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# Inequality for Gorenstein minimal 3-folds of general type

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Let $X$ be a Gorenstein minimal 3-fold of general type. We prove the optimal inequality:

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 2,$$

where $\chi(\omega_X)$ is the Euler-Poincaré characteristic of the dualizing sheaf $\omega_X$.

## 1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$.

The geography of projective varieties of general type plays a very important role in the classification of algebraic varieties. There are two important types of inequalities in studying the geographical problem: Noether inequality and Yau inequality.

For Yau inequality, we have the following results.

- In 1977, Yau ([21]) proved that the optimal inequality $(-1)^n c_1^{n-2} \cdot c_2 \geq (-1)^n \frac{n}{2(n+1)} c_1^n$ holds for all canonically polarized nonsingular varieties of dimension $n$. In [10] and [20], the same inequality is proved for more general cases.[1]
- In 1977, Miyaoka ([14]) proved that the inequality $c_1^2 \leq 3c_2$ holds for all nonsingular projective surfaces of general type. In 1985, Miyaoka ([15]) proved that $3c_2 - c_1^2$ is pseudo-effective for all nonsingular minimal projective varieties of general type.

In this paper we will restrict our interest to inequalities of Noether type.

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[1]It is pointed out by one of the referees that Yau's method can cover more general cases.

Let $S$ be a smooth minimal surface of general type. We have the classical Noether inequality: $K_S^2 \geq 2p_g(S) - 4$ and $K_S^2 \geq 2\chi(\mathcal{O}_S) - 6$ (c.f. [16]).

Let $X$ be a projective 3-fold of general type. A natural question is: does there exists an inequality of Noether type for 3-folds of general type? There have been many works dedicated to proving the 3-dimensional version of the Noether inequality:

- In 1992, M. Kobayashi (c.f. [13, Proposition 3.2]) constructed an infinite number of canonically polarized smooth 3-folds of general type satisfying the equalities:

$$K_X^3 = \frac{4}{3}p_g(X) - \frac{10}{3},\ K_X^3 = \frac{4}{3}\chi(\omega_X) - 2. \tag{1.1}$$

- In 2004, M. Chen (c.f. [7]) studied minimal 3-folds of general type and gave effective Noether type inequalities.
- In 2004, M. Chen (c.f. [8]) proved that the optimal inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ holds for all canonically polarized smooth 3-folds of general type.
- In 2006, F. Catanese, M. Chen and De-Qi Zhang (c.f. [2]) proved that the optimal inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ holds for all smooth minimal 3-folds of general type.
- In 2015, J. A. Chen and M. Chen (c.f. [5]) proved the optimal inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$ under the assumption that $X$ is Gorenstein minimal.

In this paper, a normal projective 3-fold $X$ is called Gorenstein minimal if $X$ has at worst $\mathbb{Q}$-factorial terminal singularities, the canonical divisor $K_X$ is a Cartier divisor and $K_X$ is nef.

It is interesting to know whether there exists a similar Noether type inequality between $K_X^3$ and $\chi(\omega_X)$. The following open problem was raised by M. Chen (c.f. [8, 3.9]):

**Conjecture 1.1.** [8, 3.9] Let $X$ be a Gorenstein minimal 3-fold of general type. There should be an analogue of the Noether inequality in the form:

$$K_X^3 \geq a\chi(\omega_X) - b,$$

where $a$ and $b$ are positive rational numbers.

As was pointed out by M. Chen (c.f. [8]), it is difficult to find a Noether inequality in this direction because the inter relations among $p_g(X)$, $q(X)$ and $h^2(\mathcal{O}_X)$ are not clear to us, unlike in surface case. Some partial results were proved in [19] and in [9].

· In 1997, D. K. Shin (c.f. [19]) proved that an effective inequality $K_X^3 \geq \frac{6}{7}\chi(\omega_X) - \frac{6}{7}$ holds for all smooth minimal 3-folds of general type.

· In 2006, M. Chen and C. D. Hacon (c.f. [9]) proved that an effective inequality $K_X^3 \geq \frac{8}{9}\chi(\omega_X) - \frac{10}{3}$ holds for all smooth minimal 3-folds of general type.

We restrict our attention to the situation where $X$ is a Gorenstein minimal 3-fold of general type. The aim of this paper is to prove the following.

**Theorem 1.2.** *Let $X$ be an irregular Gorenstein minimal 3-fold of general type. Then*

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X).$$

According to [5, Theorem 1.1] and Theorem 1.2, we can get our main result as follows.

**Theorem 1.3.** *Let $X$ be a Gorenstein minimal 3-fold of general type. Then*

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 2. \tag{1.2}$$

**Remark 1.4.** The inequality in Theorem 1.3 is optimal because of M. Kobayashi's examples (c.f. (1.1)). It is well known that we have $\chi(\omega_X) > 0$ if $X$ is a Gorenstein minimal 3-fold of general type. So (1.2) is meaningful. One may ask whether (1.2) is still true if $X$ is a minimal 3-fold of general type. Unfortunately, if $X$ is not Gorenstein, then $\chi(\omega_X)$ could be either positive, zero, or negative (c.f. [4, line 10-18, page 2501]). This problem does not seem possible to resolve with the methods and the techniques of the present article.

## 2. Notations and the set up

**Definition 2.1.** Let $S$ be a smooth projective surface of general type. Denote by $S_0$ its minimal model and by $(a, b) = (K_{S_0}^2, p_g(S))$. We call $S$ is a surface of type $(a, b)$.

Let $X$ be a Gorenstein minimal 3-fold of general type with $p_g(X) \geq 2$. According to [12, Lemma 5.1], $X$ is locally factorial. Write

$$|K_X| = |\overline{M}| + \overline{Z}, \tag{2.1}$$

where $|\overline{M}|$ is the movable part of $|K_X|$ and $\overline{Z}$ is the fixed part of $|K_X|$.

We shall resolve the base locus of $|\overline{M}|$ in two steps. For a linear system $\Upsilon$, we denote by $\mathrm{Bs}\Upsilon$ the base locus of $\Upsilon$. Roughly speaking, the first step is to resolve the subset $\mathrm{Bs}|\overline{M}| \cap \mathrm{Sing}(X)$.

**Lemma 2.2 (cf. [5, Section 2]).** *There is a birational morphism $\alpha\colon X_0 \to X$ satisfying the following properties.*

(a) *The morphism $\alpha$ is a composition of successive divisorial contractions to points and $X_0$ is a Gorenstein 3-fold with locally factorial terminal singularities.*

(b) *Denote by $|M_0|$ the movable part of $|\alpha^*\overline{M}|$. Then $\mathrm{Bs}|M_0| \cap \mathrm{Sing}(X_0) = \emptyset$.*

(c) *The following formulae*

$$K_{X_0} = \alpha^* K_X + \sum_{t=1}^{m} c_t D_t, \tag{2.2}$$
$$\alpha^*(\overline{M}) = M_0 + \sum_{t=1}^{m} d_t D_t, \quad \alpha^*(\overline{Z}) = Z_0 + \sum_{t=1}^{m} e_t D_t$$

*hold, where*

(i) *$Z_0$ is the strict transform of $\overline{Z}$,*

(ii) *$D_t$ is a prime divisor such that $\alpha(D_t)$ is a point for $1 \leq t \leq m$, and*

(iii) *$c_t$, $d_t$ and $e_t$ are non-negative integers such that $0 < c_t \leq d_t$ for $1 \leq t \leq m$.*

*Proof.* The birational morphism $\alpha$ is constructed in [5, p. 4–p. 5], using explicit resolutions of terminal singularities (see [3] and [5, Definition 2.2]). Then (a) and (b) follow from the construction and [12, Lemma 5.1]. Since both $X_0$ and $X$ are locally factorial, $c_t$, $d_t$ and $e_t$ are non-negative integers. The inequality $c_t \leq d_t$ follows by [5, Corollary 2.4]. □

We fix a birational morphism $\alpha\colon X_0 \to X$ as in Lemma 2.2. We may assume that the number of divisorial contractions in the construction of $\alpha$ is minimal. The second step is to resolve the base locus of $|M_0|$ without changing the singularities of $X_0$. This is possible by Lemma 2.2 (b) and by Hironaka's Theorem (cf. [11]).

**Lemma 2.3 (cf. [7, Lemma 4.2]).** *There are successive blowups*

$$\beta\colon Y = X_{n+1} \xrightarrow{\pi_n} X_n \to \cdots \to X_{i+1} \xrightarrow{\pi_i} X_i \to \cdots \to X_1 \xrightarrow{\pi_0} X_0$$

*such that $\pi_i$ is a blowup along a smooth irreducible center $W_i$, $W_i$ is contained in the base locus of the movable part of $|(\pi_0 \circ \pi_1 \circ \cdots \circ \pi_{i-1})^* M_0|$ and $W_i \cap \mathrm{Sing}(X_i) = \emptyset$. Moreover, the morphism $\beta = \pi_n \circ \cdots \circ \pi_0$ satisfies the following properties.*

(a) *Denote by $|M|$ the movable part of $|\beta^* M_0|$. Then $|M|$ is base point free.*

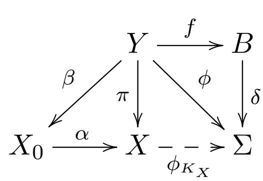
(b) *The following formulae*

$$K_Y = \beta^* K_{X_0} + \sum_{i=0}^{n} a_i E_i, \quad \beta^* M_0 = M + \sum_{i=0}^{n} b_i E_i \tag{2.3}$$

*hold, where $E_i$ is the strict transform of the exceptional divisor of $\pi_i$ for $0 \le i \le n$, and $a_i$ and $b_i$ are positive integers such that $a_i \le 2b_i$ for $0 \le i \le n$.*

*Proof.* The construction of the blowups $\pi_i$ and (a) follow by Lemma 2.2 (b) and by Hironaka's Theorem (cf. [11]). We remark that the assertion $a_i \le 2b_i$ in (b) is exactly [7, Lemma 4.2]. □

From now on, we fix a birational morphism $\beta$ as in Lemma 2.3 such that the number $n+1$ of blowups is minimal. Denote by $\phi_{K_X}$ the canonical map of $X$ and by $\Sigma$ the image of $\phi_{K_X}$. Let $\phi$ be the morphism induced by the linear system $|M|$. Then $\phi = \phi_{K_X} \circ \pi$, where $\pi = \alpha \circ \beta$. Let $Y \xrightarrow{f} B \xrightarrow{\delta} \Sigma$ be the Stein factorization of $\phi$. We have the following commutative diagram:




Note that $B$ is normal.

**Proposition 2.4.** *Keep the same notation as above. We have the following known results.*

(1) *If* $\dim B = 3$*, then* $K_X^3 \geq 2p_g(X) - 6$ *(cf. );*

(2) *If* $\dim B = 2$*, then* $K_X^3 \geq \left\lceil \frac{2}{3}(g(C)-1) \right\rceil (p_g(X) - 2)$ *where* $g(C)$ *is the genus of a general fiber* $C$ *of* $f$ *. In particular, we have* $K_X^3 \geq 2p_g(X) - 4$ *if* $g(C) \geq 3$*.*

(3) *if* $\dim B = 1$*, then either* $K_X^3 \geq 2p_g(X) - 4$ *or the general fiber of* $f$ *is a smooth projective surface of type* $(1, 2)$*.*

*Proof.* Assertion (1) follows by [13, Main Theorem]. (2) is exactly [7, Theorem 4.1 (ii)]. (3) is just [7, Theorem 4.1 (iii)]. □

## 3. Proof of Theorem 1.3

This section is devoted to proving Theorem 1.3. Throughout this section, we denote by $X$ a Gorenstein minimal 3-fold of general type. It is well known that $K_X^3$ is a positive even integer and $\chi(\omega) > 0$(c.f. [6, 2.1, 2.2]). Denote by $a\colon X \to T$ the Stein factorization of the Albanese morphism of $X$ and by $F$ a general fiber of $a$. Since 3-dimensional terminal singularities are isolated (c.f. [18]), $F$ is smooth.

The following lemma is due to [9, Propsition 2.1].

**Lemma 3.1.** *Let* $X$ *be a Gorenstein minimal* 3*-fold of general type with* $p_g(X) > 0$*. Then* $\chi(\omega_X) \leq p_g(X)$ *unless a general fiber of* $a$ *is a surface* $F$ *with* $q(F) = 0$*, in which case one has the inequality*

$$\chi(\omega_X) \leq \left(1 + \frac{1}{p_g(F)}\right) p_g(X).$$

*Proof.* After taking a resolution of $X$, Lemma 3.1 follows easily by [9, Propsition 2.1]. □

**Proposition 3.2.** *Let* $X$ *be a Gorenstein minimal* 3*-fold of general type. Keep the same notation as in the beginning of this section.*

(1) *If* $q(X) = 0$*, then* $\chi(\omega_X) \leq p_g(X) - 1$*. Therefore we have* $K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 2$*.*

(2) *If* $q(X) = 1$*, then* $\chi(\omega_X) \leq p_g(X)$ *holds and we have* $K_X^3 \geq \frac{4}{3}\chi(\omega_X) - \frac{10}{3}$*.*

(3) *If* $\mathrm{dim} T \geq 2$ *and* $p_g(X) > 0$*, then* $\chi(\omega_X) \leq p_g(X)$ *holds and we have* $K_X^3 \geq \frac{4}{3}\chi(\omega_X) - \frac{10}{3}$*.*

(4) *If* $p_g(X) = 0$*, then* $K_X^3 \geq 2\chi(\omega_X)$*.*

(5) *If* $\dim T = 1$ *and* $p_g(X) = 1$*, then* $K_X^3 \geq 2\chi(\omega_X)$*.*

*If* $X$ *satisfies one of the above conditions, then all these statements imply*

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - \frac{10}{3}. \tag{3.1}$$

*Proof.* Assertions (1) and (2) follow by [5, Theorem 1.1]. Assume that $X$ satisfies condition (3). Since $F$ is a general fiber and $X$ is of general type, $\chi(\omega_X) \leq p_g(X)$ follows by Lemma 3.1. So (3) follows by [5, Theorem 1.1]. (4) is exactly Case 2 of 1.3 of [9]. Assume that $X$ satisfies condition (5). Since $p_g(X) > 0$ and $F$ is a general fiber, we have $p_g(F) > 0$ and $F$ is of general type. According to Lemma 3.1, we have $\chi(\omega_X) \leq \frac{3}{2}p_g(X)$. So $\chi(\omega_X) = 1$. Hence $K_X^3 \geq 2\chi(\omega_X)$ for $K_X^3$ is an even positive integer. □

We now turn to the case where $X$ satisfies $p_g(X) \geq 2$, $q(X) \geq 2$ and $\dim T = 1$. In this case, $T$ is a nonsingular projective curve with $g(T) = q(X) \geq 2$ (c.f. [1, Prop V.15]) and $F$ is a smooth minimal surface of general type because $X$ is minimal. The fibration $a$ is relatively minimal because $X$ is minimal. Therefore $K_{X/T} = K_X - a^*K_T$ is nef by [17, Theorem 1.4]. Since $p_g(X) \geq 2$, we can study the nontrivial canonical map of $X$. We can take the modification $\pi\colon Y \to X$ as in Section 2 (see page 2-3). Keep the same notation as in the last section. Recall that the morphism $f\colon Y \to B$ is the Stein factorization of the canonical morphism (see page 3).

**Lemma 3.3.** *Let* $X$ *be a Gorenstein minimal* 3*-fold of general type with* $p_g(X) \geq 2$*. Keep the same notation as above. Assume that* $T$ *is a nonsingular curve of genus* $g(T) = q(X) \geq 2$*. Then*

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 6 \tag{3.2}$$

*unless the general fiber of* $f$ *is a curve of genus* $g = 2$*.*

*Proof.* **Case** 1. $\dim B \geq 2$.

According to Proposition 2.4, we have $K_X^3 \geq 2p_g(X) - 6$. Then we have (3.2) by Lemma 3.1.

**Case** 2. $\dim B = 1$.

If the general fiber of $f$ is not a surface of type $(1, 2)$, then we have $K_X^3 \geq 2p_g(X) - 4$ by Proposition 2.4. So (3.2) holds by Lemma 3.1.

Now we turn to the case where the general fiber of $f$ is a surface of type $(1, 2)$. Since $K_X^3$ is a positive even integer, (3.2) holds if $\chi(\omega_X) \leq 6$.

So we may assume that $\chi(\omega_X) \geq 7$. We have $p_g(X) \geq 5$ by Lemma 3.1. We have $q(X) \leq 1$ by [7, Lemma 4.5]. But this contradicts to our assumption $q(X) \geq 2$. We are done. □

**Proposition 3.4.** *Let $X$ be a Gorenstein minimal 3-fold of general type with $p_g(X) \geq 2$. Keep the same notation as above. Assume that $T$ is a nonsingular curve of genus $g(T) = q(X) \geq 2$ and that a general fiber $C$ of $f$ is a curve of genus $g(C) = 2$. Then*

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 6. \tag{3.3}$$

*Proof.* Since $K_X^3$ is a positive even integer, (3.3) is automatically true for $\chi(\omega_X) \leq 6$. We may assume $\chi(\omega_X) \geq 7$. So $p_g(X) \geq 5$ by Lemma 3.1. According to Proposition 3.2, we can assume that $q(X) \geq 2$ and that the general fiber $C$ of $f$ is a nonsingular curve of genus 2. Recall that $|M|$ is the movable part of $|K_Y|$ and that $|M|$ is base point free.

We have $M^2 \equiv d_\Sigma \cdot \deg \delta \cdot C$, where $\Sigma$ is the image of $\phi$ and the symbol $\equiv$ stands for numerical equivalence.

Because $\Sigma$ is non-degenerate, we have $d_\Sigma \geq p_g(X) - 2$. Since both $\pi^*K_X$ and $M$ are nef, we conclude that

$$\begin{aligned} K_X^3 &\geq (\pi^*K_X \cdot M^2) = d_\Sigma \cdot \deg \delta \cdot (\pi^*K_X \cdot C) \\ &\geq (p_g(X) - 2) \deg \delta \cdot (\pi^*K_X \cdot C). \end{aligned} \tag{3.4}$$

If $(\pi^*K_X \cdot C) \geq 2$, then $K_X^3 \geq 2p_g(X) - 4$. So (3.3) holds by (3.4) and Lemma 3.1.

From now on, we assume that $(\pi^*K_X \cdot C) = 1$.

In order to prove Proposition 3.4, we need to use the techniques in the proofs of [7, Theorem 4.3] and [5, Theorem 3.1]. Recall from (2.1), (2.2) and (2.3) that

$$\begin{aligned} K_Y &= \pi^*K_X + \left(\sum_{t=1}^{m} c_t\beta^*D_t + \sum_{i=0}^{n} a_iE_i\right), \\ \pi^*K_X &= M + \left(\sum_{t=1}^{m}(d_t + e_t)\beta^*D_t + \sum_{i=0}^{n} b_iE_i + \beta^*Z_0\right) \end{aligned} \tag{3.5}$$

In particular, we have $K_X^3 = (\pi^*K_X)^3 = ((\pi^*K_X)^2 \cdot M) + (K_X^2 \cdot \overline{Z})$. We aim to bound $((\pi^*K_X)^2 \cdot M)$ from below.

For this purpose, by Bertini's theorem, we choose a general member $S$ of $|M|$ such that $S$ is smooth and consider the fibration $f|_S$. By abuse of notation, we still denote by $C$ the general fiber of $f|_S$. We remark that $S$ is of general type since so is $X$. Also the divisors $\beta^* D_t|_S$ and $E_i|_S$ are effective for $1 \le t \le m$ and $0 \le i \le n$.

Recall that $a\colon X \to T$ is the Stein factorization of the Albanese morphism of $X$ and that $K_{X/T}$ is a nef divisor. It is easy to see that $a_Y\colon Y \to T$ is the Stein factorization of the Albanese morphism of $Y$. Take a general fiber $F$ of $a$ such that $F_Y = \pi^* F$ is also a general fiber of $a_Y$. Note that $F$ is a smooth minimal surface of general type.

Since $\dim B = 2$, we can see that $\dim f(F_Y) \ge 1$. So $S \cap F_Y$ is a nontrivial effective curve on $Y$. Thus we have $a_Y(S) = T$.

Since both $K_{X/T} = K_X - 2(q(X)-1)F$ and $F_Y$ are nef divisors, we have $(\pi^* K_X \cdot C) \ge 2(q(X)-1)(F_Y \cdot C) \ge 0$.

Therefore $(F_Y \cdot C) = 0$ because $q(X) \ge 2$ and $(\pi^* K_X \cdot C) = 1$. So $f|_S$ and $a_Y|_S$ induce the same fibration. Denote by $\gamma\colon S \to \widehat{T}$ the induced fibration. We have $g(\widehat{T}) \ge g(T) = q(X) \ge 2$.

By the definition of $M$ we can write

$$M|_S = \gamma^* A \equiv (\deg A) C, \tag{3.6}$$

where $A$ is an effective divisor on $\widehat{T}$.

Since $g(C) = 2$, we have $(K_Y \cdot C) = 2$ by the adjunction formula.

According to (3.5) and $(\pi^* K_X \cdot C) = 1$, we have

$$\left(\left(\sum_{t=1}^m c_t \beta^* D_t + \sum_{i=0}^n a_i E_i\right)\Bigg|_S \cdot C\right) = 1 \quad \text{and}$$
$$\left(\left(\sum_{t=1}^m (d_t + e_t) \beta^* D_t + \sum_{i=0}^n b_i E_i + \beta^* Z_0\right)\Bigg|_S \cdot C\right) = 1.$$

We conclude that the horizontal part of $(\sum_{t=1}^m c_t \beta^* D_t + \sum_{i=0}^n a_i E_i)|_S$ consists of an irreducible reduced curve $\Gamma$, which is also the horizontal part of $(\sum_{t=1}^m (d_t + e_t)\beta^* D_t + \sum_{i=0}^n b_i E_i + \beta^* Z_0)|_S$ and $(\Gamma \cdot C) = 1$.

We can write

$$\begin{aligned} \left(\sum_{t=1}^m c_t \beta^* D_t + \sum_{i=0}^n a_i E_i\right)\Bigg|_S &= \Gamma + D_V + E_V, \\ \left(\sum_{t=1}^m (d_t + e_t) \beta^* D_t + \sum_{i=0}^n b_i E_i + \beta^* Z_0\right)\Bigg|_S &= \Gamma + D'_V + E'_V \end{aligned} \tag{3.7}$$

where $E_V, D_V, E'_V$ and $D'_V$ are effective divisors contained in the fibers of $\gamma$.

According to Lemma 2.2 (c) and Lemma 2.3 (b), we have

$$D_V + E_V \leq 2D'_V + 2E'_V. \tag{3.8}$$

Because $\Gamma$ is a section of $\gamma$,

$$(\Gamma \cdot (2D'_V + 2E'_V - D_V - E_V)) \geq 0. \tag{3.9}$$

The adjunction formula yields

$$(K_S \cdot \Gamma) + \Gamma^2 = 2p_a(\Gamma) - 2 \geq 2q(X) - 2 \tag{3.10}$$

Note that

$$K_Y|_S = \pi^* K_X|_S + \Gamma + D_V + E_V \text{ and } \pi^* K_X|_S = M|_S + \Gamma + D'_V + E'_V$$

by (3.5). By (3.9), one has

$$\begin{aligned} 2q(X) - 2 \leq ((K_S + \Gamma) \cdot \Gamma) &= ((K_Y + M)|_S \cdot \Gamma) + \Gamma^2 \\ &= ((\pi^* K_X|_S + M|_S + D_V + E_V + 2\Gamma) \cdot \Gamma) \\ &\leq ((\pi^* K_X|_S + M|_S + 2D'_V + 2E'_V + 2\Gamma) \cdot \Gamma) \\ &= ((3\pi^* K_X|_S - M|_S) \cdot \Gamma) \\ &= 3(\pi^* K_X|_S \cdot \Gamma) - \deg A. \end{aligned} \tag{3.11}$$

The last equality holds by $M|_S \equiv (\deg A)C$ (see (3.6)) and $(\Gamma \cdot C) = 1$.

So we have

$$(\pi^* K_X \cdot \Gamma) \geq \frac{1}{3}\deg A. \tag{3.12}$$

By (3.6) and $(\pi^* K_X \cdot C) = 1$, we have

$$(\pi^* K_X|_S \cdot M|_S) = \deg A. \tag{3.13}$$

We have

$$K_X^3 \geq (\pi^* K_X|_S \cdot M|_S) + (\pi^* K_X \cdot \Gamma) \geq \frac{4}{3}\deg A \tag{3.14}$$

by (3.13) and (3.12). We will bound $\deg A$ from below.

**Case 1.** $h^1(\widehat{T}, A) > 0$.

By Clifford's inequality and (3.6), we have

$$\deg A \geq 2h^0(\widehat{T}, A) - 2 = 2h^0(S, M|_S) - 2 \geq 2p_g(X) - 4.$$

So (3.3) follows by above inequality, (3.14) and Lemma 3.1.

**Case 2.** $h^1(\widehat{T}, A) = 0$.

By Riemann-Roch formula and Lemma 3.1, we have

$$\deg A = h^0(\widehat{T}, A) + g(\widehat{T}) - 1 \geq p_g(X) + q(X) - 2 \geq \chi(\omega_X) - 1. \tag{3.15}$$

We conclude (3.3) from (3.14) and (3.15). □

We can now prove our main result.

*Proof of Theorem 1.2.* According to Proposition 3.2, Lemma 3.3 and Proposition 3.4, we have

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - 6. \tag{3.16}$$

Since $X$ is irregular, for every integer $m \geq 2$ there is a cyclic unramified covering $\tau\colon \widehat{X} \to X$ of degree $m$. We have

$$K_{\widehat{X}} = \tau^* K_X,\ K_{\widehat{X}}^3 = mK_X^3,\ \chi(\omega_{\widehat{X}}) = m\chi(\omega_X). \tag{3.17}$$

Note that $\widehat{X}$ is an irregular Gorenstein minimal 3-fold of general type. So we have $K_{\widehat{X}}^3 \geq \frac{4}{3}\chi(\omega_{\widehat{X}}) - 6$. Therefore by (3.16) and (3.17), we have

$$K_X^3 \geq \frac{4}{3}\chi(\omega_X) - \frac{6}{m}.$$

Theorem 1.2 follows by letting $m \to \infty$. □

Theorem 1.3 follows easily by Theorem 1.2 and Proposition 3.2 (1).

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## References


[1] A. Beauville, *Complex algebraic surfaces*, Cambridge Univ. Press (1983) (translation of *Surfaces algébriques complexes*, Astérisque **54**, Soc. Math. France, Paris (1978)).

[2] F. Catanese, M. Chen, and D.-Q. Zhang, *The Noether inequality for smooth minimal* 3*-folds*, Math. Res. Lett. **13** (2006), no. 4, 653–666.

[3] J. A. Chen, *Explicit resolution of three dimensional terminal singularities*, to appear in Minimal Models and Extremal Rays, Proceedings of the Conference in Honor of Shigefumi Mori's 60th Birthday, Advanced Studies in Pure Mathematics. `arXiv:1310.6445`.

[4] J. A. Chen and C. D. Hacon, *On the geography of threefolds of general type*, J. Algebra **321** (2009), no. 9, 2500–2507.

[5] J. A. Chen and M. Chen, *The Noether inequality for Gorenstein minimal* 3*-folds*, Comm. Anal. Geom. **23** (2015), no. 1, 1–9.

[6] J. A. Chen, M. Chen, and D.-Q. Zhang, *The* 5*-canonical system on* 3*-folds of general type*, J. Reine Angew. Math. **603** (2007), 165–181.

[7] M. Chen, *Inequalities of Noether type for* 3*-folds of general type*, J. Math. Soc. Japan **56** (2004), 1131–1155.

[8] M. Chen, *Minimal threefolds of small slope and the Noether inequality for canonically polarized threefolds*, Math. Res. Lett. **11** (2004), 833–852.

[9] M. Chen and C. D. Hacon, *On the geography of Gorenstein minimal* 3*-folds of general type*, Asian J. Math. **10**(2006), no. 4, 757–764.

[10] S. Y. Cheng and S. T. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of* $SU(2,1)$, Complex differential geometry and nonlinear differential equations (Brunswick, Maine, 1984), 31–44, Contemp. Math. **49**, Amer. Math. Soc., Providence, RI, 1986.

[11] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero.* I, Ann. of Math. (2) **79** (1964), 109–203; II, ibid., 205–326.

[12] Y. Kawamata, *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988), no. 1, 93–163.

[13] M. Kobayashi, *On Noether's inequality for threefolds*, J. Math. Soc. Japan **44** (1992), no. 1, 145–156.

[14] Y. Miyaoka, *On the Chern numbers of surfaces of general type*, Invent. Math. **42** (1977), 225–237.

[15] Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai, 1985. Adv. Stud. Pure Math. **10** (1987), 449–476.

[16] M. Noether, *Zur Theorie des eindeutigen Entsprechens algebraischer Ge-bilde*, Math. Ann. **2** (1870), 293-316; **8** (1875), 495–533.

[17] K. Ohno, *Some inequalities for minimal fibrations of surfaces of general type over curves*, J. Math. Soc. Japan **44** (1992), 643–646.

[18] M. Reid, *Minimal models of canonical* 3*-folds*, Adv. Stud. Pure Math. **1** (1983), 131–180.

[19] D.-K. Shin, *Some inequalities between* $K_X^3$ *and* $\chi(\mathcal{O}_X)$ *for a threefold* $X$ *of general type*, Math. Z. **225** (1997), no. 1, 133–138.

[20] G. Tian and S. T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, Mathematical aspects of string theory (San Diego, Calif. (1986)), 574–628, Adv. Ser. Math. Phys. **1**, World Sci. Publishing, Singapore (1987).

[21] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

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