t_0) where $t_0 > 0$ and $s(x_0, t_0) \neq 0$, (s, u) is a solution of (1.6) in a neighborhood of (x_0, t_0). Also, (s, u) satisfies the initial-boundary conditions, (1.7) and (1.8), in the sense of trace. Furthermore, there is a sequence t_j such that $t_j \to \infty$ as $j \to \infty$ and $(s(x, t_j), u(x, t_j))$ converges a map $(s_0(x), u_0(x))$ uniformly on compact subsets in Ω. For each point $x_0 \in \Omega$ where $s_0(x_0) \neq 0$, in a neighborhood of x_0, (s_0, u_0) is a stationary solution of the system (1.6), and (s_0, u_0) satisfies the boundary condition(1.7) in the sense of trace.*

We employ the penalization scheme in [2]. Let (s_K, u_K) be solutions of the system

$$\begin{aligned} \partial_t s_K &= \Delta s_K - 2K(s_K^2 - |u_K|^2)s - 2F_K' s_K, \\ \partial_t u_K &= \Delta u_K + 2K(s_K^2 - |u_K|^2)u_K - 2F_K' u_K, \end{aligned}$$

with initial-boundary data same as (1.7) and (1.8). We will prove that the maps (s_K, u_K) are equicontinuous on each compact subsets in $\Omega \times (0, \infty)$. Thus, a suitable sequence will converge to a map (s, u) with properties mentioned in the theorem. However, we do not obtain boundary regularity for

(s_K, u_K). Theorem 1.1 is also true when assuming (1.5) but without the assumption (1.4). In that case, we need to consider maps into the cone $\mathcal{C}(\mathcal{K})$ with $\mathcal{K} \neq 1$. The method is the same. In [1], assumption (1.5) is not needed and the assumption on the potential function W is less restricted.

We will also prove the following:

Theorem 1.2. *Let $t_0 > 0$. Either $s(x, t_0) = 0$ for all $x \in \Omega$, or $s(x, t_0)$ cannot vanish of infinite order at any point in Ω.*

The proof is a refinement of the arguments in [5]. We cannot eliminate the possibility that $s(x, t)$ may vanish identically on a time slice, even though we assume that the function $g(x)$ is not identically zero on $\partial\Omega$. It is due to the fact that we do not have the proper boundary regularity theory.

For the comparison between Ericksen's model and Oseen-Frank model for nematic liquid crystal, one may read [3]. The reader can also find the introduction section in [1] very helpful.

2. Approximating solutions and monotonicity formulas

Our assumptions on the potential function W are:

$$W(s) = F(s^2) \tag{2.1}$$

for some non-negative, C^2 function F defined on $[0, 1)$, $F(0) = 0$, $F'(\tau)$ and $F''(\tau)$ are bounded when $\tau \to 0^+$ and

$$\lim_{\tau \to 1^-} F(\tau) = \infty, \quad \lim_{\tau \to 1^-} F'(\tau) = \infty.$$

Also we assume that there is $s_* \in (0, 1)$ such that

$$F'(\tau) \geq 0 \quad \text{and} \quad F''(\tau) \geq 0 \quad \text{for} \quad \tau \in (s_*, 1).$$

Let Ω be a bounded domain in $\mathbb{R}^m$ with smooth boundary. Let (g, h) be a Lipschitz map from Ω to the cone $\mathcal{C}$. We assume that there are constants g_* such that

$$|g(x)| \leq g_* < 1 \quad \text{for} \quad x \in \Omega.$$

For each $K > 0$, we consider the heat flow problem

$$\begin{aligned} \partial_t s_K &= \Delta s_K - 2K(s_K^2 - |u_K|^2)s - 2F'_K s_K, \\ \partial_t u_K &= \Delta u_K + 2K(s_K^2 - |u_K|^2)u_K - 2F'_K u_K, \end{aligned} \tag{2.2}$$

with initial-boundary data same as (1.7) and (1.8). Here, we write

$$F'_K = F'\left(\frac{s_K^2 + |u_K|^2}{2}\right).$$

Let (s_K, u_K) be a solution of the problem on $\Omega \times (0, T_K)$. It is easy to compute that

$$\begin{aligned}(2.3)\qquad \partial_t(s_K^2 + |u_K|^2) = {} & \Delta(s_K^2 + |u_K|^2) - 2\left(|\nabla s_K|^2 + |\nabla u_K|^2\right) \\ & - 4K(s_K^2 - |u_K|^2)^2 - 4F'_K(s_K^2 + |u_K|^2).\end{aligned}$$

If $s_K^2 + |u_K|^2$ attains an interior maximum at $(\bar{x}_K, \bar{t}_K) \in \Omega \times (0, T_K)$, then by (2.3), we must have $F'\left(\frac{(s_K^2+|u_K|^2)(\bar{x}_K,\bar{t}_K)}{2}\right) \le 0$ and it implies that $(s_K^2 + |u_K|^2)(\bar{x}_K, \bar{t}_K) \le 2s_*$. Thus, we obtain

$$\begin{aligned}(2.4)\qquad & \sup\{(s_K^2 + |u_K|^2)(x,t) : (x,t) \in \Omega \times (0, T_K)\} \\ & \le \max\left\{2s_*, 2g_*^2\right\} = M_0.\end{aligned}$$

It is easy to see that $M_0 < 2$ and is independent of K. Hence, the solution (s_K, u_K) can be extended to a solution in $\Omega \times (0, \infty)$. Furthermore, it is easy to check that

$$\begin{aligned}(2.5)\qquad & \frac{d}{dt}\int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_k^2 - |u_K|^2)^2 + 2F_K\right) dx \\ & = -2\int_\Omega (|\partial_t s_K|^2 + |\partial_t u_K|^2) dx,\end{aligned}$$

where

$$F_K = F\left(\frac{s_K^2 + |u_K|^2}{2}\right).$$

It follows from (2.5) that for $t > 0$,

$$\begin{aligned}(2.6)\qquad & \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_k^2 - |u_K|^2)^2 + 2F_K\right)(x,t) dx \\ & \le \int_\Omega \left(|\nabla g(x)|^2 + |\nabla h(x)|^2 + 2F(g^2(x))\right) dx = E_0.\end{aligned}$$

Fix $x_0 \in \Omega$ and $t_0 > 0$. Choose $R > 0$ such that $B(x_0; 2R) \subset \Omega$, where $B(x_0; R) = \{x : |x - x_0| < R\}$. Let $\xi(x)$ be a cutoff function such that $\xi = 0$ outside $B(x_0; 2R)$, and $\xi = 1$ inside $B(x_0; R)$, and $|\nabla\xi| \le C/R$, and $|\nabla^2\xi| \le$

C/R^2. For $0 < t < t_0$, we define

$$e_K(x,t) = |\nabla s_K(x,t)|^2 + |\nabla u_K(x,t)|^2 + K(s_K^2(x,t) - |u_K(x,t)|^2)^2, \tag{2.7}$$

$$E_K(t; x_0, t_0) = |t - t_0| \int_\Omega e_K(x,t)\xi^2(x)G(x,t;x_0,t_0)dx, \tag{2.8}$$

$$I_K(t; x_0, t_0) = \int_\Omega (s_K^2(x,t) + |u_K(x,t)|^2)\xi^2(x)G(x,t;x_0,t_0)dx. \tag{2.9}$$

Here, $G(x,t;x_0,t_0)$ is the backward heat kernel on $\mathbb{R}^m$: for $t < t_0$,

$$G(x,t;x_0,t_0) = \frac{1}{|t-t_0|^{m/2}} \exp\left(\frac{|x-x_0|^2}{4(t-t_0)}\right).$$

We write

$$P(x_0, t_0; R) = \{(x,t) : |x - x_0| < R,\ t_0 - R^2 < t < t_0\}.$$

Proposition 2.1. *Suppose that $P(x_0,t_0;2R) \subset \Omega \times (0,\infty)$. When $t \in (t_0 - R^2, t_0)$,*

$$\begin{aligned} &- |t-t_0| \frac{d}{dt} I_K(t;x_0,t_0) \\ &\geq 2E_K(t;x_0,t_0) - C|t-t_0|I_K(t) - C\exp\left(\frac{1}{6(t-t_0)}\right), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &- |t-t_0| \frac{d}{dt} I_K(t;x_0,t_0) \\ &\leq 4E_K(t;x_0,t_0) + C|t-t_0|I_K(t) + C\exp\left(\frac{1}{6(t-t_0)}\right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} &\frac{d}{dt} E_K(t;x_0,t_0) \\ &\leq -\frac{1}{2}|t-t_0| \int_\Omega \left(\partial_t s_K + \nabla s_K \frac{x-x_0}{2(t-t_0)}\right)^2 \xi^2(x) G(x,t:x_0,t_0)dx \\ &\quad - \frac{1}{2}|t-t_0| \int_\Omega \left(\partial_t u_K + \nabla u_K \frac{x-x_0}{2(t-t_0)}\right)^2 \xi^2(x) G(x,t:x_0,t_0)dx \\ &\quad + C|t-t_0|I_K(t) + C\exp\left(\frac{1}{6(t-t_0)}\right), \end{aligned} \tag{2.12}$$

and C is a positive constant depending on R and m, M_0 and E_0.

Inequality (2.12) is a variation of the monotonicity formula in [2] and [7]. An important consequence of Proposition 2.1 is Corollary 2.2 below. Similar arguments can be found in [4] p458 and [6] p177 when considering elliptic problems.

Corollary 2.2. *Let* (x_0, t_0) *be a point in* $\Omega \times (0, \infty)$. *Let* $R > 0$ *be a constant such that* $B(x_0; 2R) \times (t_0 - (2R)^2, t_0) \subset \Omega \times (0, \infty)$. *Then there is a positive constant* $C > 0$ *such that for* $0 < r < R/2$, *we have*

$$r^{2-m} \int_{B(x_0;r)} e_K(x, t_0) dx \leq \frac{C}{\ln(R^2/r^2)}, \tag{2.13}$$

$$r^{2-m} \iint_{P(x_0,t_0;r)} \left(|\partial_t s_K|^2 + |\partial_t u_K|^2\right) dxdt \leq \frac{C}{\ln(R^2/r^2)}. \tag{2.14}$$

where e_K *is the function in (2.7). The constant* C *depends only on* m, R, M_0 *and* E_0 *only.*

We first prove Corollary 2.2.

Proof. We first note that by (2.4), $I_K(t; x_0, t_0) \leq M_0$. By (2.12), for $t_0 - R^2 \leq t < t_1 < t_0$, we have

$$E_K(t_1; x_0, t_0) \leq E_K(t; x_0, t_0) + C|t_0 - t|,$$

where C is a constant depending on m, R, M_0 and E_0. Hence, we see that

$$\int_{t_0-R^2}^{t_1} \frac{E_K(t; x_0, t_0)}{t_0 - t} dt \geq \ln(R^2/(t_0 - t_1)) E_K(t_1; x_0, t_0) - C.$$

Also, by (2.10),

$$\begin{aligned} \int_{t_0-R^2}^{t_1} \frac{E_K(t; x_0, t_0)}{t_0 - t} dt &\leq \frac{1}{2}(I_K(t_0 - R^2; x_0, t_0) - I_K(t_1; x_0, t_0)) + C \\ &\leq \frac{M_0}{2} + C. \end{aligned}$$

Thus, we see that

$$E_K(t_1; x_0, t_0) \leq \frac{C}{\ln(R^2/(t_0 - t_1))}. \tag{2.15}$$

Let t_0 be fixed and $R > 0$ be a number as in the above. For any $0 < r < R$, by (2.15), we have

$$E_K(t_0; x_0, t_0 + r^2) \leq \frac{C}{\ln(R^2/r^2)}.$$

Again, C is a constant depending on m, R, M_0 and E_0 only. We note that

$$G(x,t_0;x_0,t_0+r^2) \geq \frac{C}{r^m} \quad \text{for} \quad |x|<r. \tag{2.16}$$

This implies that

$$r^{2-m}\int_{B(x_0;r)} e_K(x,t_0)dx \leq CE_K(t_0;x_0,t_0+r^2) \leq \frac{C}{\ln(R^2/r^2)}.$$

By (2.12) and (2.15), we see that for $0<r<\frac{1}{2}R$,

$$\begin{aligned}
&\int_{t_0-3r^2}^{t_0-r^2} |t-t_0| \int_\Omega \left(\partial_t s_K + \nabla s_K \frac{x-x_0}{2(t-t_0)}\right)^2 \xi^2(x) G(x,t:x_0,t_0)dxdt \\
&+ \int_{t_0-3r^2}^{t_0-r^2} |t-t_0| \int_\Omega \left(\partial_t u_K + \nabla u_K \frac{x-x_0}{2(t-t_0)}\right)^2 \xi^2(x) G(x,t:x_0,t_0)dxdt \\
\leq\, & CE(t_0-3r^2;x_0,t_0) + Cr^2 \leq \frac{C}{\ln(R^2/r^2)}
\end{aligned}$$

Using (2.16) again, we have

$$\begin{aligned}
& r^{2-m}\int_{t_0-3r^2}^{t_0-r^2}\int_{B(x_0;r)} \left(\partial_t s_K + \nabla s_K \frac{x-x_0}{2(t-t_0)}\right)^2 dxdt \\
& + r^{2-m}\int_{t_0-3r^2}^{t_0-r^2}\int_{B(x_0;r)} \left(\partial_t u_K + \nabla u_K \frac{x-x_0}{2(t-t_0)}\right)^2 dxdt \\
\leq\, & \frac{C}{\ln(R^2/r^2)}.
\end{aligned} \tag{2.17}$$

On the other hand, by (2.13),

$$r^{2-m}\int_{t_0-3r^2}^{t_0-r^2}\int_{B(x_0;r)} \left(\left|\nabla s_K \frac{x-x_0}{(t-t_0)}\right|^2 + \left|\nabla u_K \frac{x-x_0}{(t-t_0)}\right|^2\right) dxdt \leq \frac{C}{\ln(R^2/r^2)}.$$

Using this estimate in (2.17), we obtain

$$r^{2-m}\int_{t_0-3r^2}^{t_0-r^2}\int_{B(x_0;r)} \left(|\partial_t s_K|^2 + |\partial_t u_K|^2\right) dxdt \leq \frac{C}{\ln(R^2/r^2)}.$$

Now, replace t_0 by t_0-r^2, we conclude that (2.14) holds. □

Now, we begin the proof of Proposition 2.1. After a translation, we assume that $x_0 = 0$ and $t_0 = 0$ and we write $E_K(t)$, $I_K(t)$, and $G(x,t)$ instead of $E_K(t;0,0)$, $I_K(t;0,0)$ and $G(x,t;0,0)$. We observe that

$$G_t + \Delta G = 0, \quad \text{and} \quad \nabla G = \frac{x}{2t} G.$$

We first compute I'_K:

$$\begin{aligned} I'_K(t) &= \int_\Omega \big((2s_K \partial_t s_K + 2u_K \partial_t u_K)\xi^2 G - (s_K^2 + |u_K|^2)\xi^2 \Delta G\big)\, dx \\ &= 2\int_\Omega \left(s_K \left(\partial_t s_K + \frac{x}{2t}\nabla s_K\right) + u_K \left(\partial_t u_K + \frac{x}{2t}\nabla u_K\right)\right)\xi^2 G dx \\ &\quad + 2\int_\Omega (s_K^2 + |u_K|^2)\xi \left(\frac{x}{2t}\nabla \xi\right) G dx. \end{aligned} \tag{2.18}$$

By equations (2.2), we have

$$\begin{aligned} I'_K(t) = 2&\int_\Omega (s_K \Delta s_K + u_K \Delta u_K - 2K(s_K^2 - |u_K|^2)^2)\xi^2 G dx \\ &+ 2\int_\Omega \left(s_K \left(\frac{x}{2t}\nabla s_K\right) + u_K\left(\frac{x}{2t}\nabla u_K\right)\right)\xi^2 G dx \\ &- 4\int_\Omega F'_K(s_K^2 + |u_K|^2)\xi^2 G dx + 2\int_\Omega (s_K^2 + |u_K|^2)\xi\left(\frac{x}{2t}\nabla\xi\right) G dx. \end{aligned}$$

After integrating by parts, we obtain

$$\begin{aligned} I'_K(t) = -2&\int_\Omega (|\nabla s_K|^2 + |\nabla u_K|^2 + 2K(s_K^2 - |u_K|^2)^2)\xi^2 G dx \\ &- 4\int_\Omega F'_K(s_K^2 + |u_K|^2)\xi^2 G dx \\ &+ 2\int_\Omega (s_K^2 + |u_K|^2)\xi\left(\frac{x}{2t}\nabla\xi\right) G dx \\ &- 4\int_\Omega (s_K \nabla s_K + u_K \nabla u_K)\xi\nabla\xi G dx. \end{aligned} \tag{2.19}$$

By (2.4), the term $|F'_K|$ is bounded independent of K. Thus,

$$\left|\int_\Omega F'_K(s_K^2 + |u_K|^2)\xi^2 G dx\right| \le C I(t).$$

Furthermore, since

$$
\begin{aligned}
&\nabla\xi(x) = 0 \quad \text{when} \quad |x| \le R, \\
&G(x,t) \le \exp\left(\frac{R^2}{4t}\right) \quad \text{when} \quad |x| \ge R,\ t < 0,
\end{aligned}
\tag{2.20}
$$

we have

$$
\left|\int_\Omega (s_K^2 + |u_K|^2)\xi\left(\frac{x}{2t}\nabla\xi\right) G dx\right| \le C\exp\left(\frac{1}{6t}\right).
$$

The last integral in (2.19) can be written as

$$
\begin{aligned}
&\int_\Omega 4(s_K\nabla s_K + u_K\nabla u_K)\xi\nabla\xi G dx \\
&= \int_\Omega \nabla(s_K^2 + |u_K|^2)\nabla\xi^2 G dx \\
&= -\int_\Omega (s_K^2 + |u_K|^2)\left(\Delta\xi^2 + 2\xi\frac{x}{2t}\nabla\xi\right) G dx.
\end{aligned}
$$

Thus, using similar arguments as in the above, we also have

$$
\left|\int_\Omega 4(s_K\nabla s_K + u_K\nabla u_K)\xi\nabla\xi G dx\right| \le C\exp\left(\frac{1}{6t}\right).
$$

From (2.19), we obtain inequalities (2.10) and (2.11) easily.

Next, we compute E_K'.

$$
\begin{aligned}
E_K'(t) = &-\int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right)\xi^2 G dx \\
&+ 2|t|\int_\Omega \left(\nabla s_K\nabla\partial_t s_K + \nabla u_K\nabla\partial_t u_K\right)\xi^2 G dx \\
&+ 4|t|\int_\Omega K(s_K^2 - |u_K|^2)(s_K\partial_t s_K - u_K\partial_t u_K)\xi^2 G dx \\
&- |t|\int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right)\xi^2\Delta G dx
\end{aligned}
$$

Using integration by parts, the last integral in the above can be expressed as

$$
\begin{aligned}
&- |t| \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \xi^2 \Delta G dx \\
&= 2|t| \int_\Omega \left(\nabla s_K \nabla^2 s_K \frac{x}{2t} + \nabla u_K \nabla^2 u_K \frac{x}{2t}\right) \xi^2 G dx \\
&\quad + 4|t| \int_\Omega K(s_K^2 - |u_K|^2) \left(s_K \nabla s_K \frac{x}{2t} - u_K \nabla u_K \frac{x}{2t}\right) \xi^2 G dx \\
&\quad + 4|t| \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \xi \nabla \xi \frac{x}{2t} G dx \\
&= 2|t| \int_\Omega \left(\nabla s_K \nabla \left(\nabla s_K \frac{x}{2t}\right) + \nabla u_K \nabla \left(\nabla u_K \frac{x}{2t}\right)\right) \xi^2 G dx \\
&\quad + \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2\right) \xi^2 G dx \\
&\quad + 4|t| \int_\Omega K(s_K^2 - |u_K|^2) \left(s_K \nabla s_K \frac{x}{2t} - u_K \nabla u_K \frac{x}{2t}\right) \xi^2 G dx \\
&\quad + 4|t| \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \xi \nabla \xi \frac{x}{2t} G dx.
\end{aligned}
$$

Thus, we see that

$$
\begin{aligned}
E_K'(t) = &- \int_\Omega K(s_K^2 - |u_K|^2)^2 \xi^2 G dx \\
&+ 2|t| \int_\Omega \left(\nabla s_K \nabla \left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right) + \nabla u_K \nabla \left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\right) \xi^2 G dx \\
&+ 4|t| \int_\Omega K(s_K^2 - |u_K|^2) \Big(s_K \left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right) \\
&\qquad - u_K \left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\Big) \xi^2 G dx \\
&+ 4|t| \int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \xi \nabla \xi \frac{x}{2t} G dx.
\end{aligned}
$$

We then do integration by parts for the second integral in the above,

$$
\begin{aligned}
&2|t| \int_\Omega \left(\nabla s_K \nabla \left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right) + \nabla u_K \nabla \left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\right) \xi^2 G dx \\
&= - 2|t| \int_\Omega \left(\Delta s_K + \nabla s_K \frac{x}{2t}\right) \left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right) \xi^2 G dx \\
&\quad - 2|t| \int_\Omega \left(\Delta u_K + \nabla u_K \frac{x}{2t}\right) \left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right) \xi^2 G dx \\
&\quad - 4|t| \int_\Omega \left(\left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right) \nabla s_K + \left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right) \nabla u_K\right) \nabla \xi \xi G dx
\end{aligned}
$$

Next, we apply equation (2.2) to have

$$\begin{aligned}
&2|t|\int_\Omega \left(\nabla s_K \nabla\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)+\nabla u_K\nabla\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\right)\xi^2 G dx\\
=&-2|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)^2+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)^2\right)\xi^2 G dx\\
&-4|t|\int_\Omega K(s_K^2-|u_K|^2)\left(s_K\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)\right.\\
&\qquad\left.-u_K\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\right)\xi^2 G dx\\
&-4|t|\int_\Omega F_K'\left(s_K\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)+u_K\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\right)\xi^2 G dx\\
&-4|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)\nabla s_K+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\nabla u_K\right)\nabla\xi\xi G dx
\end{aligned}$$

Now, we may write

(2.21)

$$\begin{aligned}
E_K'(t)=&-\int_\Omega K(s_K^2-|u_K|^2)^2\xi^2 G dx\\
&-2|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)^2+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)^2\right)\xi^2 G dx\\
&-4|t|\int_\Omega F_K'\left(s_K\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)+u_K\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\right)\xi^2 G dx\\
&-4|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)\nabla s_K+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\nabla u_K\right)\nabla\xi\xi G dx\\
&+4|t|\int_\Omega\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\xi\nabla\xi\frac{x}{2t} G dx.
\end{aligned}$$

Using triangle inequality, we arrive at

$$\begin{aligned}
E_K'(t)\le&-\frac{1}{2}|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)^2+\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)^2\right)\xi^2 G dx\\
&+C|t|\int_\Omega |F_K'|^2\left(s_K^2+|u_K|^2\right)\xi^2 G dx\\
&+C|t|\int_\Omega\left(|\nabla s_K|^2+|\nabla u_K|^2\right)|\nabla\xi|\xi G dx\\
&+4|t|\int_\Omega\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\xi|\nabla\xi|\frac{|x|}{2|t|} G dx.
\end{aligned}$$

As before, since $|F_K'|$ is bounded independent of K,

$$\int_\Omega |F_K'|^2 \left(s_K^2 + |u_K|^2\right) \xi^2 G dx \le C I(t).$$

By (2.6) and (2.20), for $-R^2 < t < 0$, we have

$$\int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2\right) |\nabla \xi| \xi G dx \le \frac{1}{|t|^{m/2}} \exp\left(\frac{R^2}{4t}\right) E_0 \le C \exp\left(\frac{1}{6t}\right).$$

Similarly,

$$\int_\Omega \left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \xi |\nabla \xi| \frac{|x|}{2|t|} G dx \le C \exp\left(\frac{1}{6t}\right).$$

The constant C depends only on R, m, M_0 and E_0 only. This proves (2.12).

3. Convergence of approximating solutions

We need the following proposition to prove the convergence of approximating solutions.

Proposition 3.1. *The maps $\{(s_K, u_K) : K > 0\}$ are equicontinuous on each compact subset of $\Omega \times (0, \infty)$. In fact, if $P(x_0, t_0; 2R) \subset \Omega \times (0, \infty)$, then for (x_1, t_1) and (x_2, t_2) in $P(x_0, t_0; R/4)$, we have*

$$|s_K(x_1, t_1) - s_K(x_2, t_2)| + |u_K(x_1, t_1) - u_K(x_2, t_2)| \le \left(\frac{C}{\ln(R^2/\rho^2)}\right)^{1/2},$$

where $\rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$. The constant C depends only on m, R, M_0 and E_0.

Proof. The proof is basically the same as the proof for Morrey's Lemma. Let (x_0, t_0) and $R > 0$ be fixed such that $P(x_0, t_0; 2R) \subset \Omega \times (0, \infty)$. Let (x_1, t_1) and (x_2, t_2) be points in $P(x_0, t_0; R/4)$ and $t_2 \le t_1$. Let

$$\bar{x} = (x_1 + x_2)/2, \qquad \text{and} \qquad \rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}.$$

We note that $\rho < R$. For each $x \in B(\bar{x}; r)$, we observe that

$$\begin{aligned}|s_K(x_1,t_1) - s_K(x,t_1)| &= \left|\int_0^1 (x_1 - x)\cdot \nabla s_K(x_1 + \tau(x - x_1), t_1)d\tau\right| \\ &\le 4r\int_0^1 |\nabla s_K(x_1 + \tau(x - x_1))|d\tau.\end{aligned}$$

Let $\xi(x)$ be a non-negative smooth function such that $\xi(x) = 1$ when $x \in B(\bar{x}; \rho/2)$ and $\xi(x) = 0$ when x lies outside $B(\bar{x}; r)$, and $|\nabla \xi| \le C/\rho$. After interchanging the order of integration, we obtain

$$\begin{aligned}&\frac{1}{\rho^n}\int_{B(\bar{x};\rho)} |s_K(x_1,t_1) - s_K(x,t_1)|\xi(x)dx \\ &\le \frac{1}{\rho^n}\int_{B(\bar{x};\rho)} |s_K(x_1,t_1) - s_K(x,t_1)|dx \\ &\le \frac{4}{\rho^{n-1}}\int_{B(\bar{x};\rho)}\int_0^1 |\nabla s_K(x_1 + \tau(x - x_1), t_1)|d\tau dx.\end{aligned}$$

Let $y = x_1 + \tau(x - x_1)$ and $\bar{x}_\tau = x_1 + \tau(\bar{x} - x_1)$. We note that if $x \in B(\bar{x}; \rho)$, then $|y - \bar{x}_\tau| \le \tau\rho$, and $\bar{x}_\tau \in B(x_0; R)$ for all $0 < \tau < 1$. Thus, from (2.13), we have

$$\begin{aligned}&\frac{4}{\rho^{n-1}}\int_{B(\bar{x};\rho)}\int_0^1 |\nabla s_K(x_1 + \tau(x - x_1), t_1)|d\tau dx \\ &\le C\rho^{1-n}\int_0^1\int_{B(\bar{x}_\tau;\tau\rho)} |\nabla s_K(y,t_1)|dyd\tau \\ &\le C\rho^{1-n}\int_0^1 (\tau\rho)^{n/2}\left(\int_{B(\bar{x}_\tau;\tau\rho)} |\nabla s_K(y,t_1)|^2 dy\right)^{1/2} d\tau \\ &\le C\rho^{1-n}\int_0^1 (\tau\rho)^{n-1}\left(\frac{1}{\ln(R^2/(\tau\rho)^2)}\right)^{1/2} d\tau \\ &\le C\left(\frac{1}{\ln(R^2/\rho)^2}\right)^{1/2}.\end{aligned}$$

Let

$$\bar{s}_K(\bar{x},t) = \frac{\int_{B(\bar{x};\rho)} s_K(x,t)\xi(x)dx}{\int_{B(\bar{x};\rho)} \xi(x)dx}.$$

The computations in the above implies that

$$|s_K(x_1,t_1) - \bar{s}_K(\bar{x},t_1)| \le C\left(\frac{1}{\ln(R^2/\rho^2)}\right)^{1/2}.$$

Similarly, we also have

$$|s_K(x_2,t_2) - \bar{s}_K(\bar{x},t_2)| \le C\left(\frac{1}{\ln(R^2/\rho^2)}\right)^{1/2}.$$

Since $|t_1 - t_2| \le \rho^2$, by (2.14),

$$\begin{aligned}|\bar{s}_K(\bar{x},t_1) - \bar{s}_K(\bar{x},t_2)| &\le C\rho^{-m}\int_{t_2}^{t_1}\int_{B(\bar{x};\rho)} |\partial_t s_K|\xi dxdt \\ &\le C\rho^{-m/2}\int_{t_2}^{t_1}\left(\int_{B(\bar{x};\rho)} |\partial_t s_K|^2 dx\right)^{1/2} dt \\ &\le C\left(\frac{1}{\ln(R^2/\rho^2)}\right)^{1/2}\end{aligned}$$

This implies that

$$|s_K(x_1,t_1) - s_K(x_2,t_2)| \le C\left(\frac{1}{\ln(R^2/\rho^2)}\right)^{1/2}.$$

Similarly, we can prove that

$$|u_K(x_1,t_1) - u_K(x_2,t_2)| \le C\left(\frac{1}{\ln(R^2/\rho^2)}\right)^{1/2}.$$

This completes the proof. □

By Proposition 3.1 and Arzela-Ascoli's theorem, there is a sequence K_i such that $K_i \to \infty$ as $i \to \infty$, and $(s_i, u_i) = (s_{K_i}, u_{K_i})$ converges to a continuous map (s,u) uniformly on each compact subset in $\Omega \times (0,\infty)$. Moreover, if $P(x_0,t_0;2R) \subset \Omega \times (0,\infty)$, and $(x_1,t_1),(x_2,t_2) \in P(x_0,t_0;R/4)$, then

$$(3.1) \qquad |s(x_1,t_1) - s(x_2,t_2)| + |u(x_1,t_1) - u(x_2,t_2)| \le \left(\frac{C}{\ln(R^2/\rho^2)}\right)^{1/2},$$

where $\rho = \sqrt{|x_1 - x_2|^2 + |t_1 - t_2|}$. Let $0 < t_2 < t_1$. By (2.5), we may assume that $(\nabla s_i, \nabla u_i)$ and $(\partial_t s_i, \partial_t u_i)$ converge to $(\nabla s, \nabla u)$ and (s_t, u_t) respectively weakly in $L^2(\Omega \times (t_2, t_1))$. By (2.4), the maps (s_i, u_i) are bounded, we may assume that (s_i, u_i) converges to (s, u) in $L^4(\Omega \times (t_1, t_2))$. By inequality (2.6), we have

$$\int_{t_2}^{t_1} \int_\Omega (s_i^2 - |u_i|^2)^2 dx dt \le E_0(t_1 - t_2)/K_i.$$

When $i \to \infty$, we see that

$$\int_{t_2}^{t_1} \int_\Omega (s^2 - |u|^2)^2 dx dt = 0.$$

Since the map (s, u) is continuous, for any $(x, t) \in \Omega \times (t_2, t_1)$, we have $s^2(x, t) = |u(x, t)|^2$, i.e., $(s(x, t), u(x, t))$ lies on the cone $\mathcal{C}$. Moreover, by the lower-semi-continuity theory and (2.5), for each $t > 0$,

$$\begin{aligned} &\int_\Omega \left(|\nabla s(x,t)|^2 + |\nabla u(x,t)|^2 + 2W(s(x,t))\right) dx \\ &+ \int_0^t \int_\Omega \left(s_t^2(x,\tau) + |u_t(x,\tau)|^2\right) dx d\tau \le E_0. \end{aligned} \tag{3.2}$$

Let (x_0, t_0) be a point such that $|s(x_0, t_0)| \ge 4\gamma$ for some $\gamma > 0$. We choose $R > 0$ such that $P(x_0, t_0; 2R) \subset \Omega \times (0, \infty)$, and for $(x, t) \in P(x_0, t_0; R)$, we have $|s(x, t)| \ge 2\gamma$. It then implies that $s_i^2(x, t) + |u_i(x, t)|^2 \ge \gamma^2$ when $(x, t) \in P(x_0, t_0; R)$ and when i is large enough. By straight-forward computations, we see that

$$\begin{aligned} &(-\partial_t + \Delta)\left(|\nabla s_K|^2 + |\nabla u_K|^2 + K(s_K^2 - |u_K|^2)^2\right) \\ = {} & 2|\nabla^2 s_K|^2 + 2|\nabla^2 u_K|^2 + 2K|\nabla(s_K^2 - |u_K|^2)|^2 \\ &+ 8K(s_K^2 - |u_K|^2)(|\nabla s_K|^2 - |\nabla u_K|^2) \\ &+ 8K^2(s_K^2 - |u_K|^2)^2(s_K^2 + |u_K|^2) + 8K(s_K^2 - |u_K|^2)^2 F_K' \\ &+ 8(|\nabla s_K|^2 + |\nabla u_K|^2)F_K' + 8(s_K^2|\nabla s_K|^2 + |u_K|^2|\nabla u_K|^2)F_K'', \end{aligned}$$

where $F_K' = F'(s_K^2 + |u_K|^2)$ and $F_K'' = F''(s_K^2 + |u_K|^2)$. We recalled that by (2.4), (s_K, u_K) is bounded independent of K. This makes F_K' and F_K'' bounded, independent of K. If $|s_K^2(x,t) + |u_K(x,t)|^2 \ge \gamma^2$, then there is a constant D, which may depend on γ but independent of K, such that

$$\begin{aligned}&(-\partial_t+\Delta)\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\\ \geq\ & 8\gamma^2K^2(s_K^2-|u_K|^2)^2+8K(s_K^2-|u_K|^2)(|\nabla s_K|^2-|\nabla u_K|^2)\\ &-C\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\\ \geq\ & -D\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)^2\\ &-C\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right),\end{aligned}$$

i.e., $(-\partial_t+\Delta)e_K\geq -Ce_K-De_K^2$. The constants C and D may depend on γ but do not depend on K. By the small-energy-regularity theory, see [2] Lemma 2.4, if

$$\int_{t_0-4R^2}^{t_0-R^2}\int_\Omega\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)G(x,t;x_0,t_0)dxdt<\epsilon_0 \tag{3.3}$$

and ϵ_0 is small enough, then for certain $\delta\in(0,1/8)$,

$$\sup_{P(x_0,t_0;\delta R)}\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\leq C(\delta R)^{-2}. \tag{3.4}$$

The constants ϵ_0, δ and C depend on γ, m, R, M_0 and E_0 but do not depend on K. By (2.15), if we choose R small enough, then (3.3) holds. Thus, by choosing a subsequence if necessary, we may assume that $(\nabla s_i,\nabla u_i)$ converges to $(\nabla s,\nabla u)$ strongly in $L^2(P(x_0,t_0;\delta R))$. Due to the estimate (2.15), the small energy assumption is always true by choosing R small. By repeating the arguments in [2] p94-95, one can check that, inside $P(x_0,t_0;\delta R)$, the map (s,u) is a weak solution of the system

$$\begin{aligned}s_t&=\Delta s-\frac{|\nabla u|^2-|\nabla s|^2}{2s^2}s-2F'\left(\frac{s^2+|u|^2}{2}\right)s\\ u_t&=\Delta u+\frac{|\nabla u|^2-|\nabla s|^2}{2s^2}u-2F'\left(\frac{s^2+|u|^2}{2}\right)u.\end{aligned}$$

Since $s^2=|u|^2$ and our assumption (2.1), we have $W'(s)=2F'(s^2)s$. We then see that (s,u) is a weak solution of (1.6) in $P(x_0,t_0;\delta R)$. Moreover, from (3.4), we see that

$$\sup_{P(x_0,t_0;\delta R/2)}\left(|\nabla s|^2+|\nabla u|^2\right)\leq C. \tag{3.5}$$

The constant C may depend on m, R, M_0, E_0, γ and δ. By standard arguments, (s, u) satisfies the initial-boundary conditions (1.7) and (1.8) in the sense of trace.

By (3.1) and (3.2), we may choose a sequence t_j such that $t_j \to \infty$ as $j \to \infty$ and

$$\int_\Omega \left(s_t^2(x,t_j) + |u_t(x,t_j)|^2\right) dx \to 0 \quad \text{as} \quad j \to \infty,$$

and $(s(x,t_j), u(x,t_j))$ converges to a map $(s_0(x), u_0(x))$ uniformly on compact subsets in Ω. By choosing subsequence if necessary, by (3.2), we may assume that $(\nabla s(x,t_j), \nabla u(x,t_j))$ converges weakly to $(\nabla s_0, \nabla u_0)$ in $L^2(\Omega)$. This implies that $(s_0, u_0) = (g, h)$ on $\partial\Omega$ in the sense of trace. Suppose that for some $x_0 \in \Omega$, we have $s_0(x_0) > 2\gamma > 0$. When j is large enough, we have $s(x_0, t_j) > \gamma$. We choose $R > 0$ such that $B(x_0; 2R) \subset \Omega$. By (3.5), we may assume that $(\nabla s(x,t_j), \nabla u(x,t_j))$ converges strongly to $(\nabla s_0, \nabla u_0)$ in $L^2(B(x_0; \delta R/2))$, for some $\delta > 0$. Thus, we see that (s_0, u_0) is a weak solution of the system

$$\begin{aligned} \Delta s - \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2} s - W'(s) = 0 \\ \Delta u + \frac{|\nabla u|^2 - |\nabla s|^2}{2s^2} u - \frac{W'(s)}{s} u = 0 \end{aligned}$$

inside $B(x_0; \delta R/2)$. This proves Theorem 1.1.

4. The vanishing order of the solution

In this section, we prove Theorem 1.2 stated in the Introduction. To do this, we need to further refine the monotonicity formulas obtained in section 2. Let $(x_0, t_0) \in \Omega \times (0, \infty)$, such that $P(x_0, t_0; 2R) \subset \Omega$, and $E_K(t) = E_K(t; x_0, t_0)$ and $I_K(t) = I_K(t; x_0, t_0)$ be the functions defined in (2.8) and (2.9). Again, after a translation, we let $(x_0, t_0) = (0, 0)$. From (2.21), we have

$$\begin{aligned} E_K'(t) = &- \int_\Omega K(s_K^2 - |u_K|^2)^2 \xi G dx \\ &- 2|t| \int_\Omega \left(\left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right)\xi + 2F_K' s_K \xi + \nabla s_K \nabla \xi\right)^2 \xi G dx \\ &- 2|t| \int_\Omega \left(\left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\xi + 2F_K' u_K \xi + \nabla u_K \nabla \xi\right)^2 \xi G dx \end{aligned}$$

$$+\frac{1}{2}|t|\int_\Omega\left(\left(2F_K's_K\xi+\nabla s_K\nabla\xi\right)^2+\left(2F_K'u_K\xi+\nabla u_K\nabla\xi\right)^2\right)\xi G dx$$
$$+4|t|\int_\Omega\left(|\nabla s_K|^2+|\nabla u_K|^2+K(s_K^2-|u_K|^2)^2\right)\xi\nabla\xi\frac{x}{2t}G dx.$$

By (2.4) and (2.20), we see that

$$\begin{aligned}(4.1)\quad E_K'(t)\le{}&-2|t|\int_\Omega\left(\left(\partial_t s_K+\nabla s_K\frac{x}{2t}\right)\xi+2F_K's_K\xi+\nabla s_K\nabla\xi\right)^2\xi G dx\\&-2|t|\int_\Omega\left(\left(\partial_t u_K+\nabla u_K\frac{x}{2t}\right)\xi+2F_K'u_K\xi+\nabla u_K\nabla\xi\right)^2\xi G dx\\&+CI(t)+C\exp\left(\frac{1}{6t}\right),\end{aligned}$$

where C is a constant depending on m, R, M_0 and E_0 only. By (2.10),

$$-I_K'(t)\ge -CI_K(t)-\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\quad\text{for}\quad t<0.$$

Therefore,

$$\begin{aligned}-I_K'(t)E_K(t)&=\left(-I_K'(t)+CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)E_K(t)\\&\quad-\left(CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)E_K(t)\\&\le\left(-I_K'(t)+CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)E_K(t).\end{aligned}$$

Using (2.10) again, we have

$$\begin{aligned}-I_K'(t)E_K(t)&\le\frac{|t|}{2}\left(-I_K'(t)+CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)^2\\&=\frac{|t|}{2}(I_K'(t))^2-|t|I_K'(t)\left(CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)\\&\quad+\frac{|t|}{2}\left(CI_K(t)+\frac{C}{|t|}\exp\left(\frac{1}{6t}\right)\right)^2.\end{aligned}$$

Then we apply (2.11) to obtain

$$-I'_K(t)E_K(t) \leq \frac{|t|}{2}(I'_K(t))^2 + CE_K(t)I_K(t) + CE_K(t)\exp\left(\frac{1}{8t}\right) + C\left(I_K(t)\right)^2 + C\exp\left(\frac{1}{8t}\right). \tag{4.2}$$

Also, from (2.18), we have

$$\begin{aligned} I'_K(t) &= 2\int_\Omega s_K\left(\left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right)\xi + 2F'_K s_K\xi + \nabla s_K\nabla\xi\right)\xi G dx \\ &+ 2\int_\Omega u_K\left(\left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\xi + 2F'_K u_K\xi + \nabla u_K\nabla\xi\right)\xi G dx \\ &- 2|t|\int_\Omega \left(s_K\left(2F'_K s_K\xi + \nabla s_K\nabla\xi\right) + u_K\left(2F'_K u_K\xi + \nabla u_K\nabla\xi\right)\right)\xi G dx \\ &+ 2\int_\Omega (s_K^2 + |u_K|^2)\xi\nabla\xi\frac{x}{2t} G dx. \end{aligned} \tag{4.3}$$

Again by (2.4) and (2.20), we see that

$$\begin{aligned} &\left|\int_\Omega \left(s_K\left(2F'_K s_K\xi + \nabla s_K\nabla\xi\right) + u_K\left(2F'_K u_K\xi + \nabla u_K\nabla\xi\right)\right)\xi G dx\right| \\ &\leq CI_K(t) + C\exp\left(\frac{1}{6t}\right), \\ &\left|\int_\Omega (s_K^2 + |u_K|^2)\xi\nabla\xi\frac{x}{2t} G dx\right| \leq C\exp\left(\frac{1}{6t}\right). \end{aligned}$$

Combining (4.2) and (4.3) and estimates in the above, we obtain

$$\begin{aligned} &- I'_K(t)E_K(t) \\ &\leq 2|t|\Bigg[\int_\Omega s_K\left(\left(\partial_t s_K + \nabla s_K \frac{x}{2t}\right)\xi + 2F'_K s_K\xi + \nabla s_K\nabla\xi\right)\xi G dx \\ &\quad + \int_\Omega u_K\left(\left(\partial_t u_K + \nabla u_K \frac{x}{2t}\right)\xi + 2F'_K u_K\xi + \nabla u_K\nabla\xi\right)\xi G dx\Bigg]^2 \\ &\quad + CE_K(t)I_K(t) + CE_K(t)\exp\left(\frac{1}{8t}\right) + C(I_K(t))^2 + C\exp\left(\frac{1}{8t}\right). \end{aligned} \tag{4.4}$$

By (4.1), (4.4) and Cauchy's inequality, we see that

(4.5)
$$\frac{d}{dt}\left(\frac{E_K(t)}{I_K(t)}\right) \le C + C\frac{E_K(t)}{I_K(t)} + C\frac{E_K(t)}{(I_K(t))^2}\exp\left(\frac{1}{8t}\right) + \frac{C}{(I_K(t))^2}\exp\left(\frac{1}{8t}\right).$$

Now, we are ready to prove Theorem 1.2.

Let $t_0 > 0$ and $s(x,t_0)$ is not identically zero on Ω. We claim that $s(x,t_0)$ cannot vanish in an open subset in Ω.

If it is not true, there is x_0 such that for some $R > 0$, $B(x_0; 2R) \subset \Omega$, $s(x,t_0) = 0$ when $x \in B(x_0; R/8)$ and

$$\int_{B(x_0,R/4)-B(x_0,R/8)} (s^2 + |u|^2)(x,t_0)dx = 4c_0 > 0.$$

After a translation, we assume that $(x_0,t_0) = (0,0)$. By continuity, there is r_1 such that for $|t - t_0| < (2r_1)^2$, we have

$$\int_{B(x_0,R/4)-B(x_0,R/8)} (s^2 + |u|^2)(x,t)dx \ge 2c_0 > 0.$$

Since (s_i,u_i) converges uniformly to (s,u) on compact subsets, we may assume that for each $i = 1,2,3...$, for $|t - t_0| < (2r_1)^2$,

$$\int_{B(x_0,R/4)-B(x_0,R/8)} (s_i^2 + |u_i|^2)(x,t)dx \ge c_0 > 0.$$

It is easy to compute that

$$I_K(t) \ge c_0 C \exp\left(\frac{1}{20t}\right) \quad \text{for} \quad -(2r_1)^2 \le t < 0, \tag{4.6}$$

and C is a constant depending on m and R only. Then by (4.5) and (4.6), we have

$$\frac{d}{dt}\left(\frac{E_K(t)}{I_K(t)}\right) \le C\left(1 + \frac{E_K(t)}{I_K(t)}\right) \quad \text{for} \quad -r_1^2 < t < 0,$$

where C is a constant depending on c_0, r_1, m, R, M_0 and E_0 only. This implies that

$$\frac{E_K(t)}{I_K(t)} \le e^{Cr_1^2}\left(1 + \frac{E_K(-r_1^2)}{I_K(-r_1^2)}\right) \quad \text{for} \quad -r_1^2 < t < 0.$$

By (2.15) and (4.6),

$$\frac{E_K(-r_1^2)}{I_K(-r_1^2)} \le \frac{C}{\ln(R^2/r_1^2)} \exp\left(\frac{1}{20r_1^2}\right).$$

Thus,

$$\frac{E_K(t)}{I_K(t)} \le N_0 \quad \text{for} \quad -r_1^2 < t < 0,$$

where N_0 is a constant depending on c_0, r_1, m, R, M_0 and E_0 only. By (2.11) and (4.6), we have

$$-\frac{I_K'(t)}{I_K(t)} \le \frac{4N_0 + C}{|t|} \quad \text{for} \quad -r_1^2 < t < 0.$$

After integrating from $-r_1^2$ to t, we obtain

$$I_K(t) \ge I_K(-r_1^2)|t|^{2N_0+C} \quad \text{for} \quad -r_1^2 < t < 0.$$

Thus, using (4.6) again, we see that

$$I_K(t_0 - r^2; x_0, t_0) \ge Dr^{2N_1} \quad \text{for} \quad 0 < r < r_1.$$

Moreover, the constants N_1 and D depend on c_0, r_1, m, R, M_0 and E_0 only. We may replace t_0 by $t_0 + r^2$ in the above arguments to have

$$I_K(t_0; x_0, t_0 + r^2) \ge Dr^{2N_1} \quad \text{for} \quad 0 < r < \frac{r_1}{4}.$$

Since (s_i, u_i) converges uniformly to (s, u) on compact subsets, the same is true for (s, u), i.e.,

$$\text{(4.7)} \qquad \int_\Omega (s^2 + |u|^2)\xi^2 G(x, t_0; x_0, t_0 + r^2)dx \ge Dr^{2N_1} \quad \text{for} \quad 0 < r < \frac{r_1}{4}.$$

It contradicts our assumption that $s(x, t_0) = 0$ when $x \in B(x_0; R/8)$ and the claim is proved.

Finally, by our claim, for any $x_0 \in \Omega$ and $B(x_0; 2R) \subset \Omega$, $s(x, t_0)$ is not zero somewhere inside $B(x_0; R/4)$, i.e., (4.6) holds. By repeating the arguments in the above, we see that (4.7) is also true. This proves the Theorem.

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References

[1] M. C. Calderer, D. Golovaty, F. H. Lin, and C. Liu, *Time evolution of nematic liquid crystals with variable degree of orientation*, SIAM J. Math. Anal. **33** (2002), no. 5, 1033–1047.

[2] Y. Chen and M. Struwe, *Existence and partial regularity results for the heat flow for harmonic maps*, Math. Z. **201** (1989), 83–103.

[3] F. H. Lin, *Nonlinear theory of defects in nematic liquid crystals, phase transition and flow phenomena*, Comm. Pure Appl. Math. **42** (1989), 789–814.

[4] F. H. Lin, *On nematic liquid crystals with variable degree of orientation*, Comm. Pure Appl. Math. **44** (1991), 453–468.

[5] C. C. Poon, *Unique continuation for parabolic equations*, Comm. in P.D.E. **21** (1996), 521–539.

[6] R. Schoen and S. T. Yau, Lectures on Harmonic Maps, International Press, 1997.

[7] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geometry **38** (1988), 485–502.

Department of Mathematics, National Chung Cheng University
Minghsiung 621, Chiayi, Taiwan
E-mail address: ccpoon@math.ccu.edu.tw

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