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Moduli spaces of sheaves on K3 surfaces and symplectic stacks

Ziyu Zhang

We view a moduli space of semistable sheaves on a K3 surface as a global quotient stack, and compute its cotangent complex in terms of the universal sheaf on the Quot scheme using the classical and reduced Atiyah classes. As an application, we define the notion of a symplectic stack, and show that it includes moduli stacks of semistable sheaves on K3 surfaces.

1. Introduction

This paper grows out of the attempt of studying moduli spaces of semistable sheaves on K3 surfaces and holomorphic symplectic manifolds from the point of view of stacks.

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Holomorphic symplectic manifolds are complex manifolds with nowhere degenerate holomorphic 2-forms. They have very rich geometry and beautiful properties, mainly due to the interaction of two structures on the second cohomology group, namely, the weight 2 Hodge decomposition and the Beauville-Bogomolov pairing. For example, they have unobstructed deformations [4, 18, 30, 32, 33], local and global Torelli theorems [1, 11, 35]. Furthermore, birational irreducible holomorphic symplectic manifolds are always deformation equivalent [10]. Looking for examples of holomorphic symplectic manifolds is always one of the central problems in this area.

On the other hand, moduli spaces of Gieseker semistable sheaves on a projective variety [7, 20, 21] have been a very popular research area in differential geometry, algebraic geometry, gauge theory and theoretical physics for a long time. When the underlying variety is a K3 surface, Mukai [22] constructed a non-degenerate holomorphic 2-form on the smooth locus of the moduli space. Therefore, the smooth moduli spaces of sheaves on K3 surfaces provide a whole series of examples of irreducible holomorphic symplectic manifolds. A similar result on abelian surfaces [1] yields another series of examples, which are the so called generalized Kummer varieties. For quite a long time these are the only known examples of irreducible symplectic varieties. A natural question to ask at this stage is: since Mukai has showed the existence of a holomorphic 2-form on the smooth locus of any singular moduli space of semistable sheaves on K3 surfaces, is there any way to turn these singular moduli spaces into "symplectic objects"?

O'Grady's work [26, 27] partly answered this question. He studied such a 10-dimensional singular moduli space, as well as a 6-dimensional moduli space of sheaves over an abelian surface, and constructed their symplectic resolutions. A comparison of topological invariants shows that they are two new examples of irreducible symplectic manifolds. Some work was done along this route, and eventually, Kaledin, Lehn and Sorger showed that, O'Grady's example was the only one which could arise by desingularizing moduli spaces of sheaves on K3 surfaces [17, Theorem 6.2]. However, this result excludes many of the moduli spaces from the game, although they are very close to symplectic manifolds.

At the same time, people are trying to generalize the notion of symplectic manifolds to allow singularities. Beauville defined the notion of symplectic singularities in [2]. After that a lot of work was extensively done by many other people, such as [16, 23, 24]. In particular, in [17, Theorem 6.2], it is proved that all singular moduli spaces of sheaves on K3 surfaces are (singular) symplectic varieties in the sense of [2].

Another reason why we should enlarge the notion of holomorphic symplectic manifolds to include all singular moduli spaces of sheaves on K3 surfaces roots in enumerative geometry and theoretical physics. In recent years, the study of Donaldson-Thomas type invariants has grown into a large area involving many modern techniques in many different fields in algebraic geometry, such as deformation theory, stacks, derived categories and motives. Although a lot of work about Donaldson-Thomas type invariants on Calabi-Yau 3-folds is done, not so much is known on a K3 surface. On the other hand, Vafa and Witten in [34] predicted from S-duality that the generating function of the Euler characteristics of instanton moduli spaces on K3 surfaces has a modularity property. As a consequence, the Euler characteristics of singular moduli spaces could be all determined by those of the smooth ones. To the best of the author's knowledge, mathematically there's no convincing definition of the Euler characteristics (which are presumbly Donaldson-Thomas type invariants) needed for this conjecture so far, and the contribution of the singularities of the moduli spaces to the denominators remains a mystery.

In this paper, we are trying to generalize the notion of holomorphic symplectic manifolds into the stacky world. So that one has the possibility of dealing with all moduli spaces of semistable sheaves on K3 surfaces, when considered as Artin stacks, in a uniform way, without the necessity of distinguishing them by the existence of symplectic resolutions. The role of the holomorphic symplectic form in the definition of the holomorphic symplectic manifolds, is to provide an isomorphism of the tangent bundle, or rather the cotangent bundle, with its dual, such that the isomorphism is anti-symmetric. By thinking of these moduli spaces as stacks, we will replace the cotangent bundle by the cotangent complex. Therefore, motivated by the work on symmetric obstruction theories in [3], we propose the definition of a *symplectic stack* as follows:

Definition 1.1. A *symplectic stack* is an algebraic stack, whose cotangent complex is a *symplectic complex*, namely, a complex equipped with a non-degenerate anti-symmetric bilinear pairing.

The precise definitions can be found in Definitions 6.1, 6.5 and 6.6.

Besides the trivial examples of symplectic manifolds and quotients of symplectic manifolds by finite subgroups of symplectomorphisms, the major part of this paper is devoted to study the question whether the moduli spaces of semistable sheaves on K3 surfaces can be understood as symplectic stacks. For this purpose, we fix a K3 surface X, a Mukai vector $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$.

There exists a system of hyperplanes in the ample cone of X with respect to $\mathbf{v}$; see [12, Section 4.C]. We fix an ample class H which is generic with respect to $\mathbf{v}$, namely, not contained in any of these hyperplanes. The advantage of H being generic lies in the fact that the Mukai vector of any direct summand of an H-polystable sheaf of Mukai vector $\mathbf{v}$ is proportional to $\mathbf{v}$.

For technical reasons, we assume further that $\mathbf{v}^2 > 0$ to guarantee the existence of stable sheaves. We consider the global quotient stack $\mathcal{M} = [Q/G]$ as the quotient of the GIT-semistable locus Q of Quot scheme by the gauge group $G = PGL(N)$. We will call it the *reduced moduli stack* of semistable sheaves on a K3 surface (see Remark 6.9), and prove that it is a symplectic stack (see Theorem 6.10).

The main difficulty during establishing this result lies in the computation of the cotangent complexes of the quotient stack. We obtain explicit formulas for the cotangent complexes in terms of the universal quotients, which we view as the main result of this paper (see Theorems 4.5 and 5.10):

Theorem 1.2. *Under the above assumptions, the cotangent complexes of the GIT-semistable locus of the Quot scheme Q and the reduced moduli stack of semistable sheaves $\mathcal{M}$ can be expressed explicitly by the universal quotient sequence on Q. More precisely, using the notations given at the end of this section, we have quasi-isomorphisms*

$$\begin{aligned} R\pi_* RHom(\mathcal{K}, \mathcal{F})_0^\vee &\xrightarrow{\cong} \mathbb{L}_Q \\ R\pi_* RHom(\mathcal{F}, \mathcal{F})_0^\vee[-1] &\xrightarrow{\cong} q^*\mathbb{L}_{\mathcal{M}}. \end{aligned}$$

By using Serre duality, we can prove the following theorem (see Theorem 6.10):

Theorem 1.3. *Let (X, H) be a polarized K3 surface, $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ a Mukai vector with respect to which H is generic. Assume $\mathbf{v}^2 > 0$. Then the reduced moduli stack $\mathcal{M}$ of semistable sheaves on X is a symplectic stack.*

The main techniques used in this presentation were adopted from the paper [13] of Huybrechts and Thomas on the application of Atiyah class on deformation theory of complexes, and the paper [8] of Gillam on the application of reduced Atiyah class on the deformation theory of quotients. The paper is organized as follows:

In Section 2, we first of all briefly recall some properties of cotangent complexes which will be used later. Then we turn to a short summary of classical Atiyah classes and reduced Atiyah classes, including their definitions and properties in the deformation-obstruction theory. We will also show that the reduced Atiyah class is a lift of the classical Atiyah class.

Section 3 is mainly a technical point. Since we will eventually be interested in the moduli space of sheaves with fixed determinant, we have to remove the effect of the trace map. This section uses techniques in derived categories to create "traceless version" of all complexes involved in the following sections.

Section 4 contains the first half of the central computation, which is on the cotangent complex of the GIT-semistable locus of the Quot scheme. Following [8, 13], we use the reduced Atiyah class to establish a morphism from a complex constructed only from the universal family on the Quot scheme, to the cotangent complex of Quot scheme. Then we show that this morphism induces isomorphisms on all cohomology groups.

Section 5 provides the other half of the central computation. We use the transitivity property of the cotangent complex, together with the cotangent complex of the Quot scheme computed in previous section to obtain the cotangent complex of the quotient stack. The commutativity of the diagram (13) is the major obstacle that we have to overcome in this section.

In Section 6, we take the definition of symmetric obstruction theories in [3] as a model, and formally introduce the notion of a symplectic stack. As an application of the computations in previous sections, we show that the moduli stacks we studied in previous sections are indeed examples of symplectic stacks.

After the first version of this manuscript was finished, the author was informed that similar results were obtained independently by Pantev, Toën, Vaquié and Vezzosi in [29]. Using the powerful machinery in derived algebraic geometry, the authors of [29] were able to construct symplectic structures under much more general settings. However, the language used in the present paper is mostly classical and much less technical, therefore this paper can be viewed as a more elementary approach to the computation of cotangent complexes and the construction of symplectic structures.

Notations. Throughout this paper, X will always be a projective K3 surface, and $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ is a fixed Mukai vector. Moreover, H is an ample line bundle on X generic with respect to $\mathbf{v}$, which is used to determine the stability of sheaves in Gieseker's sense. We also assume that $\mathbf{v}^2 > 0$ under the Mukai pairing, so that there is at least one H-stable sheaf on X with Mukai vector $\mathbf{v}$. We always use Q for the GIT-semistable locus of the Grothendieck's Quot scheme used in the GIT construction of the moduli space. We denote the two projections from $Q \times X$ by

$$\begin{tikzcd} & Q \times X \arrow[dl, "\pi"'] \arrow[dr, "\pi_X"] & \\ Q & & X. \end{tikzcd}$$

The gauge group $PGL(N)$ in the GIT construction will be denoted by G. It acts on Q and the global quotient stack $[Q/G]$ will be denoted by $\mathcal{M}$. We use

$$q : Q \longrightarrow \mathcal{M}$$

for the structure morphism from Q to the global quotient stack $\mathcal{M}$. We always assume that the stable locus is non-empty. Since the stability is an open condition, we denote the open dense subscheme of Q over which the quotient sheaf is stable by Q^s, and the corresponding image of Q^s under q by $\mathcal{M}^s$.

We also fix the universal quotient sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0 \tag{1}$$

on the GIT-semistable locus of the Quot scheme, or more precisely, on $Q \times X$. Here we should note that $\mathcal{E}$ is obtained by pulling back a vector bundle on X via the projection π_X, therefore is a trivial family over Q.

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2. Cotangent complexes and Atiyah classes

2.1. Cotangent complexes

We first of all recall some properties of cotangent complexes, which will be important for our later discussions. The classical reference for cotangent complexes is [14]. For cotangent complexes of stacks, one can see [19, Chapter 17] and [28].

Lemma 2.1. *[14, 19] The cotangent complex of $\mathcal{X}$ is an object in the derived category $D^b(\mathcal{X})$, which*

1) *is quasi-isomorphic to a single locally free sheaf in degree 0 if $\mathcal{X}$ is a smooth scheme;*
2) *has perfect amplitude in $[-1,0]$ and is quasi-isomorphic to a single sheaf in degree 0 if $\mathcal{X}$ is a scheme of locally complete intersection;*
3) *has perfect amplitude in $(-\infty,0]$ if $\mathcal{X}$ is a scheme or a Deligne-Mumford stack;*
4) *has perfect amplitude in $(-\infty,1]$ if $\mathcal{X}$ is an Artin stack.*

The computation of the cotangent complex is in general very difficult, however the following functorial property turns out to be very helpful in some cases.

Proposition 2.2. *[14, 19] Let*

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$$

be two morphisms of schemes or stacks. Then we have the following exact triangle in $D^b(\mathcal{X})$:

$$f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \longrightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \longrightarrow \mathbb{L}_{\mathcal{X}/\mathcal{Y}}. \tag{2}$$

Furthermore, this exact triangle is functorial. Namely, if we have a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "f"] \arrow[d, "u"'] & \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\
\mathcal{X}' \arrow[r, "f'"] & \mathcal{Y}' \arrow[r] & \mathcal{Z}',
\end{tikzcd}$$

then there are morphisms between two exact triangles (vertical arrows in the following diagram) which make the diagram commute

$$\begin{tikzcd}
u^* f'^* \mathbb{L}_{\mathcal{Y}'/\mathcal{Z}'} \arrow[r] \arrow[d] & u^* \mathbb{L}_{\mathcal{X}'/\mathcal{Z}'} \arrow[r] \arrow[d] & u^* \mathbb{L}_{\mathcal{X}'/\mathcal{Y}'} \arrow[d] \\
f^* \mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \arrow[r] & \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \arrow[r] & \mathbb{L}_{\mathcal{X}/\mathcal{Y}}
\end{tikzcd}$$

Another important property which is helpful in understanding the cotangent complex of a scheme is

Lemma 2.3. *[14] Let X be a scheme. Then*

$$H^0(\mathbb{L}_X) = \Omega_X,$$

the cotangent sheaf of X. In particular, if X is a local complete intersection, we have a quasi-isomorphism

$$\mathbb{L}_X \xrightarrow{\cong} \Omega_X.$$

Proof. See [14, Proposition 1.2.4.2]. □

The reason why cotangent complexes are important is that they play a central role in deformation theory, which is the whole essence of [14, 15]. For our purpose, we need two properties of cotangent complexes concerning deformation theory.

Let X be any scheme, and I be any coherent $\mathcal{O}_X$-module. We use the notion

$$X[I] = \operatorname{Spec}_X(\mathcal{O}_X \oplus I)$$

for the trivial square zero extension of X by I. In other words, $X[I]$ can be viewed as a first order deformation of X, and we denote the corresponding natural inclusion by $\iota_I : X \longrightarrow X[I]$. A retract of $X[I]$ is defined to be a morphism $r : X[I] \longrightarrow X$ such that the composition $r \circ \iota_I = \mathrm{id}_X$.

Proposition 2.4. *[14] Under the above notations, all retracts from $X[I]$ to X are parametrized by $\operatorname{Hom}(\mathbb{L}_X, I)$. Or in other words, the automorphism group of the deformation $X[I]$ is canonically given by $\operatorname{Hom}(\mathbb{L}_X, I)$.*

Proof. This is part of [14, Theorem III.2.1.7]. Since X is a scheme, by the above lemmas, we have

$$\operatorname{Hom}(\mathbb{L}_X, I) = \operatorname{Hom}(\Omega_X, I) = \operatorname{Der}(\mathcal{O}_X, I).$$

However every retract $r : X[I] \longrightarrow X$ corresponds to a splitting φ of the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & \mathcal{O}_X \oplus I \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[l, bend left, "\varphi"] & 0, \end{tikzcd}$$

which is an algebra homomorphism, therefore corresponds to a derivation into I. □

A priori, $X[I]$ may not be the only first order thickening of X by the ideal I. The following proposition gives a parameter space for all such thickenings. It's an application of the so-called "Fundamental Theorem of Cotangent Complex" in [14].

Proposition 2.5. *[14] Under the above notations, all isomorphism classes of first order thickenings of X by the ideal I are parametrized by* $\operatorname{Ext}^1(\mathbb{L}_X, I)$.

Proof. See [14, Theorem III.1.2.3, III.1.2.7]. □

The class corresponding to a particular thickening of X is called the *Kodaira-Spencer class* which represents the thickening. Obviously, the zero element in $\operatorname{Ext}^1(\mathbb{L}_X, I)$ represents the trivial thickening $X[I]$.

Now we apply the general theory of cotanget complexes to Q; i.e., the GIT-semistable locus of the Quot scheme. Due to the existence of strictly semistable sheaves, Q is in general singular. However, the cotangent complex of Q is very simple. In fact, it is quasi-isomorphic to a single sheaf concentrated in degree 0. The following lemma is part of [17, Proposition 3.11]. We nevertheless give an elementary proof.

Lemma 2.6. *The GIT-semistable locus of the Quot scheme Q in the GIT construction of the moduli space of semistable sheaves on a K3 surface is a local complete intersection. In particular, the cotangent complex $\mathbb{L}_Q$ has perfect amplitude in $[-1, 0]$, and is quasi-isomorphic to the cotangent sheaf Ω_Q.*

Proof. By [13, Proposition 2.2.8], at every closed point $q \in Q$, which is by the above notation represented by a quotient

$$0 \longrightarrow K \longrightarrow E \longrightarrow F \longrightarrow 0,$$

there is an inequality concerning the dimension of Q at the point

$$\dim \operatorname{Hom}(K,F) \geqslant \dim_q Q \geqslant \dim \operatorname{Hom}(K,F) - \dim \operatorname{Ext}^1(K,F)_0,$$

where $\operatorname{Ext}^1(K,F)_0$ is the kernel of the composition map

$$\operatorname{Ext}^1(K,F) \xrightarrow{\cong} \operatorname{Ext}^2(F,F) \xrightarrow{tr} H^2(\mathcal{O}_X) = \mathbb{C}.$$

And the Quot scheme Q is a local complete intersection if and only if the second equality holds at every closed point $q \in Q$.

By [17, Theorem 4.4] (see also [36, Theorem 3.18]), we know that the GIT-semistable locus of the Quot scheme Q is irreducible. Therefore, $\dim_q Q$ is constant on the only connected component of Q. Furthermore, it is easy to check that for every $i \geqslant 2$, we have

$$\operatorname{Ext}^i(K,F) = 0,$$

which implies

$$\dim \operatorname{Hom}(K,F) - \dim \operatorname{Ext}^1(K,F)_0 = \chi(K,F) + 1$$

which is a topological number only depending on the Chern classes of F, hence is also constant. Therefore it suffices to check the equality of both sides at one closed point of Q.

However by the assumption, there exists at least one point in the Quot scheme Q which is represented by a stable quotient sheaf F. At such a point the obstruction space $\operatorname{Ext}^1(K,F)_0$ vanishes, therefore both equalities hold at the same time. By the above discussion we conclude that the Quot scheme Q is a local complete intersection. And the statement about $\mathbb{L}_Q$ follows from that. □

2.2. Classical Atiyah classes

In [14], the Atiyah class were defined in two different ways. We follow the second approach using the exact sequence of principal parts, which itself was defined in [14, III.1.2.6].

Let $A \to B$ be a ring homomorphism, and I the kernel of the surjective ring homomorphism $B \otimes_A B \to B$. Then we have an exact sequence

$$0 \longrightarrow I \longrightarrow B \otimes_A B \longrightarrow B \longrightarrow 0$$

which splits by either of the ring homomorphisms

$$j_1, j_2 : B \longrightarrow B \otimes_A B$$

where

$$j_1(x) = x \otimes 1, \quad j_2(x) = 1 \otimes x.$$

After dividing by I^2 we obtained

$$0 \longrightarrow I/I^2 \longrightarrow B \otimes_A B/I^2 \longrightarrow B \longrightarrow 0$$

which we denoted by

$$0 \longrightarrow \Omega_{B/A} \longrightarrow P^1_{B/A} \longrightarrow B \longrightarrow 0.$$

Note that $P^1_{B/A}$ is a B-B-bimodule. Let M be a B-module. We tensor the above exact sequence by M from right side and obtain

$$0 \longrightarrow \Omega_{B/A} \otimes_B M \longrightarrow P^1_{B/A} \otimes_B M \longrightarrow M \longrightarrow 0,$$

which defines a class in $\mathrm{Ext}^1_B(M, M \otimes \Omega_{B/A})$. This class is called the *Atiyah class of M*.

In our settings, we let A be $\mathcal{O}_X$ and B be $\mathcal{O}_{Q\times X}$. For any sheaf $\mathcal{F}$ on $Q \times X$, we obtained the Atiyah class of $\mathcal{F}$, which we denote by $At(\mathcal{F})$. Note that, a priori, what we defined above is only a truncation of the full Atiyah class. For the definition of the full Atiyah class, we need to replace B by a simplicial resolution in the sequence of principal parts. However, because of Lemma 2.6, there's no difference in this case. More precisely, since

$$\Omega_{Q\times X/X} = \mathbb{L}_{Q\times X/X} = \pi^*\mathbb{L}_Q,$$

we have actually defined the (full) Atiyah class

$$At(\mathcal{F}) \in \mathrm{Ext}^1_{Q\times X}(\mathcal{F}, \mathcal{F} \otimes \pi^*\mathbb{L}_Q).$$

Now we study the deformation properties of the Atiyah class. We have already seen from Lemma 2.4 that, for any coherent sheaf I on Q, the space

$\mathrm{Hom}_Q(\mathbb{L}_Q, I)$ parametrizes all retracts $\iota : Q[I] \longrightarrow Q$. On the other hand, from classical sheaf deformation theory, we also know that $\mathrm{Ext}^1_{Q\times X}(\mathcal{F}, \mathcal{F} \otimes \pi^* I)$ parametrizes all flat deformations of $\mathcal{F}$ from Q to $Q[I]$ (see for example [31, Lemma 3.4]). The Atiyah class gives a nice relation of the two spaces as follows:

Proposition 2.7. *Let I be any coherent $\mathcal{O}_Q$-module and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{Q\times X}$-module which is flat over Q. Let $At(\mathcal{F})$ be its Atiyah class defined as above. Then the map*

$$At(\mathcal{F}) \cup (\mathcal{F} \otimes -) : \mathrm{Hom}_Q(\mathbb{L}_Q, I) \longrightarrow \mathrm{Ext}^1_{Q\times X}(\mathcal{F}, \mathcal{F} \otimes \pi^* I)$$

given by precomposing with the Atiyah class of $\mathcal{F}$ can be interpreted as

$$\{\text{retracts of } \iota_I\} \longrightarrow \{\text{flat deformations of the } \mathcal{F} \text{ over } Q[I]\},$$

where the arrow is given by pulling back the sheaf $\mathcal{F}$ via the chosen retract.

The proof of the proposition is very straightforward and is just a matter of unwinding the definitions. However, to the best of my knowledge, it doesn't seem to appear anywhere in this form. So we include a proof here.

Proof. From Lemma 2.4, we actually know that, for any

$$u \in \mathrm{Hom}_Q(\mathbb{L}_Q, I) = \mathrm{Hom}_Q(\Omega_Q, I),$$

the corresponding retract is given by the splitting of the second row in the following diagram via the arrow $(\mathrm{id}, u \circ d_Q)$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_Q \arrow[r] \arrow[d, "u"] & \mathcal{O}_Q \oplus \Omega_Q \arrow[r, shift right] \arrow[d] & \mathcal{O}_Q \arrow[r] \arrow[l, shift right, "(\mathrm{id},d_Q)"'] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] & \mathcal{O}_Q \oplus I \arrow[r, shift left] & \mathcal{O}_Q \arrow[r] \arrow[l, shift left, "(\mathrm{id},u\circ d_Q)"] & 0,
\end{tikzcd}$$

where d_Q is the universal derivation defined by

$$d_Q(x) = 1 \otimes x - x \otimes 1$$

for every $x \in \mathcal{O}_Q$.

Note that $\mathcal{O}_Q \oplus \Omega_Q$ is exactly the principal part P^1_Q together with the left $\mathcal{O}_Q$-module structure, while the splitting (id, d_Q) is exactly the right

$\mathcal{O}_Q$-module structure. Similarly, $\mathcal{O}_Q \oplus I$ can also be identified with $\mathcal{O}_{Q[I]}$ such that the splitting $(\mathrm{id}, u \circ d_Q)$ corresponds exactly to the retract.

We pullback the diagram to $Q \times X$ and tensor every term with the sheaf $\mathcal{F}$. Using the splittings of both rows, we get

$$\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega_Q \otimes \mathcal{F} \arrow[r] \arrow[d] & P^1_{Q\times X/X} \otimes \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi^*I \otimes \mathcal{F} \arrow[r] & \pi^*Q[I] \otimes \mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & 0,
\end{tikzcd}$$

From the above construction we see exactly that the second row is the class $At(\mathcal{F}) \cup (\mathcal{F} \otimes u)$, which finishes the proof. □

The above proposition concerns the relation of the Atiyah class with deformations of $\mathcal{F}$ over the trivial first order deformation $Q[I]$ of Q. Next proposition relates the Atiyah class with obstructions of deforming $\mathcal{F}$ to an arbitrary first order deformation of Q. More precisely, we have seen from Lemma 2.5 that $\mathrm{Ext}^1(\mathbb{L}_Q, I)$ parametrizes all isomorphism classes of first order thickenings of Q by the ideal sheaf I. And classical deformation theory tells us that the obstruction of deforming $\mathcal{F}$ to any first order extension of the base lies in the space $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \pi^*I)$ (see for example [31, Proposition 3.13]). Via the Atiyah class, we can make a precise formation of their relation:

Proposition 2.8. *[14] Let I be any coherent $\mathcal{O}_Q$-module and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_{Q\times X}$-module which is flat over Q. Let $At(\mathcal{F})$ be its Atiyah class defined as above. The map*

$$At(\mathcal{F}) \cup (\mathcal{F} \otimes -) : \mathrm{Ext}^1_Q(\mathbb{L}_Q, I) \longrightarrow \mathrm{Ext}^2_{Q\times X}(\mathcal{F}, \mathcal{F} \otimes \pi^*I)$$

given by precomposing with the Atiyah class can be interpreted as

$$\{\textit{thickenings } Q' \textit{ of } Q \textit{ by } I\} \longrightarrow \{\textit{obstructions to the existence of flat deformations of } \mathcal{F} \textit{ over } Q'\}.$$

Proof. This proposition lies under the general principle of "the composition of Atiyah class and Kodaira-Spencer class is the obstruction". The proof can be found in [14, Proposition IV.3.1.8]. □

2.3. Reduced Atiyah classes

Now we turn to the reduced Atiyah class, which was defined and extensively studied in [8]. Two definitions were given, one using graded cotangent complex, the other using more classical language. We follow the second approach in [8] and give a brief description of the reduced Atiyah class in our context, under the additional property that Q is a local complete intersection, just to avoid any simplicial resolution of algebras.

Recall that we have the short exact sequence of sheaves on $Q \times X$ given by (1), where

$$\mathcal{E} = \pi_X^* E_0$$

is a trivial family of vector bundles over Q. We have the following commutative diagram with all rows and columns exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{K} \otimes \pi^*\Omega_Q \arrow[r] \arrow[d] & \mathcal{E} \otimes \pi^*\Omega_Q \arrow[r] \arrow[d] & \mathcal{F} \otimes \pi^*\Omega_Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P^1(\mathcal{K}) \arrow[r] \arrow[d] & P^1(\mathcal{E}) \arrow[r] \arrow[d] & P^1(\mathcal{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] \arrow[u, bend right, "\sigma"'] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The exactness of the three columns are trivial, because they are all exact sequence of principal parts. The exactness of the third row is part of given data. The exactness of the first row is due to the flatness of $\mathcal{F}$ and exactness of the middle row follows from that of the other two rows.

As observed in [8], the middle column naturally splits, due to the fact that $\mathcal{E}$ is a trivial family over Q. The reversed arrow σ in the above diagram is chosen as follows: we have

$$\mathcal{E} = \mathcal{O}_Q \otimes_{\mathbb{C}} E_0$$

and

$$P^1(\mathcal{E}) = (\mathcal{O}_{Q\times Q}/I^2) \otimes_{\mathbb{C}} E_0 = (\mathcal{O}_Q \otimes_{\mathbb{C}} \mathcal{O}_Q/I^2) \otimes_{\mathbb{C}} E_0,$$

where I is the ideal sheaf of the diagonal in $Q \times Q$. Then we define

$$\begin{array}{rccl} \sigma: & \mathcal{E} & \longrightarrow & P^1(\mathcal{E}) \\ & a \otimes e & \longmapsto & a \otimes 1 \otimes e. \end{array}$$

It's obvious that σ is indeed a splitting.

After all of the preparation, we define the reduced Atiyah class $at \in \mathrm{Hom}(\mathcal{K}, \mathcal{F} \otimes \pi^* \mathbb{L}_Q)$ as the composition of the following arrows from the above diagram, starting from $\mathcal{K}$:

(3)
$$\begin{tikzcd}
 & & F \otimes \pi^* \mathbb{L}_Q \arrow[d] \\
 & P^1(\mathcal{E}) \arrow[r] & P^1(\mathcal{F}) \\
\mathcal{K} \arrow[r] & \mathcal{E} \arrow[u, "\sigma"'] &
\end{tikzcd}$$

where the downward arrow in the right column means that, everything in $P^1(\mathcal{F})$ which comes from $\mathcal{K}$ via the composition of the other three arrows is in the image of this downward arrow, therefore can be uniquely lifted to $\mathcal{F} \otimes \pi^* \mathbb{L}_Q$. The reason is that if we maps it further down to $\mathcal{F}$ as in the above diagram, we get the zero section. Hence the composition of the four arrows is well-defined.

The reduced Atiyah class behaves compatibly with the classical Atiyah class. In fact, it is a lifting of the classical Atiyah class in the following sense

Proposition 2.9. *Let* $at \in \mathrm{Hom}(\mathcal{K}, \mathcal{F} \otimes \pi^* \mathbb{L}_Q)$ *be the reduced Atiyah class, and* $e \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{K})$ *be the extension class represented by the universal quotient sequence* (1) *on* Q. *Then*

1) *The composition*

$$(at[1]) \circ e : \mathcal{F} \longrightarrow \mathcal{K}[1] \longrightarrow \mathcal{F} \otimes \pi^* \mathbb{L}_Q[1]$$

is the classical Atiyah class $At(\mathcal{F})$*;*

2) *The composition*

$$(e \otimes \pi^* \mathbb{L}_Q) \circ at : \mathcal{K} \longrightarrow \mathcal{F} \otimes \pi^* \mathbb{L}_Q \longrightarrow \mathcal{K} \otimes \pi^* \mathbb{L}_Q[1]$$

is the classical Atiyah class $At(\mathcal{K})$.

Proof. We will only use the first half of the proposition, so only this part will be proved in details. However, the proof of the second part of the proposition is completely parallel to that of the first part.

To prove the first part of the proposition, it suffices to show that

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \otimes \pi^*\mathbb{L}_Q \arrow[r] & P^1(\mathcal{F}) \arrow[r] & \mathcal{F} \arrow[r] & 0
\end{tikzcd}$$

is a pushout diagram, where the first row is the universal family (1), while the second row is the principal part sequence for $\mathcal{F}$. By the construction of the pushout, in fact we just need to show the sequence

$$0 \longrightarrow \mathcal{K} \xrightarrow{\varphi_1} \mathcal{E} \oplus (\mathcal{F} \otimes \pi^*\mathbb{L}_Q) \xrightarrow{\varphi_2} P^1(\mathcal{F}) \longrightarrow 0 \tag{4}$$

is an exact complex, where the map φ_1 is the pair of the first arrow in diagram (3) and the negation of the reduced Atiyah class $-at$, and the map φ_2 is the sum of composition of the middle two arrows in (3) and the downward arrow.

To verify this claim, we observe that

- φ_1 is injective, which is obvious because the first component is injective;
- φ_2 is surjective. In fact, $\mathrm{Im}(\varphi_2)$ is a submodule of $P^1(\mathcal{F})$, and obviously $\mathcal{F} \otimes \pi^*\mathbb{L}_Q$ lies in $\mathrm{Im}(\varphi_2)$. Furthermore, $\mathrm{Im}(\varphi_2)/(\mathcal{F} \otimes \pi^*\mathbb{L}_Q) = \mathcal{F}$ because the image of $\mathcal{E}$ hits everything in $\mathcal{F}$;
- $\mathrm{Im}(\varphi_1) \subset \ker(\varphi_2)$, which is due to the construction of the reduced Atiyah class;
- $\ker(\varphi_2) \subset \mathrm{Im}(\varphi_1)$. In fact, if $(e, f') \in \mathcal{E} \oplus (\mathcal{F} \otimes \pi^*\mathbb{L}_Q)$ satisfies $\alpha_2(e, f') = 0$, then e comes from a certain $k \in \mathcal{K}$, because its image in $\mathcal{F}$ is the negation of the image of f' in $\mathcal{F}$, which is 0. Then it is clear that $\varphi_1(k) = (e, f')$.

The above observations finish the proof of the exact sequence (4). □

Similar to the discussion of the classical Atiyah class, we will also need to use some deformation interpretations of the reduced Atiyah class. We know from the deformation theory of quotients (for example, in Chapter 2 of [12]) that, for any coherent $\mathcal{O}_Q$-module I, the space $\mathrm{Hom}(\mathcal{K}, \mathcal{F} \otimes \pi^*I)$

parametrizes all first order flat deformations of the universal quotient (1) to the square-zero extension $Q[I]$. We also know that $\mathrm{Ext}^1_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*I)$ contains all obstruction classes of lifting the quotient (1) to the first order.

The following two propositions from [8] concerning the relation between the reduced Atiyah class and the deformation theory. The statements are parallel to similar results in previous section about classical Atiyah class. The first one relates the reduced Atiyah class with deformation of quotients on the trivial square free deformation of the base:

Proposition 2.10. *[8] Let I be any coherent $\mathcal{O}_Q$-module and at be the reduced Atiyah class of the quotient* (1)*. The map*

$$at\cup(\mathcal{F}\otimes\pi^*(-)):\mathrm{Hom}_Q(\mathbb{L}_Q,I)\longrightarrow\mathrm{Hom}_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*I)$$

given by precomposing with the reduced Atiyah class can be interpreted as

$$\{\textit{retracts of } \iota_I\}\longrightarrow\{\textit{flat deformations of the quotient } (1) \textit{ over } Q[I]\},$$

where the arrow is given by pullback.

Proof. See the proof in [8, Lemma 3.2]. □

Next proposition concerns the relation between the Atiyah class and the obstruction of deformation of the quotient map, which is also under the essence of "the product of Atiyah class and Kodaira-Spencer class is the obstruction class".

Proposition 2.11 ([8]). *Let I be any coherent $\mathcal{O}_Q$-module and at be the reduced Atiyah class of the quotient* (1)*. The map*

$$at\cup(\mathcal{F}\otimes\pi^*(-)):\mathrm{Ext}^1_Q(\mathbb{L}_Q,I)\longrightarrow\mathrm{Ext}^1_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*I)$$

given by precomposing with the reduced Atiyah class can be interpreted as

$$\begin{aligned}&\{\textit{thickenings } Q' \textit{ of } Q \textit{ by } I\}\\ \longrightarrow&\{\textit{obstructions to the existence}\\ &\ \textit{of flat deformations of the quotient } (1) \textit{ over } Q'\}.\end{aligned}$$

Proof. See [8, Lemma 1.13]. □

3. Trace maps and trace-free parts of complexes

On the Quot scheme Q, we apply the derived functor $\mathrm{R}\pi_*\mathcal{RH}om(-,\mathcal{F})$ to the exact sequence (1) and get an exact triangle of complexes

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})[-1] \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F}). \tag{5}$$

In this section we will construct the "traceless" version of this exact triangle.

First of all it is easy to see that the trace map $tr : \mathcal{RH}om(F,F) \longrightarrow \mathcal{O}_{Q\times X}$ splits by a reverse map of scaling. Therefore we have

$$\mathcal{RH}om(F,F) = \mathcal{RH}om(F,F)_0 \oplus \mathcal{O}_{Q\times X}, \tag{6}$$

where the first summand is the kernel of the above trace map. The splitting leads to

$$\mathrm{R}\pi_*\mathcal{RH}om(F,F) = \mathrm{R}\pi_*\mathcal{RH}om(F,F)_0 \oplus \mathrm{R}\pi_*\mathcal{O}_{Q\times X}.$$

However, the second component above can be further decomposed into two direct summands as

$$\begin{aligned} \mathrm{R}\pi_*\mathcal{O}_{Q\times X} &= \mathrm{R}\pi_*\pi_X^*\mathcal{O}_X \\ &= R\Gamma(\mathcal{O}_X)\otimes\mathcal{O}_Q \\ &= (H^0(\mathcal{O}_X)\oplus H^2(\mathcal{O}_X)[-2])\otimes\mathcal{O}_Q \\ &= \pi_*\mathcal{O}_{Q\times X}\oplus \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]. \end{aligned}$$

Therefore we have a decomposition of the middle complex of (5)

$$\mathrm{R}\pi_*\mathcal{RH}om(F,F) = \mathrm{R}\pi_*\mathcal{RH}om(F,F)_0 \oplus \pi_*\mathcal{O}_{Q\times X}\oplus \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2], \tag{7}$$

in which we also keep in mind that

$$\begin{aligned} \pi_*\mathcal{O}_{Q\times X} &= \mathcal{O}_Q, \\ \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X} &= \mathcal{O}_Q. \end{aligned}$$

We combine the equation (5) and the obvious morphisms of embeddings and splittings from the equation (7) to get two morphisms

$$\begin{aligned} &\alpha:\ \pi_*\mathcal{O}_{Q\times X} \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F}); \\ &\beta:\ \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})[-1] \longrightarrow \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]. \end{aligned}$$

Now we define the "traceless" version of the other two complexes by completing the exact triangles. More precisely, we define

$$\begin{aligned}\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0 &= \mathrm{Cone}(\alpha)\\ \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0 &= \mathrm{Cone}(\beta).\end{aligned}$$

Then we have the following

Proposition 3.1. *We naturally get an exact triangle*

$$\begin{aligned}R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0[-1] &\longrightarrow R\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0\\ &\longrightarrow R\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0.\end{aligned}\tag{8}$$

Proof. We observe two exact triangles from the above cone construction:

$$\pi_*\mathcal{O}_{Q\times X} \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F}) \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0,\tag{9}$$

and

$$\begin{aligned}\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0[-1] &\longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})[-1]\\ &\longrightarrow \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2].\end{aligned}\tag{10}$$

Then this proposition is just a direct consequence of next lemma. □

Lemma 3.2. *Let*

$$A \longrightarrow B \longrightarrow C$$

be an exact triangle in a triangulated category, where $B = B_1 \oplus B_2$.

1) *If we complete the natural morphism* $A \longrightarrow B_2$ *into an exact triangle*

$$A_0 \longrightarrow A \longrightarrow B_2,$$

then we get a new exact triangle

$$A_0 \longrightarrow B_1 \longrightarrow C;$$

2) *If we complete the natural morphism* $B_1 \longrightarrow C$ *into an exact triangle*

$$B_1 \longrightarrow C \longrightarrow C_0,$$

then we get a new exact triangle

$$A \longrightarrow B_2 \longrightarrow C_0.$$

Proof. Both statements are immediate applications of the octahedral axiom of triangulated categories. □

Next we analyze the fiberwise behaviour of the exact triangle (8) and the corresponding cohomology groups. Let $p \in Q$ be a closed point. Let X_p be the corresponding fiber in the product $Q \times X$, and

$$0 \longrightarrow K_p \longrightarrow E_p \longrightarrow F_p \longrightarrow 0 \tag{11}$$

be the corresponding quotient represented by p. Then the restriction of the decomposition (7) becomes

$$\mathrm{RHom}(F_p, F_p) = \mathrm{RHom}(F_p, F_p)_0 \oplus H^0(\mathcal{O}_{X_p}) \oplus H^2(\mathcal{O}_{X_p})[-2].$$

When we restrict the exact triangle (9) to the closed point p, we get the exact triangle

$$H^0(\mathcal{O}_{X_p}) \longrightarrow \mathrm{RHom}(E_p, F_p) \longrightarrow \mathrm{RHom}(E_p, F_p)_0.$$

When we consider the corresponding long exact sequence of the cohomology groups, we realize that the complexes $\mathrm{RHom}(E_p, F_p)$ and $\mathrm{RHom}(E_p, F_p)_0$ have the same cohomology groups except in degree 0, where we get an exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{X_p}) \xrightarrow{\alpha_p} \mathrm{Hom}(E_p, F_p) \longrightarrow \mathrm{Hom}(E_p, F_p)_0 \longrightarrow 0.$$

From the above construction we see that the arrow α_p factor through $\mathrm{Hom}(F_p, F_p)$ by a scalar map $H^0(\mathcal{O}_{X_p}) \longrightarrow \mathrm{Hom}(F_p, F_p)$ and a natural map induced by the quotient (11), therefore $\mathrm{Hom}(E_p, F_p)_0$ is obtained by "removing" the 1-dimensional vector space generated by the map in the quotient (11).

Similarly, we can analyze the exact triangle (10) and get a parallel conclusion. Summarizing the discussion we obtain the following lemma

Lemma 3.3. *The "traceless" complexes*

$$R\pi_* \mathcal{RH}om(\mathcal{E}, \mathcal{F})_0 \quad \textit{and} \quad R\pi_* \mathcal{RH}om(\mathcal{K}, \mathcal{F})_0$$

have the following pointwise behaviour:

1) *The restriction of the complex* $R\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0$ *to any closed point* $p\in Q$ *computes the cohomology groups*

$$\mathrm{Ext}^i(E_p,F_p)_0=\begin{cases}\mathrm{Ext}^i(E_p,F_p) & \text{if } i\neq 0;\\ \mathrm{coker}(\alpha_p) & \text{if } i=0,\end{cases}$$

where α_p *is the following composition of the scalar and the natural map induced by* (11)

$$\alpha_p : H^0(\mathcal{O}_{X_p})\longrightarrow \mathrm{Hom}(F_p,F_p)\longrightarrow \mathrm{Hom}(E_p,F_p).$$

2) *The restriction of the complex* $R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0$ *to any closed point* $p\in Q$ *computes the cohomology groups*

$$\mathrm{Ext}^i(K_p,F_p)_0=\begin{cases}\mathrm{Ext}^i(K_p,F_p) & \text{if } i\neq 1;\\ \ker(\beta_p) & \text{if } i=1,\end{cases}$$

where β_p *is the following composition of the natural map induced by* (11) *and the trace map*

$$\beta_p : \mathrm{Ext}^1(K_p,F_p)\longrightarrow \mathrm{Ext}^2(F_p,F_p)\longrightarrow H^2(\mathcal{O}_{X_p}).$$

4. Cotangent complex of the quot scheme

The goal of this section is to compute the cotangent complex of the GIT-semistable locus of the Quot scheme Q.

Lemma 4.1. *The reduced Atiyah class at induces a morphism from* $R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee$ *to the cotangent complex* $\mathbb{L}_Q$.

Proof. In Section 2.3, we defined the reduced Atiyah class

$$at\in \mathrm{Hom}_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*\mathbb{L}_Q).$$

By Grothendieck-Verdier duality and Serre duality, we have

$$\begin{aligned}\mathrm{Hom}_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*\mathbb{L}_Q)&=\mathrm{Hom}_{Q\times X}(\mathcal{RH}om(\mathcal{F},\mathcal{K}),\pi^*\mathbb{L}_Q)\\&=\mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{K})[2],\mathbb{L}_Q)\\&=\mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F}\otimes\omega_\pi)^\vee,\mathbb{L}_Q)\\&=\mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee,\mathbb{L}_Q).\end{aligned}$$

Therefore the class *at* induces a morphism between the two complexes, denoted by

$$\gamma : \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee \longrightarrow \mathbb{L}_Q.$$

□

The rest of the section is aiming to prove that, although γ itself is not a quasi-isomorphism, if we replace the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee$ by its "traceless" counterpart $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$ constructed in previous section, then we get a quasi-isomorphism. We start from the following lemma comparing the degree 0 cohomology groups.

Lemma 4.2. *The morphism γ defined as above induces an isomorphism on the 0-th cohomology groups of the two complexes.*

This lemma was proved in [8, Theorem 4.2]. For the sake of completeness we include the proof here.

Proof. For simplicity, in this proof we denote

$$C := \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee.$$

We are aiming to show that

$$H^0(\gamma) : H^0(C) \longrightarrow H^0(\mathbb{L}_Q)$$

is an isomorphism. By Yoneda's lemma for the abelian category of coherent sheaves, it suffices to show that, for every coherent sheaf I on Q, the induces morphism

$$H^0(\gamma)_I : \mathrm{Hom}_Q(H^0(\mathbb{L}_Q), I) \longrightarrow \mathrm{Hom}_Q(H^0(C), I)$$

is an isomorphism. However, notice that both complexes $\mathbb{L}_Q$ and C have non-trivial cohomology only in non-positive degrees. Therefore, we have

$$\mathrm{Hom}_Q(H^0(\mathbb{L}_Q), I) = \mathrm{Hom}_Q(\mathbb{L}_Q, I),$$
$$\mathrm{Hom}_Q(H^0(C), I) = \mathrm{Hom}_Q(C, I).$$

Hence it suffices to show that the pullback morphism

$$\gamma_I : \mathrm{Hom}_Q(\mathbb{L}_Q, I) \longrightarrow \mathrm{Hom}_Q(C, I)$$

is an isomorphism. Again by Grothendieck and Serre duality theorems, we have

$$\begin{aligned}\mathrm{Hom}_Q(C, I) &= \mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K}, \mathcal{F})^\vee, I)\\ &= \mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{K})[2], I)\\ &= \mathrm{Hom}_{Q\times X}(\mathcal{RH}om(\mathcal{F}, \mathcal{K}), \pi^* I)\\ &= \mathrm{Hom}_{Q\times X}(\mathcal{K}, \mathcal{F}\otimes \pi^* I).\end{aligned}$$

Therefore the morphism γ_I becomes

$$\gamma_I : \mathrm{Hom}_Q(\mathbb{L}_Q, I) \longrightarrow \mathrm{Hom}_{Q\times X}(\mathcal{K}, \mathcal{F}\otimes \pi^* I),$$

which is given by the product with the reduced Atiyah class at. The deformation interpretation of this morphism is given by Proposition 2.10, namely, for any retract $r : Q[I] \to Q$, $\gamma_I(r)$ is a deformation of quotient of $\mathcal{E}$, given by pulling back the universal quotient (1) from Q to $Q[I]$ via r.

However, because of the universal property of Q, any deformation of the universal quotient is obtained by pulling back from Q. Therefore, the above morphism γ_I is an isomorphism, which concludes that $H^0(\gamma)$ is also an isomorphism. □

We take the dual of the exact triangle (10) and get another exact triangle

$$\mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[1] \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K}, \mathcal{F})^\vee \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K}, \mathcal{F})_0^\vee. \tag{12}$$

Lemma 4.3. *The morphism*

$$\gamma : R\pi_*\mathcal{RH}om(\mathcal{K}, \mathcal{F})^\vee \longrightarrow \mathbb{L}_Q$$

can be extended to a morphism

$$\gamma_0 : R\pi_*\mathcal{RH}om(\mathcal{K}, \mathcal{F})_0^\vee \longrightarrow \mathbb{L}_Q,$$

which induces an isomorphism of the 0-th cohomology groups of the two complexes.

Proof. For the first statement, it suffices to prove that

$$\mathrm{Hom}_Q(\mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[1], \mathbb{L}_Q) = 0.$$

In fact, by Lemma 2.6, $\mathbb{L}_Q$ is quasi-isomorphic to a single sheaf in degree 0. Also notice that $\mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[1] = \mathcal{O}_Q[1]$ is a single sheaf lying in degree -1.

Due to degree reason, the above equation holds, hence γ can be extended to γ_0.

From the long exact sequence induced by (12) we realise that the complexes $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee$ and $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$ have the same 0-th cohomology group. The second statement follows from this observation and Lemma 4.2. ☐

Lemma 4.4. *The complex* $R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$ *has perfect amplitude in* $[-1,0]$, *and is quasi-isomorphic to a single sheaf in degree 0.*

Proof. We first look at the fiber cohomology of $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0$ before taking the dual. From the second part of Lemma 3.3, we already know all cohomology groups of the restriction of the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0$ to any closed point $p \in Q$. Moreover, by the long exact sequence induced by the restriction of the universal quotient at the point p, we can easily tell which of them vanish. It's not hard to find out that

$$\mathrm{Ext}^i(K_p,F_p)_0 = \begin{cases} \mathrm{Hom}(K_p,F_p) & \text{if } i=0; \\ \mathrm{Ext}^2(F_p,F_p)_0 & \text{if } i=1; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we know that the only possible non-trivial fiber cohomology lies in degree 0 and 1. Furthermore, over the locus where F_p is stable, the only non-trivial fiber cohomology lies in degree 0.

Now we turn to the dual complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$. Since the fiber cohomology respect the operation of taking duals, we conclude that, the only possible non-trivial fiber cohomology lies in degree -1 and 0. Furthermore, on the open subset of Q where F_p is stable, the fiber cohomology in degree -1 is even trivial.

Now we are ready to prove the two statements in the lemma. First of all, we can always resolve the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$ by a perfect complex of finite length. We denote this perfect resolution by

$$A^s \longrightarrow A^{s+1} \longrightarrow \cdots \longrightarrow A^{t-1} \longrightarrow A^t.$$

We prove the first statement. If $s < -1$, we can actually truncate the complex at the position $s+1$, by replacing A^s by 0 and A^{s+1} by the cokernel of the map $A^s \longrightarrow A^{s+1}$. We claim that this cokernel is again a locally free sheaf over Q. In fact, for any closed point $p \in Q$, the kernel of the fiber map $A^s_p \longrightarrow A^{s+1}_p$ is the fiber cohomology group $\mathrm{Ext}^{-s}(K_p,F_p)_0^\vee$, which by the above discussion vanishes when $s < -1$. Therefore the morphism between the two locally free sheaves A^s and A^{s+1} is fiberwise injective, hence has a locally free cokernel. This operation increases the lowest degree of the locally free resolution by 1. We can repeat this procedure until we have $s = -1$.

Similarly, if $t > 0$, we can always truncate the resolution step by step from the highest degree, while keeping every term in the complex locally free, until we reach $t = 0$, by using the fact that the fiberwise cohomology groups vanish in positive degrees. Therefore, we know tha the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$ is quasi-isomorphic to a perfect complex in degree $[-1,0]$, which we still denote by

$$A^{-1} \longrightarrow A^0.$$

Finally, to prove the second statement, we only need to show that the cohomology of this 2-term complex in degree -1 vanishes. In fact, we already know that over an open dense subset Q^s of Q where the quotient sheaf is stable, the fiberwise cohomology of this 2-term cohomology is 0 in degree -1, which implies that the morphism of locally free sheaves

$$A^{-1} \longrightarrow A^0$$

is injective over the stable locus Q^s. However any subsheaf of a locally free sheaf is torsion free, hence we conclude that the kernel sheaf is 0, which proves the second statement. □

Finally, we can state the main result of this section

Theorem 4.5. *The morphism*

$$\gamma_0 : R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee \xrightarrow{\cong} \mathbb{L}_Q$$

is a quasi-isomorphism.

Proof. By Lemma 4.3, the morphism γ_0 is well-defined and $H^0(\gamma_0)$ is an isomorphism. from which the proposition is clear. By Lemmas 2.6 and 4.4, the only non-trivial cohomology groups in both complexes are in degree 0. Therefore γ_0 is a quasi-isomorphim. □

5. Cotangent complex of the moduli stack

The goal of this section is to compute the cotangent complex of the moduli stack $\mathcal{M} = [Q/G]$. To achieve this goal, we will first build up the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee \arrow[r] \arrow[d] & \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee \arrow[d, dashed] \\
\mathbb{L}_Q \arrow[r] & \mathbb{L}_{Q/\mathcal{M}}
\end{tikzcd} \tag{13}$$

such that the two vertical arrows are isomorphisms. We should notice that the two horizontal arrows are both functorial morphisms, while the left vertical arrow was constructed in the previous section and proved to be an isomorphism. It only remains to construct the right vertical arrow to make the diagram commute, and prove that it is an isomorphism.

Before we get into the main business, we need to study the fiber product of the quotient map $q : Q \longrightarrow \mathcal{M}$ with itself. More precisely, we have the following lemma:

Lemma 5.1. *The following diagram commutes:*

$$\begin{tikzcd}
G\times Q \arrow[dr, "j"] \arrow[drr, bend left, "m"] \arrow[ddr, bend right, "pr_2"'] & & \\
& Q\times_{\mathcal{M}} Q \arrow[r, "p_1"'] \arrow[d, "p_2"] & Q \arrow[d, "q"'] \\
& Q \arrow[r, "q"] & \mathcal{M},
\end{tikzcd} \tag{14}$$

where G is the gauge group $PGL(N)$, whose action on the Quot scheme Q is the upper horizontal arrow m, pr_i is the projection from $G\times Q$ to the i-th factor, p_i is the projection from $Q\times_{\mathcal{M}} Q$ to the i-th factor, and $j=(m,pr_2)$. Moreover, j is an isomorphism of schemes.

Proof. The commutativity is straightforward. In fact, the commutativity of the square is due to the fiber product, and the commutativity of the two triangles is due to the definition of the map j. And the statement that j is an isomorphism is a standard fact. For example, see [6, Part I, Proposition 4.43]. □

We will also need the following two facts related to the fiber product described in diagram (14).

Lemma 5.2. *Notations are the same as in Lemma 5.1. Then we have*

$$m^*\mathbb{L}_{Q/\mathcal{M}} = pr_1^*\mathbb{L}_G.$$

Proof. From the above lemma we know that the outer square of the diagram (14) is also a fiber product. Since the quotient map q is smooth, we apply [19, Theorem 17.3 (4)] and get

$$m^*\mathbb{L}_{Q/\mathcal{M}} = \mathbb{L}_{G\times Q/Q} = pr_1^*\mathbb{L}_G,$$

which proves the claim. □

Lemma 5.3. *Notations are the same as in Lemma 5.1. Let*

$$\mathbb{L}_Q \longrightarrow \mathbb{L}_{Q/\mathcal{M}}$$

be the functorial map induced by the right vertical arrow q, and

$$m^*\mathbb{L}_Q \longrightarrow m^*\mathbb{L}_{Q/\mathcal{M}}$$

be its pullback via the multiplication. Then we have the following commutative diagram

(15)
$$\begin{tikzcd} m^*\mathbb{L}_Q \arrow[r] \arrow[d] & \mathbb{L}_{G\times Q} \arrow[d] \\ m^*\mathbb{L}_{Q/\mathcal{M}} \arrow[r, "\cong"] & pr_1^*\mathbb{L}_G \end{tikzcd}$$

where the upper horizontal arrow is the funtorial map induced by the multiplication map m, the lower horizontal arrow is the one constructed in Lemma 5.2, and the right vertical arrow is the projection into the first factor.

Proof. This is a direct application of the functoriality of the transitivity sequence in Lemma 2.2. □

Lemma 5.4. *Notations are the same as in Lemma 5.1. Recall that $\mathcal{F}$ is the universal quotient sheaf on $Q \times X$. Then we have*

$$p_1^*\mathcal{F} = p_2^*\mathcal{F}.$$

Or equivalently, by further pulling back via the isomorphism j, we have

$$m^*\mathcal{F} = pr_2^*\mathcal{F}.$$

Proof. The two identities are equivalent because j is an isomorphism and the diagram (14) commutes. Now we show the second identity. By the general construction of moduli spaces of semistable sheaves, in the universal exact sequence (1), we have

$$\mathcal{E} = V \otimes_k \mathcal{O}_{Q\times X}(-m)$$

for some large positive integer m; see [12, Section 4.3].

If we pullback the morphism $u : \mathcal{E} \longrightarrow \mathcal{F}$ along pr_2, then we get

$$\widetilde{u} : V \otimes_k \mathcal{O}_G \otimes_k \mathcal{O}_{Q\times X}(-m) \longrightarrow \widetilde{\mathcal{F}}$$

where $\widetilde{\mathcal{F}} = pr_2^*\mathcal{F}$.

On the other hand, since $G = PGL(V)$, we have a universal isomorphism

$$h : V \otimes_k \mathcal{O}_G \longrightarrow V \otimes_k \mathcal{O}_G$$

such that at any point $g \in G$ we have the automorphism of V given by g. The action of G on Q shows that the pullback of the morphism $u : \mathcal{E} \longrightarrow \mathcal{F}$ along m is given by the composition

$$V \otimes_k \mathcal{O}_G \otimes_k \mathcal{O}_{Q\times X}(-m) \xrightarrow{h} V \otimes_k \mathcal{O}_G \otimes_k \mathcal{O}_{Q\times X}(-m) \xrightarrow{\widetilde{u}} \widetilde{\mathcal{F}}.$$

It follows that $m^*\mathcal{F} = \widetilde{\mathcal{F}} = pr_2^*\mathcal{F}$. $\square$

Lemma 5.5. *The composition of the reduced Atiyah class and the functorial morphism of cotangent complexes factor through $\mathcal{E}$. In other words, the dotted arrows in the following diagram exist and make the diagram commute:*

$$\begin{tikzcd}
\mathcal{K} \arrow[r, dashed] \arrow[d] & \mathcal{E} \arrow[d, dashed] \\
\mathcal{F} \otimes \pi^*\mathbb{L}_Q \arrow[r] & \mathcal{F} \otimes \pi^*\mathbb{L}_{Q/\mathcal{M}}
\end{tikzcd}$$

Proof. To prove the composition of the two solid arrows factors through $\mathcal{E}$, we only need to show that the composition of the following three maps is a zero map:

$$\begin{tikzcd}
\mathcal{F}[-1] \arrow[r] & \mathcal{K} \arrow[d] & \\
& \mathcal{F} \otimes \pi^*\mathbb{L}_Q \arrow[r] & \mathcal{F} \otimes \pi^*\mathbb{L}_{Q/\mathcal{M}}
\end{tikzcd}$$

However, by Proposition 2.9, we know that the composition of the first two maps in the above diagram is exactly the classical Atiyah class. Therefore the problem becomes to show the composition of the classical Atiyah class and the functorial morphism between the cotangent complexes, i.e., the following two morphism, is a zero map:

$$\mathcal{F}[-1] \longrightarrow \mathcal{F} \otimes \pi^*\mathbb{L}_Q \longrightarrow \mathcal{F} \otimes \pi^*\mathbb{L}_{Q/\mathcal{M}}. \tag{16}$$

We can pull back the maps in equation (16) via the multiplication m. If we denote $m^*\mathcal{F}$ by $\widetilde{\mathcal{F}}$, by applying Lemma 5.2, we get

$$\widetilde{\mathcal{F}}[-1] \longrightarrow \widetilde{\mathcal{F}} \otimes m^*\mathbb{L}_Q \longrightarrow \widetilde{\mathcal{F}} \otimes pr_1^*\mathbb{L}_G. \tag{17}$$

We will first show that the compositions of these two maps is zero.

By Lemma 5.3, we can further replace the above maps into the composition of three

$$\widetilde{\mathcal{F}}[-1] \longrightarrow \widetilde{\mathcal{F}} \otimes m^*\mathbb{L}_Q \longrightarrow \widetilde{\mathcal{F}} \otimes \mathbb{L}_{G\times Q} \longrightarrow \widetilde{\mathcal{F}} \otimes pr_1^*\mathbb{L}_G. \tag{18}$$

Since $\widetilde{\mathcal{F}}$ is obtained by the pullback via m, by the functorial property of Atiyah classes, we realized that the composition of the first two maps in (18) is simply the Atiyah class of the sheaf $\widetilde{\mathcal{F}}$ itself!

However, by Lemma 5.4, we see that the universal sheaf $\widetilde{\mathcal{F}}$ can also be realized as $pr_2^*\mathcal{F}$, therefore by the functorial property again, its Atiyah class can also be viewed as the pull back of the Atiyah class of $\mathcal{F}$ via the projection pr_2. In particular, it lies in the second component of

$$\mathrm{Ext}^1(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}} \otimes \mathbb{L}_{G\times Q}) = \mathrm{Ext}^1(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}} \otimes pr_1^*\mathbb{L}_G) \oplus \mathrm{Ext}^1(\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}} \otimes pr_2^*\mathbb{L}_Q).$$

Therefore its projection into the first factor is 0, which implies the composition of the three maps in (18) is 0.

Now we define the diagonal map

$$\begin{aligned} \Delta : Q &\longrightarrow G \times Q, \\ q &\longmapsto (1, q) \end{aligned}$$

and it is easy to see that the composition $m \circ \Delta$ is the identity map on Q. Because of this, we can pull back the maps in (17) via the map Δ and obtain the original maps in (16). The above discussion implies that the composition in (16) is 0. Therefore the dotted arrows in the statement exist and make the whole diagram commutative, where the horizontal dotted arrow is simply the one in the exact sequence of the universal quotient over the Quot scheme Q. □

We can translate the above lemma into the language of cotangent complexes as follows.

Lemma 5.6. *We have the commutative diagram*

$$\begin{tikzcd}
R\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee \arrow[r] \arrow[d] & R\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})^\vee \arrow[d] \\
\mathbb{L}_Q \arrow[r] & \mathbb{L}_{Q/\mathcal{M}},
\end{tikzcd} \tag{19}$$

where the upper horizontal arrow is the natural map from the universal quotient, the lower horizontal arrow is the functorial map given by the quotient, and the left vertical arrow is given by the reduced Atiyah class.

Proof. This is just a literal translation of Lemma 5.5. In the proof of Lemma 4.1, we use the Grothendieck-Verdier duality and Serre duality to construct a canonical isomorphism (by abuse of notation we simply use equalities)

$$\mathrm{Hom}_{Q\times X}(\mathcal{K},\mathcal{F}\otimes\pi^*\mathbb{L}_Q)=\mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee,\mathbb{L}_Q).$$

Following exactly the same steps, we can construct another canonical isomorphism

$$\mathrm{Hom}_{Q\times X}(\mathcal{E},\mathcal{F}\otimes\pi^*\mathbb{L}_{Q/\mathcal{M}})=\mathrm{Hom}_Q(\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})^\vee,\mathbb{L}_{Q/\mathcal{M}}).$$

If we denote the two vertical arrows in Lemma 5.5 by u and v, which are elements of the spaces on the left hand side of the two equations respectively. We denote the corresponding elements on the right hand side by u' and v'.

The previous lemma claims that, the composition of u with the canonical map $\mathbb{L}_Q\longrightarrow\mathbb{L}_{Q/\mathcal{M}}$ agrees with the precomposition of v with the map $\mathcal{K}\longrightarrow\mathcal{E}$ in the universal quotient sequence (1). Therefore, by the above canonical isomorphisms, we know that, the composition of u' with the canonical map $\mathbb{L}_Q\longrightarrow\mathbb{L}_{Q/\mathcal{M}}$ also agrees with the precomposition of v' with the map $\mathcal{K}\longrightarrow\mathcal{E}$ in the universal quotient sequence (1), which is exactly the conclusion of this lemma. □

If we compare the above result with the one we stated at the beginning of the section, we need to replace the upper two complexes by their traceless counterparts. Therefore we have the following lemma.

Lemma 5.7. *We have the commutative diagram* (13)*.*

Proof. From the discussion in Section 3 we had the following two exact triangles, which are the dual of the exact triangles (9) and (10):

$$\mathcal{O}_Q[1] \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee \tag{20}$$

and

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})^\vee \longrightarrow \mathcal{O}_Q. \tag{21}$$

First of all we claim that both arrows coming out of $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee$ in (19) factor through $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee$. For this purpose it suffices to show that the pre-composition of these two arrows by the first arrow in (20) is 0. In fact, since $\mathcal{O}_Q[1]$ is a single sheaf lying in degree -1, while both $\mathbb{L}_Q$ and $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee$ are both single sheaves lying in degree 0, there is only the zero map from a sheaf in degree -1 to a sheaf in degree 0. This allows us to replace the upper left corner of (19) by its traceless counterpart.

Next we claim that the upper horizontal arrow in (19) can be lifted to $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee$. For this we only need to show that the composition of this arrow with the second arrow in (21)

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})^\vee \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})^\vee \longrightarrow \mathcal{O}_Q$$

is a zero map, or equivalently, its dual composition

$$\mathcal{O}_Q \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F}) \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})$$

is a zero map on Q. Since we assume that there is at least one stable sheaf in the moduli space, the stable locus Q^s in the Quot scheme is open and dense. Therefore it suffices to check the above claim at every closed point in Q^s.

Pick any closed point $p \in Q^s$. By the construction of the traceless complexes, the restriction of the above two maps at p becomes

$$\mathrm{Hom}(F_p,F_p) \longrightarrow \mathrm{Hom}(E_p,F_p) \longrightarrow \mathrm{Hom}(K_p,F_p),$$

whose composition of 0, as expected. Therefore we can as well replace the upper right corner of (19) by its traceless counterpart and obtain the commutative diagram (13). $\square$

Finally we are aiming to prove that the right vertical map in (13) is a quasi-isomorphism, or more precisely, an isomorphism between two single sheaves in degree 0. First we compute the two sheaves explicitly to see if they have a chance to be isomorphic.

Lemma 5.8. *Both* $R\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee$ *and* $\mathbb{L}_{Q/\mathcal{M}}$ *are quasi-isomorphic to a trivial vector bundle of rank equal to* $\dim G$ *concentrated in degree 0.*

Proof. We recall the construction of the Quot scheme. There exists a sufficient large integer n, such that for every semistable sheaf F with the prescribed Mukai vector, $F \otimes \mathcal{O}(n)$ has trivial cohomology in positive degrees, and $E \otimes cO(n)$ is the trivial bundle generated by the global sections of $F \otimes \mathcal{O}(n)$. If we assume the dimension of the global sections is N, then the gauge group $G = PGL(N)$. Therefore we have

$$\begin{aligned}\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F}) &= \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E}(n),\mathcal{F}(n))\\ &= \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{O}^{\oplus N},\mathcal{F}(n))\\ &= \mathrm{R}\pi_*\mathcal{F}(n)\otimes\mathcal{O}^{\oplus N\vee}\\ &= \pi_*\mathcal{F}(n)\otimes\mathcal{O}^{\oplus N\vee}\\ &= \mathcal{O}^{\oplus N\vee}\otimes\mathcal{O}^{\oplus N} = \mathcal{E}nd(\mathcal{O}^{\oplus N}).\end{aligned}$$

And from the construction of exact triangle (9), we see that the map $\mathcal{O}_Q \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})$ is at every closed point $p \in Q$ given by the identity map

$$\mathbb{C}\mathrm{Id} \hookrightarrow \mathrm{Hom}(F_p,F_p) \longrightarrow \mathrm{Hom}(E_p,F_p)$$

which is injective. Therefore the exactly triangle (9) actually becomes an exact sequence of sheaves on Q

$$0 \longrightarrow \mathcal{O}_Q \longrightarrow \mathcal{E}nd(\mathcal{O}_Q^{\oplus N}) \longrightarrow \mathcal{E}nd(\mathcal{O}_Q^{\oplus N})_0 \longrightarrow 0.$$

So $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0$ is quasi-isomorphic to $\mathcal{E}nd(\mathcal{O}_Q^{\oplus N})_0$ which is a trivial bundle of rank N^2-1 concentrated in degree 0.

On the other hand, by noticing that

$$m \circ \Delta = \mathrm{Id},$$

together with Lemma 5.2, we have

$$\mathbb{L}_{Q/\mathcal{M}} = \Delta^* m^* \mathbb{L}_{Q/\mathcal{M}} = \Delta^* pr_1^* \mathbb{L}_G = \mathfrak{g} \otimes \mathcal{O}_Q$$

which is also a trivial bundle of rank equal to $\dim \mathfrak{g} = N^2 - 1$. □

Finally, we are ready to prove the following lemma.

Lemma 5.9. *The right vertical arrow constructed in* (13)

$$\varphi : R\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee \longrightarrow \mathbb{L}_{Q/\mathcal{M}}$$

is an isomorphism of two sheaves.

Proof. From the above discussion we know that this arrow is a map between two locally free sheaves of the same rank.

First of all we will show that, on the stable locus Q^s, φ is an isomorphism. For this purpose, it suffices to show that φ_p is surjective on the stable locus Q^s. However, due to the commutativity of the diagram (13), whose left vertical arrow is an isomorphism, it suffices to show that the functorial map

$$\mathbb{L}_{Q^s} \longrightarrow \mathbb{L}_{Q^s/\mathcal{M}^s} \tag{22}$$

is surjective on Q^s, where $\mathcal{M}^s$ is as a substack of $\mathcal{M}$ the quotient of Q^s by the group G.

To show that the map (22) is surjective, we only need to show that the pullback of the map via

$$m_s : G \times Q^s \longrightarrow Q^s$$

is surjective. By applying Lemma 5.2, we just need to prove that

$$m_s^*\mathbb{L}_{Q^s} \longrightarrow pr_1^*\mathbb{L}_G$$

is surjective. Here by abuse of notation, we use pr_1 for the projection of $G \times Q^s$ to its first factor.

Since both Q^s and G are smooth, we can consider the dual of the above map

$$pr_1^*T_G \longrightarrow m^*T_{Q^s}.$$

We need to show that it is injective on fibers at every closed point $p \in Q^s$. Or in other words, we need to show that the pushforward of the tangent spaces

$$m_{s*}(pr_1^*T_G) \longrightarrow T_{Q^s}$$

is injective at every closed point $p \in Q^s$. However, this is equivalent of saying that the G-action is free on the stable locus Q^s, which is obvious.

So far we have proved that the map φ is an isomorphism of two locally free sheaves of the same rank on Q^s. Next we claim that φ is actually an isomorphism over Q. In fact, the locus in Q where φ is not an isomorphism is the zero locus of the corresponding map of determinant line bundles, therefore is a Cartier divisor. In particular, if it is not an empty set, it should have dimension 1. However, by [17, Proposition 6.1], the strictly semistable locus $Q\backslash Q^s$ has codimension at least 2. Therefore the degeneracy locus must be empty, and φ is an isomorphism everywhere. ☐

Now we get our key result on the cotangent complex of the moduli stack.

Theorem 5.10. *We have a quasi-isomorphism*

$$R\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0^\vee[-1] \xrightarrow{\cong} q^*\mathbb{L}_{\mathcal{M}}.$$

Proof. From the above discussion on the diagram (13), and two functorial exact triangles, we obtain the following diagram (in which the first exact triangle follows from equation (8)):

(23)
$$\begin{tikzcd}
\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{K},\mathcal{F})_0^\vee \arrow[r] \arrow[d] & \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{E},\mathcal{F})_0^\vee \arrow[r] \arrow[d] & \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0^\vee \arrow[d, dashed] \\
\mathbb{L}_Q \arrow[r] & \mathbb{L}_{Q/\mathcal{M}} \arrow[r] & q^*\mathbb{L}_{\mathcal{M}}[1]
\end{tikzcd}$$

Since the left square commutes, by the axioms of triangulated categories, the dotted arrow exists and is a quasi-isomorphism. ☐

6. Symplectic stacks

In this section we give an application of Theorem 5.10. Motivated by the symmetric obstruction theory in [3], we want to study bilinear pairings on complexes. The following notion of anti-symmetric forms is completely parallel to [3, Definition 1.1]:

Definition 6.1. Let $\mathcal{X}$ be a scheme, and $E^\bullet \in D^b(\mathcal{X})$ be a perfect complex. A *non-degenerate anti-symmetric bilinear form* on $E^\bullet$ is a morphism

$$\beta : E^\bullet \otimes E^\bullet \longrightarrow \mathcal{O}_{\mathcal{X}}$$

in $D^b(\mathcal{X})$, which is

1) anti-symmetric, i.e. the following diagram is commutative

$$\begin{tikzcd} E^{\bullet} \otimes E^{\bullet} \arrow[r, "\beta"] \arrow[d, "\iota"] & \mathcal{O}_{\mathcal{X}} \arrow[d, "-\mathrm{id}"] \\ E^{\bullet} \otimes E^{\bullet} \arrow[r, "\beta"] & \mathcal{O}_{\mathcal{X}}, \end{tikzcd} \tag{24}$$

where ι is the isomorphism switching the two factors of the tensor product;

2) non-degenerate, which means that β induces an isomorphism

$$\theta : E^{\bullet} \longrightarrow E^{\bullet\vee}.$$

In such a case, we call $E^{\bullet}$ a *symplectic complex* and β a *symplectic pairing* on $E^{\bullet}$.

Remark 6.2. Note that there are other equivalent ways of phrasing this definition (c.f. [3, Remark 1.2]). In fact, we can avoid using the tensor product and use only the isomorphism β. Then the condition of anti-symmetry becomes $\theta^{\vee} = -\theta$, or more precisely, the following diagram commutes:

$$\begin{tikzcd} E^{\bullet} \arrow[r, "\theta"] \arrow[d, "i_E"] & E^{\bullet\vee} \arrow[d, "-\mathrm{id}"] \\ E^{\bullet\vee\vee} \arrow[r, "\theta^{\vee}"] & E^{\bullet\vee}, \end{tikzcd} \tag{25}$$

where i is the naturally isomorphism of the perfect complex E and its double dual.

Similar to the situation in [3], it is usually easier to work with θ only. Then an anti-symmetric pairing on the complex E is simply an isomorphism $\theta : E^{\bullet} \longrightarrow E^{\bullet\vee}$ satisfying $\theta^{\vee} = -\theta$.

Remark 6.3. Note that here we adopted the sign conventions in [5, Section 1.3]. The sign conventions which are most relevant to the above definition are the ones related to switching the two factors in a tensor product and to the identification of a complex with its dual. More precisely, we should keep in mind that the definition of the natural isomorphism

$$E_1^{\bullet} \otimes E_2^{\bullet} \cong E_2^{\bullet} \otimes E_1^{\bullet} \tag{26}$$

uses a sign of $(-1)^{pq}$ on the component $E_1^p \otimes E_2^q$ [5, page 11]. Moreover, from the definition of $\mathcal{H}om$ complex in [5, page 10], we can easily find that if E is a perfect complex represented by

$$\cdots \longrightarrow E^{i-1} \xrightarrow{\varphi_{i-1}} E^{i} \xrightarrow{\varphi_i} E^{i+1} \longrightarrow \cdots ,$$

then the dual complex $E^\vee$ can be represented by

$$\cdots \longrightarrow (E^{i+1})^\vee \xrightarrow{(-1)^i\varphi_i^\vee} (E^{i})^\vee \xrightarrow{(-1)^{i-1}\varphi_{i-1}^\vee} (E^{i-1})^\vee \longrightarrow \cdots ,$$

and the double dual complex $E^{\vee\vee}$ becomes

$$\cdots \longrightarrow E^{i-1} \xrightarrow{-\varphi_{i-1}} E^{i} \xrightarrow{-\varphi_i} E^{i+1} \longrightarrow \cdots .$$

Note that the extra sign is induced in all the morphisms in the complex. To get compatible with this, according to [5, page 14], the isomorphism $i_E : E^\bullet \longrightarrow E^{\bullet\vee\vee}$ is chosen to involve a sign of $(-1)^n$ in degree n.

An obvious example of a symplectic complex is a single vector bundle equipped with a symplectic metric sitting in degree 0. However, to get a better feeling of a symplectic complex, especially the tricky sign conventions, we can see the following example:

Example 6.4. Let $X = \mathbb{C}^{2n}$ with $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ as coordinates. Let E be the complex of locally free sheaves

$$\mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2n} \xrightarrow{\beta} \mathcal{O}_X,$$

where the morphisms are

$$\begin{aligned}\alpha &= (x_1, \ldots, x_n, -y_1, \ldots, -y_n)^T,\\ \beta &= (y_1, \ldots, y_n, x_1, \ldots, x_n),\end{aligned}$$

where the letter "T" in the upper right corner denotes the transpose of the matrix. Then the dual complex $E^\vee$ becomes

$$\mathcal{O}_X \xrightarrow{\beta^T} \mathcal{O}_X^{\oplus 2n} \xrightarrow{-\alpha^T} \mathcal{O}_X,$$

and we can define a morphism $\theta : E \longrightarrow E^\vee$ by

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_X \arrow[r, "\beta^T"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "-\alpha^T"] & \mathcal{O}_X,
\end{tikzcd}$$

where γ is the standard $2n \times 2n$ symplectic matrix

$$\begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

The dual isomorphism $\theta^\vee : E^{\vee\vee} \longrightarrow E^\vee$ now becomes

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "-\alpha"] \arrow[d, "\mathrm{id}"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "-\beta"] \arrow[d, "\gamma^T"] & \mathcal{O}_X \arrow[d, "\mathrm{id}"] \\
\mathcal{O}_X \arrow[r, "\beta^T"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "-\alpha^T"] & \mathcal{O}_X,
\end{tikzcd}$$

We also mentioned above that the natural isomorphism $i_E : E \longrightarrow E^{\vee\vee}$ is defined to be

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "\alpha"] \arrow[d, "-\mathrm{id}"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "\beta"] \arrow[d, "\mathrm{id}"] & \mathcal{O}_X \arrow[d, "-\mathrm{id}"] \\
\mathcal{O}_X \arrow[r, "-\alpha"] & \mathcal{O}_X^{\oplus 2n} \arrow[r, "-\beta"] & \mathcal{O}_X.
\end{tikzcd}$$

The above diagrams verify the required symplectic condition in 25. Therefore the complex E in this example is a symplectic complex.

Now we define a symplectic complex on an algebraic stack, by using an atlas of a stack.

Definition 6.5. Let $\mathcal{X}$ be an algebraic stack, and $u : U \longrightarrow \mathcal{X}$ an atlas of the stack $\mathcal{X}$, where U is a scheme. Let $\mathcal{G} \in D^b(\mathcal{X})$ be a perfect complex. We say $\mathcal{G}$ is a *symplectic complex*, if there exists a symplectic pairing

$$\beta : u^*\mathcal{G} \otimes u^*\mathcal{G} \longrightarrow \mathcal{O}_U,$$

satisfying that

$$q_1^*\beta = q_2^*\beta,$$

where q_1 and q_2 are the projections in the following fiber diagram

$$\begin{tikzcd}
U \times_{\mathcal{X}} U \arrow[r, "q_1"] \arrow[d, "q_2"] & U \arrow[d, "u"] \\
U \arrow[r, "u"'] & \mathcal{X}.
\end{tikzcd}$$

In the special case that $\mathcal{X}$ is a global quotient stack $[U/G]$ for some group G acting on U, the above definition can be understood as the G-equivariance of the pairing β. Based on the definition of symplectic complex, we can now define the following notion of symplectic stacks:

Definition 6.6. Let $\mathcal{X}$ be a scheme or an algebraic stack. We call $\mathcal{X}$ a *symplectic stack*, if its cotangent complex $\mathbb{L}_{\mathcal{X}}$ is a symplectic complex.

From this definition we immediately see:

Example 6.7. Any holomorphic symplectic manifold X is a symplectic stack, because a nowhere degenerate holomorphic 2-form defines a symplectic pairing on the tangent bundle T_X, or equivalently the cotangent bundle Ω_X.

A slightly more general situation is the following:

Example 6.8. Let X be a holomorphic symplectic manifold with a nowhere degenerate holomorphic 2-form σ, and G is a finite group acting on X preserving the symplectic form σ. Let $q : X \longrightarrow \mathcal{X} = [X/G]$ be the stacky quotient map. Then the Deligne-Mumford stack $\mathcal{X}$ is a symplectic stack.

In fact, by Proposition 2.2, we know that

$$q^*\mathbb{L}_{\mathcal{X}} = \mathbb{L}_X = \Omega_X,$$

because G is finite. The holomorphic symplectic form σ defines a symplectic pairing on Ω_X. Since the G-action preserves σ, this symplectic pairing descends to $\mathbb{L}_{\mathcal{X}}$, which shows $\mathcal{X}$ is a symplectic stack.

We mentioned that the cotangent complex of a stack could lie over all degrees not larger than 1. However, for a symplectic stack, due to the isomorphism between the cotangent complex and its dual, its perfect amplitude could only be within the interval $[-1, 1]$. Therefore, the cotangent complex could have only two types: either a single locally free sheaf sitting in degree 0, or a perfect complex in degree $[-1, 1]$. All the above examples fall in the first type. Next we will show that the global quotient stack $\mathcal{M} = [Q/G]$ that we studied in previous sections provides an example of the second type, in which the cotangent complex has non-trivial cohomology in degrees $-1, 0$ and 1.

Remark 6.9. Before we prove $\mathcal{M}$ is a symplectic stack, we briefly mention the relation of the global quotient stack $\mathcal{M} = [Q/G]$ with the moduli stack of semistable sheaves on the K3 surface X with any fixed Mukai vector $\mathbf{v}$. It was proved in [25, Theorem 5.1] that the algebraic stack of semistable sheaves on X with Mukai vector $\mathbf{v}$ is a global quotient stack $[Q/GL(N)]$. However, since the non-zero scalar matrices act trivially on Q, it is natural to replace the group $GL(N)$ by the quotient group $G = PGL(N)$, so that the G-action on the stable locus in Q is free (assuming the stable locus is non-empty). In this sense, we can call the global quotient stack $\mathcal{M} = [Q/G]$ the *reduced moduli stack of semistable sheaves* on the K3 surface with Mukai vector $\mathbf{v}$.

Theorem 6.10. *Let (X, H) be a polarized K3 surface, $\mathbf{v} \in H^*_{\mathrm{alg}}(X, \mathbb{Z})$ a Mukai vector with respect to which H is generic. Assume $\mathbf{v}^2 > 0$. Then the reduced moduli stack $\mathcal{M}$ of H-semistable sheaves on X with Mukai vector $\mathbf{v}$ is a symplectic stack.*

Proof. By Proposition 5.10, we know that the pullback of the cotangent complex via the quotient map is

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{F})_0^\vee[-1] \xrightarrow{\cong} q^*\mathbb{L}_{\mathcal{M}}.$$

To prove the cotangent complex $\mathbb{L}_{\mathcal{M}}$ is a symplectic complex, we first show that there exists a symplectic pairing on $q^*\mathbb{L}_{\mathcal{M}}$, then show that the symplectic pairing satisfies the compatibility condition in Definition 6.5 for a symplectic complex on a stack.

Step 1. We show that the relative Serre duality defines a non-degenerate anti-symmetric bilinear form on the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{F})$.

The relative Serre duality tells us that the composition of the derived Yoneda product and the trace map

$$\begin{aligned} & \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{F}) \otimes \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{F} \otimes \omega_\pi) \\ \xrightarrow{\cup}\ & \mathrm{R}^2\pi_*\mathcal{RH}om(\mathcal{F}, \mathcal{F} \otimes \omega_\pi)[-2] \\ \xrightarrow{\mathrm{tr}}\ & \mathrm{R}^2\pi_*\omega_\pi[-2] \cong \mathcal{O}_Q[-2] \end{aligned}$$

is a non-degenerate bilinear form.

Due to the fact that X is a K3 surface, the relative dualizing sheaf ω_π of the projection $\pi : Q \times X \longrightarrow Q$ has a trivialization given by the pullback of generator of $H^{2,0}(X)$ via the second projection. We denote the isomorphism by

$$\sigma : \mathcal{O}_{Q\times X} \longrightarrow \omega_\pi.$$

Then we can also write the above non-degenerate bilinear form as

$$\begin{aligned} & \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F}) \otimes \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F}) \\ \xrightarrow{\cup} \; & \mathrm{R}^2\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})[-2] \\ \xrightarrow{\mathrm{tr}} \; & \mathcal{O}_Q[-2] \end{aligned} \tag{27}$$

Furthermore, this trace map also satisfies the symmetry condition ([12, Equation 10.3])

$$\mathrm{tr}(e \cup e') = (-1)^{\deg(e)\deg(e')}\mathrm{tr}(e' \cup e).$$

After a degree shift we get a bilinear form

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})[1] \otimes \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})[1] \longrightarrow \mathcal{O}_Q$$

which is still non-degenerate, and the symmetry condition becomes

$$\begin{aligned} \mathrm{tr}(e \cup e') &= (-1)^{(\deg(e)+1)(\deg(e')+1)}\ \mathrm{tr}(e' \cup e) \\ &= (-1)^{\deg(e)\deg(e')+\deg(e)+\deg(e')+1}\ \mathrm{tr}(e' \cup e) \\ &= (-1)^{\deg(e)\deg(e')+1}\ \mathrm{tr}(e' \cup e). \end{aligned}$$

The reason for the last equality is that: for $\mathrm{tr}(e \cup e')$ to lie in the only non-trivial degree of the complex $\mathcal{O}_Q$, we must have

$$\deg(e) + \deg(e') = 0.$$

Comparing the above equation with the sign convention in the equation (26), we realize that, switching the two factors in the trace map actually introduces an extra negative sign. This verifies the condition in equation (24), therefore the bilinear pairing on $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})[1]$ is anti-symmetric.

Step 2. We show that the bilinear form defined in step 1 restricts to a bilinear form on $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0[1]$, which remains to be non-degenerate and anti-symmetric.

By the decomposition (7), the bilinear form defined by Serre duality in step 1 restricts to a bilinear form on a direct summand $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0[1]$, which is still anti-symmetric. It remains to show that it is non-degenerate.

We claim that the sum of the other two components $\pi_*\mathcal{O}_{Q\times X} \oplus \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]$ in (7) is orthogonal to the component $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0$ under Serre duality. Indeed, as the Serre duality (27) is given by the composition of Yoneda product and trace map, it is clear that the component $\pi_*\mathcal{O}_{Q\times X}$ is orthogonal to the kernel of the trace map; i.e. the sum of the components $\pi_*\mathcal{O}_{Q\times X} \oplus \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0$. On the other hand, we know that the natural inclusion $\mathcal{O}_Q \longrightarrow \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})$ in degree 0 is Serre dual to the trace map $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F}) \longrightarrow \mathcal{O}_Q[-2]$ in degree 2; see [22, Section 1]. Hence $\mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]$ is orthogonal to the components $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0 \oplus \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]$.

Therefore the traceless component $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0$ is orthogonal to the direct sum $\pi_*\mathcal{O}_{Q\times X} \oplus \mathrm{R}^2\pi_*\mathcal{O}_{Q\times X}[-2]$. Hence the restriction of the Serre duality pairing discussed in step 1 remains non-degenerate on $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0$.

We conclude from step 2 that $q^*\mathbb{L}_{\mathcal{M}}$ is a symplectic complex.

Step 3. Finally, we want to show that the symplectic pairing on the complex $\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0$ is G-equivariant.

For this purpose we just need to show that, the pull back of the symplectic pairing via the maps m and pr_2 in diagram (14) agree with each other.

By flatness and [9, Propositions 5.8, 5.9], as well as the fact that the decomposition (6) is natural under pullback, we conclude that the pullback of the Serre duality pairing

$$\mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0 \otimes \mathrm{R}\pi_*\mathcal{RH}om(\mathcal{F},\mathcal{F})_0 \longrightarrow \mathrm{R}^2\pi_*\mathcal{O}_Q[-2] \tag{28}$$

via m is

$$\mathrm{R}\pi_*\mathcal{RH}om(m^*\mathcal{F},m^*\mathcal{F})_0 \otimes \mathrm{R}\pi_*\mathcal{RH}om(m^*\mathcal{F},m^*\mathcal{F})_0 \longrightarrow \mathrm{R}^2\pi_*\mathcal{O}_{G\times Q}[-2],$$

which is again the Serre duality pairing on $G \times Q$ by the functoriality of Serre duality.

Similarly, if we pull back the pairing (28) via the other map pr_2, we again get a Serre duality pairing

$$\mathrm{R}\pi_*\mathcal{RH}om(pr_2^*\mathcal{F},pr_2^*\mathcal{F})_0 \otimes \mathrm{R}\pi_*\mathcal{RH}om(pr_2^*\mathcal{F},pr_2^*\mathcal{F})_0 \longrightarrow \mathrm{R}^2\pi_*\mathcal{O}_{G\times Q}[-2].$$

In Lemma 5.4, we have showed that the pullback of the universal sheaf $\mathcal{F}$ via m and pr_2 are canonically isomorphic, denoted by

$$\widetilde{\mathcal{F}} = m^*\mathcal{F} = pr_2^*\mathcal{F}.$$

Therefore the above two pullback maps agree with each other, and we conclude that the moduli stack $\mathcal{M}$ of the semistable sheaves on a K3 surface is a symplectic stack in the sense of Definition 6.6. □

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ENDENICHER ALLEE 60, MATHEMATICAL INSTITUTE
UNIVERSITY OF BONN, 53115 BONN, GERMANY
Current address:
INSTITUTE FOR ALGEBRAIC GEOMETRY, LEIBNIZ UNIVERSITY HANNOVER
WELFENGARTEN 1, 30167 HANNOVER, GERMANY
E-mail address: `zhangzy@math.uni-hannover.de`

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