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Cohomogeneity one coassociative submanifolds in the bundle of anti-self-dual 2-forms over the 4-sphere

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Coassociative submanifolds are 4-dimensional calibrated submanifolds in G_2-manifolds. In this paper, we construct explicit examples of coassociative submanifolds in $\Lambda^2_- S^4$, which is the complete G_2-manifold constructed by Bryant and Salamon. Classifying the Lie groups which have 3- or 4-dimensional orbits, we show that the only homogeneous coassociative submanifold is the zero section of $\Lambda^2_- S^4$ up to the automorphisms and construct many cohomogeneity one examples explicitly. In particular, we obtain examples of non-compact coassociative submanifolds with conical singularities and their desingularizations.

1. Introduction

In 1996, Strominger, Yau and Zaslow [21] presented a conjecture explaining mirror symmetry of compact Calabi-Yau 3-folds in terms of dual fibrations by special Lagrangian 3-tori, including singular fibers. Analogously, fibrations of coassociative 4-folds in compact G_2-manifolds are expected to play the same role as special Lagrangian fibrations in Calabi-Yau manifolds. In this paper, we focus on the construction of coassociative 4-folds in a non-compact G_2-manifold. By constructing these examples, we will gain a greater understanding of coassociative geometry and local models for coassociative submanifolds in compact G_2-manifolds.

In $\mathbb{R}^7$, Harvey and Lawson gave SU(2)-invariant coassociative submanifolds in their pioneering paper [6]. Lotay [14, 15] constructed 2-ruled examples and ones with the $T^2 \times \mathbb{R}_{>0}$ symmetries using evolution equations. Fox [4] obtained a family of non-2-ruled, non-conical examples from a 2-ruled coassociative cone. Ionel, Karigiannis and Min-Oo [11] gave examples in $\Lambda^2_- \mathbb{R}^4 \cong \mathbb{R}^7$, which are the total spaces of certain rank 2 subbundles over immersed surfaces in $\mathbb{R}^4$. Karigiannis and Leung [12] generalized this method by twisting the bundles by a special section of a complementary bundle. Karigiannis and Min-Oo [13] applied the method in [11] to $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$ and obtained some examples. Here, $\Lambda^2_- S^4$ and $\Lambda^2_- \mathbb{C}P^2$ admit complete G_2-metrics constructed by Bryant and Salamon [2].

In this paper, we focus on the case of $\Lambda^2_- S^4$ and construct many explicit examples of coassociative submanifolds in $\Lambda^2_- S^4$. There exists a family of torsion-free G_2-structures $\{(\varphi_\lambda, g_\lambda)\}_{\lambda>0}$ on $\Lambda^2_- S^4$ (Proposition 3.1). For each $\lambda > 0$, the automorphism group of $(\Lambda^2_- S^4, \varphi_\lambda, g_\lambda)$ is SO(5) acting on $\Lambda^2_- S^4$ by the lift of the standard action on S^4 ([19]).

First, by classifying the Lie subgroups of SO(5) which have 4-dimensional orbits in $\Lambda^2_- S^4$, we obtain the following result.

Theorem 1.1. *Let $\{(\varphi_\lambda, g_\lambda)\}_{\lambda>0}$ be the family of torsion-free G_2-structures on $\Lambda^2_- S^4$ in Proposition 3.1. For each $\lambda > 0$, every homogeneous coassociative submanifold in $(\Lambda^2_- S^4, \varphi_\lambda, g_\lambda)$ is congruent under the action of* SO(5) *to the zero section $S^4 \subset \Lambda^2_- S^4$.*

Next, we prove that the Lie subgroup of SO(5) which have 3-dimensional orbits in $\Lambda^2_- S^4$ is one of the following (Proposition 4.15).

$$\begin{aligned} &\mathrm{SO}(4) = \mathrm{SO}(4) \times \{1\}, \qquad \mathrm{SO}(3) \times \mathrm{SO}(2), \quad \mathrm{U}(2),\ \mathrm{SU}(2) \subset \mathrm{SO}(4) \times \{1\}, \\ &\mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}, \qquad \mathrm{SO}(3) \text{ acting irreducibly on } \mathbb{R}^5. \end{aligned}$$

We derive O.D.E.s which give coassociative submanifolds by the cohomogeneity one method of Hsiang and Lawson [10] in each case. In many cases, O.D.E.s are solved explicitly and we obtain the following new examples.

Let $(x_1, x_2, x_3, x_4, x_5)$ be standard coordinates of $\mathbb{R}^5$ and regard S^4 as the unit sphere in $\mathbb{R}^5$. Let (a_1, a_2, a_3) be the local fiber coordinates of $\Lambda^2_- S^4$ by choosing a local frame for $\Lambda^2_- S^4$ as in Section 3.2.1.

Theorem 1.2 (Case of $\mathrm{SO}(3) \times \mathrm{SO}(2)$**).** *Let* $\mathrm{SO}(3) \times \mathrm{SO}(2)$ *act on* $\Lambda^2_- S^4$ *by the lift of the standard* $\mathrm{SO}(3) \times \mathrm{SO}(2)$*-action on* S^4*. For any* $C \in \mathbb{R}$*, set*

$$\begin{aligned} M_C = \mathrm{SO}(3) \times \mathrm{SO}(2) \cdot \Bigg\{ \left({}^t\!\left(x_1, 0, 0, \sqrt{1 - x_1^2}, 0\right), {}^t(a_1, 0, 0) \right); \\ G(a_1, x_1) = C,\ a_1 \in \mathbb{R}, 0 < x_1 \leq 1 \Bigg\}, \end{aligned}$$

where $G(a_1, x_1)$ *is defined in (5.2). Then* M_C *is coassociative and it is homeomorphic to*

$$\begin{cases} (S^2 \times \mathbb{R}^2) \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) & \text{for } C \neq 0, \\ S^4 \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) & \text{for } C = 0, \end{cases}$$

where S^4 *is the zero section of* $\Lambda^2_- S^4$*.*

Theorem 1.3 (Case of $\mathrm{SU}(2) \subset \mathrm{SO}(4) \times \{1\}$**).** *Let* $\mathrm{SU}(2)$ *act on* $\Lambda^2_- S^4$ *by the lift of the standard action of* $\mathrm{SU}(2) \subset \mathrm{SO}(4) \times \{1\}$ *on* S^4*. For any* $C \in \mathbb{R}$ *and* $v \in S^2 \subset \mathbb{R}^3$*, set*

$$\begin{aligned} M_{C,v} := \mathrm{SU}(2) \cdot \Bigg\{ \left({}^t\!\left(\sqrt{1 - x_5^2}, 0, 0, 0, x_5\right), rv \right); \\ F(r, x_5) = C, r \geq 0, -1 \leq x_5 \leq 1 \Bigg\}, \end{aligned}$$

where $F(r, x_5)$ *is defined in (5.7). Then* $M_{C,v}$ *is coassociative and it is homeomorphic to*

$$\begin{cases} \mathbb{R}^4 & \text{for } C > 0, \\ S^4 \sqcup (S^3 \times \mathbb{R}_{>0}) & \text{for } C = 0, \\ \mathcal{O}_{\mathbb{C}P^1}(-1) & \text{for } C < 0, \end{cases}$$

where S^4 *is the zero section of* $\Lambda^2_- S^4$ *and* $\mathcal{O}_{\mathbb{C}P^1}(-1)$ *is the tautological line bundle over* $\mathbb{C}P^1 \cong S^2$*.*

By using the stereographic local coordinates, the SU(2)-action is described as in the case of $\mathbb{R}^7$. See (5.8). In this sense, the above example is an analogue of an SU(2)-invariant coassociative submanifold in $\mathbb{R}^7$ given by Harvey and Lawson [6].

Theorem 1.4 (Case of $\mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}$**).** *Let* SO(3) *act on* $\Lambda^2_- S^4$ *by the lift of the standard* $\mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}$*-action on* S^4*. For* $C, D \geq 0$ *and* $E \in \mathbb{R}$*, set*

$$M_{C,D,E} = \mathrm{SO}(3) \cdot \left\{ ({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, 0)); \begin{array}{r} x_4^4(\lambda + r^2) = C, \\ x_5^4(\lambda + r^2) = D, \\ a_1 x_1 = E \end{array} \right\}.$$

Then $M_{C,D,E}$ *is coassociative and the topology of* $M_{C,D,E}$ *is given in Lemma 5.12. In particular, we obtain examples of non-compact coassociative submanifolds with conical singularities for* $(C, D) \neq (0, 0), E = 0, \sqrt{C} + \sqrt{D} = \sqrt{\lambda}$ *and their desingularizations.*

Theorem 1.5 (Case of irreducible SO(3)**).** *Let* SO(3) *act on* $\Lambda^2_- S^4$ *by the lift of the irreducible* SO(3)*-action on* S^4*. Let* $x_1(t)$*,* $x_5(t)$*,* $a_1(t)$*,* $a_2(t)$*,* $a_3(t)$ *be smooth functions on an open interval* $I \subset \mathbb{R}$ *satisfying* $0 \leq x_1(t) \leq \sqrt{3}/2$*,* $|x_5(t)| < 1/2$*,* $x_1^2(t) + x_5^2(t) = 1$*,*

$$\begin{aligned} 4\left\{(2\sqrt{3}x_1 + 4x_5 + 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_1 + 8x_1(-x_1 + \sqrt{3}x_5)\dot{a}_1& \\ -(\sqrt{3}x_1 + x_5 + 1)(1 - 2x_5)a_1 \frac{d}{dt}\log(\lambda + r^2) &= 0, \\ 4\left\{(2\sqrt{3}x_1 - 4x_5 - 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_2 + 8x_1(x_1 + \sqrt{3}x_5)\dot{a}_2& \\ +(-\sqrt{3}x_1 + x_5 + 1)(1 - 2x_5)a_2 \frac{d}{dt}\log(\lambda + r^2) &= 0, \\ 4\{-(x_5 + 1)\dot{x}_5 + 3x_1\dot{x}_1\} a_3 + 2(x_1^2 - 3x_5^2)\dot{a}_3& \\ +(1 + x_5)(1 - 2x_5)a_3 \frac{d}{dt}\log(\lambda + r^2) &= 0, \end{aligned}$$

where $r^2(t) = \sum_{j=1}^3 a_j^2(t)$ *and* $\dot{x}_1 = dx_1/dt$*, etc. Then*

$$\mathrm{SO}(3) \cdot \left\{ ({}^t(x_1(t), 0, 0, 0, x_5(t)), {}^t(a_1(t), a_2(t), a_3(t))); t \in I \right\},$$

is a coassociative submanifold invariant under the irreducible SO(3)*-action.*

This paper is organized as follows. In Section 2, we review the fundamental facts of calibrated geometry and G_2 geometry and introduce the cohomogeneity one method of Hsiang and Lawson [10]. In Section 3, we introduce the G_2-structure on $\Lambda^2_- S^4$ given by Bryant and Salamon [2]. In Section 4, we classify the connected closed subgroups of SO(5), which is the automorphism group of the G_2-manifold $\Lambda^2_- S^4$, and study their orbits. Classifying Lie subgroups which have 3- or 4-dimensional orbits, we prove Theorem 1.1. In Section 5, according to the classification in Section 4, we construct cohomogeneity one coassociative submanifolds and prove Theorems 1.2, 1.3, 1.4 and 1.5.

2. Preliminaries

Definition 2.1. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0 = e^{123} + e^{145} - e^{167} + e^{246} + e^{257} + e^{347} - e^{356}, \tag{2.1}$$

where $(e^1, \dots, e^7)$ is the standard dual basis on $\mathbb{R}^7$ and wedge signs are omitted. The stabilizer of φ_0 is the exceptional Lie group G_2:

$$G_2 = \{g \in GL(7, \mathbb{R}); g^* \varphi_0 = \varphi_0\}.$$

This is a 14-dimensional compact simply-connected simple Lie group.

The Lie group G_2 also fixes the standard metric $g_0 = \sum_{i=1}^7 (e^i)^2$, the orientation on $\mathbb{R}^7$ and the 4-form

$$*\varphi_0 = e^{4567} + e^{2367} - e^{2345} + e^{1357} + e^{1346} + e^{1256} - e^{1247},$$

where $*$ means the Hodge dual. They are uniquely determined by φ_0 via

$$-6 g_0(v_1, v_2) \mathrm{vol}_{g_0} = i(v_1)\varphi_0 \wedge i(v_2)\varphi_0 \wedge \varphi_0, \tag{2.2}$$

where vol_{g_0} is the volume form of g_0, $i(\cdot)$ is the interior product, and $v_i \in T(\mathbb{R}^7)$.

Definition 2.2. Let Y be a 7-dimensional oriented manifold and φ be a 3-form on Y. A 3-form φ is called a G_2-**structure** on Y if for each $y \in Y$, there exists an oriented isomorphism between $T_y Y$ and $\mathbb{R}^7$ identifying φ_y with φ_0. From (2.2), φ induces the metric g on Y, volume form on Y and $*\varphi \in \Omega^4(Y)$.

A G_2-structure φ is called **torsion-free** if φ is closed and coclosed: $d\varphi = d * \varphi = 0$. We call a triple (Y, φ, g) a G_2-**manifold** if $\varphi \in \Omega^3(Y)$ is a torsion-free G_2-structure on Y and g is the associated metric.

Lemma 2.3 ([3]). *Let (Y, φ, g) be a manifold with a G_2-structure. Then the holonomy group of g is contained in G_2 if and only if $d\varphi = d * \varphi = 0$.*

Recall the notion of a calibration introduced by Harvey and Lawson [6].

Definition 2.4. Let (Y, g) be an n-dimensional Riemannian manifold. A closed k-form φ on Y, where $1 \leq k \leq n$, is called a **calibration** on Y if $\varphi|_V \leq \mathrm{vol}_V$ for each point $p \in Y$ and every oriented k-dimensional subspace $V \subset T_pY$. We say that an oriented k-dimensional submanifold L of Y is a **calibrated submanifold** of Y (or calibrated by φ) if $\varphi|_{TL} = \mathrm{vol}_L$.

There are canonical calibrations on a G_2-manifold.

Lemma 2.5 ([6]). *Let (Y, φ, g) be a G_2-manifold. Then the G_2-structure φ and its Hodge dual $*\varphi$ define calibrations on Y.*

Definition 2.6 ([6]). An oriented 3-dimensional submanifold is called an **associative submanifold** of Y if it is calibrated by φ. An oriented 4-dimensional submanifold is called a **coassociative submanifold** of Y if it is calibrated by $*\varphi$.

Lemma 2.7 ([6]). *If $L \subset Y$ is an oriented 4-dimensional submanifold, then L is a coassociative submanifold of Y up to a possible change of orientation for L if and only if $\varphi|_{TL} = 0$.*

This description is often more useful and easier to work with.

2.1. Cohomogeneity one method

Let L be a coassociative submanifold of a G_2-manifold (Y, φ, g). The symmetry group K of L is defined to be the Lie subgroup of the automorphism group which fixes L. If the principal orbits of K are of codimension one in L, we call L a **cohomogeneity one** coassociative submanifold. The action of K on L is called a **cohomogeneity one action**.

Coassociative submanifolds are defined by first order nonlinear P.D.E.s, which are difficult to solve in general. By the cohomogeneity one action of the Lie group, we reduce the P.D.E.s of the coassociative condition to nonlinear O.D.E.s, which are easier to solve. This method was introduced in [10] for minimal submanifolds. We give a summary in our coassociative settings based on [8].

Lemma 2.8. *Let (Y, φ, g) be a G_2-manifold and G be a Lie subgroup of the automorphism group of (Y, φ, g). Let $\Sigma \subset Y$ be a subset which is transverse to the G-orbits and satisfies $G \cdot \Sigma = Y$. Suppose that G has 3-dimensional orbits on Y.*

Then the solution of the first order nonlinear O.D.E.s $\varphi|_{G \cdot \mathrm{Image}(c)} = 0$, where $c : I \to \Sigma$ is a path and $I \subset \mathbb{R}$ is an open interval, gives a G-invariant coassociative submanifold $G \cdot \mathrm{Image}(c)$.

Note that there is a similar construction by using evolution equations. This method was introduced by Lotay [15] for associative, coassociative and Cayley submanifolds.

3. Geometry in $\Lambda^2_- S^4$

3.1. G_2-structure on $\Lambda^2_- S^4$

We introduce the complete metric on the bundle $\Lambda^2_- S^4$ of anti-self-dual 2-forms over the 4-sphere S^4 obtained by Bryant and Salamon [2]. We also refer to [13, 20]. Since $\Lambda^2_- S^4$ has a connection induced by the Levi Civita connection on S^4, the tangent space $T_\omega(\Lambda^2_- S^4)$ has a canonical splitting $T_\omega(\Lambda^2_- S^4) \cong \mathcal{H}_\omega \oplus \mathcal{V}_\omega$ into horizontal and vertical subspaces for each $\omega \in \Lambda^2_- S^4$.

Proposition 3.1 (Bryant and Salamon [2]). *For $\lambda > 0$, define the 3-form $\varphi_\lambda \in \Omega^3(\Lambda^2_- S^4)$ and the metric g_λ on $\Lambda^2_- S^4$ by*

$$\varphi_\lambda = 2 s_\lambda d\tau + \frac{1}{s_\lambda^3} \mathrm{vol}_{\mathcal{V}}, \qquad g_\lambda = 2 s_\lambda^2 g_{\mathcal{H}} + \frac{1}{s_\lambda^2} g_{\mathcal{V}},$$

where $s_\lambda = (\lambda + r^2)^{1/4}$, r is the distance function measured by the fiber metric induced by that on S^4, τ is a tautological 2-form and $\mathrm{vol}_{\mathcal{V}}$ is the volume form of $g_{\mathcal{V}}$ on the vertical fiber.

Then for each $\lambda > 0$, $(\Lambda^2_- S^4, \varphi_\lambda, g_\lambda)$ is a G_2-manifold and g_λ is the complete metric with holonomy equal to G_2.

Remark 3.2. A complete holonomy G_2 metric is constructed not only on $\Lambda^2_- S^4$ but also on $\Lambda^2_- \mathbb{C}P^2$ in [2]. Of course, we can also apply the method in Section 2.1 to $\Lambda^2_- \mathbb{C}P^2$ and construct examples in theory.

By using a local frame, φ_λ is described as follows. Let $\{e^1, e^2, e^3, e^4\}$ be a local oriented orthonormal coframe with respect to the standard metric and the standard orientation on S^4. Define 2-forms ω_i on S^4 by

$$\omega_1 = e^{12} - e^{34}, \qquad \omega_2 = e^{13} - e^{42}, \qquad \omega_3 = e^{14} - e^{23}.$$

Then $\{\omega_1, \omega_2, \omega_3\}$ is a local oriented coframe of $\Lambda^2_- S^4$, which is orthogonal but not normalized to unit length, and induces the local fiber coordinates (a_1, a_2, a_3) of $\Lambda^2_- S^4$. Write $\nabla \omega_i = \sum_{j=1}^3 \gamma_{ij} \otimes \omega_j$, where ∇ is the induced connection from the Levi-Civita connection of the standard metric on S^4 and γ_{ij} is a local 1-form. Let $\pi : \Lambda^2_- S^4 \to S^4$ be the projection. Denoting $b_i = da_i + \sum_{j=1}^3 a_j \pi^* \gamma_{ji}$, we have

$$r^2 = \sum_{i=1}^3 a_i^2, \qquad \tau = \sum_{i=1}^3 a_i \pi^* \omega_i, \qquad d\tau = \sum_{i=1}^3 b_i \wedge \pi^* \omega_i, \qquad \mathrm{vol}_{\mathcal{V}} = b_{123},$$

where $b_{123} = b_1 \wedge b_2 \wedge b_3$. Thus the G_2-structure φ_λ is described as

$$\varphi_\lambda = 2 s_\lambda \sum_{i=1}^3 b_i \wedge \pi^* \omega_i + \frac{1}{s_\lambda^3} b_{123}. \tag{3.1}$$

Remark 3.3. For $\lambda = 0$, the metric g_0 is a cone metric on $\Lambda^2_- S^4 - \{\text{zero section}\} \cong \mathbb{C}P^3 \times \mathbb{R}_{>0}$. The metric $g_{\mathbb{C}P^3}$ on $\mathbb{C}P^3$ induced from g_0 is not the standard metric, but a 3-symmetric Einstein, non-Kähler metric. The metric g_0 is not complete because of the singularity at 0, while its holonomy group is equal to G_2.

3.2. Local frames of $\Lambda^2_- S^4$

We use the following local frames of $\Lambda^2_- S^4$ for the convenience of computations.

3.2.1. Local frame on $S^4 - \{x_5 = \pm 1\}$. Define a local oriented orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on $S^4 - \{x_5 = \pm 1\}$ by

$$(e_1, e_2, e_3, e_4) = \frac{1}{\sqrt{1 - x_5^2}} \left(\begin{pmatrix} -x_2 \\ x_1 \\ -x_4 \\ x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_3 \\ x_4 \\ x_1 \\ -x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_4 \\ -x_3 \\ x_2 \\ x_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_1 x_5 \\ -x_2 x_5 \\ -x_3 x_5 \\ -x_4 x_5 \\ 1 - x_5^2 \end{pmatrix} \right).$$

Let $\{e^i\}$ be the dual coframe of $\{e_i\}$. Set the local orthogonal trivialization $\{\omega_1, \omega_2, \omega_3\} = \{e^{12} - e^{34}, e^{13} - e^{42}, e^{14} - e^{23}\}$ of $\Lambda^2_- S^4$ and denote by

(a_1, a_2, a_3) local fiber coordinates with respect to $\{\omega_1, \omega_2, \omega_3\}$. Recall 1-forms γ_{ij} and b_i are defined by $\nabla \omega_i = \sum_{j=1}^3 \gamma_{ij} \otimes \omega_j$, $b_i = da_i + \sum_{j=1}^3 a_j \pi^* \gamma_{ji}$. Denote by ∇^{S^4} the Levi-Civita connection of the standard metric on S^4. Then we see the following by a straightforward computation.

Lemma 3.4.

$$(\nabla^{S^4}_{e_i} e^j) = \frac{1}{\sqrt{1-x_5^2}} \begin{pmatrix} x_5 e^4 & -e^3 & e^2 & -x_5 e^1 \\ e^3 & x_5 e^4 & -e^1 & -x_5 e^2 \\ -e^2 & e^1 & x_5 e^4 & -x_5 e^3 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(\gamma_{ij}) = \frac{1+x_5}{\sqrt{1-x_5^2}} \begin{pmatrix} 0 & -e^1 & e^2 \\ e^1 & 0 & e^3 \\ -e^2 & -e^3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} da_1 \\ da_2 \\ da_3 \end{pmatrix} + \frac{1+x_5}{\sqrt{1-x_5^2}} \begin{pmatrix} a_2 e^1 - a_3 e^2 \\ -a_1 e^1 - a_3 e^3 \\ a_1 e^2 + a_2 e^3 \end{pmatrix}.$$

3.2.2. Frame at $\underline{p_0} = {}^t(0,0,0,0,\pm 1)$. Set an oriented orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of $T_{\underline{p_0}} S^4$ by

$$f_1 = {}^t(1,0,0,0,0), \qquad f_2 = {}^t(0,1,0,0,0),$$
$$f_3 = {}^t(0,0,1,0,0), \qquad f_4 = {}^t(0,0,0,\pm 1,0).$$

Note that the induced orientation on $T_{{}^t(0,0,0,0,1)} S^4$ is opposite to that of $T_{{}^t(0,0,0,0,-1)} S^4$. Let $\{f^i\}$ be the dual coframe of $\{f_i\}$. Then a basis $\{\Omega_1, \Omega_2, \Omega_3\} = \{f^{12} - f^{34}, f^{13} - f^{42}, f^{14} - f^{23}\}$ of $\Lambda^2_- S^4|_{\underline{p_0}}$ gives fiber coordinates (A_1, A_2, A_3) of $\Lambda^2_- S^4|_{\underline{p_0}}$.

4. Orbits of closed Lie subgroups of SO(5)

For each $\lambda > 0$, the automorphism group of $(\Lambda^2_- S^4, \varphi_\lambda, g_\lambda)$ is SO(5) acting on $\Lambda^2_- S^4$ as the lift of the standard action on S^4 ([19]). We study the Lie subgroups of SO(5) to obtain homogeneous and cohomogeneity one coassociative submanifolds. By the classification of compact Lie groups, we obtain the following.

Lemma 4.1. *The k-dimensional connected closed Lie subgroup of* SO(5)*, where* $3 \le k \le 10$*, is one of the following.*

$$\begin{array}{ll} \mathrm{SO}(5), & \\ \mathrm{SO}(4) = \mathrm{SO}(4) \times \{1\}, & \mathrm{SU}(2) \subset \mathrm{SO}(4) \times \{1\}, \\ \mathrm{SO}(3) \times \mathrm{SO}(2), & \mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}, \\ \mathrm{U}(2) \subset \mathrm{SO}(4) \times \{1\}, & \mathrm{SO}(3) \text{ acting irreducibly on } \mathbb{R}^5. \end{array}$$

The proof is given in Appendix B. According to Lemma 4.1, we study the orbits on $\Lambda^2_- S^4$ of Lie subgroups of SO(5) above.

4.1. SO(4) = SO(4) × {1} and SO(5)-actions

In this subsection, We consider both the $\mathrm{SO}(4) = \mathrm{SO}(4) \times \{1\}$ and the SO(5)-orbits.

Lemma 4.2 (Orbits of the SO(4)**-action).** *By the* SO(4)*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5)$ *where* $x_1 \ge 0$*. The* SO(4)*-orbit through* $p_0 \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} \mathrm{SO}(4)/\mathrm{SO}(2) & \text{for } x_1 > 0, p_0 \neq 0, \\ S^3 & \text{for } x_1 > 0, p_0 = 0, \\ S^2 & \text{for } x_5 = \pm 1, p_0 \neq 0, \\ * & \text{for } x_5 = \pm 1, p_0 = 0. \end{cases}$$

Corollary 4.3. *Let* $\mathcal{O}$ *be an* SO(5)*-orbit. Then* $\dim \mathcal{O} \le 4$ *if and only if* $\mathcal{O}$ *is the zero section* S^4*.*

Proof of Lemma 4.2. It is obvious that any point in $\Lambda^2_- S^4$ is congruent to a point in the fiber of $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5)$, where $x_1 \ge 0$, by the $\mathrm{SO}(4) = \mathrm{SO}(4) \times \{1\}$-action.

Suppose that $x_1 > 0$. Since the stabilizer of the SO(4)-action on S^4 at $\underline{p_0}$ is $\mathrm{SO}(3) = \{1\} \times \mathrm{SO}(3) \times \{1\} \subset \mathrm{SO}(5)$, we consider this SO(3)-action on $\Lambda^2_- S^4|_{\underline{p_0}}$. Use the notation in Section 3.2.1. Since $x_2 = x_3 = x_4 = 0$, the action of $g = (g_{ij}) \in \mathrm{SO}(3) = \{1\} \times \mathrm{SO}(3) \times \{1\}$ is given by

$$g_* e_i = \sum_{j=1}^{3} g_{ji} e_j \quad \text{for } i = 1, 2, 3, \qquad g_* e_4 = e_4$$

at $\underline{p_0}$. Then the induced action of $g = (g_{ij}) \in \mathrm{SO}(3)$ on $\Lambda^2_- S^4|_{\underline{p_0}}$ is described as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto \begin{pmatrix} g_{33} & -g_{32} & -g_{31} \\ -g_{23} & g_{22} & g_{21} \\ -g_{13} & g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

Thus the stabilizer of the SO(4)-action on $\Lambda^2_- S^4$ at $p_0 = ({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3))$ is SO(2) when ${}^t(a_1, a_2, a_3) \neq 0$. It is SO(3) when ${}^t(a_1, a_2, a_3) = 0$.

Next, suppose that $x_5 = \pm 1$. Then the stabilizer of the SO(4)-action on S^4 at $\underline{p_0} = {}^t(0, 0, 0, 0, \pm 1)$ is SO(4). By using the frame in Section 3.2.2, the induced action of SO(4) on $\Lambda^2_- S^4|_{\underline{p_0}}$ is equivalent to that of $\mathrm{SO}(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$ on $\Lambda^2_\mp \mathbb{R}^4 = \mathbb{R}^3 = \mathrm{Im}\mathbb{H}$, which is described as

$$\begin{aligned} [(p, q)] \cdot a &= qa\overline{q} \qquad \text{if } \underline{p_0} = {}^t(0, 0, 0, 0, 1), \\ [(p, q)] \cdot a &= pa\overline{p} \qquad \text{if } \underline{p_0} = {}^t(0, 0, 0, 0, -1). \end{aligned}$$

This is the standard action of $\mathrm{Sp}(1)/\mathbb{Z}_2 = \mathrm{SO}(3)$ on $\mathbb{R}^3$, and hence we obtain the lemma. □

Proof of Corollary 4.3. It is obvious that any point in $\Lambda^2_- S^4$ is congruent to a point in the fiber of $\underline{p_0} = {}^t(1, 0, 0, 0, 0)$ by the SO(5)-action. By Lemma 4.2, the subgroup $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ has 5-dimensional orbits on each point of $\Lambda^2_- S^4|_{\underline{p_0}} - \{0\}$. Hence $\mathcal{O}$ must be the zero section S^4. □

4.2. SO(3) × SO(2)-action

Use the notation in Section 3.2.1.

Lemma 4.4 (Orbits of the SO(3) × SO(2)**-action).** *By the* SO(3) × SO(2)*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1, 0, 0, x_4, 0) \in S^4$*, where* $x_1, x_4 \geq 0$*. The* SO(3) × SO(2)*-orbit through* $p_0 = ({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3)) \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} \mathrm{SO}(3) \times \mathrm{SO}(2) & \textit{for } 0 < x_1 < 1, (a_2, a_3) \neq 0, \\ S^2 \times S^1 & \textit{for } 0 < x_1 < 1, (a_2, a_3) = 0, \\ (\mathrm{SO}(3) \times \mathrm{SO}(2))/\mathrm{SO}(2) & \textit{for } x_1 = 1, (a_2, a_3) \neq 0, \\ S^2 & \textit{for } x_1 = 1, (a_2, a_3) = 0, \\ S^2 \times S^1 & \textit{for } x_1 = 0, (a_1, a_2, a_3) \neq 0, \\ S^1 & \textit{for } x_1 = 0, (a_1, a_2, a_3) = 0. \end{cases}$$

When $x_1 = 1, (a_2, a_3) \neq 0$*, the dividing group* $\mathrm{SO}(2)$ *is identified with*

$$\left\{ \left(\begin{pmatrix} 1 & \\ & h \end{pmatrix}, h \right) \in \mathrm{SO}(3) \times \mathrm{SO}(2); h \in \mathrm{SO}(2) \right\}.$$

Proof. A direct computation gives the following descriptions. When $x_5 \neq \pm 1$, the action of $g = (g_{ij}) \in \mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}$ is given by

(4.1)

$$\begin{aligned} & g \cdot \left({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, a_3) \right) \\ = & \left({}^t(g_{11}x_1, g_{21}x_1, g_{31}x_1, x_4, x_5), \begin{pmatrix} g_{11} & g_{12} & -g_{13} \\ g_{21} & g_{22} & -g_{23} \\ -g_{31} & -g_{32} & g_{33} \end{pmatrix} {}^t(a_1, a_2, a_3) \right). \end{aligned}$$

When $x_4 \neq \pm 1$, the action of

$$h = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \in \mathrm{SO}(2) = \{I_3\} \times \mathrm{SO}(2)$$

is given by

$$\begin{aligned} & h \cdot \left({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3) \right) \\ = & \left({}^t(x_1, 0, 0, x_4 \cos\alpha, x_4 \sin\alpha), \begin{pmatrix} 1 & \\ & A \end{pmatrix} {}^t(a_1, a_2, a_3) \right), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} A = & \frac{1}{1 - x_4 \sin\alpha} \\ & \times \begin{pmatrix} x_4^2(1-\cos\alpha) - x_4 \sin\alpha + \cos\alpha & x_1 x_4 (1-\cos\alpha) - x_1 \sin\alpha \\ -x_1 x_4 (1-\cos\alpha) + x_1 \sin\alpha & x_4^2(1-\cos\alpha) - x_4 \sin\alpha + \cos\alpha \end{pmatrix}. \end{aligned}$$

At $\underline{p_0} = {}^t(0, 0, 0, 1, 0)$, set the orthonormal basis $\{f_1, f_2, f_3, f_4\}$ of $T_{\underline{p_0}} S^4$ by

$$\begin{aligned} f_1 &= {}^t(0, 0, -1, 0, 0), & f_2 &= {}^t(0, 1, 0, 0, 0), \\ f_3 &= {}^t(-1, 0, 0, 0, 0), & f_4 &= {}^t(0, 0, 0, 0, 1). \end{aligned}$$

Let $\{f^i\}$ be the dual coframe of $\{f_i\}$. Then the local trivialization $\{\Omega_1, \Omega_2, \Omega_3\} = \{f^{12} - f^{34}, f^{13} - f^{42}, f^{14} - f^{23}\}$ of $\Lambda^2_- S^4$ gives local fiber coordinates (A_1, A_2, A_3) of $\Lambda^2_- S^4$. The action of $g = (g_{ij}) \in \mathrm{SO}(3) = \mathrm{SO}(3) \times$

$\{I_2\}$ on $\Lambda^2_- S^4|_{p_0}$ is given by

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \mapsto \begin{pmatrix} g_{11} & g_{12} & -g_{13} \\ g_{21} & g_{22} & -g_{23} \\ -g_{31} & -g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

By these computations, we see the lemma as in the proof of Lemma 4.2. □

Define the basis $\{E_i\}_{1\leq i\leq 3}$ of $\mathfrak{so}(3)$ by

(4.3)
$$E_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and set $E_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathfrak{so}(2)$. Via the identifications $\mathfrak{so}(3) = \mathfrak{so}(3) \oplus \{0\}$ and $\mathfrak{so}(2) = \{0\} \oplus \mathfrak{so}(2)$, $\{E_i\}_{1\leq i\leq 4}$ form a basis of $\mathfrak{so}(3) \oplus \mathfrak{so}(2)$. By (4.1) and (4.2), the vector fields $\tilde{E}_i^*$ on $\Lambda^2_- S^4$ generated by E_i are described as

$$\begin{aligned}
\tilde{E}_1^* &= x_1(x_1 e_1 + x_4 e_2) - a_2 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2}, \\
\tilde{E}_2^* &= x_1(-x_4 e_1 + x_1 e_2) + a_3 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_3}, \\
\tilde{E}_3^* &= a_3 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial a_3}, \\
\tilde{E}_4^* &= x_4 e_4 + x_1 \left(-a_3 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_3} \right),
\end{aligned}$$

at $p_0 = \left({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3)\right)$. A straightforward computation gives the following.

Lemma 4.5. *At $p_0 = \left({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3)\right)$, we have*

$$\begin{aligned}
&\left(\pi^*\omega_j(\tilde{E}_1^*, \tilde{E}_2^*)\right) = (x_1^2, 0, 0), && \left(\pi^*\omega_j(\tilde{E}_1^*, \tilde{E}_4^*)\right) = (0, x_1 x_4^2, x_1^2 x_4), \\
&\left(\pi^*\omega_j(\tilde{E}_1^*, \tilde{E}_3^*)\right) = 0, && \left(\pi^*\omega_j(\tilde{E}_2^*, \tilde{E}_4^*)\right) = (0, x_1^2 x_4, -x_1 x_4^2), \\
&\left(\pi^*\omega_j(\tilde{E}_2^*, \tilde{E}_3^*)\right) = 0, && \left(\pi^*\omega_j(\tilde{E}_3^*, \tilde{E}_4^*)\right) = 0,
\end{aligned}$$

$$\left(b_i(\tilde{E}_j^*)\right) = \begin{pmatrix} -x_4(a_2x_4+a_3x_1) & x_4(-a_2x_1+a_3x_4) & 0 & 0 \\ a_1x_4^2 & a_1x_1x_4 & a_3 & -a_3x_1 \\ a_1x_1x_4 & -a_1x_4^2 & -a_2 & a_2x_1 \end{pmatrix}.$$

4.3. Action of $\mathbf{U(2)} \subset \mathbf{SO(4)} \times \{\mathbf{1}\}$

Use the notation in Section 3.2.1 and 3.2.2.

Lemma 4.6 (Orbits of the $\mathrm{U}(2)$**-action).** *By the* $\mathrm{U}(2)$*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5)$ *for some* $x_1 \geq 0$*. The* $\mathrm{U}(2)$*-orbit through* $p_0 = ({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)) \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} \mathrm{U}(2) & \textit{for } \ x_5 \neq \pm 1, (a_1, a_2) \neq 0, \\ S^3 & \textit{for } \ x_5 \neq \pm 1, (a_1, a_2) = 0, \\ S^2 & \textit{for } \ x_5 = 1, p_0 \neq 0, \\ S^1 & \textit{for } \ x_5 = -1, (A_2, A_3) \neq 0, \\ * & \textit{for } \ x_5 = 1, p_0 = 0, \ \textit{or } \ x_5 = -1, (A_2, A_3) = 0. \end{cases}$$

Proof. Suppose that $x_5 \neq \pm 1$. Denoting $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$, we have

$$(e_1, e_2, e_3, e_4) = \frac{1}{\sqrt{1-x_5^2}} \left(\begin{pmatrix} iz_1 \\ iz_2 \\ 0 \end{pmatrix}, \begin{pmatrix} -\overline{z}_2 \\ \overline{z}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i\overline{z}_2 \\ i\overline{z}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_5z_1 \\ -x_5z_2 \\ 1-x_5^2 \end{pmatrix} \right) \subset \mathbb{C}^2 \oplus \mathbb{R}.$$

We see that e_1, e_2, e_3 and e_4 are SU(2)-invariant. Namely, $g_* e_i = e_i$ for any $1 \leq i \leq 4$ and $g \in \mathrm{SU}(2)$. Then the 2-forms ω_i are all SU(2)-invariant, and hence $g \in \mathrm{SU}(2)$ acts on $\Lambda^2_- S^4$ by

$$g \cdot ({}^t(z_1, z_2, x_5), {}^t(a_1, a_2, a_3)) = ({}^t(g\,{}^t(z_1, z_2), x_5), {}^t(a_1, a_2, a_3)). \tag{4.4}$$

The action of $k(\theta) = \begin{pmatrix} 1 & \\ & e^{i\theta} \end{pmatrix} \in \mathrm{U}(2)$, where $\theta \in \mathbb{R}$, is given by

$$\begin{aligned} &(k(\theta)_* e_1, k(\theta)_* e_2, k(\theta)_* e_3, k(\theta)_* e_4) \\ &= (e_1, e_2 \cos\theta + e_3 \sin\theta, -e_2 \sin\theta + e_3 \cos\theta, e_4), \end{aligned}$$

which induces the action of $k(\theta)$ on $\Lambda^2_- S^4$ described as

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}. \tag{4.5}$$

Since any element in U(2) is described as $k(\theta)g$ for some θ and $g \in \mathrm{SU}(2)$, we see the case $x_5 \neq \pm 1$.

Next, suppose that $x_5 = \pm 1$. Then the stabilizer of the U(2)-action on S^4 at $\underline{p_0} = {}^t(0,0,0,0,\pm 1)$ is U(2). By using the notation in Section 3.2.2, the induced action of $k(\theta)g$, where $\theta \in \mathbb{R}, g \in \mathrm{SU}(2)$, on $\Lambda^2_- S^4|_{\underline{p_0}}$ is described as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \varpi'(g) \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},$$

where ϖ' is a a double covering $\varpi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ (resp. a trivial representation) when $x_5 = 1$ (resp. $x_5 = -1$). This gives the proof in the case $x_5 = \pm 1$. □

Note that the double covering $\varpi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is given by

$$\varpi\left(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right) = \begin{pmatrix} |a|^2 - |b|^2 & 2\mathrm{Im}(ab) & -2\mathrm{Re}(ab) \\ -2\mathrm{Im}(\bar{a}b) & \mathrm{Re}(a^2 + b^2) & \mathrm{Im}(a^2 + b^2) \\ 2\mathrm{Re}(\bar{a}b) & \mathrm{Im}(-a^2 + b^2) & \mathrm{Re}(a^2 - b^2) \end{pmatrix}, \tag{4.6}$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$.

Define the basis $\{E_i\}_{1 \leq i \leq 4}$ of $\mathfrak{u}(2)$ by

$$\begin{aligned} E_1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \\ E_3 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad E_4 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \end{aligned} \tag{4.7}$$

By (4.4) and (4.5), the vector fields $\tilde{E_i^*}$ on $\Lambda^2_- S^4$ generated by E_i are described as

$$\begin{aligned} &\tilde{E_1^*} = -x_1 e_2, \qquad \tilde{E_2^*} = x_1 e_3, \qquad \tilde{E_3^*} = x_1 e_1, \\ &\tilde{E_4^*} = x_1 e_1 - 2a_2 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} \end{aligned}$$

at $p_0 = \left({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)\right)$. A straightforward computation gives the following.

Lemma 4.7. *At* $p_0 = \left({}^t(x_1,0,0,0,x_5), {}^t(a_1,a_2,a_3)\right)$, *we have*

$$\left(\pi^*\omega_j(\tilde{E_1^*},\tilde{E_2^*})\right) = (0,0,x_1^2), \qquad \left(\pi^*\omega_j(\tilde{E_1^*},\tilde{E_4^*})\right) = (x_1^2,0,0),$$
$$\left(\pi^*\omega_j(\tilde{E_1^*},\tilde{E_3^*})\right) = (x_1^2,0,0), \qquad \left(\pi^*\omega_j(\tilde{E_2^*},\tilde{E_4^*})\right) = (0,-x_1^2,0),$$
$$\left(\pi^*\omega_j(\tilde{E_2^*},\tilde{E_3^*})\right) = (0,-x_1^2,0), \qquad \left(\pi^*\omega_j(\tilde{E_3^*},\tilde{E_4^*})\right) = 0,$$
$$\left(b_i(\tilde{E_j^*})\right) = \begin{pmatrix} (1+x_5)a_3 & 0 & (1+x_5)a_2 & (-1+x_5)a_2 \\ 0 & -(1+x_5)a_3 & -(1+x_5)a_1 & (1-x_5)a_1 \\ -(1+x_5)a_1 & (1+x_5)a_2 & 0 & 0 \end{pmatrix}.$$

4.4. Action of SU(2) $\subset$ SO(4) $\times$ {1}

The next lemma follows easily from the proof of Lemma 4.6.

Lemma 4.8 (Orbits of the SU(2)**-action).** *By the* SU(2)*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1,0,0,0,x_5)$ *with* $x_1 \geq 0$. *The* SU(2)*-orbit through* $p_0 \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} S^3 & \text{for } x_5 \neq \pm 1, \\ S^2 & \text{for } x_5 = 1, p_0 \neq 0, \\ * & \text{for } x_5 = 1, p_0 = 0 \text{ or } x_5 = -1. \end{cases}$$

Define the basis $\{E_1,E_2,E_3\}$ of the Lie algebra $\mathfrak{su}(2)$ of SU(2) by

$$E_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{4.8}$$

which satisfies $[E_j, E_{j+1}] = 2E_{j+2}$ for $j \in \mathbb{Z}/3$. Note that via the inclusion $\mathrm{SU}(2) \hookrightarrow \mathrm{SO}(4) \times \{1\}$, E_1, E_2 and E_3 correspond to

$$\begin{pmatrix} & I_2 & \\ -I_2 & & \\ & & 0 \end{pmatrix}, \qquad \begin{pmatrix} & J & \\ J & & \\ & & 0 \end{pmatrix}, \qquad \begin{pmatrix} J & & \\ & -J & \\ & & 0 \end{pmatrix}, \tag{4.9}$$

where $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, respectively. Since E_i in (4.8) agrees with E_i in (4.7) for $i = 1,2,3$, we have the same formula as Lemma 4.7.

4.5. SO(3) = SO(3) $\times$ $\{I_2\}$-action

Use the notation in Section 3.2.1.

Lemma 4.9 (Orbits of the SO(3)**-action).** *By the* SO(3)*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1, 0, 0, x_4, x_5)$ *for some* $x_1 \geq 0$*. The* SO(3)*-orbit through* $p_0 = ({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, a_3)) \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} \mathrm{SO}(3) & \text{for } x_1 > 0, (a_2, a_3) \neq 0, \\ S^2 & \text{for } x_1 > 0, (a_2, a_3) = 0 \text{ or } x_1 = 0, p_0 \neq 0, \\ * & \text{for } x_1 = 0, p_0 = 0. \end{cases}$$

Proof. We easily see the cases $x_1 > 0$ and $x_1 = 0, x_5 \neq \pm 1$ from (4.1). Suppose that $x_1 = 0, x_5 = \pm 1$. Then the stabilizer of the SO(3)-action on S^4 at $\underline{p_0} = {}^t(0, 0, 0, 0, \pm 1)$ is SO(3). By using the notation in Section 3.2.2, the action of $g = (g_{ij}) \in \mathrm{SO}(3)$ is given by

$$(g_* f_1, g_* f_2, g_* f_3, g_* f_4) = (f_1, f_2, f_3, f_4) \begin{pmatrix} g & \\ & 1 \end{pmatrix}$$

at $\underline{p_0}$. The induced action of $g = (g_{ij}) \in \mathrm{SO}(3)$ on $\Lambda^2_- S^4|_{\underline{p_0}}$ is described as

$$\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \mapsto \begin{pmatrix} g_{33} & -g_{32} & -g_{31} \\ -g_{23} & g_{22} & g_{21} \\ -g_{13} & g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix},$$

which gives the proof in the case $x_5 = \pm 1$. □

By Lemma 4.9, an SO(3)-orbit through $({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, a_3))$ is 3-dimensional when $x_1 > 0, (a_2, a_3) \neq 0$. By the fact that the stabilizer at its point is SO(2) and (4.1), its SO(3)-orbit contains a point $({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, 0))$, where $x_1 > 0, a_2 > 0$. Thus we may assume that $x_1 > 0, a_2 > 0, a_3 = 0$.

Let $\{E_i\}_{1 \leq i \leq 3}$ be the basis of $\mathfrak{so}(3)$ in (4.3). At $p_0 = \big({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, 0)\big)$, the vector fields $\tilde{E}_i^*$ on $\Lambda^2_- S^4$ generated by E_i are described as

$$\begin{aligned} \tilde{E}_1^* &= \frac{x_1}{\sqrt{1 - x_5^2}}(x_1 e_1 + x_4 e_2) - a_2 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_2}, \\ \tilde{E}_2^* &= \frac{x_1}{\sqrt{1 - x_5^2}}(-x_4 e_1 + x_1 e_2) - a_1 \frac{\partial}{\partial a_3}, \\ \tilde{E}_3^* &= -a_2 \frac{\partial}{\partial a_3}, \end{aligned}$$

by (4.1). A straightforward computation gives the following.

Lemma 4.10. *At* $p_0 = \left({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, 0)\right)$*, we have*

$$
\begin{aligned}
&\left(\pi^*\omega_j(\tilde{E}_1^*, \tilde{E}_2^*)\right) = (x_1^2, 0, 0),\\
&\left(\pi^*\omega_j(\tilde{E}_1^*, \tilde{E}_3^*)\right) = 0,\\
&\left(\pi^*\omega_j(\tilde{E}_2^*, \tilde{E}_3^*)\right) = 0,\\
&\left(b_i(\tilde{E}_j^*)\right) = \begin{pmatrix} \left(-1 + \frac{x_1^2}{1-x_5}\right) a_2 & -\frac{x_1 x_4}{1-x_5} a_2 & 0 \\ \left(1 - \frac{x_1^2}{1-x_5}\right) a_1 & \frac{x_1 x_4}{1-x_5} a_1 & 0 \\ \frac{x_1 x_4}{1-x_5} a_1 & \left(-1 + \frac{x_1^2}{1-x_5}\right) a_1 & -a_2 \end{pmatrix}.
\end{aligned}
$$

4.6. Irreducible SO(3)-action

The irreducible representation of SO(3) on $\mathbb{R}^5$ is described as follows.

Let V be the space of all 3×3 real symmetric traceless matrices, which is isomorphic to $\mathbb{R}^5$. Let SO(3) act on V by $g \cdot X = gXg^{-1}$, where $X \in V, g \in \mathrm{SO}(3)$. This action preserves the norm $|X|^2 = \mathrm{tr}(X^2)/2$, and hence induces the action on the unit sphere $S^4 = \{X \in V; |X| = 1\} \subset V$. We identify $V \cong \mathbb{R}^5$ by

$$
\begin{pmatrix} \lambda_1 & \mu_1 & \mu_2 \\ \mu_1 & \lambda_2 & \mu_3 \\ \mu_2 & \mu_3 & -\lambda_1 - \lambda_2 \end{pmatrix} \mapsto {}^t\!\left(\lambda_1 + \frac{\lambda_2}{2}, -\mu_2, \mu_3, \mu_1, -\frac{\sqrt{3}}{2}\lambda_2\right). \tag{4.10}
$$

Remark 4.11. We can also describe the irreducible representation of $\mathrm{SO}(3) = \mathrm{SU}(2)/\mathbb{Z}_2$ on $\mathbb{R}^5$ by the method in Appendix A. We use the description above because it is easier to work with.

Use the notation in Section 3.2.1.

Lemma 4.12 (Orbits of the irreducible SO(3)-action). *By the* SO(3)*-action, any point in* $\Lambda^2_- S^4$ *is mapped to a point in the fiber of* $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5)$ *where* $x_1 > 0, |x_5| \leq 1/2$.

When $|x_5| < 1/2$*, the* SO(3)*-orbit through*

$$
p_0 = ({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)) \in \Lambda^2_- S^4|_{\underline{p_0}}
$$

is diffeomorphic to

$$\begin{cases} \mathrm{SO}(3) & \textit{when} \ \ a_1a_2a_3 \neq 0 \ \textit{or one of} \ \{a_1, a_2, a_3\} \ \textit{is} \ 0, \\ \mathrm{SO}(3)/\mathbb{Z}_2 & \textit{when} \ \ \textit{two of} \ \{a_1, a_2, a_3\} \ \textit{are} \ 0, \\ \mathrm{SO}(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2) & \textit{when} \ \ a_1 = a_2 = a_3 = 0. \end{cases}$$

When $x_5 = 1/2$ *(resp.* $x_5 = -1/2$*), the* SO(3)*-orbit through* $p_0 = ({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)) \in \Lambda^2_- S^4|_{\underline{p_0}}$ *is diffeomorphic to*

$$\begin{cases} \mathrm{SO}(3) & \textit{for} \ \ a_2 \neq 0, (a_1, a_3) \neq 0, \\ S^2 & \textit{for} \ \ a_2 \neq 0, (a_1, a_3) = 0, \\ \mathrm{SO}(3)/\mathbb{Z}_2 & \textit{for} \ \ a_2 = 0, (a_1, a_3) \neq 0, \\ \mathbb{R}P^2 & \textit{for} \ \ a_1 = a_2 = a_3 = 0. \end{cases}$$

$$\left(\textit{resp.} \ \begin{cases} \mathrm{SO}(3) & \textit{for} \ \ a_1 \neq 0, (a_2, a_3) \neq 0, \\ S^2 & \textit{for} \ \ a_1 \neq 0, (a_2, a_3) = 0, \\ \mathrm{SO}(3)/\mathbb{Z}_2 & \textit{for} \ \ a_1 = 0, (a_2, a_3) \neq 0, \\ \mathbb{R}P^2 & \textit{for} \ \ a_1 = a_2 = a_3 = 0. \end{cases} \right)$$

Remark 4.13. The SO(3)-orbit in S^4 through ${}^t(\sqrt{3}, 0, 0, 0, \pm 1)/2$ is a superminimal surface called a Veronese surface. For example, see [9].

Proof. The first statement is well-known. See, for example, [1]. Set

$$\Sigma = \left\{ \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3); \lambda_1 \geq \lambda_2 \geq \lambda_3, \lambda_1 + \lambda_2 + \lambda_3 = 0, \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 2 \right\}.$$

Since every symmetric matrix is diagonalizable by an orthogonal matrix, unique up to the order of its diagonal elements, we see that every orbit of the SO(3)-action on S^4 intersects Σ at precisely one point. Via (4.10), Σ corresponds to

$$\left\{ {}^t(x_1, 0, 0, 0, x_5) \in S^4; x_1 > 0, -\frac{1}{2} \leq x_5 \leq \frac{1}{2} \right\}.$$

The stabilizer at $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5)$, where $|x_5| < 1/2$, is given by

$$\{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2); \epsilon_1 = \pm 1, \epsilon_2 = \pm 1\}.$$

Note that

$$e_1 = {}^t(0,1,0,0,0), \qquad e_2 = {}^t(0,0,1,0,0),$$
$$e_3 = {}^t(0,0,0,1,0), \qquad e_4 = {}^t(-x_5,0,0,0,x_1)$$

at $\underline{p_0}$. Then via the identification (4.10), the action of $k = \mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2)$ is given by

$$(k_* e_1, k_* e_2, k_* e_3, k_* e_4) = (\epsilon_2 e_1, \epsilon_1 e_2, \epsilon_1\epsilon_2 e_3, e_4),$$

which induces the action of k on $\Lambda^2_- S^4|_{\underline{p_0}}$ described as

$${}^t(a_1, a_2, a_3) \mapsto {}^t(\epsilon_1\epsilon_2 a_1, \epsilon_1 a_2, \epsilon_2 a_3).$$

The stabilizer at $\underline{p_0} = {}^t(\sqrt{3}/2, 0, 0, 0, \pm 1/2)$ is given by

$$\left\{ \begin{pmatrix} \det A & \\ & A \end{pmatrix} ; A \in \mathrm{O}(2) \right\}, \qquad \left\{ \begin{pmatrix} A & \\ & \det A \end{pmatrix} ; A \in \mathrm{O}(2) \right\},$$

respectively. The induced action of $\begin{pmatrix} \det A & \\ & A \end{pmatrix}$ (resp. $\begin{pmatrix} A & \\ & \det A \end{pmatrix}$), where $A = (a_{ij}) \in \mathrm{O}(2)$, on $\Lambda^2_- S^4|_{\underline{p_0}}$ is given by

$$\det A \begin{pmatrix} a_{11}(a_{11}^2 - 3a_{12}^2) & 0 & a_{12}(3a_{11}^2 - a_{12}^2) \\ 0 & 1 & 0 \\ a_{21}(3a_{11}^2 - a_{12}^2) & 0 & a_{22}(a_{11}^2 - 3a_{12}^2) \end{pmatrix}$$
$$\left(\text{resp. } \det A \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{21} \\ 0 & a_{12} & a_{11} \end{pmatrix} \right).$$

Hence we obtain the statement. □

The irreducible representation of $\mathrm{SO}(3)$ on $\mathbb{R}^5$ gives the inclusion $\mathrm{SO}(3) \hookrightarrow \mathrm{SO}(5)$. Via this inclusion, the basis $\{E_1, E_2, E_3\}$ of $\mathfrak{so}(3)$ in (4.3) correspond to

(4.11)
$$\begin{pmatrix} & J & \\ J & & \begin{matrix} 0 \\ \sqrt{3} \end{matrix} \\ & \begin{matrix} 0 & -\sqrt{3} \end{matrix} & \end{pmatrix}, \begin{pmatrix} -2J & & \\ & -J & \\ & & 0 \end{pmatrix}, \begin{pmatrix} & -I_2 & \\ I_2 & & \begin{matrix} -\sqrt{3} \\ 0 \end{matrix} \\ & \begin{matrix} \sqrt{3} & 0 \end{matrix} & \end{pmatrix},$$

where $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, respectively. Let E_i^* be the vector field on S^4 generated by E_i. Then we have at $\underline{p_0} = {}^t(x_1, 0, 0, 0, x_5) \in S^4$, where $x_1 > 0, |x_5| \leq 1/2$,

$$([E_i^*, e_j]) = \frac{\sqrt{3}}{x_1} \begin{pmatrix} -x_5 e_2 & x_5 e_1 & e_4 & -e_3 \\ 0 & \sqrt{3} x_1 e_3 & -\sqrt{3} x_1 e_2 & 0 \\ -x_5 e_3 & -e_4 & x_5 e_1 & e_2 \end{pmatrix},$$

$$(L_{E_i^*} \omega_j) = \begin{pmatrix} 0 & \frac{\sqrt{3}(1+x_5)}{x_1} \omega_3 & -\frac{\sqrt{3}(1+x_5)}{x_1} \omega_2 \\ 3\omega_2 & -3\omega_1 & 0 \\ -\frac{\sqrt{3}(1+x_5)}{x_1} \omega_3 & 0 & \frac{\sqrt{3}(1+x_5)}{x_1} \omega_1 \end{pmatrix}.$$

Hence at $p_0 = \left({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)\right) \in \Lambda_-^2 S^4$, where $x_1 > 0, |x_5| \leq 1/2$, the vector fields $\tilde{E_i^*}$ on $\Lambda_-^2 S^4$ generated by E_i are described as

$$\begin{aligned} \tilde{E_1^*} &= (x_1 + \sqrt{3} x_5) e_3 + \frac{\sqrt{3}(1 + x_5)}{x_1} \left(a_3 \frac{\partial}{\partial a_2} - a_2 \frac{\partial}{\partial a_3} \right), \\ \tilde{E_2^*} &= -2x_1 e_1 + 3 \left(a_2 \frac{\partial}{\partial a_1} - a_1 \frac{\partial}{\partial a_2} \right), \\ \tilde{E_3^*} &= (x_1 - \sqrt{3} x_5) e_2 + \frac{\sqrt{3}(1 + x_5)}{x_1} \left(-a_3 \frac{\partial}{\partial a_1} + a_1 \frac{\partial}{\partial a_3} \right). \end{aligned}$$

A straightforward computation gives the following.

Lemma 4.14. *At $p_0 = \left({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)\right)$, we have*

$$\begin{aligned} \left(\pi^* \omega_j (\tilde{E_1^*}, \tilde{E_2^*})\right) &= (0, 2x_1(x_1 + \sqrt{3} x_5), 0), \\ \left(\pi^* \omega_j (\tilde{E_1^*}, \tilde{E_3^*})\right) &= (0, 0, x_1^2 - 3x_5^2), \\ \left(\pi^* \omega_j (\tilde{E_2^*}, \tilde{E_3^*})\right) &= (2x_1(-x_1 + \sqrt{3} x_5), 0, 0), \end{aligned}$$

$$\left(b_i(\tilde{E_j^*})\right) = \begin{pmatrix} 0 & (1 - 2x_5) a_2 & -(\sqrt{3} x_1 + x_5 + 1) a_3 \\ (\sqrt{3} x_1 - x_5 - 1) a_3 & (-1 + 2x_5) a_1 & 0 \\ (-\sqrt{3} x_1 + x_5 + 1) a_2 & 0 & (\sqrt{3} x_1 + x_5 + 1) a_1 \end{pmatrix}.$$

4.7. Classification of homogeneous coassociative submanifolds

Summarizing the results in Section 4, we obtain the following.

Proposition 4.15. *The connected closed Lie subgroup of* SO(5) *which has a 4-dimensional orbit on* $\Lambda^2_- S^4$ *is either* SO(5)*, whose only 4-dimensional orbit is the zero section,* SO(3) $\times$ SO(2)*, or* U(2)*.*

The connected closed Lie subgroup of SO(5) *which has a 3-dimensional orbit on* $\Lambda^2_- S^4$ *is one of the following.*

$$\begin{array}{ll} \mathrm{SO}(4) = \mathrm{SO}(4) \times \{1\}, & \mathrm{SO}(3) \times \mathrm{SO}(2), \quad \mathrm{U}(2),\ \mathrm{SU}(2) \subset \mathrm{SO}(4) \times \{1\}, \\ \mathrm{SO}(3) = \mathrm{SO}(3) \times \{I_2\}, & \mathrm{SO}(3) \text{ acting irreducibly on } \mathbb{R}^5. \end{array}$$

By Proposition 4.15, we prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 4.15, we consider the actions of $\mathrm{SO}(5), \mathrm{SO}(3) \times \mathrm{SO}(2)$, and U(2). A 4-dimensional SO(5)-orbit, which is the zero section, is obviously coassociative.

Consider the SO(3) $\times$ SO(2)-action. Use the notation in Section 4.2. By Lemma 4.4, the SO(3) $\times$ SO(2) orbits through

$$p_0 = \left({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3)\right),$$

where $0 < x_1 < 1, (a_2, a_3) \neq 0$, are 4-dimensional. By (3.1) and Lemma 4.5, we compute

$$\begin{aligned} \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \tilde{E}_3^*) &= \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \tilde{E}_4^*) = 0, \\ \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_3^*, \tilde{E}_4^*) &= 2 s_\lambda x_1 x_4 (a_2 x_1 - a_3 x_4), \\ \varphi_\lambda(\tilde{E}_2^*, \tilde{E}_3^*, \tilde{E}_4^*) &= -2 s_\lambda x_1 x_4 (a_2 x_4 + a_3 x_1), \end{aligned}$$

at p_0. Hence the orbit is coassociative if and only if $x_1 = 0$ or $x_4 = 0$ or $a_2 = a_3 = 0$, which implies that the orbit is not 4-dimensional.

Consider the U(2)-action. Use the notation in Section 4.3. By Lemma 4.6, the U(2) orbits through $p_0 = \left({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3)\right)$, where $x_5 \neq \pm 1, (a_1, a_2) \neq 0$, are 4-dimensional. By (3.1) and Lemma 4.7, we compute

$$\begin{aligned} \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \tilde{E}_3^*) &= \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \tilde{E}_4^*) = 0, \\ \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_3^*, \tilde{E}_4^*) &= -4 s_\lambda x_1^2 a_2, \\ \varphi_\lambda(\tilde{E}_2^*, \tilde{E}_3^*, \tilde{E}_4^*) &= -4 s_\lambda x_1^2 a_1, \end{aligned}$$

at p_0. Hence the orbit is coassociative if and only if $x_1 = 0$ or $a_1 = a_2 = 0$, which implies that the orbit is not 4-dimensional. □

5. Cohomogeneity one coassociative submanifolds

The connected Lie subgroups which have 3-dimensional orbits are classified in Proposition 4.15. We construct cohomogeneity one coassociative submanifolds in each case. In this section, denote by $I \subset \mathbb{R}$ an open interval.

5.1. SO(4) = SO(4) × {1}-action

By Lemma 4.2, an SO(4)-orbit through $({}^t(x_1, 0, 0, 0, x_5), {}^t(0, 0, 0))$, where $x_1 > 0$, is 3-dimensional. We may find a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(x_1(t), 0, 0, 0, x_5(t)), {}^t(0, 0, 0)\right)$$

satisfying $x_1(t) > 0, \varphi_\lambda|_{\mathrm{SO}(4)\cdot\mathrm{Image}(c)} = 0$. However, since $\mathrm{SO}(4) \cdot \mathrm{Image}(c)$ is contained in the zero section which is an obvious coassociative submanifold, we cannot find new examples in this case.

5.2. SO(3) × SO(2)-action

We give a proof of Theorem 1.2. Recall the notation in Section 4.2. By Lemma 4.4, an $\mathrm{SO}(3) \times \mathrm{SO}(2)$-orbit through $({}^t(x_1, 0, 0, x_4, 0), {}^t(a_1, a_2, a_3))$ is 3-dimensional when

1) $0 < x_1 < 1, (a_2, a_3) = 0$,
2) $x_1 = 1, (a_2, a_3) \neq 0$, or
3) $x_1 = 0, (a_1, a_2, a_3) \neq 0$.

Consider <u>case 1</u>. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(x_1(t), 0, 0, x_4(t), 0), {}^t(a_1(t), 0, 0)\right),$$

where $x_1(t), x_4(t) > 0$. Note that $\tilde{E_3^*} = 0$ at $c(t)$. We find a path c satisfying $\varphi_\lambda|_{(\mathrm{SO}(3)\times\mathrm{SO}(2))\cdot\mathrm{Image}(c)} = 0$, where φ_λ is given by (3.1). We easily see that $\varphi_\lambda(\tilde{E_i^*}, \tilde{E_j^*}, \tilde{E_k^*})|_c = 0$ for $1 \leq i, j, k \leq 4$ by Lemma 4.5. Since $\dot{c} = (-\dot{x}_1 x_4 + x_1 \dot{x}_4) e_3 + \dot{a}_1 \frac{\partial}{\partial a_1}$ and

$$(\pi^* \omega_i(\tilde{E_j^*}, \dot{c})) = \begin{pmatrix} 0 & 0 & 0 & -\dot{x}_1 \\ x_1 \dot{x}_4 & -x_4 \dot{x}_4 & 0 & 0 \\ -x_4 \dot{x}_4 & -x_1 \dot{x}_4 & 0 & 0 \end{pmatrix}, \qquad (b_i(\dot{c})) = \begin{pmatrix} \dot{a}_1 \\ 0 \\ 0 \end{pmatrix},$$

we have at $c(t)$

$$\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \dot{c}) = 2s_\lambda\left(-2a_1x_4\dot{x}_4 + \dot{a}_1x_1^2\right) - s_\lambda^{-3}\dot{a}_1a_1^2x_4^2,$$
$$\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \tilde{E_4^*}) = \varphi_\lambda(\tilde{E_1^*}, \tilde{E_4^*}, \dot{c}) = \varphi_\lambda(\tilde{E_2^*}, \tilde{E_4^*}, \dot{c}) = 0.$$

Thus the condition $\varphi_\lambda|_{(\mathrm{SO}(3)\times\mathrm{SO}(2))\cdot\mathrm{Image}(c)} = 0$ is equivalent to

$$4a_1x_1\dot{x}_1 + \frac{1}{\lambda + a_1^2}\left\{-a_1^2 + (2\lambda + 3a_1^2)x_1^2\right\}\dot{a}_1 = 0.$$

This equation is solved explicitly as

$$G(a_1, x_1) = C \tag{5.1}$$

for $C \in \mathbb{R}$, where $G : \mathbb{R} \times (0, 1] \to \mathbb{R}$ is defined by

$$G(a_1, x_1) = a_1(\lambda + a_1^2)^{1/4}(2x_1^2 - 1) + \frac{1}{2}\int_0^{a_1} \frac{x^2 + 2\lambda}{(\lambda + x^2)^{3/4}} dx. \tag{5.2}$$

This solution is obtained by Maple 16 [17].

Remark 5.1. We give some remarks on the domain of G. Since we take a path $c(t) = \left({}^t(x_1(t), 0, 0, x_4(t), 0), {}^t(a_1(t), 0, 0)\right)$ satisfying $0 < x_1(t) < 1$, G is defined on $\mathbb{R} \times (0, 1)$ in the first place. Though G extends to a map $\mathbb{R} \times [0, 1] \to \mathbb{R}$ formally, it is not appropriate to define G on $x_1 = 0$.

In fact, by (4.1) and (4.2), the $\mathrm{SO}(3) \times \mathrm{SO}(2)$-orbit through

$$\left({}^t(0, 0, 0, 1, 0), {}^t(a_1, 0, 0)\right)$$

coincides with that through $\left({}^t(0, 0, 0, 1, 0), {}^t(-a_1, 0, 0)\right)$. Thus we should have $G(-a_1, 0) = G(a_1, 0)$. However, we easily see that $G(-a_1, 0) = -G(a_1, 0)$.

Such a problem does not occur when $x_1 = 1$. Hence we regard G as a map $\mathbb{R} \times (0, 1] \to \mathbb{R}$.

Set

$$M_C = \mathrm{SO}(3) \times \mathrm{SO}(2) \cdot \left\{ \left({}^t(x_1, 0, 0, \sqrt{1 - x_1^2}, 0), {}^t(a_1, 0, 0)\right); \right.$$
$$\left. G(a_1, x_1) = C,\ a_1 \in \mathbb{R}, 0 < x_1 \leq 1 \right\},$$
$$M_C^{\pm} = \mathrm{SO}(3) \times \mathrm{SO}(2) \cdot \left\{ \left({}^t(x_1, 0, 0, \sqrt{1 - x_1^2}, 0), {}^t(a_1, 0, 0)\right); \right.$$
$$\left. G(a_1, x_1) = C,\ \pm a_1 > 0, 0 < x_1 \leq 1 \right\}.$$

Then M_C is coassociative and $M_C = M_C^+ \sqcup M_C^-$ when $C \neq 0$ and $M_0 = M_0^+ \sqcup M_0^- \sqcup S^4$.

Lemma 5.2. *The coassociative submanifold M_C is homeomorphic to*

$$\begin{cases} (S^2 \times \mathbb{R}^2) \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) & \text{for } C \neq 0, \\ S^4 \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) \sqcup (S^2 \times S^1 \times \mathbb{R}_{>0}) & \text{for } C = 0, \end{cases}$$

where S^4 is the zero section of $\Lambda^2_- S^4$.

Proof. Since we have

$$\begin{aligned} \frac{\partial G}{\partial x_1} &= 4a_1(\lambda + a_1^2)^{1/4} x_1, \\ \frac{\partial G}{\partial a_1} &= 2^{-1}(\lambda + a_1^2)^{-3/4}\left\{(2x_1^2 - 1)(3a_1^2 + 2\lambda) + a_1^2 + 2\lambda\right\}, \end{aligned}$$

$G(a_1, \cdot)$ is monotonically increasing (resp. decreasing) on $(0, 1]$ for a fixed $a_1 > 0$ (resp. $a_1 < 0$) and $\lim_{x_1 \to 0} G(\cdot, x_1)$ (resp. $G(\cdot, 1)$) is monotonically decreasing (resp. increasing) on $\mathbb{R}$. We compute

$$G(0, \cdot) = 0, \qquad \lim_{a_1 \to \pm\infty} G(a_1, 1) = \pm\infty, \qquad \lim_{a_1 \to \pm\infty} \lim_{x_1 \to 0} G(a_1, x_1) = \mp\infty,$$

where we use the estimate

$$\int_0^{a_1} \frac{x^2 + 2\lambda}{(\lambda + x^2)^{3/4}} dx \leq a_1 \left\{(\lambda + a_1^2)^{1/4} + \lambda^{1/4}\right\} \qquad \text{for } a_1 \geq 0.$$

Thus for any $C \in \mathbb{R}$, there exists a unique $\alpha_C \in \mathbb{R}$ (resp. $\beta_C \in \mathbb{R}$) such that $C = G(\alpha_C, 1)$ (resp. $C = \lim_{x_1 \to 0} G(\beta_C, x_1)$). Note that C and α_C (resp. β_C) have the same (resp. opposite) sign. Now, define a function $g_C : \mathbb{R} - \{0\} \to \mathbb{R}$ by

$$g_C(a_1) = a_1^{-1}(\lambda + a_1^2)^{-1/4}\left(C - \frac{1}{2}\int_0^{a_1} \frac{x^2 + 2\lambda}{(\lambda + x^2)^{3/4}} dx\right).$$

Note that $G(a_1, x_1) = C$ is equivalent to $2x_1^2 - 1 = g_C(a_1)$. We may find the condition on a_1 so that $-1 < g_C(a_1) \leq 1$.

First, suppose that $\underline{C > 0}$.

Lemma 5.3. *When $a_1 > 0$, $g_C(a_1) > -1$ holds and $g_C(a_1) \leq 1$ is equivalent to $a_1 \geq \alpha_C$. When $a_1 < 0$, $g_C(a_1) < 1$ holds and $g_C(a_1) > -1$ is equivalent to $a_1 < \beta_C$.*

Proof. Suppose that $a_1 > 0$. Then $g_C(a_1) > -1$ is equivalent to $C > \lim_{x_1 \to 0} G(a_1, x_1)$, which holds for any $a_1 > 0$ since $\lim_{x_1 \to 0} G(a_1, x_1) < 0$. The condition that $g_C(a_1) \leq 1$ is equivalent to $C = G(\alpha_C, 1) \leq G(a_1, 1)$. Since $G(\cdot, 1)$ is monotonically increasing, this is equivalent to $a_1 \geq \alpha_C$. We can prove similarly when $a_1 < 0$. □

Set $\Gamma(C)^{\pm} = \{(a_1, x_1) \in \mathbb{R} \times (0, 1]; G(a_1, x_1) = C, \pm a_1 > 0\}$. By Lemma 5.3, we have homeomorphisms $[\alpha_C, \infty) \cong \Gamma(C)^+$ and $(-\infty, \beta_C) \cong \Gamma(C)^-$ via $a_1 \mapsto (a_1, \sqrt{(g_C(a_1) + 1)/2}$. Then from (4.1) and (4.2), it follows that

$$M_C^+ = \left\{ \left(\begin{pmatrix} g_{11}x_1 \\ g_{21}x_1 \\ g_{31}x_1 \\ \sqrt{1 - x_1^2} \cos\alpha \\ \sqrt{1 - x_1^2} \sin\alpha \end{pmatrix}, \begin{pmatrix} g_{11}a_1 \\ g_{21}a_1 \\ -g_{31}a_1 \end{pmatrix} \right); \begin{array}{c} 0 < x_1 = \sqrt{(g_C(a_1) + 1)/2} \leq 1 \\ a_1 \in [\alpha_C, \infty), \\ (g_{ij}) \in \mathrm{SO}(3), \\ \alpha \in \mathbb{R} \end{array} \right\},$$

which implies that M_C^+ is homeomorphic to

$$S^2 \times (S^1 \times [\alpha_C, \infty)/(S^1 \times \{\alpha_C\})) \cong S^2 \times \mathbb{R}^2.$$

In the same way, we see that M_C^- is homeomorphic to $S^2 \times S^1 \times \mathbb{R}_{>0}$. We can prove the case $\underline{C < 0}$ similarly.

When $\underline{C = 0}$, we see that $|g_C(a_1)| < 1$ holds for any $a_1 \neq 0$ as Lemma 5.3. Then we have homeomorphisms $(0, \infty) \cong \Gamma(0)^+$ and $(-\infty, 0) \cong \Gamma(0)^-$, and by Lemma 4.4, we obtain $M_0^{\pm} \cong S^2 \times S^1 \times \mathbb{R}_{>0}$. □

Remark 5.4. When $\lambda = 0$, the equation (5.1) is given by

$$a_1 |a_1|^{\frac{1}{2}} \left(2x_1^2 - \frac{2}{3} \right) = C. \tag{5.3}$$

We exhibit the graph of (5.3). The solid curve indicates the case $C > 0$, the dashed curve indicates the case $C = 0$ and the dotted curve indicates the case $C < 0$. We see that the solution (5.1) is asymptotic to this graph as $\lambda \to 0$. The vertical line gives a coassociative cone in $\Lambda^2_- S^4 - \{\text{zero section}\} \cong \mathbb{C}P^3 \times \mathbb{R}_{>0}$, which corresponds to a Lagrangian submanifold in the nearly Kähler $\mathbb{C}P^3$.

Consider case 2. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(1, 0, 0, 0, 0), {}^t(a_1(t), a_2(t), a_3(t)) \right).$$

We may assume that $a_2 > 0, a_3 = 0$ so that $c(t)$ is transverse to the $\mathrm{SO}(3) \times \mathrm{SO}(2)$-orbit. We find a path c satisfying $\varphi_\lambda|_{(\mathrm{SO}(3) \times \mathrm{SO}(2)) \cdot \mathrm{Image}(c)} = 0$, where

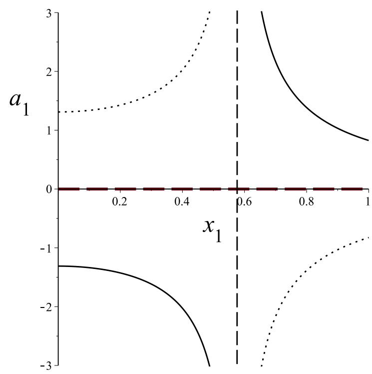

Figure 1: the graph of (5.3).

φ_λ is given by (3.1). Since $\dot{c} = \sum_{i=1}^{2} \dot{a}_i \frac{\partial}{\partial a_i}$, we have at $c(t)$

$$\begin{aligned}
(\pi^*\omega_i(\tilde{E}_j^*, \dot{c})) &= 0, \\
(b_i(\dot{c})) &= (\dot{a}_1, \dot{a}_2, 0)\,, \\
\varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= 2s_\lambda \dot{a}_1, \\
\varphi_\lambda(\tilde{E}_p^*, \tilde{E}_q^*, \dot{c}) &= 0 \qquad \text{for} \quad (p,q) \neq (1,2), (2,1).
\end{aligned}$$

Thus the condition $\varphi_\lambda|_{(\mathrm{SO}(3)\times\mathrm{SO}(2))\cdot\mathrm{Image}(c)} = 0$ is equivalent to $a_1 = C$ for $C \in \mathbb{R}$. Set $M_C = \mathrm{SO}(3) \times \mathrm{SO}(2) \cdot \{({}^t(1,0,0,0,0), {}^t(C, r, 0)); r \in \mathbb{R}\}$. By (4.1) and (4.2), M_C is explicitly described as

(5.4)

$$\begin{aligned}
M_C &= \left\{ \left(\begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} g_{11} & g_{12} & -g_{13} \\ g_{21} & g_{22} & -g_{23} \\ -g_{31} & -g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & u & -v \\ 0 & v & u \end{pmatrix} \begin{pmatrix} C \\ r \\ 0 \end{pmatrix} \right) ; \begin{array}{c} (g_{ij}) \in \mathrm{SO}(3) \\ u^2 + v^2 = 1 \\ r \in \mathbb{R} \end{array} \right\} \\
&= \left\{ \left({}^t(x_1, x_2, x_3, 0, 0), {}^t(a_1, a_2, a_3) \right) \in S^4 \times \mathbb{R}^3 ; a_1 x_1 + a_2 x_2 - a_3 x_3 = C \right\}.
\end{aligned}$$

Thus M_C is canonically identified with $\{(v, w) \in S^2 \times \mathbb{R}^3; \langle v, w \rangle = C\}$, which is homeomorphic to $\{(v, w) \in S^2 \times \mathbb{R}^3; \langle v, w \rangle = 0\} = TS^2$ via $(v, w) \mapsto (v, w - Cv)$.

Remark 5.5. Let $\Sigma^2 \subset S^4$ be an oriented 2-submanifold. Let $L \to \Sigma$ be a line bundle over Σ spanned by $\mathrm{vol}_\Sigma - *\mathrm{vol}_\Sigma$, where vol_Σ is a volume form of Σ and $*$ is a Hodge star in S^4. Denote by $L^\perp$ the orthogonal complement bundle of L in $\Lambda^2_- S^4$ and take a section η of L over Σ. By the argument in [13] and [12],

$$\eta + L^\perp = \left\{ (x, \eta_x + \sigma) \in \Lambda^2_- S^4|_\Sigma ; x \in \Sigma, \sigma \in L^\perp_x \right\}$$

is coassociative if and only if Σ is superminimal and $\eta \in \mathbb{R}(\mathrm{vol}_\Sigma - *\mathrm{vol}_\Sigma)$.

The submanifold M_C is a special case of these examples. In fact, $\pi(M_C)$ is a totally geodesic $S^2 = \{{}^t(x_1, x_2, x_3, 0, 0) \in S^4\}$ and define

$$\tau \in C^\infty(S^2, \Lambda^2_- S^4|_{S^2})$$

by

$$\tau_x = x_1(\omega_1)_x + x_2(\omega_2)_x - x_3(\omega_3)_x,$$

where $x = {}^t(x_1, x_2, x_3, 0, 0) \in S^2$. Note that τ is not the restriction of the tautological 2-form to S^2. We easily see that $M_C = C\tau + (\mathbb{R}\tau)^\perp$ and $\tau = \mathrm{vol}_{S^2} - *\mathrm{vol}_{S^2}$ by the SO(3)-invariance of τ.

Consider <u>case 3</u>. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(0, 0, 0, 1, 0), {}^t(a_1(t), a_2(t), a_3(t)) \right).$$

We may assume that $a_1 > 0, a_2 = a_3 = 0$ so that $c(t)$ is transverse to the $\mathrm{SO}(3) \times \mathrm{SO}(2)$-orbit. Since $\tilde{E_1^*} = a_1 \frac{\partial}{\partial a_2}$, $\tilde{E_2^*} = -a_1 \frac{\partial}{\partial a_3}$, $\tilde{E_3^*} = 0, \tilde{E_4^*} = e_4$ at $c(t)$, we compute $\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \dot{c}) = -a_1^2 \dot{a}_1 / s_\lambda^3$, which implies that a_1 is constant. Hence we cannot obtain a 4-submanifold.

5.3. Action of U(2) $\subset$ SO(4) $\times$ {1}

By Lemma 4.6, a U(2)-orbit through $p_0 = ({}^t(x_1, 0, 0, 0, x_5), {}^t(0, 0, a_3))$, where $x_1 > 0$, is 3-dimensional. At p_0, the stabilizer of the U(2)-action is U(1). Thus a U(2)-orbit through p_0 agrees with an SU(2)-orbit through p_0. The case of SU(2) is considered in the next subsection.

5.4. Action of SU(2) $\subset$ SO(4) $\times$ {1}

We give a proof of Theorem 1.3. Recall the notation in Section 4.3 and 4.4. By Lemma 4.8, an SU(2)-orbit through $({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3))$ is

3-dimensional when $x_5 \neq 0$. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(x_1(t), 0, 0, 0, x_5(t)), {}^t(a_1(t), a_2(t), a_3(t))\right),$$

where $x_1(t) > 0$. We find a path c satisfying $\varphi_\lambda|_{\mathrm{SU}(2)\cdot \mathrm{Image}(c)} = 0$, where φ_λ is given by (3.1). The condition $\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \tilde{E_3^*}) = 0$ is always satisfied. In fact, since the G_2-structure φ_λ is preserved by the SU(2)-action, we have $d(\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \tilde{E_3^*})) = 0$ by Cartan's formula. Since the action of SU(2) is not free, we have $\tilde{E_1^*} \wedge \tilde{E_2^*} \wedge \tilde{E_3^*} = 0$ at some point. Thus we have $\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \tilde{E_3^*}) = 0$.

Lemma 5.6. *The condition* $\varphi_\lambda|_{\mathrm{SU}(2)\cdot \mathrm{Image}(c)} = 0$ *is equivalent to*

$$4\dot{a}_i \frac{1 - x_5}{1 + x_5} - a_i \frac{d}{dt}\left\{\log(\lambda + r^2) + 8\log(1 + x_5)\right\} = 0 \qquad \textit{for } i = 1, 2, 3. \tag{5.5}$$

Proof. Since $\dot{c}(t) = (-\dot{x}_1 x_5 + x_1 \dot{x}_5)\, e_4 + \sum_{j=1}^3 \dot{a}_j \frac{\partial}{\partial a_j}$, we have

$$\left(\pi^*\omega_i(\tilde{E_j^*}, \dot{c})\right) = \begin{pmatrix} 0 & -\dot{x}_5 & 0 \\ -\dot{x}_5 & 0 & 0 \\ 0 & 0 & \dot{x}_5 \end{pmatrix},$$
$$b_j(\dot{c}) = \dot{a}_j \qquad \text{for } j = 1, 2, 3.$$

Then we compute

$$\begin{aligned}
&\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \dot{c})|_c \\
&= 2s_\lambda \sum_{i=1}^3 b_i \wedge \pi^*\omega_i(\tilde{E_1^*}, \tilde{E_2^*}, \dot{c}) + \frac{1}{s_\lambda^3} b_{123}(\tilde{E_1^*}, \tilde{E_2^*}, \dot{c}) \\
&= 2s_\lambda \left(-2(1 + x_5)\dot{x}_5 a_3 + \dot{a}_3 x_1^2\right) - \frac{(1 + x_5)^2 a_3}{2s_\lambda^3}\frac{d(r^2)}{dt} \\
&= \frac{s_\lambda(1 + x_5)^2}{2}\left\{4\dot{a}_3 \frac{1 - x_5}{1 + x_5} - a_3 \frac{d}{dt}\left(\log(\lambda + r^2) + 8\log(1 + x_5)\right)\right\}.
\end{aligned}$$

We compute $\varphi_\lambda(E_i^*, E_{i+1}^*, \dot{c})|_c$ in the same way and we see the lemma. □

By (5.5), we have

$$\frac{d}{dt}{}^t(a_1(t), a_2(t), a_3(t)) = f(t)\,{}^t(a_1(t), a_2(t), a_3(t))$$

for some function $f(t)$. The solution is given by

$$ {}^t(a_1(t), a_2(t), a_3(t)) = \exp\left(\int^t f(s)ds \right) v $$

for some $v \in \mathbb{R}^3$. Thus we may assume that ${}^t(a_1(t), a_2(t), a_3(t)) = r(t)v$ for a smooth function $r : I \to \mathbb{R}_{\geq 0}$ and $v \in S^2 \subset \mathbb{R}^3$. Then (5.5) is solved explicitly as

$$ F(r, x_5) = C \tag{5.6} $$

for $C \in \mathbb{R}$, where $F : [0, \infty) \times [-1, 1] \to \mathbb{R}$ is defined by

$$ F(r, x_5) = (1 - 3x_5)(\lambda + r^2)^{1/8}\sqrt{r} + \int_0^{\sqrt{r}} \frac{2\lambda}{(\lambda + x^4)^{7/8}} dx. \tag{5.7} $$

This solution is obtained by Maple 16 [17]. Though the definition of c implies that the domain of F is $[0, \infty) \times (-1, 1)$, F extends to a map $[0, \infty) \times [-1, 1] \to \mathbb{R}$ as in Remark 5.1. Thus we obtain the coassociative submanifold

$$ \begin{aligned} M_{C,v} := \mathrm{SU}(2) \cdot \Bigg\{ \left({}^t(\sqrt{1 - x_5^2}, 0, 0, 0, x_5), rv \right); \\ F(r, x_5) = C, r \geq 0, -1 \leq x_5 \leq 1 \Bigg\}, \end{aligned} $$

where $C \in \mathbb{R}$ and $v \in S^2 \subset \mathbb{R}^3$. We study the topology of $M_{C,v}$ now.

Lemma 5.7. *The coassociative submanifold M_C is homeomorphic to*

$$ \begin{cases} \mathbb{R}^4 & \text{for } C > 0, \\ S^4 \sqcup (S^3 \times \mathbb{R}_{>0}) & \text{for } C = 0, \\ \mathcal{O}_{\mathbb{C}P^1}(-1) & \text{for } C < 0, \end{cases} $$

where S^4 is the zero section of $\Lambda^2_- S^4$ and $\mathcal{O}_{\mathbb{C}P^1}(-1)$ is the tautological line bundle over $\mathbb{C}P^1 \cong S^2$.

Proof. Since we have

$$ \begin{aligned} \frac{\partial F}{\partial x_5} &= -3(\lambda + r^2)^{1/8}\sqrt{r}, \\ \frac{\partial F}{\partial r} &= 4^{-1}r^{-1/2}(\lambda + r^2)^{-7/8} \left\{ (1 - 3x_5)(3r^2 + 2\lambda) + 4\lambda \right\}, \end{aligned} $$

$F(r, \cdot)$ is monotonically decreasing on $[-1, 1]$ for a fixed $r > 0$ and $F(\cdot, -1)$ (resp. $F(\cdot, 1)$) is monotonically increasing (resp. decreasing) on $\mathbb{R}_{\geq 0}$. We compute

$$F(0, \cdot) = 0, \qquad \lim_{r\to\infty} F(r, \mp 1) = \pm\infty.$$

Thus for any $C > 0$ (resp. $C < 0$), there exists a unique $\alpha_C > 0$ (resp. $\beta_C > 0$) such that $C = F(\alpha_C, -1)$ (resp. $C = F(\beta_C, 1)$).

Now, define a function $f_C : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$f_C(r) = r^{-1/2}(\lambda + r^2)^{-1/8} \left(C - \int_0^{\sqrt{r}} \frac{2\lambda}{(\lambda + x^4)^{7/8}} dx \right).$$

Note that $F(x_5, t) = C$ is equivalent to $1 - 3x_5 = f_C(r)$. Since $-1 \leq x_5 \leq 1$, we may find the condition on a_1 so that $-2 \leq f_C(r) \leq 4$.

Lemma 5.8. *When $C > 0$, $f_C(r) > -2$ holds for any $r > 0$ and $f_C(r) \leq 4$ is equivalent to $r \geq \alpha_C$. When $C < 0$, $f_C(r) < 4$ holds for any $r > 0$ and $f_C(r) \geq -2$ is equivalent to $r \geq \beta_C$. When $C = 0$, $-2 < f_C(r) < 4$ holds for any $r > 0$.*

Proof. Suppose that $C > 0$. Then $f_C(r) > -2$ is equivalent to $C > F(r, 1)$, which holds for any $r > 0$ since $F(r, 1) < 0$. The condition that $f_C(r) \leq 4$ is equivalent to $C = F(\alpha_C, -1) \leq F(r, -1)$. Since $F(\cdot, -1)$ is monotonically increasing, this is equivalent to $r \geq \alpha_C$. We can prove similarly when $C \leq 0$. $\square$

Remark 5.9. Set $\Gamma(C) = \{(x_5, r) \in [-1, 1] \times [0, \infty); F(x_5, r) = C\}$. By Lemma 5.8, we have homeomorphisms $[\alpha_C, \infty) \cong \Gamma(C)$ when $C > 0$, $[\beta_C, \infty) \cong \Gamma(C)$ when $C < 0$, and $(0, \infty) \cong \Gamma(0) \cap \{r \neq 0\}$ via $r \mapsto ((1 - f_C(r))/3, r)$. Note that $\Gamma(C) \cap \{x_5 = 1\} = \emptyset$ when $C > 0$, $\Gamma(C) \cap \{x_5 = -1\} = \emptyset$ when $C < 0$, and $\Gamma(0) \cap \{r \neq 0\} \cap \{x_5 = \pm 1\} = \emptyset$.

Hence we see that

$$\begin{aligned} &M_{0,v} \cap \{r > 0\} \\ &= \left\{ \left(\begin{pmatrix} \sqrt{1 - x_5^2}\, a \\ \sqrt{1 - x_5^2}\, b \\ x_5 \end{pmatrix}, rv \right) \in \mathbb{C}^2 \oplus \mathbb{R} \oplus \mathbb{R}^3; \begin{array}{c} -1 < x_5 = \frac{1 - f_C(r)}{3} < 1, \\ r > 0, \\ a, b \in \mathbb{C}, \\ |a|^2 + |b|^2 = 1 \end{array} \right\}, \end{aligned}$$

which is homeomorphic to $S^3 \times \mathbb{R}_{>0}$.

When $C \neq 0$, $M_{C,v}$ intersects with $\Lambda^2_- S^4|_{{}^t(0,0,0,0,\pm 1)}$. To study the topology of $M_{C,v}$, we use the stereographic local coordinates.

First, suppose that $\underline{C > 0}$. By Remark 5.9, $M_{C,v}$ does not intersect with $\Lambda^2_- S^4|_{{}^t(0,0,0,0,1)}$. Take the stereographic local coordinates of $\Phi : S^4 - \{x_5 = 1\} \to \mathbb{R}^4$ given by

$$\Phi(x_1, \dots, x_5) = \frac{(x_1, x_2, x_3, -x_4)}{1 - x_5},$$
$$\Phi^{-1}(y_1, \dots, y_4) = \frac{\left(2y_1, 2y_2, 2y_3, -2y_4, -1 + |y|^2\right)}{1 + |y|^2},$$

where $|y|^2 = \Sigma_{i=1}^4 y_i^2$. The standard metric on S^4 is given by $4 \sum_{j=1}^4 dy_j^2/(1 + |y|^2)^2$, and hence we see that $\left\{2dy_i/(1 + |y|^2)\right\}_{i=1,\dots,4}$ is a local oriented orthonormal coframe. The trivialization

$$4(1 + |y|^2)^{-2}\{dy_{12} - dy_{34}, dy_{13} - dy_{42}, dy_{14} - dy_{23}\}$$

of $\Lambda^2_- S^4$ induces the local fiber coordinates $(\alpha_1, \alpha_2, \alpha_3)$. Setting $\zeta_1 = y_1 + iy_2, \zeta_2 = y_3 + iy_4$, the action of SU(2) is described as

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \cdot ({}^t(\zeta_1, \zeta_2), {}^t(\alpha_1, \alpha_2, \alpha_3)) = ({}^t(a\zeta_1 - \overline{b\zeta_2}, \overline{\beta\zeta_1} + \alpha\zeta_2), {}^t(\alpha_1, \alpha_2, \alpha_3)),$$

where $a, b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. Then we obtain

$$M_{C,v} = \left\{ \left({}^t(y_1 a, y_1 b)\,, rv'\right) \in \mathbb{C}^2 \oplus \mathbb{R}^3; \begin{array}{c} r \in [\alpha_C, \infty), y_1 = \sqrt{\frac{6}{f_C(r)+2} - 1}, \\ a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \end{array} \right\},$$

where $v' \in S^2$ is a corresponding element to v under the change of local coordinates. Then it follows that $M_{C,v}$ is homeomorphic to $(S^3 \times [\alpha_C, \infty))/(S^3 \times \{\alpha_C\}) \cong \mathbb{R}^4$.

Next, suppose that $\underline{C < 0}$. By Remark 5.9, $M_{C,v}$ does not intersect with $\Lambda^2_- S^4|_{{}^t(0,0,0,0,-1)}$. Take the stereographic local coordinates of $\Psi : S^4 - \{x_5 = -1\} \to \mathbb{R}^4$ given by

$$\Psi(x_1, \dots, x_5) = \frac{(x_1, x_2, x_3, x_4)}{1 + x_5},$$
$$\Psi^{-1}(u_1, \dots, u_4) = \frac{\left(2u_1, 2u_2, 2u_3, 2u_4, 1 - |u|^2\right)}{1 + |u|^2},$$

where $|u|^2 = \Sigma_{i=1}^4 u_i^2$. The standard metric on S^4 is given by $4 \sum_{j=1}^4 du_j^2/(1 + |u|^2)^2$, and hence we see that $\left\{2du_i/(1 + |u|^2)\right\}_{i=1,\dots,4}$ is a local oriented orthonormal coframe. The trivialization $4(1+|u|^2)^{-2}\{du_{12}-du_{34}, du_{13}-du_{42}, du_{14}-du_{23}\}$ of $\Lambda^2_- S^4$ induces the local fiber coordinates $(\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3})$. Setting $\underline{\zeta_1} = u_1 + iu_2, \underline{\zeta_2} = u_3 + iu_4$, the action of SU(2) is described as

$$g \cdot ({}^t(\underline{\zeta_1}, \underline{\zeta_2}), {}^t(\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3})) = (g\,{}^t(\underline{\zeta_1}, \underline{\zeta_2}), \varpi(g)\,{}^t(\underline{\alpha_1}, \underline{\alpha_2}, \underline{\alpha_3})), \tag{5.8}$$

where $g \in \mathrm{SU}(2)$ and $\varpi : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a double covering given by (4.6). Then we obtain

(5.9)

$$M_{C,v} = \left\{ \left(g\,{}^t(u_1, 0)\,, r\varpi(g)v'\right) \in \mathbb{C}^2 \oplus \mathbb{R}^3; \begin{array}{c} r \in [\beta_C, \infty), u_1 = \sqrt{\frac{6}{4-f_C(r)} - 1}, \\ g \in \mathrm{SU}(2) \end{array} \right\},$$

where $v' \in S^2$ is a corresponding element to v under the change of local coordinates.

Note that the topology of $M_{C,v}$ is independent of v. In fact, fix $v_0 \in S^2$ and let v'_0 be a corresponding element to v_0 under the change of local coordinates.

For any $v \in S^2$, there exists $g_0 \in \mathrm{SU}(2)$ such that $v' = \varpi(g_0)v'_0$. Then $M_{C,v} \cong M_{C,v_0}$ via $(g\,{}^t(u_1, 0), r\varpi(g)v') \mapsto (gg_0\,{}^t(u_1, 0), r\varpi(g)v')$. Thus we only have to consider the case $v'_0 = {}^t(1, 0, 0)$. Setting $v'_0 = {}^t(1, 0, 0)$ in (5.9), we obtain

$$\left\{ \left({}^t(u_1 a, u_1 b)\,, r\,{}^t(|a|^2 - |b|^2, 2\mathrm{Im}(a\bar{b}), 2\mathrm{Re}(a\bar{b})\right); \begin{array}{c} r \in [\beta_C, \infty), u_1 = \sqrt{\frac{6}{4-f_C(r)} - 1}, \\ a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \end{array} \right\},$$

which is homeomorphic to

$$\left\{(v, [w, r]) \in S^2 \times (S^3 \times [\beta_C, \infty))/(S^3 \times \{\beta_C\}); w \in p^{-1}(v)\right\},$$

where $p : S^3 \to \mathbb{C}P^1 = S^2$ is the Hopf fibration. This is the tautological line bundle $\mathcal{O}_{\mathbb{C}P^1}(-1)$ over $\mathbb{C}P^1$. □

Remark 5.10. When $\lambda = 0$, (5.6) is given by

$$(1 - 3x_5)r^{3/4} = C. \tag{5.10}$$

We exhibit the graph of (5.10). The solid curve indicates the case $C > 0$, the dashed curve indicates the case $C = 0$ and the dotted curve indicates the case $C < 0$. We see that the solution (5.6) is asymptotic to this graph as $\lambda \to 0$. The vertical line gives a coassociative cone in $\Lambda^2_- S^4 - \{\text{zero section}\} \cong$

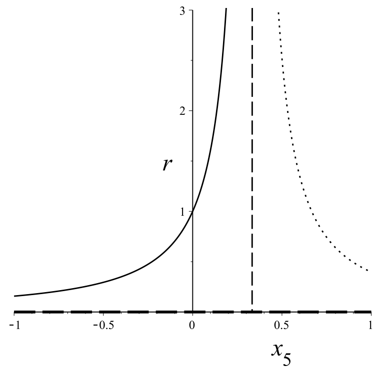

Figure 2: The graph of (5.10).

$\mathbb{C}P^3 \times \mathbb{R}_{>0}$, which corresponds to a Lagrangian submanifold in the nearly Kähler $\mathbb{C}P^3$.

5.5. $\mathbf{SO(3) = SO(3) \times \{I_2\}}$-action

We give a proof of Theorem 1.4. Recall the notation in Section 4.5. By Lemma 4.9, an SO(3)-orbit through $({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, a_3))$ is 3-dimensional when $x_1 > 0, (a_2, a_3) \neq 0$. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(x_1(t), 0, 0, x_4(t), x_5(t)), {}^t(a_1(t), a_2(t), 0)\right),$$

where $x_1(t) > 0, a_2(t) > 0$. We assume that $a_3 = 0$ so that $c(t)$ is transverse to the SO(3)-orbits. We find a path c satisfying $\varphi_\lambda|_{\mathrm{SO(3)\cdot Image}(c)} = 0$, where φ_λ is given by (3.1). We see that $\varphi_\lambda(\tilde{E_1^*}, \tilde{E_2^*}, \tilde{E_3^*}) = 0$ as in Section 5.4.

Lemma 5.11. *The condition* $\varphi_\lambda|_{\mathrm{SO(3)\cdot Image}(c)} = 0$ *is equivalent to*

$$4(2x_1\dot{x}_1a_1 + \dot{a}_1x_1^2) - (1 - x_1^2)a_1\frac{d}{dt}\log(\lambda + r^2) = 0, \tag{5.11}$$

$$4x_1\dot{x}_1 - \frac{d}{dt}\log(\lambda + r^2) + \frac{x_1^2}{1 - x_5}\left(4\dot{x}_5 + \frac{d}{dt}\log(\lambda + r^2)\right) = 0, \tag{5.12}$$

$$4\dot{x}_4 + \frac{x_4}{1 - x_5}\left(4\dot{x}_5 + \frac{d}{dt}\log(\lambda + r^2)\right) = 0, \tag{5.13}$$

where $r^2 = a_1^2 + a_2^2$.

Proof. Since $\dot{c} = \frac{-\dot{x}_1 x_4 + x_1 \dot{x}_4}{\sqrt{1-x_5^2}} e_3 + \frac{\dot{x}_5}{\sqrt{1-x_5^2}} e_4 + \dot{a}_1 \frac{\partial}{\partial a_1} + \dot{a}_2 \frac{\partial}{\partial a_2}$, we have

$$(\pi^* \omega_i(\tilde{E}_j^*, \dot{c})) = \begin{pmatrix} 0 & 0 & 0 \\ x_1 \dot{x}_4 + \frac{x_1 x_4 \dot{x}_5}{1-x_5} & x_1 \dot{x}_1 + \frac{x_1^2 \dot{x}_5}{1-x_5} & 0 \\ x_1 \dot{x}_1 + \frac{x_1^2 \dot{x}_5}{1-x_5} & -x_1 \dot{x}_4 - \frac{x_1 x_4 \dot{x}_5}{1-x_5} & 0 \end{pmatrix},$$
$$(b_i(\dot{c})) = \begin{pmatrix} \dot{a}_1 \\ \dot{a}_2 \\ \frac{-\dot{x}_1 x_4 + x_1 \dot{x}_4}{1-x_5} a_2 \end{pmatrix}.$$

Then we compute

$$\begin{aligned} \sum_{i=1}^{3} b_i \wedge \pi^* \omega_i(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= 2x_1 a_1 \left\{ -\frac{x_1 x_4 \dot{x}_4}{1-x_5} + \left(1 - \frac{x_1^2 + x_4^2}{1-x_5}\right) \frac{x_1 \dot{x}_5}{1-x_5} \right. \\ &\quad \left. + \left(1 - \frac{x_1^2}{1-x_5}\right) \dot{x}_1 \right\} + \dot{a}_1 x_1^2, \\ b_{123}(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= -\left\{ \left(1 - \frac{x_1^2}{1-x_5}\right)^2 + \frac{x_1^2 x_4^2}{(1-x_5)^2} \right\} a_1 (a_1 \dot{a}_1 + a_2 \dot{a}_2). \end{aligned}$$

Since $x_1^2 + x_4^2 + x_5^2 = 1, x_1 \dot{x}_1 + x_4 \dot{x}_4 + x_5 \dot{x}_5 = 0$, it follows that

$$\begin{aligned} \sum_{i=1}^{3} b_i \wedge \pi^* \omega_i(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= 2x_1 \dot{x}_1 a_1 + \dot{a}_1 x_1^2, \\ b_{123}(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= -\frac{(1-x_1^2) a_1}{2} \frac{d(r^2)}{dt}, \end{aligned}$$

which implies (5.11). In the same way, we compute

$$\begin{aligned} \sum_{i=1}^{3} b_i \wedge \pi^* \omega_i(\tilde{E}_1^*, \tilde{E}_3^*, \dot{c}) &= a_2 \left(x_1 \dot{x}_1 + \frac{x_1^2 \dot{x}_5}{1-x_5} \right), \\ b_{123}(\tilde{E}_1^*, \tilde{E}_3^*, \dot{c}) &= \left(-1 + \frac{x_1^2}{1-x_5} \right) \frac{a_2}{2} \frac{d(r^2)}{dt}, \\ \sum_{i=1}^{3} b_i \wedge \pi^* \omega_i(\tilde{E}_2^*, \tilde{E}_3^*, \dot{c}) &= -a_2 \left(x_1 \dot{x}_4 + \frac{x_1 x_4 \dot{x}_5}{1-x_5} \right), \\ b_{123}(\tilde{E}_2^*, \tilde{E}_3^*, \dot{c}) &= -\frac{x_1 x_4 a_2}{2(1-x_5)} \frac{d(r^2)}{dt}, \end{aligned}$$

and obtain (5.12) and (5.13). □

Next, we solve (5.11), (5.12), (5.13). Calculating (5.12) $+ x_4 \cdot$ (5.13), we have

$$4\dot{x}_5 + x_5 \frac{d}{dt}\log(\lambda + r^2) = 0. \tag{5.14}$$

Substitution of (5.14) into (5.13) gives

$$4\dot{x}_4 + x_4 \frac{d}{dt}\log(\lambda + r^2) = 0. \tag{5.15}$$

From (5.14) and (5.15), we have

$$\begin{aligned}(1 - x_1^2)\frac{d}{dt}\log(\lambda + r^2) &= (x_4^2 + x_5^2)\frac{d}{dt}\log(\lambda + r^2) \\ &= -4(x_4\dot{x}_4 + x_5\dot{x}_5) \\ &= 4x_1\dot{x}_1.\end{aligned}$$

which implies that (5.11) is equivalent to

$$x_1 \frac{d}{dt}(a_1 x_1) = 0. \tag{5.16}$$

Equations (5.14), (5.15), (5.16) are solved easily and we obtain

$$x_4^4(\lambda + r^2) = C, \qquad x_5^4(\lambda + r^2) = D, \qquad a_1 x_1 = E$$

for $C, D \geq 0, E \in \mathbb{R}$. Thus

$$M_{C,D,E} = \mathrm{SO}(3) \cdot \left\{ ({}^t(x_1, 0, 0, x_4, x_5), {}^t(a_1, a_2, 0));\ \begin{array}{c} x_4^4(\lambda + r^2) = C, \\ x_5^4(\lambda + r^2) = D, \\ a_1 x_1 = E \end{array} \right\}$$

is a coassociative submanifold for $C, D \geq 0, E \in \mathbb{R}$.

Next, we consider the topology of $M_{C,D,E}$.

Lemma 5.12. *Set* $N = (\mathbb{R}_{\geq 0} \times \mathrm{SO}(3))/(\{0\} \times \mathrm{SO}(3))$, *which is the cone over* $\mathrm{SO}(3)$ *with the apex. Then the topology of* $M_{C,D,E}$ *is given by the following.*

condition	*topology of $M_{C,D,E}$*
$C > 0, D > 0, E = 0, \sqrt{C} + \sqrt{D} \neq \sqrt{\lambda}$	$TS^2 \sqcup TS^2 \sqcup TS^2 \sqcup TS^2$
$C > 0, D > 0, E = 0, \sqrt{C} + \sqrt{D} = \sqrt{\lambda}$	$N \sqcup N \sqcup N \sqcup N$
$C > 0, D > 0, E \neq 0$	$TS^2 \sqcup TS^2 \sqcup TS^2 \sqcup TS^2$
$C = 0, D = 0$	TS^2
$C > 0, D = 0, E = 0, \sqrt{C} \neq \sqrt{\lambda}$	$TS^2 \sqcup TS^2$
$C > 0, D = 0, E = 0, \sqrt{C} = \sqrt{\lambda}$	$N \sqcup N$
$C > 0, D = 0, E \neq 0$	$TS^2 \sqcup TS^2$
$C = 0, D > 0, E = 0, \sqrt{D} \neq \sqrt{\lambda}$	$TS^2 \sqcup TS^2$
$C = 0, D > 0, E = 0, \sqrt{D} = \sqrt{\lambda}$	$N \sqcup N$
$C = 0, D > 0, E \neq 0$	$TS^2 \sqcup TS^2$

Lemma 5.13. *For any convergent sequence $\{(C_j, D_j)\} \subset (\mathbb{R}_{>0})^2$ satisfying $\sqrt{C_j} + \sqrt{D_j} < \sqrt{\lambda}$ for any j (or $\sqrt{C_j} + \sqrt{D_j} > \sqrt{\lambda}$ for any j) and $\sqrt{C_\infty} + \sqrt{D_\infty} = \sqrt{\lambda}$, where $C_\infty = \lim_{j\to\infty} C_j$, $D_\infty = \lim_{j\to\infty} D_j$, $M_{C_j,D_j,0}$ converges to $M_{C_\infty,D_\infty,0}$ in the sense of currents.*

Similarly, for any convergent sequence $\{C_j\} \subset \mathbb{R}_{>0}$ satisfying $\sqrt{C_j} < \sqrt{\lambda}$ for any j (or $\sqrt{C_j} > \sqrt{\lambda}$ for any j) and $\sqrt{C_\infty} = \sqrt{\lambda}$, where $C_\infty = \lim_{j\to\infty} C_j$, $M_{C_j,0,0}$ converges to $M_{C_\infty,0,0}$ and $M_{0,C_j,0}$ converges to $M_{0,C_\infty,0}$ in the sense of currents.

Proof of Lemma 5.12. First, suppose that $M_{C,D,E}$ does not intersect with $\Lambda^2_- S^4|_{{}^t(0,0,0,0,\pm 1)}$. Then by (4.1) we see that

$$M_{C,D,E} = \left\{ \left(\begin{pmatrix} g_{11}x_1 \\ g_{21}x_1 \\ g_{31}x_1 \\ x_4 \\ x_5 \end{pmatrix}, \begin{pmatrix} a_1 g_{11} + a_2 g_{12} \\ a_1 g_{21} + a_2 g_{22} \\ -a_1 g_{31} - a_2 g_{32} \end{pmatrix} \right); \begin{array}{c} x_4^4(\lambda + r^2) = C, \\ x_5^4(\lambda + r^2) = D, \\ a_1 x_1 = E, \\ (g_{ij}) \in \mathrm{SO}(3) \end{array} \right\}$$
$$= \left\{ \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \in S^4 \times \mathbb{R}^3; \begin{array}{c} x_4^4(\lambda + r^2) = C, \\ x_5^4(\lambda + r^2) = D, \\ (r^2 = \sum_{i=1}^3 a_i^2) \\ a_1 x_1 + a_2 x_2 - a_3 x_3 = E \end{array} \right\}.$$

We study the topology of $M_{C,D,E}$ in the following cases:

1) $C > 0, D > 0,$

a) $E = 0, \sqrt{C} + \sqrt{D} < \sqrt{\lambda}$, c) $E = 0, \sqrt{C} + \sqrt{D} = \sqrt{\lambda}$,
b) $E = 0, \sqrt{C} + \sqrt{D} > \sqrt{\lambda}$, d) $E \neq 0$,

2) $C = 0, D = 0$, 3) $C > 0, D = 0$, 4) $C = 0, D > 0$.

Consider case 1. Then $M_{C,D,E}$ does not intersect with $\Lambda^2_- S^4|_{{}^t(0,0,0,0,\pm 1)}$. Set

$$M^{\pm,+}_{C,D,E} = M_{C,D,E} \cap \{\pm x_4 > 0\} \cap \{x_5 > 0\},$$
$$M^{\pm,-}_{C,D,E} = M_{C,D,E} \cap \{\pm x_4 > 0\} \cap \{x_5 < 0\}.$$

Each $M^{\pm,\pm}_{C,D,E}$ is a connected component of $M_{C,D,E}$ and is homeomorphic to

(5.17)
$$N_{C,D,E} = \left\{ (v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 ; \langle v, w \rangle = E, (1 - |v|^2)\sqrt{\lambda + |w|^2} = \sqrt{C} + \sqrt{D} \right\}.$$

We only have to consider the topology of $N_{C,D,E}$.

Consider case 1-(a). We have $|v|^2 = 1 - (\sqrt{C} + \sqrt{D})/\sqrt{\lambda + |w|^2} \geq 1 - (\sqrt{C} + \sqrt{D})/\sqrt{\lambda} > 0$. Hence there is an homeomorphism $N_{C,D,0} \to \{(v, w) \in S^2 \times \mathbb{R}^3 ; \langle v, w \rangle = 0\} = TS^2$ via $(v, w) \mapsto (v/|v|, w)$.

Consider case 1-(b). We have $|w|^2 = (\sqrt{C} + \sqrt{D})^2/(1 - |v|^2)^2 - \lambda \geq (\sqrt{C} + \sqrt{D})^2 - \lambda > 0$. Hence there is an homeomorphism $N_{C,D,0} \to \{(w, v) \in S^2 \times \mathbb{R}^3 ; \langle w, v \rangle = 0, |v| < 1\} \cong TS^2$ via $(v, w) \mapsto (w/|w|, v)$.

Consider case 1-(c). A map $N = (\mathbb{R}_{\geq 0} \times \mathrm{SO}(3))/(\{0\} \times \mathrm{SO}(3)) \to N_{C,D,0}$ defined by $[(r, (g_1, g_2, g_3))] \mapsto (f(r) g_1, r g_2)$, where $g_i \in \mathbb{R}^3$, $\langle g_i, g_j \rangle = \delta_{ij}$, and $f(r) = \sqrt{1 - (\sqrt{C} + \sqrt{D})/\sqrt{\lambda + r^2}}$, gives a homeomorphism.

Consider case 1-(d). Since $N_{C,D,E} \cong N_{C,D,-E}$ via $(v, w) \mapsto (v, -w)$, we may assume that $E > 0$. Since $E \neq 0$, we have $v, w \neq 0$ for any $(v, w) \in N_{C,D,E}$. Define $c_0 \in \mathbb{R}$ and a function $f : (c_0, \infty) \to (f(c_0), 1)$ by

$$c_0 = \begin{cases} 0 & \text{when } (\sqrt{C} + \sqrt{D})^2 - \lambda \leq 0, \\ \sqrt{(\sqrt{C} + \sqrt{D})^2 - \lambda} & \text{when } (\sqrt{C} + \sqrt{D})^2 - \lambda \geq 0, \end{cases}$$
$$f(r) = \sqrt{1 - \frac{\sqrt{C} + \sqrt{D}}{\sqrt{\lambda + r^2}}}.$$

Then f is bijective and monotonically increasing. Note that for $(v, w) \in N_{C,D,E}$, we have $f(|w|) = |v|$. Since $r f(r) : (c_0, \infty) \to (0, \infty)$ is bijective and monotonically increasing, there exists a unique $d_0 > c_0 > 0$ such that $d_0 f(d_0) = E$. Now define a function

$$g : [d_0, \infty) \to [0, \infty) \quad \text{by } g(r) = \sqrt{r^2 - (E^2/f(r)^2)}.$$

Note that for $(v, w) \in N_{C,D,E}$, we have $|w - (Ev/|v|^2)| = g(|w|)$.

Define a map $\Phi : N_{C,D,E} \to \{(v', w') \in S^2 \times \mathbb{R}^3; \langle v, w\rangle = 0\} = TS^2$ by $\Phi(v, w) = (v/|v|, w - (Ev/|v|^2))$. Then Φ is a homeomorphism and the inverse map Φ^{-1} is given by

$$\Phi^{-1}(v', w') = (f(g^{-1}(|w'|))v', w' + (Ev'/f(g^{-1}(|w'|)))).$$

Consider <u>case 2</u>. By definition, we have $x_4 = x_5 = 0$. Then

$$M_{0,0,E} = \left\{\left({}^t(x_1, x_2, x_3, 0, 0)\,, {}^t(a_1, a_2, a_3)\right) \in S^4 \times \mathbb{R}^3; a_1x_1 + a_2x_2 - a_3x_3 = E\right\},$$

which is obtained in (5.4) and is homeomorphic to TS^2.

Consider <u>case 3</u>. By definition, we have $x_5 = 0$ and

$$M_{C,0,E} = \left\{\left({}^t(x_1, x_2, x_3, x_4, 0)\,, {}^t(a_1, a_2, a_3)\right) \in S^4 \times \mathbb{R}^3; \begin{array}{c} x_4^4(\lambda + r^2) = C, \\ a_1x_1 + a_2x_2 - a_3x_3 = E \end{array}\right\}.$$

Set $M^{\pm}_{C,0,E} = M_{C,0,E} \cap \{\pm x_4 > 0\}$. Each $M^{\pm}_{C,0,E}$ is a connected component of $M_{C,0,E}$ and is homeomorphic to $N_{C,0,E}$ defined in (5.17).

Consider <u>case 4</u>. By (4.2), $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SO}(2) = \{I_3\} \times \mathrm{SO}(2) \subset \mathrm{SO}(5)$ gives a homeomorphism from $N_{0,D,E}$ to $N_{D,0,E}$. Hence this case is reduced to <u>case 3</u>. $\square$

Proof of Lemma 5.13. We only have to prove that $N_{C_j,D_j,0}$ converges to $N_{C_\infty,D_\infty,0} - \{(0, 0)\}$ in the sense of currents. Note that sets differing only a set of measure zero are identified in the theory of currents.

Suppose that $\sqrt{C_j} + \sqrt{D_j} < \sqrt{\lambda}$ for any j. Then by the proof of Lemma 5.12, there is a homeomorphism

$$h_{C_j,D_j} : N_{C_j,D_j,0} \to \{(v, w) \in S^2 \times \mathbb{R}^3; \langle v, w\rangle = 0\} = TS^2$$

via $(v, w) \mapsto (v/|v|, w)$. Note that $h^{-1}_{C_j,D_j}$ is given by

$$(v', w') \mapsto (f_{C_j,D_j}(|w'|)v', w'),$$

where $f_{C_j,D_j}(r) = \sqrt{1 - (\sqrt{C_j} + \sqrt{D_j})/\sqrt{\lambda + r^2}}$.

On the other hand, $N_{C_\infty,D_\infty,0} - \{(0,0)\}$ is homeomorphic to $TS^2 - \{0\}$ via $h_{C_\infty,D_\infty} : (v,w) \mapsto (v/|v|, w)$ and $h^{-1}_{C_\infty,D_\infty}$ is given by

$$(v',w') \mapsto (f_{C_\infty,D_\infty}(|w'|)v', w').$$

Then we see that for any compactly supported 4-form α on $\mathbb{R}^3 \times \mathbb{R}^3$

$$\begin{aligned}\int_{N_{C_j,D_j,0}} \alpha = \int_{TS^2-\{0\}} (h^{-1}_{C_j,D_j})^* \alpha \\ \to \int_{TS^2-\{0\}} (h^{-1}_{C_\infty,D_\infty})^* \alpha = \int_{N_{C_\infty,D_\infty,0}-\{(0,0)\}} \alpha,\end{aligned}$$

which implies that $N_{C_j,D_j,0}$ converges to $N_{C_\infty,D_\infty,0} - \{(0,0)\}$ in the sense of currents. We can prove the other cases similarly and obtain the statement. □

Remark 5.14. Use the notation in [16]. By Lemma 5.12, $M_{C,D,E}$ is a coassociative submanifold with conical singularities when (i) $C > 0$, $D > 0$, $E = 0$, $\sqrt{C} + \sqrt{D} = \sqrt{\lambda}$, (ii)$C > 0$, $D = 0$, $E = 0$, $\sqrt{C} = \sqrt{\lambda}$, or (iii)$C = 0$, $D > 0$, $E = 0$, $\sqrt{D} = \sqrt{\lambda}$. In each case, the tangent cone is modeled on $C(L) = \mathbb{R}_{>0} \times L$, where L is given by

$$\begin{aligned}L = \left\{ {}^t(0, z_1, z_2, z_3) \in \mathbb{R} \oplus \mathbb{C}^3 ; z_1^2 + z_2^2 + \overline{z}_3^2 = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1 \right\} \\ \cong \mathrm{SO}(3).\end{aligned}$$

We calculate the rate at singular points as follows. For simplicity, we only consider the case of $M^+_{\lambda,0,0}$ which is singular at $p_0 = \left({}^t(0,0,0,1,0), {}^t(0,0,0)\right)$.

Let $B(0,r) \subset \mathbb{R}^4$ be an open ball of radius r. Set $D = \{x_4 > 0\} \subset S^4$ and $k = 2^{-1/2}\lambda^{-1/4}$. Define $\chi : B(0,1/k) \times \mathbb{R}^3 \to D \times \mathbb{R}^3$ by

$$\begin{aligned}&\left({}^t(u_1,u_2,u_3,u_4), {}^t(a_1,a_2,a_3)\right) \\ &\mapsto \left({}^t\!\left(-ku_3, ku_2, -ku_1, \sqrt{1-k^2|u|^2}, ku_4\right), \lambda^{1/4}\,{}^t(a_1,a_2,a_3)\right),\end{aligned}$$

where $|u|^2 = \sum_{j=1}^4 u_j^2$. Since $(d\chi)_0(\frac{\partial}{\partial u_i})_0 = k(e_i)_{p_0}$, $(d\chi)_0(\frac{\partial}{\partial a_i})_0 = \lambda^{1/4}(\frac{\partial}{\partial a_i})_{p_0}$, we see that $(d\chi)_0^*(\varphi_\lambda)_{p_0} = \varphi_0$, where φ_0 is a 3-form on $\mathbb{R}^7$ given by (2.1). Note that

$$\chi^{-1}(M^+_{\lambda,0,0}) = \left\{ {}^t(u_1,u_2,u_3,0,a_1,a_2,a_3);\ \begin{array}{l} (1-k^2|u|^2)^2\left(1 + \sum_{j=1}^3 a_j^2\right) = 1, \\ -u_3a_1 + u_2a_2 - u_1a_3 = 0 \end{array} \right\}.$$

Define $\Phi : \mathbb{R}_{>0} \times L \to \mathbb{R}^7$ by

$$\begin{aligned} &\left(r, {}^t(0, x_1 + iy_1, x_2 + iy_2, x_3 + iy_3)\right) \\ &\mapsto {}^t(f(r)x_1, f(r)x_2, f(r)x_3, 0, ry_3, ry_2, -ry_1)\,, \end{aligned}$$

where $f(r) = 2\lambda^{1/4}\sqrt{1 - \sqrt{\frac{2}{2+r^2}}}$. This gives the diffeomorphism $\chi \circ \Phi : \mathbb{R}_{>0} \times L \to M^+_{\lambda,0,0} - \{p_0\}$. Since we see that $f(r) = \lambda^{1/4} r + O(r^3)$ as $r \to 0$, we see that the rate at p_0 is equal to 3 in these coordinates.

5.6. Irreducible SO(3)-action

We give a proof of Theorem 1.5. Recall the notation in Section 4.6. By Lemma 4.12, an SO(3)-orbit through $({}^t(x_1, 0, 0, 0, x_5), {}^t(a_1, a_2, a_3))$ is 3-dimensional when

1) $-1/2 < x_5 < 1/2$,
2) $x_5 = 1/2, (a_1, a_3) \neq 0$, or
3) $x_5 = -1/2, (a_2, a_3) \neq 0$.

Consider case 1. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(x_1(t), 0, 0, 0, x_5(t)), {}^t(a_1(t), a_2(t), a_3(t))\right),$$

where $x_1(t) > 0, |x_5(t)| < 1/2$. We find a path c satisfying $\varphi_\lambda|_{\mathrm{SO}(3)\cdot \mathrm{Image}(c)} = 0$, where φ_λ is given by (3.1). We see that $\varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \tilde{E}_3^*) = 0$ as in Section 5.4.

Lemma 5.15. *The condition* $\varphi_\lambda|_{\mathrm{SO}(3)\cdot \mathrm{Image}(c)} = 0$ *is equivalent to*

$$\begin{aligned} &4\left\{(2\sqrt{3}x_1 + 4x_5 + 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_1 + 8x_1(-x_1 + \sqrt{3}x_5)\dot{a}_1 \\ &\qquad -(\sqrt{3}x_1 + x_5 + 1)(1 - 2x_5)a_1 \frac{d}{dt}\log(\lambda + r^2) = 0, \end{aligned} \tag{5.18}$$

$$\begin{aligned} &4\left\{(2\sqrt{3}x_1 - 4x_5 - 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_2 + 8x_1(x_1 + \sqrt{3}x_5)\dot{a}_2 \\ &\qquad +(-\sqrt{3}x_1 + x_5 + 1)(1 - 2x_5)a_2 \frac{d}{dt}\log(\lambda + r^2) = 0, \end{aligned} \tag{5.19}$$

$$\begin{aligned} &4\left\{-(x_5 + 1)\dot{x}_5 + 3x_1\dot{x}_1\right\} a_3 + 2(x_1^2 - 3x_5^2)\dot{a}_3 \\ &\qquad +(1 + x_5)(1 - 2x_5)a_3 \frac{d}{dt}\log(\lambda + r^2) = 0, \end{aligned} \tag{5.20}$$

where $r^2 = \sum_{j=1}^{3} a_j^2$.

This lemma implies Theorem 1.5. In general, it is hard to solve the equations (5.18), (5.19), (5.20) explicitly.

Proof. Since $\dot{c} = (-\dot{x}_1 x_5 + x_1 \dot{x}_5)e_4 + \sum_{j=1}^{3} \dot{a}_j \frac{\partial}{\partial a_j}$, we have

$$(\pi^*\omega_i(\tilde{E}_j^*, \dot{c})) = \begin{pmatrix} -\dot{x}_5 + \sqrt{3}\dot{x}_1 & 0 & 0 \\ 0 & 0 & \dot{x}_5 + \sqrt{3}\dot{x}_1 \\ 0 & -2\dot{x}_5 & 0 \end{pmatrix},$$
$$b_j(\dot{c}) = a_j \qquad \text{for } j = 1, 2, 3.$$

Then we compute

$$\begin{aligned} \sum_{i=1}^{3} b_i \wedge \pi^*\omega_i(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= \left\{(2\sqrt{3}x_1 - 4x_5 - 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_2 \\ &\quad + 2x_1(x_1 + \sqrt{3}x_5)\dot{a}_2, \\ b_{123}(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= -(\sqrt{3}x_1 - x_5 - 1)(1 - 2x_5)\frac{a_2}{2}\frac{d(r^2)}{dt}, \end{aligned}$$

which implies (5.19). In the same way, we compute

$$\begin{aligned} \sum_{i=1}^{3} b_i \wedge \pi^*\omega_i(\tilde{E}_1^*, \tilde{E}_3^*, \dot{c}) &= \{-2(x_5 + 1)\dot{x}_5 + 6x_1\dot{x}_1\} a_3 + (x_1^2 - 3x_5^2)\dot{a}_3, \\ b_{123}(\tilde{E}_1^*, \tilde{E}_3^*, \dot{c}) &= (1 + x_5)(1 - 2x_5)a_3\frac{d(r^2)}{dt}, \\ \sum_{i=1}^{3} b_i \wedge \pi^*\omega_i(\tilde{E}_2^*, \tilde{E}_3^*, \dot{c}) &= \left\{(2\sqrt{3}x_1 + 4x_5 + 1)\dot{x}_5 + \sqrt{3}(2x_5 - 1)\dot{x}_1\right\} a_1 \\ &\quad + 2x_1(\sqrt{3}x_5 - x_1)\dot{a}_1, \\ b_{123}(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= -(\sqrt{3}x_1 + x_5 + 1)(1 - 2x_5)\frac{a_1}{2}\frac{d(r^2)}{dt}, \end{aligned}$$

and obtain (5.18) and (5.20). □

Consider <u>case 2</u>. Take a path $c : I \to \Lambda^2_- S^4$ given by

$$c(t) = \left({}^t(\sqrt{3}/2, 0, 0, 0, 1/2), {}^t(a_1(t), a_2(t), a_3(t))\right).$$

We may assume that $a_3 = 0$ so that $c(t)$ is transverse to the SO(3)-orbit. We find a path c satisfying $\varphi_\lambda|_{\mathrm{SO}(3)\cdot\mathrm{Image}(c)} = 0$, where φ_λ is given by (3.1).

Since $\dot{c} = \sum_{i=1}^{2} \dot{a}_i \frac{\partial}{\partial a_i}$, we have at $c(t)$

$$\begin{aligned}(\pi^*\omega_i(\tilde{E}_j^*, \dot{c})) &= 0,\\ (b_i(\dot{c})) &= (\dot{a}_1, \dot{a}_2, 0)\,,\\ \varphi_\lambda(\tilde{E}_1^*, \tilde{E}_2^*, \dot{c}) &= 6 s_\lambda \dot{a}_2,\\ \varphi_\lambda(\tilde{E}_p^*, \tilde{E}_q^*, \dot{c}) &= 0 \qquad \text{for} \quad (p,q) \neq (1,2), (2,1).\end{aligned}$$

Thus the condition $\varphi_\lambda|_{\mathrm{SO}(3)\cdot\mathrm{Image}(c)} = 0$ is equivalent to $a_2 = C$ for $C \in \mathbb{R}$. Then as Remark 5.5, we see that

$$M_C = \mathrm{SO}(3) \cdot \{({}^t(\sqrt{3}/2, 0, 0, 0, 1/2), {}^t(r, C, 0)); r \in \mathbb{R}\},$$

where $C \in \mathbb{R}$, is a coassociative submanifold described as

$$M_C = C\tau + (\mathbb{R}\tau)^\perp,$$

where $\tau = \mathrm{vol}_\Sigma - *\mathrm{vol}_\Sigma$ and $\Sigma = \mathrm{SO}(3) \cdot {}^t(\sqrt{3}/2, 0, 0, 0, 1/2) \subset S^4$ is a Veronese surface. In Case 3, we obtain the similar coassociative submanifold, and hence we cannot obtain new examples in Case 2 and Case 3.

5.7. Cohomogeneity two coassociative submanifolds

When $\lambda \to 0$, $\varphi_0 = \varphi_\lambda|_{\lambda=0}$ defines a G_2-structure on $\Lambda^2_- S^4 - \{\text{zero section}\} \cong \mathbb{C}P^3 \times \mathbb{R}_{>0}$ by Remark 3.3. On $\Lambda^2_- S^4 - \{\text{zero section}\}$, $\mathbb{R}_{>0}$ acts by dilations preserving φ_0 up to scalar multiplication. Thus by using the $\mathbb{R}_{>0}$-action, we can apply the same method as the cohomogeneity one case and we derive some systems of O.D.E.s. However, we can find no explicit solutions which give new coassociative examples. In some cases, we obtain some explicit solutions, all of which turn out to be congruent to examples in Section 5 up to the SO(5)-action.

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Appendix A. Real irreducible representations

We give a summary about real irreducible representations in [5, 18].

Definition A.1. Let G be a compact Lie group and (V, ρ) be a $\mathbb{C}$-irreducible representation of G. We call (V, ρ) **self-conjugate** if V has a conjugate linear map J on V satisfying

$$J^2 = \pm 1, \qquad J \circ \rho(g) = \rho(g) \circ J \quad \text{for } g \in G.$$

This map is called a **structure map**. A self-conjugate representation (V, ρ) is said to be of **index** ± 1 if $J^2 = \pm 1$.

Proposition A.2. *Let (V, ρ) be a $\mathbb{C}$-irreducible representation of G. Then one of the following is satisfied.*

1) *(V, ρ) is a self-conjugate representation of index 1. In this case, (V, ρ) is a complexification of a real representation.*
2) *(V, ρ) is a self-conjugate representation of index -1. In this case, (V, ρ) is a quaternionic representation.*
3) *(V, ρ) is not a self-conjugate representation.*

Proposition A.3. *Let (V, ρ) be a $\mathbb{C}$-irreducible representation of G. As a real representation, ρ is reducible (resp. irreducible) if and only if 1. (resp. 2. or 3.) in Proposition A.2 is satisfied.*

Proposition A.4. *All $\mathbb{R}$-irreducible representations of G are given as follows.*

- *A $\mathbb{R}$-irreducible component of a $\mathbb{C}$-irreducible representation which is reducible as a $\mathbb{R}$-representation. This is an eigenspace of 1 or -1 of the structure map J in 1. of Proposition A.2. (Note that an eigenspace of 1 and that of -1 are mutually equivalent real irreducible representations of G.)*
- *A $\mathbb{C}$-irreducible representation which is also irreducible as a $\mathbb{R}$-representation. This corresponds 2. or 3. in Proposition A.2.*

In many cases, we know $\mathbb{C}$-irreducible representations, from which we can deduce $\mathbb{R}$-irreducible representations by Proposition A.4.

All equivalence classes of finite dimensional $\mathbb{C}$-irreducible representations of $\mathrm{SU}(2)$ are represented by $\{(V_n, \rho_n)\}_{n \geq 0}$, where V_n is a $\mathbb{C}$-vector space of all complex homogeneous polynomials with two variables z_1, z_2 of degree n and ρ_n is the induced action from the standard action of SU(2) on $\mathbb{C}^2$. By Proposition A.4, we deduce the following.

Lemma A.5 ([18]). *Let V be a $\mathbb{R}$-irreducible representation of* SU(2). *Then* $\dim_{\mathbb{R}} V = 4m$ *or* $2n-1$, *where* $m, n \geq 1$.

For compact Lie groups H_1 and H_2, any $\mathbb{C}$-irreducible representation of $H_1 \times H_2$ is given by $\sigma_1 \otimes \sigma_2$, where σ_i is a irreducible $\mathbb{C}$-representation of H_i. Thus in the same way, we obtain the following.

Lemma A.6. *Let V be a $\mathbb{R}$-irreducible representation of* $\mathrm{SU}(2) \times \mathrm{SU}(2)$. *Then*

$$\dim_{\mathbb{R}} V = \begin{cases} 2(k+1)(l+1) & \text{when } k, l \geq 0, k+l\text{: odd,} \\ (k+1)(l+1) & \text{when } k, l \geq 0, k+l\text{: even.} \end{cases}$$

If $k = 0$ or $l = 0$, the representation reduces to that of SU(2).

Lemma A.7. *Let V be a $\mathbb{R}$-irreducible representation of* $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. *Then*

$$\dim_{\mathbb{R}} V = \begin{cases} 2(k+1)(l+1)(m+1) & \text{when } k, l, m \geq 0, k+l+m\text{: odd,} \\ (k+1)(l+1)(m+1) & \text{when } k, l, m \geq 0, k+l+m\text{: even.} \end{cases}$$

If one of $\{k, l, m\}$ is equal to 0, the representation reduces to that of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. *If two of $\{k, l, m\}$ are equal to 0, the representation reduces to that of* SU(2).

Appendix B. Proof of Lemma 4.1

First, we prove the following.

Lemma B.1. *Let $\mathfrak{g} \subset \mathfrak{so}(5)$ be a compact Lie subalgebra with $\dim_{\mathbb{R}} \mathfrak{g} \geq 3$. Then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras:*

$$\mathfrak{so}(5), \qquad \mathfrak{so}(4), \qquad \mathfrak{su}(2) \oplus \mathbb{R}, \qquad \mathfrak{su}(2).$$

For the proof of Lemma B.1, we need the $\mathbb{R}$-irreducible representations of compact Lie groups in Appendix A. By Lemma B.1 and its proof, we obtain Lemma 4.1.

Proof. By the classification of compact Lie algebras, the possible k-dimensional Lie subalgebra $\mathfrak{g}$ of $\mathfrak{so}(5)$, where $3 \leq k \leq 10$, is isomorphic to one of the following:

$$\begin{array}{llll} \mathfrak{so}(5) & \text{for } k=10, & \mathfrak{su}(2)^2 & \text{for } k=6, \\ \mathbb{R}\oplus\mathfrak{su}(3),\ \mathfrak{su}(2)^3 & \text{for } k=9, & \mathbb{R}^2\oplus\mathfrak{su}(2) & \text{for } k=5, \\ \mathfrak{su}(3),\ \mathbb{R}^2\oplus\mathfrak{su}(2)^2 & \text{for } k=8, & \mathbb{R}\oplus\mathfrak{su}(2) & \text{for } k=4, \\ \mathbb{R}\oplus\mathfrak{su}(2)^2 & \text{for } k=7, & \mathfrak{su}(2) & \text{for } k=3. \end{array}$$

We check whether the Lie subalgebras in this list are actually contained in $\mathfrak{so}(5)$.

First, we show that $\mathfrak{su}(3), \mathbb{R}\oplus\mathfrak{su}(3) \not\subset \mathfrak{so}(5)$. By Theorem 5.10 of [7], the dimension of the $\mathbb{C}$-irreducible representation of $\mathfrak{su}(3)$ is of the form

$$\frac{1}{2}(m_1+1)(m_2+1)(m_1+m_2+2),$$

where $m_j \in \mathbb{Z}_{\geq 0}$. Since any representation of the compact Lie algebra $\mathfrak{su}(3)$ is completely reducible, we see that $\mathfrak{su}(3) \not\subset \mathfrak{so}(5)$ by Proposition A.4, which implies that $\mathbb{R}\oplus\mathfrak{su}(3) \not\subset \mathfrak{so}(5)$.

Similarly, by Lemma A.7, we see that $\mathfrak{su}(2)^3 \not\subset \mathfrak{so}(5)$. By Lemma A.6, the only inclusion $\mathfrak{su}(2)^2 \hookrightarrow \mathfrak{so}(5)$ is the standard inclusion $\mathfrak{su}(2)^2 = \mathfrak{so}(4) \hookrightarrow \mathfrak{so}(5)$. We may assume that $\mathfrak{so}(4) = \begin{pmatrix} \mathfrak{so}(4) & \\ & 0 \end{pmatrix} \hookrightarrow \mathfrak{so}(5)$. Since

$$\{Y \in \mathfrak{so}(5); [X,Y]=0 \text{ for any } X \in \mathfrak{so}(4)\} = \{0\},$$

we see that $\mathbb{R}^2\oplus\mathfrak{su}(2)^2, \mathbb{R}\oplus\mathfrak{su}(2)^2 \not\subset \mathfrak{so}(5)$.

By Lemma A.5, we have 3 types of inclusions $\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(5)$ given by

$$\mathfrak{su}(2) = \mathfrak{so}(3) \hookrightarrow \mathfrak{so}(5), \tag{B.1}$$

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(4) \hookrightarrow \mathfrak{so}(5), \tag{B.2}$$

$$\mathfrak{su}(2) \hookrightarrow \mathfrak{so}(5)\text{: irreducibly.} \tag{B.3}$$

Note that the basis of $\mathfrak{su}(2)$ of (B.1) is given by $\left\{ \begin{pmatrix} E_i & \\ & O_2 \end{pmatrix} \right\}_{i=1,2,3}$, where E_i is defined in (4.3). The basis of $\mathfrak{su}(2)$ of (B.2) is given by (4.9), and that of (B.3) is given by (4.11). We easily see that $Z := \{Y \in \mathfrak{so}(5); [X,Y] =$

0 for any $X \in \mathfrak{su}(2)\}$ is spanned by

$$\begin{pmatrix} O_3 & \\ & J \end{pmatrix} \text{ for (B.1)}, \ O_5 \text{ for (B.3)},$$

$$\begin{pmatrix} & I' & \\ -I' & & \\ & & 0 \end{pmatrix}, \begin{pmatrix} & -J' & \\ J' & & \\ & & 0 \end{pmatrix}, \begin{pmatrix} J & & \\ & J & \\ & & 0 \end{pmatrix} \text{ for (B.2)},$$

where $J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, $I' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $J' = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$.

From these computations, we see that $\mathbb{R}^2 \oplus \mathfrak{su}(2) \not\subset \mathfrak{so}(5)$. In fact, for (B.1) and (B.3), we have $\dim_{\mathbb{R}} Z \leq 1$, which implies that $\mathbb{R}^2 \oplus \mathfrak{su}(2) \not\subset \mathfrak{so}(5)$. For (B.2), we have $Z \cong \mathfrak{su}(2)$, which has no nontrivial commutative Lie subalgebras. □

References

[1] G. Bor, *Yang-Mills fields which are not self-dual*, Comm. Math. Phys. **145** (1992), no. 2, 393–410.

[2] R. L. Bryant and S. M. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829–850.

[3] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. (4) **32** (1982), 19–45.

[4] D. Fox, *Coassociative Cones Ruled by 2-planes*, Asian J. Math. **11** (2007), no. 4, 535–554.

[5] M. Goto and F. Grosshans, *Semisimple Lie algebras*, Marcel-Dekker, New York, 1978.

[6] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[7] B. Hall, *Lie groups, Lie algebras, and representations: An elementary introduction*, Grad. Texts in Math. **222**, Springer-Verlag, New York, 2003.

[8] K. Hashimoto and T. Sakai, *Cohomogeneity one special Lagrangian submanifolds in the cotangent bundle of the sphere*, Tohoku Math. J. **64** (2012), 141–169.

[9] N. Hitchin, *A new family of Einstein metrics*, Manifolds and geometry (Pisa, 1993), 190–222, Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge, 1996.

[10] W. Y. Hsiang and H. B. Lawson, *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971), 1–38.

[11] M. Ionel, S. Karigiannis, and M. Min-Oo, *Bundle constructions of calibrated submanifolds in* $\mathbb{R}^7$ *and* $\mathbb{R}^8$, Math. Res. Lett. **12** (2005), 493–512.

[12] S. Karigiannis and N. C.-H. Leung, *Deformations of calibrated subbundles of Euclidean spaces via twisting by special sections*, Ann. Global Anal. Geom. **42** (2012), 371–389.

[13] S. Karigiannis and M. Min-Oo, *Calibrated subbundles in noncompact manifolds of special holonomy*, Ann. Global Anal. Geom. **28** (2005), 371–394.

[14] J. D. Lotay, 2-*Ruled Calibrated 4-folds in* $\mathbb{R}^7$ *and* $\mathbb{R}^8$, J. Lond. Math. Soc. (2) **74** (2006), 219–243.

[15] J. D. Lotay, *Calibrated Submanifolds of* $\mathbb{R}^7$ *and* $\mathbb{R}^8$ *with Symmetries*, Q. J. Math. **58** (2007), 53–70.

[16] J. D. Lotay, *Coassociative* 4-*folds with conical singularities*, Comm. Anal. Geom. **15** (2007), 891–946.

[17] Maple 16, Waterloo Maple Inc. (Maplesoft).

[18] K. Mashimo, *Minimal immersions of* 3-*dimensional spheres into spheres*, Osaka J. Math. **21** (1984), 721–732.

[19] S. M. Salamon, *Riemannian geometry and holonomy groups*, Longman Group UK Limited, Harlow, 1989.

[20] S. Salamon, *Explicit metric with holonomy* G_2, Proc. UK-Japan Winter School (2004).

[21] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.

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