Surjectivity of a gluing construction in special Lagrangian geometry

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This paper is motivated by a relatively recent work by Joyce [12–16] in special Lagrangian geometry, but the basic idea of the present paper goes back to an earlier pioneering work of Donaldson [5] (explained also by Freed and Uhlenbeck [7]) in Yang–Mills gauge theory; Donaldson discovered a global structure of a (compactified) moduli space of Yang–Mills instantons, and a key step to that result was the proof of surjectivity of Taubes' gluing construction [23].

In special Lagrangian geometry we have currently no such a global understanding of (compactified) moduli spaces, but in the present paper we determine a neighbourhood of a 'boundary' point. It is locally similar to Donaldson's result, and in particular as Donaldson's result implies the surjectivity of Taubes' gluing construction so our result implies the surjectivity of Joyce's gluing construction in a certain simple case.

1. Introduction

The main result of the present paper may be stated briefly as the surjectivity of Joyce's gluing construction in a certain simple case, and we begin therefore with a review of Joyce's work [12–16].

Let M be a Calabi–Yau (or more generally almost[1]Calabi–Yau) manifold of complex dimension m, and let X be a compact special Lagrangian m-fold in M with finitely many singular points $x, y, \ldots, z \in X$ modelled on multiplicity-one special Lagrangian cones $C_x, C_y, \ldots, C_z \subset \mathbb{C}^m$ with isolated singularity.

Joyce [14, 15] studied a *smoothing*[2]of X by the gluing technique, which may be sketched as follows: as local smoothing models for $C_x, C_y, \ldots, C_z$ let $L_x, L_y, \ldots, L_z$ be non-compact special Lagrangian submanifolds properly-embedded in $\mathbb{C}^m$ and asymptotic at infinity to $C_x, C_y, \ldots, C_z$ with multi-plicity 1 respectively; for each $t > 0$ let $tL_x := \{t\boldsymbol{z} \in \mathbb{C}^m : \boldsymbol{z} \in L_x\}$ and define $tL_y, \ldots, tL_z$ likewise; then under some hypotheses we can glue $tL_x, tL_y, \ldots,$ tL_z to X at $x, y, \ldots, z$ respectively into a family of compact special La-grangian submanifolds N_t of M with small $t > 0$ and tending to X as $t \to +0$ as *varifolds*[3], which are a notion of 'singular' submanifolds in geometric mea-sure theory.

Consider now the space $\mathcal{V}$ consisting of all compactly-supported special Lagrangian integral varifolds with no boundary in M, so that $X \in \mathcal{V}$, where X has isolated singular points and multiplicity-one tangent cones but gen-eral elements of $\mathcal{V}$ may have *non-isolated* singularity and *higher-multiplicity* tangent cones.

We wish to detemine a *neighbourhood* of X in $\mathcal{V}$. For some simple X indeed we can determine a neighbourhood of X in $\mathcal{V}$ in a way similar to Donaldson's work (explained also by Freed and Uhlenbeck [7]) in Yang–Mills gauge theory.

Donaldson compactifies a moduli space $\mathcal{M}$ of Yang–Mills instantons by adding some objects with *isolated* singularity (by Uhlenbeck's theorem [24])

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[1] This is a terminology of Joyce [12–16] which means that M need *not* be Ricci-flat.

[2] Joyce [14, 15] calls it a *desingularization* but in algebraic geometry it means a *resolution* of singularity which does not fit our context, and we shall therefore call it a smoothing.

[3] Another well-known notion in geometric measure theory are *currents* and there is some difference between varifolds and currents, but the difference will not matter in special Lagrangian geometry (or more generally *calibrated* geometry) as we shall explain in §2 below.

and those singular objects form the boundary $\partial\mathcal{M}$ of the compactified moduli space $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$.

Donaldson determines indeed a neighbourhood $\mathcal{N}$ of $\partial\mathcal{M}$ in $\overline{\mathcal{M}}$ and its proof is based on the bubbling-off (or blowing-up) analysis using a technique of Uhlenbeck [24] and on the classification of local models by Atiyah, Hitchin and Singer [3].

In our situation the basic tool necessary for the blowing-up analysis is already prepared in the preceding paper [11], which will be explained in §3 below. On the other hand the classification of local models $L_x, L_y, \ldots, L_z$ will be difficult in general. Consider therefore the simple case where $m = 3$ and the cones $C_x, C_y, \ldots, C_z$ are all equal to

$$(1.1) \qquad C := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{0\} : |z_1| = |z_2| = |z_3|, z_1 z_2 z_3 \in (0, \infty)\}$$

which is a special Lagrangian cone in $\mathbb{C}^3$ discovered by Harvey and Lawson [8, Chapter III.3.A, Theorem 3.1]. It is also *stable* in the sense of Joyce [13, §3.2]. In the unit sphere $S^5 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ it is easy to see that $C \cap S^5$ is diffeomorphic to T^2 and so C is a T^2-cone. Haskins [9] proves that C is, as a stable T^2-cone, unique up to SU_3-rotation.

To state our classification theorem we also define:

$$(1.2) \quad L_1 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 - 1 = |z_2|^2 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\},$$
$$(1.3) \quad L_2 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 - 1 = |z_3|^2, z_1 z_2 z_3 \in [0, \infty)\},$$
$$(1.4) \quad L_3 := \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 = |z_2|^2 = |z_3|^2 - 1, z_1 z_2 z_3 \in [0, \infty)\},$$

which are all non-compact special Lagrangian submanifolds properly-embedded in $\mathbb{C}^3$ and asymptotic at infinity to C with multiplicity 1. In particular L_1, L_2, L_3 are all *non-singular*, despite of the condition $z_1 z_2 z_3 \in [0, \infty)$; for instance the map $S^1 \times \mathbb{C} \to L_1 \subset \mathbb{C}^3$ given by $(e^{i\theta}, z) \mapsto (e^{i\theta}\sqrt{|z|^2 + 1}, z, e^{-i\theta}\bar{z})$ is a diffeomorphism; permuting the co-ordinates of $\mathbb{C}^3$ we also get diffeomorphisms onto L_2 and L_3.

Our classification theorem is:

Theorem 1.1. *Let W be a special Lagrangian varifold in $\mathbb{C}^3$ with no boundary asymptotic at infinity to C with multiplicity 1. Then W is a mutiplicity-one varifold represented by C or $sL + b$ for some $L \in \{L_1, L_2, L_3\}$, $t > 0$ and $b \in \mathbb{C}^3$ where $sL + b := \{tz + b \in \mathbb{C}^3 : z \in L\}$.*

The proof of Theorem 1.1 will be given in §4 and at the moment we only point out that the proof uses a *symmetry* of C.

We return now to the special Lagrangian geometry in the (almost) Calabi–Yau manifold M. Let X be a compact special Lagrangian 3-fold X with *only one* singular point x modelled on C so that we can apply Theorem 1.1 above. One may also consider of course two or more singular points in which case however one has to consider their *interaction* as Joyce does [16, §10.3]—we shall not discuss it in the present paper.

To state our main results we also introduce the subspace $\mathcal{X} \subset \mathcal{V}$ consisting of those Y which are singular only at one point and modelled on C with multiplicity 1, so that $X \in \mathcal{X}$. Our main results may then be summarized briefly into the following single statement:

Theorem 1.2. *There exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that any element of $\mathcal{U} \setminus \mathcal{X}$ may be obtained by Joyce's gluing construction.*

Here $\mathcal{U} \setminus \mathcal{X}$ may be *empty*, in which case we have $\mathcal{U} \subset \mathcal{X}$ and so X is *unsmoothable*.

We shall state a more precise meaning of 'Joyce's gluing construction' in Theorem 1.2. As in (1.2)–(1.4) we have three local models L_1, L_2, L_3 and so there are apparently three distinct ways of smoothing X but in fact there is *at most one*[4]way, which may be explained as follows.

For any $L \in \{L_1, L_2, L_3\}$ we can certainly glue L to X at x into a compact submanifold of M but to make it *Lagrangian* in M we need a topological condition between L and X given by Joyce [16, Theorem 10.4 (see also Theorem 7.3)]. There may be no $L \in \{L_1, L_2, L_3\}$ for which L and X satisfy the topological condition, which will be recalled in §6.1 below. Joyce [12, Proposition 10.3] proves indeed:

Lemma 1.3. There is *at most one* $L \in \{L_1, L_2, L_3\}$ for which L and X satisfy the topological condition.

Theorem 1.2 includes the following statement:

Theorem 1.4. *If there is no $L \in \{L_1, L_2, L_3\}$ for which L and X satisfy the topological condition of Joyce [16, Theorem 10.4] then there exists a neighbourhood of X in $\mathcal{V}$ contained in $\mathcal{X}$.*

It is easy to see that that if there is $L \in \{L_1, L_2, L_3\}$ for which L and X satisfy the topological condition, then so do L and Y, where Y is not

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[4] There will be possibly more than one way of smoothing X if one considers a family of (almost) Calabi–Yau manifolds as Joyce does [16, §10.2].

exactly X but in a *neighbourhood* $\mathcal{Y}$ of X in $\mathcal{X}$. Consequently L and Y may be glued together into a compact special Lagrangian submanifold of M.

More precisely we re-scale L by small $t > 0$ and glue tL to X at x. Making $\mathcal{Y}$ smaller if necessary we can indeed find a real number $\tau > 0$ and define a continuous map $G : [0, \tau) \times \mathcal{Y} \to \mathcal{V}$ with the following two properties:

(i) $G(0, Y) = Y$ for all $Y \in \mathcal{Y}$;

(ii) if $(t, Y) \in (0, \tau) \times \mathcal{Y}$ then $G(t, Y)$ is a compact special Lagrangian submanifold of M obtained by gluing Y and tL together.

Theorem 1.2 may be then refined as follows:

Theorem 1.5. If there is $L \in \{L_1, L_2, L_3\}$ for which L and X satisfy the topological condition of Joyce [16, Theorem 10.4] then $G : [0, \tau) \times \mathcal{Y} \to \mathcal{V}$ is a *homeomorphism onto a neighbourhood* $\mathcal{U}$ of X in $\mathcal{V}$ with $\mathcal{U} \cap \mathcal{X} = \mathcal{Y}$.

Thus $\mathcal{U}$ is a *collar* neighbourhood of $\mathcal{Y}$ in $\mathcal{V}$ and so locally similar to Donaldson's situation [5] which also contains a collar neighbourhood of the boundary $\partial \mathcal{M}$ of the compactified moduli space $\overline{\mathcal{M}}$ in the notation above.

We also note that $\mathcal{Y}$ and $\mathcal{U} \setminus \mathcal{Y}$ are *manifolds* of finite dimension, and that G maps $(0, \tau) \times \mathcal{Y}$ *diffeomorphically* onto $\mathcal{U} \setminus \mathcal{Y}$. These facts may be proven in the following three steps:

(i) As $\mathcal{U} \setminus \mathcal{Y}$ consists of compact special Lagrangian (non-singular) submanifolds of M it follows from McLean's theorem [18, Theorem 3.6] that $\mathcal{U} \setminus \mathcal{Y}$ is a manifold but with respect to the C^∞-topology. By Allard's regularity theorem [1, Theorem 8.19] however the C^∞-topology on $\mathcal{U} \setminus \mathcal{Y}$ is equal to the varifold topology induced from $\mathcal{V}$.

(ii) Joyce [13, Corollary 6.11] extended McLean's theorem to the *stable-cone* singularity and as C is stable (see [13, §3.2]) we can apply Joyce's theorem so that $\mathcal{Y}$ will be a manifold with respect to a strong topology (given by Joyce [13, Definition 5.6]). It is again equal to the varifold topology, which we prove in Theorem 6.7 below.

(iii) The fact that G maps $(0, \tau) \times \mathcal{Y}$ diffeomorphically onto $\mathcal{U} \setminus \mathcal{Y}$ is already observed by Joyce (see a discussion after [16, Definition 8.9]); there are in fact natural co-ordinate systems on $(0, \tau) \times \mathcal{Y}$ and $\mathcal{U} \setminus \mathcal{Y}$ with respect to which G is a *product* map.

Thus Theorems 1.4 and 1.5 are the precise meaning of Theorem 1.2. Their proof will be given in §§5–7, the last three sections of the present paper. It will be again similar to the corresponding part of Donaldson's

proof [5, III.4] (see also Freed and Uhlenbeck [7, §9, Connectivity of the Collar]).

We point out that Theorem 1.5 implies the *connectivity* of a neighbourhood of X in $\mathcal{V}$, which part requires a careful treatment in particular. It is also the case in Donaldson's situation.

The remaining sections may be summarized as follows. In §2 we give a review of geometric measure theory and *calibrated* geometry including special Lagrangian geometry. There are two notions of 'singular' submanifolds in geometric measure theory, varifolds and currents, and there is some difference between them in general, but it does not matter in calibrated geometry as we shall explain in §2.

We shall mainly use varifolds rather than currents so that we can directly use Allard's regularity theorem [1, Theorem 8.19] which for instance we have used above in the proof that the two topologies on $\mathcal{U} \setminus \mathcal{Y}$, the C^∞-one and the varifold one, are the same.

In §3 we analyse the blowing-up of special Lagrangian varifolds in a general situation, concerning a multiplicity-one special Lagrangian *Jacobi-integrable* cone with isolated singularity, which fits Joyce's framework [16, Definition 6.7].

In §4 we prove Theorem 1.1 as we have mentioned above.

Some material of §§2–4 is not directly relevant to the main results (Theorems 1.4 and 1.5) but may be of independent interest and of potential use in more general situations.

Acknowledgements

2. Calibrated geometry and geometric measure theory

In this section we shall give a review of special Lagrangian geometry (a kind of calibrated geometry) and geometric measure theory. First of all we shall define almost Calabi–Yau manifolds and their special Lagrangian submanifolds.

Let (M, ω) be a symplectic manifold of dimension $2m$, and let J be a complex structure on M such that if we put $\hat{g}(v, w) = \omega(v, Jw)$ then $\hat{g}$

will be a Kähler metric on (M, J). Let Ω be a holomorphic $(m, 0)$-form on (M, J) with $\Omega|_x \neq 0$ for every $x \in M$. Then we shall call (M, ω, J, Ω) an almost Calabi–Yau manifold, and (ω, J, Ω) an almost Calabi–Yau structure on M. We can define a smooth function $\psi : M \to (0, \infty)$ such that

$$(2.1) \qquad \frac{\psi^{2m}}{m!}\omega^{\wedge m} = (-1)^s \left(\frac{i}{2}\right)^m \Omega \wedge \overline{\Omega} \text{ where } s = \frac{m(m-1)}{2}.$$

We put $g = \psi^2 \hat{g}$. We shall call g the almost Calabi–Yau metric on (M, ω, J, Ω). Here g need not be a Kähler metric on (M, J). If we have $\psi(x) = 1$ for every $x \in M$ then g will be a Kähler metric of Ricci curvature 0. In that case we shall call (ω, J, Ω) a Calabi–Yau structure on M, and g the Calabi–Yau metric on (M, ω, J, Ω).

We define a Calabi–Yau structure on $\mathbb{R}^{2m} = \mathbb{C}^m$ as follows. Let $(z^1, \ldots, z^m)$ be the co-ordinates of $\mathbb{C}^m$. Let $\omega_0 = \frac{i}{2}(dz^1 \wedge d\overline{z^1} + \cdots + dz^m \wedge d\overline{z^m})$, $g_0 = dz^1 \otimes d\overline{z^1} + \cdots + dz^m \otimes d\overline{z^m}$ and $\Omega_0 = dz^1 \wedge \cdots \wedge dz^m$. Let J_0 be the complex structure of $\mathbb{C}^m$. Then $(\omega_0, J_0, \Omega_0)$ is a Calabi–Yau structure on $\mathbb{R}^{2m}$, and g_0 is the almost Calabi–Yau metric on $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

Let (M, ω, J, Ω) be an almost Calabi–Yau manifold, and let g be the almost Calabi–Yau metric on (M, ω, J, Ω). Then $\operatorname{Re}\Omega$ will be a calibration of degree m on (M, g) in the sense of Harvey and Lawson [8]; i.e. for any point $x \in M$ and any $\mathbb{R}$-linear subspace $S \subset T_x M$ with $\dim_{\mathbb{R}} S = m$ we have $|(\operatorname{Re}\Omega)_x|_S| \leqslant 1$ where the norm is induced from the metric g on M.

Special Lagrangian submanifolds of (M, ω, J, Ω) are defined as real m-dimensional submanifolds N of M with $|\operatorname{Re}\Omega|_N| = 1$ where the norm is again induced from the metric g on M. Special Lagrangian submanifolds of (M, ω, J, Ω) will be Lagrangian with respect to ω and area-minimizing with respect to g.

We can also define $\operatorname{Re}\Omega$-varifolds and currents in (M, g), which we shall call special Lagrangian varifolds and currents in (M, ω, J, Ω), respectively. In the remainder of this section we shall consider calibrated geometry not limited to special Lagrangian geometry.

We suppose that M is a manifold. For each $x \in M$ we denote by $G_p(T_x M)$ the Grassmann manifold of all vector subspaces of $T_x M$ of dimension p. We put $G_p(TM) = \bigcup_{x \in M} G_p(T_x M)$. By a varifold of dimension p in M we shall mean a Radon measure on $G_p(TM)$.

We suppose that g is a Riemannian metric on M, and ϕ is a calibration of degree p on M. For each $x \in M$ we put $G_\phi(T_x M) = \{S \in G_p(T_x M) : |\phi|_S|_g = 1\}$. We also put $G_\phi(TM) = \bigcup_{x \in M} G_\phi(T_x M)$. By a ϕ-varifold in M we shall mean a Radon measure on $G_\phi(TM)$.

For each $x \in M$ and $S \in G_\phi(T_x M)$ we define $\overrightarrow{S} \in \bigwedge^p T_x M$ as follows. We take an orthonormal basis $(e_1, \ldots, e_p)$ for $T_x M$ with respect to $g|_x$ such that $\langle \phi|_x, e_1 \wedge \cdots \wedge e_p \rangle = 1$. We set $\overrightarrow{S} = e_1 \wedge \cdots \wedge e_p$. It is easy to see that $\overrightarrow{S}$ is independent of the choice of $(e_1, \ldots, e_p)$, and so well-defined.

Let V be a ϕ-varifold in M. Then we can define a p-current $\overrightarrow{V}$ in M by setting

$$\overrightarrow{V}(\chi) = \int_{G_\phi(TM)} \langle \chi|_x, \overrightarrow{S} \rangle dV(x, S)$$

for every compactly-supported p-form χ on M.

Harvey and Lawson [8, Chapter II.1, Definition 1.4] define positive ϕ-currents in M, which we shall explain next. First of all we recall a definition of Harvey and Lawson [8, Chapter II.A, Definition A.1]: by a ϕ-non-negative p-form on M we shall mean a p-form χ on M with $\langle \chi|_x, \overrightarrow{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_x M)$. Harvey and Lawson [8, Chapter II.A, Proposition A.2] prove that a p-current T in M is a positive ϕ-current if (and only if) we have $T(\chi) \geq 0$ for every compactly-supported ϕ-non-negative p-form on M. We have:

Theorem. *Let V be a ϕ-varifold in M. Then $\overrightarrow{V}$ is a positive ϕ-current in M in the sense of Harvey and Lawson.*

Proof. We have only to prove that if χ is a compactly-supported ϕ-non-negative p-form on M then we have $\overrightarrow{V}(\chi) \geq 0$. By the definition of ϕ-non-negative p-forms we have $\langle \chi|_x, \overrightarrow{S} \rangle \geq 0$ for every $x \in M$ and $S \in G_\phi(T_x M)$. Hence we get $\overrightarrow{V}(\chi) = \int \langle \chi|_x, \overrightarrow{S} \rangle dV(x, S) \geq 0$, completing the proof. $\qquad\square$

We suppose now that M is compact. We take $a \in H_p(M; \mathbb{R})$. We denote by $\mathcal{V}$ the set of all compactly-supported integral ϕ-varifolds V in M, $\partial \overrightarrow{V} = 0$ and $[\overrightarrow{V}] = a$. Here V integral means that there exists an integer-valued $\|V\|$-measurable function Θ_V (called the *multiplicity* function of V) such that for any compactly-supported continuous function $f : M \to [0, \infty)$ we have $\|V\|(f) = \int_M f \Theta_V \mathcal{H}^p$ where $\mathcal{H}^p$ denotes the p-dimensional Hausdorff measure in (M, g).

We give $\mathcal{V}$ the weak topology in the sense of Allard [1, Definition 2.6(2)], i.e. the topology of the Radon measures on $G_p(TM)$. We have then:

Theorem. *$\mathcal{V}$ is compact.*

Proof. It is easy to see that $\mathcal{V}$ is a metrizable space. It suffices therefore to prove that if $V_1, V_2, V_3, \ldots \in \mathcal{V}$ then there exists a subsequence of $(V_n)_{n=1}^\infty$

converging in $\mathcal{V}$. Since M is compact it is clear that ϕ is compactly-supported, and for each $n = 1, 2, 3, \ldots$ therefore we have area $V_n = \overrightarrow{V_n}(\phi) = a \cdot [\phi]$ where $[\phi]$ denotes the de Rham cohomology class of ϕ. This implies that area V_n is bounded with respect to n. By an integral compactness theorem of Allard [1, Theorem 6.4], therefore, we can find a subsequence of $(V_n)_{n=1}^{\infty}$ converging as Radon measures on $G_p(TM)$. We may identify $(V_n)_{n=1}^{\infty}$ with the subsequence. We denote its limit by V. It suffices then to prove $V \in \mathcal{V}$. Allard's [1, Theorem 6.4] integral compactness theorem implies that V is an integral varifold. Since V_n tends to V as varifolds we see that $\overrightarrow{V_n}$ tends to $\overrightarrow{V}$ as p-currents in M. As $\partial \overrightarrow{V_n} = 0$ for every $n = 1, 2, 3, \ldots$ so we have $\partial \overrightarrow{V} = 0$. As $[\overrightarrow{V_n}] = a$ for every $n = 1, 2, 3, \ldots$ so we have $[\overrightarrow{V}] = a$. We have therefore $V \in \mathcal{V}$ as we want. $\qquad\square$

Let V be a varifold of dimension p in M. Then we shall denote by $\|V\|$ the Radon measure on M defined by setting $\|V\|(f) = V(f \circ \pi)$ for every $f \in C_c(M; \mathbb{R})$, where π denotes the projection of $G_p(TM)$ onto M, and $C_c(M; \mathbb{R})$ denotes the set of all compactly-supported continuous functions on M.

We denote by $\mathcal{R}$ the set of all Radon measures on M which may be expressed as $\|V\|$ for some $V \in \mathcal{V}$. We give $\mathcal{R}$ the topology of the Radon measures on M. We have then:

Theorem. *The mapping $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$.*

Proof. From the definition of $\mathcal{R}$ it is clear that $V \mapsto \|V\|$ maps $\mathcal{V}$ onto $\mathcal{R}$. We claim that the mapping $V \mapsto \|V\|$ is one-to-one. As V is area-minimizing it follows from Allard's theorem [1, Theorem 5.5] that V is *rectifiable* so that for each $V \in \mathcal{V}$ and $f \in C_c(M; \mathbb{R})$ we have

$$(2.2) \qquad V(f) = \int_{x \in M} f(x, T_x\|V\|) d\|V\|(x)$$

where $T_x\|V\|$ denotes the (approximate) tangent space to $\|V\|$ at x, which exists for $\|V\|$-almost every $x \in M$ (the terminology 'approximate' tangent space is used by Simon [22, see Remark 38.2]). The expression (2.2) shows that V is determined by $\|V\|$ so that the mapping $V \mapsto \|V\|$ is one-to-one.

From the definition of $\|V\|$, moreover, we see that $V \mapsto \|V\|$ is continuous. We have thus proved that $V \mapsto \|V\|$ is a continuous bijection of $\mathcal{V}$ onto $\mathcal{R}$. Notice also that $\mathcal{V}$ is compact and $\mathcal{R}$ is Hausdorff. Then we see that $V \mapsto \|V\|$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{R}$. $\qquad\square$

We denote by $\mathcal{T}$ the set of all positive ϕ-currents in M which may be expressed as $\overrightarrow{V}$ for some $V \in \mathcal{V}$. We give $\mathcal{T}$ the topology of p-currents in M. We have then:

Theorem. *The mapping $V \mapsto \overrightarrow{V}$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{T}$.*

Proof. From the definition of $\mathcal{T}$ it is clear that $V \mapsto \overrightarrow{V}$ maps $\mathcal{V}$ onto $\mathcal{T}$. We claim that the mapping $V \mapsto \overrightarrow{V}$ is one-to-one. For each $V \in \mathcal{V}$ and $f \in C_c(M; \mathbb{R})$ we have

$$\overrightarrow{V}(f\phi) = \int_{(x,S)\in G_\phi(TM)} \langle f(x)\phi|_x, S\rangle dV(x,S) = \int_{G_\phi(TM)} f(x)dV = \|V\|(f).$$

Thus $\overrightarrow{V}$ determines $\|V\|$. On the other hand we have already seen in (2.2) that $\|V\|$ determines V. Consequently $\overrightarrow{V}$ determines V; more precisely the mapping $V \mapsto \overrightarrow{V}$ is one-to-one.

It is easy to see that $V \mapsto \overrightarrow{V}$ is continuous. We have thus proved that $V \mapsto \overrightarrow{V}$ is a continuous bijection of $\mathcal{V}$ onto $\mathcal{T}$. Notice also that $\mathcal{V}$ is compact and $\mathcal{T}$ is Hausdorff. Then we see that $V \mapsto \overrightarrow{V}$ is a homeomorphism of $\mathcal{V}$ onto $\mathcal{T}$. $\square$

3. Analysis of blowing-up

In this section we shall analyse blowing-up near multiplicity-one special Larangian cones with isolated singularity. We can summarize our results as follows.

Let (M, ω, J, Ω) be an almost Calabi–Yau manifold of complex dimension m, and X a special Lagrangian m-fold in (M, ω, J, Ω) with only one singular point x modelled on a multiplicity-one special Lagrangian cone $C \subset \mathbb{C}^m$ with isolated singularity, where C need not be the stable T^2-cone as in §1 and so we are in a more general situation. The problems in the present section are *local* near x and so we may take M to be an open ball about x. We consider special Lagrangian varifolds $V_1, V_2, \ldots, V_n, \ldots$ tending to X.

We begin in §3.1 by recalling the definition of an *energy* functional for V_n introduced in the preceding paper [11]. We prove that if the energy of V_n is small then V_n has singularity only at one point y_n and asymptotic at y_n to a multiplicity-one special Lagrangian cone C^1-close to C.

We also prove in §3.2 that if the energy of V_n is large for all n then V_n *blows up*; i.e. there exist points y_n near x_n and small numbers $s_n > 0$ such that if we re-scale V_n about y_n by s_n then it will tend to a special

Lagrangian varifold W with no boundary in $\mathbb{C}^m$ asymptotic at infinity to a multiplicity-one special Lagrangian cone C^1-close to C.

3.1. Energy functional

We define an energy functional as follows. For each varifold V of dimension m in $\mathbb{R}^n$ we put

$$\mathcal{E}(V) = \int_{\mathbb{R}^n \times G_m(\mathbb{R}^n)} \frac{|S^\perp y|^2}{|y|^{m+2}} dV(y, S) \in [0, \infty]$$

where $S^\perp$ denotes the orthogonal complement to S in $\mathbb{R}^n$, and $S^\perp y$ denotes the projection of y onto $S^\perp$.

We shall recall a monotonocity formula for stationary varifolds. For each $\rho > 0$ let B_ρ denote the open ball of radius ρ about $\mathbf{0}$ in $\mathbb{C}^m$, i.e. $B_\rho := \{y \in \mathbb{R}^n : |y| < \rho\}$. For each $\rho > \sigma > 0$ let $A_{\sigma,\rho}$ denote the open annulus of inner radius σ and outer radius ρ about $\mathbf{0}$ in $\mathbb{C}^m$, i.e. $A_{\sigma,\rho} = B_\rho \setminus \overline{B_\sigma}$. For each $Z \subset \mathbb{R}^n$ we put $\widetilde{Z} = Z \times G_m(\mathbb{R}^n)$. If V is a stationary varifold of dimension m in (B, g_0) and if $0 < \sigma < \rho < 1$ then we have

$$(3.1) \qquad \frac{\|V\|(B_\rho)}{\rho^m} - \frac{\|V\|(B_\sigma)}{\sigma^m} = \mathcal{E}(V \llcorner \widetilde{A}_{\sigma,\rho});$$

for the proof we refer e.g. to Allard [1, Theorem 5.1(1)] or Simon [22, Equation 17.4]. It is easy to extend (3.1) to Riemannian metrics in place of g_0:

Proposition 3.1. *There exist constants $\epsilon_0 \in (0, 1)$ and $k > 1$ depending only on m, n and satisfying the following property: let g is a Riemannian metric on B_1 with $\epsilon := |g - g_0|_{C^1(B_1)} < \epsilon_0$. and V a stationary varifold of dimension m in (B_1, g); then for every $\sigma, \rho \in \mathbb{R}$ with $0 < \sigma < \rho < 1$ we have*

$$(3.2) \qquad e^{k\epsilon\rho} \frac{\|V\|(B_\rho)}{\rho^m} - e^{k\epsilon\sigma} \frac{\|V\|(B_\sigma)}{\sigma^m} \geq \mathcal{E}(V \llcorner \widetilde{A}_{\sigma,\rho}).$$

Remark 3.2. The co-efficients $e^{k\epsilon\rho}$ and $e^{k\epsilon\sigma}$ on the left-hand side come from the following computation: modifying the proof of Simon [22, Equation 17.4] we get indeed

$$k\epsilon \frac{\|V\|(B_\rho)}{\rho^m} + \frac{d}{d\rho} \frac{\|V\|(B_\rho)}{\rho^m} \geqslant \frac{d}{d\rho} \int_{y \in \widetilde{B}_\rho} \frac{|S^\perp y|^2}{|y|^{m+2}} dV$$

the left-hand side of which is not greater than $\frac{d}{d\rho}\left(e^{k\epsilon\rho}\frac{\|V\|(B_\rho)}{\rho^m}\right)$ and so integration over the interval (σ, ρ) implies (3.2).

We suppose now that $n = 2m$ and (ω, J, Ω) is an almost Calabi–Yau structure on B_1. We denote by $\mathcal{V}_0$ the space of all special Lagrangian integral varifolds with no boundary in (B_1, ω, J, Ω).

We define $r : \mathbb{R}^{2m} \to [0, \infty)$ by setting $r(y) = |y|$ for each $y \in \mathbb{R}^{2m}$. We put $g_{\mathrm{cyl}} = r^{-2}g_0$, which we shall call the cylindrical metric on $\mathbb{R}^{2m} \setminus \{0\}$.

We regard $(0, \infty)$ as a multiplicative group acting upon $\mathbb{R}^{2m}$ as re-scaling. By a smooth cone in $\mathbb{R}^{2m}$ we shall mean a closed submanifold of $\mathbb{R}^{2m} \setminus \{0\}$ invariant under the re-scaling by $(0, \infty)$. We denote by $\mathcal{C}$ the set of all special Lagrangian smooth cones in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

We suppose $C \in \mathcal{C}$. We denote by NC the normal bundle to C in $\mathbb{R}^{2m}$ with respect to g_0. We get the same bundle even if we use g_{cyl} in place of g_0.

For each $u \in C^k(C; NC)$ and $\rho < \sigma$ we define $|u|^{k,\mathrm{cyl}}_{[\rho,\sigma]}$ as follows. We put $t = -\log r$ and $\partial_t = \frac{\partial}{\partial t}$. We put $\Sigma = C \cap S^{2m-1}$. By the definition of smooth cones Σ is a compact submanifold of S^{2m-1}. We denote by ∇_Σ the Levi-Civita connexion over Σ induced from g_0. We put

$$|u|^{k,\mathrm{cyl}}_{[\rho,\sigma]} = \sup_{C \cap A_{\sigma,\rho}} \sum_{i,j \geq 0, i+j \leq k} |\partial_t^i \nabla_\Sigma^j u|.$$

If $|u|^{1,\mathrm{cyl}}_{[\rho,\sigma]}$ is sufficient small, then we can define the exponential map $\exp u : C \cap A_{\sigma,\rho} \to A_{\sigma,\rho}$ with respect to the metric g_{cyl}, and the image of $\exp u$ will be a submanifold of $A_{\sigma,\rho}$, which we shall denote by $\mathrm{Graph}_{\mathrm{cyl}}\, u$. We put $|\mathrm{Graph}_{\mathrm{cyl}}\, u| = \mathcal{H}^m \llcorner \mathrm{Graph}_{\mathrm{cyl}}\, u$ where $\mathcal{H}^m$ denotes the Hausdorff m-dimensional measure with respect to the almost Calabi–Yau metric g.

We suppose that (ω, J, Ω) is an almost Calabi–Yau structure on B_1, and g is the almost Calabi–Yau metric on B_1. From the proof of the author [11, Theorem 2.2] we get:

Theorem 3.3. *There exists $\epsilon > 0$ depending only on m and C such that if we have*

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \epsilon, \ V \in \mathcal{V}_0, \ 0 < \rho < 1, \ \mathcal{E}(V \llcorner \widetilde{A}_{\rho,1}) < \epsilon$$

and $v \in C^\infty(C \cap A_{1/2,1}; NC)$, $|v|^{2,\mathrm{cyl}}_{[1/2,1]} < \epsilon$, $|\mathrm{Graph}_{\mathrm{cyl}}\, v| = \|V\| \llcorner A_{1/2,1}$

then we can extend v to $C \cap A_{\rho,1}$ so that $\|V\| \llcorner A_{\rho,1} = |\mathrm{Graph}_{\mathrm{cyl}}\, v|$ and $|v|^{2,\mathrm{cyl}}_{[\rho,1]} < k\epsilon^\alpha$ for some $k > 0$ and $\alpha \in (0,1)$ depending only on m and C.

Remark. For the proof one has to use a result of Łojasiewicz [17], following Simon [20].

We shall give a corollary to the theorem above. We give $\mathcal{C}$ the C^∞-topology. Let $\mathcal{C}'$ be a neighbourhood of C in $\mathcal{C}$. Then we have:

Corollary 3.4. *There exists $\epsilon > 0$ depending only on m, C and $\mathcal{C}'$ such that if we have*

$$|\Omega - \Omega_0|_{C^0(B_1)} + |g - g_0|_{C^1(B_1)} < \epsilon, \ V \in \mathcal{V}_0, \ \mathcal{E}(V) < \epsilon$$
$$and \ v \in C^\infty(C \cap A_{1/2,1}; NC), \ |v|^{2,\text{cyl}}_{[1/2,1]} < \epsilon, \ |\text{Graph}_{\text{cyl}} v| = \|V\| \llcorner A_{1/2,1}$$

then V is singular only at $\mathbf{0}$ and asymptotic at $\mathbf{0}$ to some $C' \in \mathcal{C}'$ with multiplicity 1.

Remark 3.5. By a result of Simon [20, Theorem 5] C' will be a unique tangent cone to V at $\mathbf{0}$.

Proof. Let C' be a tangent cone to V at $\mathbf{0}$. For each $\delta > 0$ let $\delta^{-1}V$ denote the re-scaling of V by δ^{-1}. Then by the definition of tangent cones we can take $\delta_1 > \delta_2 > \delta_3 > \cdots$ tending to 0 with $\delta_n^{-1}V$ tending to C' as $n \to \infty$. By definition $\mathcal{C}'$ is a C^1-neighbourhood of C in $\mathcal{C}$ and so we can take $\eta > 0$ such that if $v \in C^\infty(\Sigma; NC)$ and $|v|_{C^1(\Sigma)} \leq \eta$ then we have $\text{Graph}_{\text{cyl}} v \in \mathcal{C}'$.

Let k, α be as in Theorem 3.3 and make ϵ so small that $k\epsilon^\alpha < \eta$. Applying Theorem 3.3 with $\rho = \delta/2$ we find $v_n \in C^\infty(C \cap A_{\delta_n/2,1}; NC)$ such that

$$(3.3) \qquad \|V\| \llcorner A_{\delta_n/2,1} = |\text{Graph}_{\text{cyl}} v_n| \text{ with } |v_n|^{2,\text{cyl}}_{[\delta_n/2,1]} \leq \eta.$$

Define $v'_n \in C^\infty(C \cap A_{1/2,1}; NC)$ by $v'_n(y) = v_n(\delta_n y)$ for $y \in C \cap A_{1/2,1}$. Then by (3.3) we have

$$(3.4) \qquad \|\delta_n^{-1}V\| \llcorner A_{1/2,1} = |\text{Graph}_{\text{cyl}} v'_n| \text{ with } |v'_n|^{2,\text{cyl}}_{[1/2,1]} \leq \eta.$$

The definition of $|\bullet|^{2,\text{cyl}}_{[1/2,1]}$ also implies that v'_n is C^2-bounded and so we can find a subsequence of v'_n converging in the C^1-topology as $n \to \infty$ to some $w \in C^1(C \cap A_{1/2,1}; NC)$ with $|w|^{1,\text{cyl}}_{[1/2,1]} \leqslant \eta$. On the other hand since $\delta_n^{-1}V$

tends to C' as $n \to \infty$ it follows that

(3.5) $$C' \cap A_{1/2,1} = \operatorname{Graph} w.$$

This implies that C' is a multiplicity-one cone with isolated singularity, and a result of Simon [20, Theorem 5] implies in particular that V is singular only at $\mathbf{0}$.

The equation (3.5) also implies that w is C^∞-differentiable and invariant under the re-scaling of C. In particular since $|w|^{1,\text{cyl}}_{[1/2,1]} \leqslant \eta$ it follows that $C' \in \mathcal{C}'$, completing the proof of Corollary 3.4. $\qquad\square$

3.2. Bubbling-off

We suppose that (ω, J, Ω) is an almost Calabi–Yau structure on B_1 with $J|_0 = J_0$ and $\Omega|_0 = \Omega_0$. We denote by g the almost Calabi–Yau metric on (B_1, ω, J, Ω). We write $g = \sum_{i,j=1}^{2m} g_{ij} dy^i dy^j$ and suppose

$$g_{ij}(0) = \delta_{ij} \quad \text{and} \quad \frac{\partial g_{ij}}{\partial y^k}(0) = 0 \quad \text{for each } i, j, k = 1, \ldots, 2m.$$

If W is a varifold in $\mathbb{R}^n$ and if $\delta > 0$ then we can define $\delta^{-1}W$ by re-scaling W by δ^{-1} (as in the proof of Corollary 3.4 above). If $b \in \mathbb{R}^n$ then we can also define $W - b$ by translating W by $-b$.

We take $C \in \mathcal{C}$ and a neighbourhood $\mathcal{C}'$ of C in $\mathcal{C}$. We denote by $\mathcal{X}'$ the space of all elements of $\mathcal{V}_0$ with singularity only at one point y and asymptotic at y to some element of $\mathcal{C}'$ with multiplicity 1. Let $X \in \mathcal{X}'$ and let X be singular at $\mathbf{0}$ and asymptotic at $\mathbf{0}$ to C with multiplicity 1. We have then:

Theorem 3.6. *Let $(X_n)_{n=1}^\infty$ be a sequence in $\mathcal{X}'$ converging to X. For each $n = 1, 2, 3, \ldots$ let x_n be the singular point of X_n, and let C_n be the multiplicity 1 smooth tangent cone to X_n at x_n. Then x_n tends to $\mathbf{0}$ and C_n tends to C as $n \to \infty$.*

Proof. By Allard's regularity theorem x_n tends to $\mathbf{0}$ as $n \to \infty$, and so we have only to prove that for each neighbourhood $\mathcal{C}'$ of C in $\mathcal{C}$ there exists an integer $N > 0$ such that for $n > N$ we have $C_n \in \mathcal{C}$. Let $\epsilon > 0$ be so small that we may apply Proposition 3.1 and Corollary 3.4 where Corollary 3.4

involves the choice of $\mathcal{C}'$ and so ϵ depends on $\mathcal{C}'$. By Proposition 3.1 we have

$$\mathcal{E}\big((X_n - x_n) \llcorner \widetilde{A}_{\sigma,\rho}\big) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - e^{k\epsilon\sigma} \frac{\|X_n - x_n\|(B_\sigma)}{\sigma^m}$$

where $0 < \sigma < \rho < 1$. Letting $\sigma \to 0$ we get

(3.6) $$\mathcal{E}(X_n - x_n) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \mathrm{area}(C_n \cap B_1).$$

Making $\mathcal{C}'$ smaller if necessary we may suppose that $|\mathrm{area}(C' \cap B_1) - \mathrm{area}(C \cap B_1)| < \epsilon/2$ for all $C' \in \mathcal{C}'$ so that by (3.6) we have

(3.7) $$\mathcal{E}(X_n - x_n) \leq e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \mathrm{area}(C \cap B_1) + \frac{\epsilon}{2}.$$

Making ρ smaller if necessary we may suppose

$$e^{k\epsilon\rho} \frac{\|X\|(B_\rho)}{\rho^m} - \mathrm{area}(C \cap B_1) < \frac{\epsilon}{2}.$$

For n sufficiently large, therefore, we have

$$e^{k\epsilon\rho} \frac{\|X_n - x_n\|(B_\rho)}{\rho^m} - \mathrm{area}(C \cap B_1) < \frac{\epsilon}{2}.$$

This combined with (3.7) implies

$$\mathcal{E}(X_n - x_n) < \epsilon.$$

On the other hand Allard's regularity theorem implies that

$$\|X_n - x_n\| \llcorner A_{1/2,1} = \mathrm{Graph}\, v_n$$

for some $v_n \in C^\infty(C \cap A_{1/2,1}; NC)$ with $|v_n|_{[1/2,1]}^{2,\mathrm{cyl}} < \epsilon$. Hence applying Corollary 3.4 and Remark 3.5 to $X_n - x_n$ we find that $C_n \in \mathcal{C}'$, completing the proof of Theorem 3.6. $\qquad\square$

We have also:

Theorem 3.7. *Let $(V_n)_{n=1}^\infty$ be a sequence in $\mathcal{V}_0 \setminus \mathcal{X}'$ converging to X. Then there exists a sequence $(\delta_n)_{n=1}^\infty$ of positive real numbers converging to 0, a sequence $(y_n)_{n=1}^\infty$ in B_1 converging to 0, and a subsequence of $\big(\delta_n^{-1}(V_n - y_n)\big)_{n=1}^\infty$ converging to some varifold W in $\mathbb{R}^{2m}$ asymptotic at infinity to*

some element of $\mathcal{C}'$ with multiplicity 1 and satisfying $\mathcal{E}(W - b) > 0$ for every $b \in \mathbb{R}^{2m}$.

Remark. W will automatically be a special Lagrangian integral varifold with $\partial \overrightarrow{W} = 0$ in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$.

Proof. We take positive real numbers σ, ρ and ϵ, which we shall make smaller if necessary. Allard's regularity theorem implies that for each $y \in B_{\sigma\rho}$ and n large enough there exists $v_{n,y} \in C^\infty(C \cap A_{1/2,1}; NC)$ such that

$$(3.8) \qquad \|V_n - y\| \llcorner A_{1/2,1} = |\operatorname{Graph} v_{n,y}|, \; |v_{n,y}|^{2,\mathrm{cyl}}_{[1/2,1]} < \epsilon$$

where ϵ is as in Corollary 3.4. Hence by Corollary 3.4 we find that $\mathcal{E}(V_n - y) \geq \epsilon$; otherwise V_n will be singular only at y and asymptotic at y to some element of $\mathcal{C}'$ with multiplicity 1, which contradicts $V_n \in \mathcal{V}_0 \setminus \mathcal{X}'$. We put

$$\delta_n(y) = \inf\left\{\delta \in (0, \rho) : \mathcal{E}\big((V_n - y) \llcorner \widetilde{A}_{\delta,\rho}\big) = \frac{\epsilon}{2}\right\}.$$

Since $\mathcal{E}(V_n - y) \geq \epsilon$ we see $\delta_n(y) > 0$. It is also easy to see that $y \mapsto \delta_n(y)$ is lower semi-continuous. Hence we can find $y_n \in B_{\sigma\rho}$ with $\delta_n(y_n) = \inf_{y \in B_{\sigma\rho}} \delta_n(y)$. We put $\delta_n = \delta_n(y_n) > 0$. We have then:

Lemma. δ_n *tends to 0 as $n \to \infty$.*

Proof. Making ρ smaller if necessary we may suppose $\mathcal{E}(X \llcorner \widetilde{B}_\rho) < \epsilon/2$. We take $\delta > 0$. We have then $\mathcal{E}(X \llcorner \widetilde{A}_{\delta,\rho}) < \epsilon/2$. Since V_n tends to X as $n \to \infty$ we see that for n sufficiently large we have $\mathcal{E}(V_n \llcorner \widetilde{A}_{\delta,\rho}) < \epsilon/2$. This implies $\delta_n(0) < \delta$ and so $\delta_n(0)$ tends to 0 as $n \to \infty$. Since $\delta_n \leq \delta_n(0)$ we see that δ_n also tends to 0 as $n \to \infty$. $\qquad\square$

We have also:

Lemma. $\big(\delta_n^{-1}(V_n - y_n)\big)_{n=1}^\infty$ *has a subsequence converging to some varifold W in $\mathbb{R}^{2m}$.*

Proof. Let $R > 0$. Then we have only to prove

$$(3.9) \qquad \sup_{n=1,2,3,\dots} \|\delta_n^{-1}(V_n - y_n)\|(B_R) < \infty.$$

Notice that $\|\delta_n^{-1}(V_n - y_n)\|(B_R) = \delta_n^{-m}\|V_n - y_n\|(B_{\delta_n R})$. Notice also that by Proposition 3.1 we can find $\epsilon' > 0$ such that

$$e^{k\epsilon'\delta_n R}\frac{\|V_n - y_n\|(B_{\delta_n R})}{(\delta_n R)^m} \le e^{k\epsilon'}\|V_n - y_n\|(B_1)$$

which tends to $e^{k\epsilon'}\|X\|(B_1)$ as $n \to \infty$. Then we get (3.9). $\qquad\square$

We take W as above. From the definition of δ_n it is easy to see $\mathcal{E}(W \llcorner \widetilde{A}_{1,\infty}) \le \epsilon/2$. Using (3.8) with $y = y_n$ we can also apply Theorem 3.3 to find $v_n \in C^\infty(C \cap A_{\delta_n,1}; NC)$ with

$$(3.10) \qquad \|V_n - y_n\| \llcorner A_{\delta_n,1} = \text{Graph } v_n, \ |v_n|_{[\delta_n,1]}^{2,\text{cyl}} < k\epsilon^\alpha.$$

This implies that there exists $w \in C^\infty(C \cap A_{1/2,1}; NC)$ such that

$$(3.11) \qquad \|W\| \llcorner A_{1,2} = \text{Graph } w, \ |w|_{[1,2]}^{2,\text{cyl}} \leqslant k\epsilon^\alpha$$

and so the situation is similar to that of Corollary 3.4 in the sense that W is away from infinity close to the multiplicity-one cone with isolated singularity and that W satisfies the energy estimate $\mathcal{E}(W \llcorner \widetilde{A}_{1,\infty}) \le \epsilon/2$. In a way similar to the proof of Corollary 3.4 we can prove indeed that W is asymptotic at infinity to a multiplicity-one cone C' with isolated singularity. Making ϵ smaller if necessary we may suppose that C' is so C^1-close to C that

$$(3.12) \qquad |\text{area}(C' \cap B_1) - \text{area}(C \cap B_1)| < \frac{\epsilon}{8}.$$

It remains to prove $\mathcal{E}(W - b) > 0$ for every $b \in \mathbb{R}^{2m}$. We put $V_n' = V_n - y_n - \delta_n b$. By Proposition 3.1 we have

$$(3.13) \quad \exp(k\epsilon'\rho)\frac{\|V_n'\|(B_\rho)}{\rho^m} - \exp(k\epsilon'\delta_n)\frac{\|V_n'\|(B_{\delta_n})}{\delta_n^m} \ge \mathcal{E}(V_n' \llcorner \widetilde{A}_{\delta_n,\rho}) \ge \frac{\epsilon}{2}.$$

We note that V_n' tends to X and $\delta_n^{-1}V_n'$ tends to $W - b$ as $n \to \infty$. By (3.13) therefore we have

$$(3.14) \qquad e^{k\epsilon'\rho}\frac{\|X\|(B_\rho)}{\rho^m} - \|W - b\|(B_1) \ge \frac{\epsilon}{2}.$$

Making $\rho > 0$ smaller if necessary we may suppose

$$e^{k\epsilon'\rho}\frac{\|X\|(B_\rho)}{\rho^m} \le \text{area}(C \cap B_1) + \frac{\epsilon}{4}.$$

This combined with (3.14) implies

$$\text{area}(C \cap B_1) - \|W - b\|(B_1) \geqslant \frac{\epsilon}{4}.$$

Hence by (3.12) we get

(3.15) $$\text{area}(C' \cap B_1) - \|W - b\|(B_1) > \frac{\epsilon}{8}.$$

On the other hand by (3.1) we have

$$\frac{\|W - b\|(B_R)}{R^m} - \|W - b\|(B_1) = \mathcal{E}\big((W - b) \llcorner \widetilde{A}_{1,R}\big)$$

for each $R > 1$. Letting $R \to \infty$ we get

$$\text{area}(C' \cap B_1) - \|W - b\|(B_1) = \mathcal{E}\big((W - b) \llcorner \widetilde{A}_{1,\infty}\big).$$

This combined with (3.15) implies $\mathcal{E}\big((W - b) \llcorner \widetilde{A}_{1,\infty}\big) > \epsilon/8$ and so $\mathcal{E}(W - b) > 0$, completing the proof. $\square$

4. Classification of local models

Let C be as in (1.1), and let L_1, L_2, L_3 be as in (1.2)–(1.4). We begin by recalling the statement of our classification result:

Theorem 4.1 (Re-statement of Theorem 1.1). *Let W be a special Lagrangian integral varifold with no boundary in $(\mathbb{R}^6, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to C with multiplicity 1. Then we have $\|W\| = |sL + b|$ for some $s > 0$, $L \in \{C, L_1, L_2, L_3\}$ and $b \in \mathbb{R}^6$.*

Here $|sL + b|$ denotes the Radon measure on $\mathbb{R}^6$ associated to $sL + b$ with multiplicity 1 with respect to the Euclidean metric g_0.

In what follows we give a sketch of the proof of Theorem 4.1.

First of all we analyse the asymptotic behaviour of W at infinity. From the stability of C we find $b \in \mathbb{R}^6$ such that $W - b$ will decay rapidly to C at infinity. For the sake of simplicity, therefore, we may suppose that W decays rapidly to C at infinity.

We define a T^2-action on $\mathbb{R}^6 = \mathbb{C}^3$ by setting

$$(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2, z_3) = (e^{i\theta} z_1, e^{i\phi} z_2, e^{-i\theta - i\phi} z_3).$$

We note that ω_0, J_0, Ω_0 and C are invariant under the T^2-action. We take a moment map $\mu : \mathbb{C}^3 \to \mathbb{R}^2$. We prove that μ is constant on $\mathrm{Spt}\, \|W\|$. The idea of the proof is as follows.

We suppose for a moment that W is a submanifold of $\mathbb{C}^3$. By the T^2-action, then, we can deform W as a special Lagrangian submanifold of $\mathbb{C}^3$. The infinitesimal deformation may be identified with $d\mu|_W$, and so $d\mu|_W$ will be a harmonic 1-form on W, by the theory of McLean. Therefore $\mu|_W$ will be a harmonic function on W. Since W decays rapidly to C at infinity we shall see that $\mu|_W$ decays rapidly at infinity to some constant. By the maximum principle, therefore, $\mu|_W$ will be a constant function.

Actually W need not be a submanifold of $\mathbb{R}^6$. Using some basic results on varifolds, however, we can modify the argument above. Thus we see that μ is constant on $\mathrm{Spt}\, \|W\|$.

Harvey and Lawson [8, Chapter III.3.A, Theorem 3.1] construct a special Lagrangian fibration $F : \mathbb{C}^3 \to \mathbb{R}^3$ invariant under the T^2-action. From the proof of Harvey and Lawson we see that every T^2-invariant special Lagrangian submanifold of $\mathbb{C}^3$ is contained in a fibre of F.

We suppose again that W is a submanifold of $\mathbb{C}^3$. Since μ is constant on W we see that W is then invariant under the T^2-action, and so contained in a fibre of F. Actually W need not be a submanifold of $\mathbb{C}^3$, but we can again modify the argument so that $\mathrm{Spt}\, \|W\|$ will be contained in a fibre of F.

For each fibre of F we have an explicit description of the topology and asymptotic behaviour at infinity, and so we shall be able to complete the proof by an elementary argument.

We begin now with a review of some basic properties of Laplacians over cones.

We suppose that Σ is a compact manifold of dimension $m - 1$. We put $C = (0, \infty) \times \Sigma$. We denote by r the projection of C onto $(0, \infty)$. We suppose that g_Σ is a Riemannian metric on Σ. We put $g_C = dr^2 + r^2 g_\Sigma$. With respect to g_C we can define the Laplacian $\Delta_C : C^\infty(C; \mathbb{R}) \to C^\infty(C; \mathbb{R})$.

We have also the Laplacian $\Delta_\Sigma : C^\infty(\Sigma; \mathbb{R}) \to C^\infty(\Sigma; \mathbb{R})$ with respect to g_Σ. We put $t = \log r : C \to \mathbb{R}$ and $\partial_t = \frac{\partial}{\partial t}$. It is easy then to see $-e^{2t} \Delta_C = \partial_t^2 + (m - 2)\partial_t - \Delta_\Sigma$. We denote by $0 = \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \cdots$ the eigenvalues of Δ_Σ. For each integer $i \geq 0$ we consider the equation $x^2 + (m - 2)x - \gamma_i = 0$ in x. We denote by α_i and β_i the two solutions with $\alpha_i \geq \beta_i$. We suppose $m \geq 2$. Since $\gamma_0 = 0$ we get $\beta_0 = 2 - m \leq 0 = \alpha_0$. Since $\gamma_0 \leq \gamma_1 \leq \gamma_2 \cdots$ we

get $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$ and $\beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots$. Since γ_i tends to ∞ as $i \to \infty$ we see that α_i tends to ∞ and β_i tends to $-\infty$ as $i \to \infty$. We also put $\Lambda = \{\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2, \dots\}$. For each $\lambda \in \mathbb{R}$ we put

$$E_\lambda = \{f \in C^\infty(\Sigma; \mathbb{R}) : \Delta_\Sigma f = \lambda(\lambda + m - 2)f\}.$$

By the definition of the eigenvalues we have $E_\lambda \neq \{0\}$ if and only if $\lambda \in \Lambda$.

We can also take a complete orthonormal basis $\{v_0, v_1, v_2, \dots\}$ for $L^2(\Sigma; \mathbb{R})$ such that $\Delta_\Sigma v_i = \gamma_i v_i$ for each integer $i \geq 0$. We have then:

Proposition 4.2. *Let I be an open interval in $\mathbb{R}$, let $u \in C^\infty(I \times \Sigma; \mathbb{R})$ and suppose $\Delta_C u = 0$. Then there exist $a_0, b_0, a_1, b_1, a_2, b_2 \cdots \in \mathbb{R}$ such that $u = \sum_{i=0}^\infty (a_i r^{\alpha_i} + b_i r^{\beta_i})v_i$ where the series converges in the local C^∞-topology.*

We give a proof for the sake of clarity:

Proof. We put $\log I = \{t \in \mathbb{R} : e^t \in I\}$. For each $t \in \log I$ and $i \in \{0, 1, 2, \dots\}$ we define $u_i(t)$ as the inner product of $u|_{\{e^t\} \times \Sigma}$ and v_i in $L^2(\Sigma; \mathbb{R})$. Then $t \mapsto u_i(t)$ is a smooth function on $\log I$. Since $\Delta_C u = 0$ we get $(\partial_t^2 + (m - 2)\partial_t - \lambda_i)u_i = 0$. Hence we find $a_i, b_i \in \mathbb{R}$ such that $u_i = a_i e^{\alpha_i t} + b_i e^{\beta_i t}$. It suffices therefore to prove that $\sum_{i=0}^\infty u_i v_i$ converges to u in the local C^∞-topology. We denote by $d\mu_\Sigma$ the Riemannian measure on Σ with respect to g_Σ, and by $\|\bullet\|$ the norm of $L^2(\Sigma; \mathbb{R})$. We have then

$$\int_{\log I \times \Sigma} |u|^2 dt d\mu_\Sigma = \int_{\log I} \sum_{i=0}^\infty \|u_i(t)\|^2 dt = \sum_{i=0}^\infty \int_{\log I} \|u_i(t)\|^2 dt < \infty.$$

Putting $w_n = \sum_{i=0}^n u_i v_i$ we get

$$\int_{\log I \times \Sigma} |u - w_n|^2 dt d\mu_\Sigma = \sum_{i=n+1}^\infty \int_{\log I} \|u_i(t)\|^2 dt$$

which tends to 0 as $n \to \infty$. Thus w_n tends to u in the L^2-topology. Applying elliptic regularity to $u - w_n$ we see that w_n tends to u in the local C^∞-toplogy. $\square$

Let $u \in C^\infty(C; \mathbb{R})$ and $\alpha \in \mathbb{R}$. Then we shall write $u = O(r^\alpha)$ as $r \to \infty$ if there exists $R > 0$ such that

$$\sup_{(R,\infty) \times \Sigma} |r^{-\alpha+k} \nabla^k u| < \infty \quad \text{for every } k = 0, 1, 2, 3, \dots$$

where ∇ denotes the Levi-Civita connexion with respect to g_C, and $|\bullet|$ denotes the norm with respect to g_C.

By a result of Simon [21, Part I, Lemma 5.9] we have:

Proposition. *Suppose* $q \in (2 - m, \infty) \setminus \Lambda$ *and* $f \in C^\infty\big((R, \infty) \times \Sigma; \mathbb{R}\big)$ *with* $f = O(r^{q-2})$. *Then there exists* $u \in C^\infty\big((R, \infty) \times \Sigma; \mathbb{R}\big)$ *with* $u = O(r^q)$ *such that* $\Delta_C u = f$.

We give a corollary to this:

Corollary 4.3. *Suppose* $R > 0$, $u \in C^\infty\big((R, \infty) \times \Sigma; \mathbb{R}\big)$, $p, q \in \mathbb{R}$, $2 - m < q < p$, $q \notin \Lambda$, $u = O(r^p)$ *and* $\Delta_C u = O(r^{q-2})$. *Then there exists* $(f_\lambda)_{\lambda \in \Lambda} \in \bigoplus_{\lambda \in \Lambda} E_\lambda$ *such that* $u = \sum_{\lambda \in \Lambda \cap (q, p]} f_\lambda r^\lambda + O(r^q)$ *where* $\Lambda \cap (q, p]$ *is possibly empty and in that case we set* $\bigoplus_{\lambda \in \emptyset} E_\lambda = \{0\}$.

Proof. Applying the proposition above to $\Delta_C u$ in place of f we find $u' \in C^\infty\big((R, \infty) \times \Sigma; \mathbb{R}\big)$ with $u' = O(r^q)$ such that $\Delta_C u' = \Delta_C u$. By Proposition 4.2 we can find some $a_0, b_0, a_1, b_1, a_2, b_2, \cdots \in \mathbb{R}$ such that $u - u' = \sum_{i=0}^{\infty} (a_i r^{\alpha_i} + b_i r^{\beta_i}) v_i$.

Since $u' = O(r^q)$ we get $u = \sum_{i=0}^{\infty} (a_i r^{\alpha_i} + b_i e^{\beta_i}) v_i + O(r^q)$. Since $u = O(r^p)$ and $p > q$ we get $a_i = 0$ if $\alpha_i > p$. Since $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \cdots$ tend to ∞ we can find a unique integer $i(p)$ such that $\alpha_{i(p)+1} > p$ and $\alpha_{i(p)} \leq p$. Since $a_i = 0$ for every $i > i(p)$ we see that $\sum_{i=0}^{\infty} b_i r^{\beta_i} v_i$ converges in the local C^∞-topology. We put $w = \sum_{i=0}^{\infty} b_i r^{\beta_i} v_i$.

Since $0 \geq 2 - m = \beta_0 \geq \beta_1 \geq \beta_2 \geq \cdots$ it follows that for every $\rho > R$ we have

$$\int_{\rho}^{2\rho} \int_{\Sigma} |w|^2 \frac{dr}{r} d\mu_{\Sigma} \leq \left(\frac{\rho}{R}\right)^{2-m} \int_{R}^{2R} \int_{\Sigma} |w|^2 \frac{dr}{r} d\mu_{\Sigma}.$$

Applying elliptic regularity to w we get $w = O(r^{2-m})$. Since

$$u = \sum_{i=0}^{i(p)} a_i r^{\alpha_i} v_i + w + O(r^q) \quad \text{and} \quad q > 2 - m$$

we get $u = \sum_{i=0}^{i(p)} a_i r^{\alpha_i} v_i + O(r^q)$, completing the proof. $\qquad \square$

We suppose now that C is a smooth special Lagrangian cone in $(\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \Omega_0)$ and $\Sigma = C \cap S^{2m-1}$. For each $\rho > 0$ we put $B_\rho = \{y \in \mathbb{R}^n : |y| < \rho\}$. We denote by $i_C : C \to \mathbb{R}^{2m}$ the inclusion map of C into $\mathbb{R}^{2m}$.

Joyce [12, Definition 7.1] defines special Lagrangian submanifolds of $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$ asymptotic to C with multiplicity 1 at infinity with some

rate < 2. We extend it to varifolds as follows. We denote by $\mathcal{W}$ the set of all special Lagrangain integral varifolds W with $\partial \overrightarrow{W} = 0$ in $(\mathbb{R}^{2m}, \omega_0, J_0, \Omega_0)$ asymptotic at infinity to C with multiplicity 1. For each $\lambda < 2$ we denote by $\mathcal{W}_\lambda$ the set of all $W \in \mathcal{W}$ such that we can find a compact subset K of W, an $R > 0$ and a diffeomorphism $f : C \setminus \overline{B_R} \to \mathrm{Spt}\,\|W\| \setminus K$ such that $f - i_C = O(r^{\lambda - 1})$.

We suppose that C is Jacobi-integrable in the sense of Joyce [12, Definition 6.7]. In a way similar to Simon [21, Part II, §§5 and 6], then, we can prove that if $W \in \mathcal{W}$ then there exists $\lambda < 2$ such that $W \in \mathcal{W}_\lambda$.

Joyce [12, Theorem 4.3] proves a version of Weinstein's theorem [25, Corollary 6.2], which we shall recall next. We denote by T^*C the cotangent bundle over C, by 0_C the zero-section of T^*C, and by ω_C the canonical symplectic form on T^*C. We regard $(0, \infty)$ as a multiplicative group acting upon C and $\mathbb{R}^{2m}$ as re-scaling. We can lift the $(0, \infty)$-action uniquely to T^*C so that for each $t \in (0, \infty)$ we shall have $t^* \omega_C = t^2 \omega_C$ (on the left-hand side we regard t as a map of C into itself). We have then:

Lemma 4.4. *There exist a neighbourhood UC of $\mathrm{Im}\, 0_C$ in T^*C invariant under $(0, \infty)$, and a diffeomorphism Φ_C of UC into $\mathbb{R}^{2m}$ equivariant under $(0, \infty)$ with $\Phi_C \circ 0_C = i_C$ and $\Phi_C^* \omega_0 = \omega_C$.*

Let $W \in \mathcal{W}_\lambda$. Then we can take a compact subset K of W and a closed 1-form w on $C \setminus \overline{B_R}$ such that $\mathrm{Spt}\,\|W\| \setminus K_W = \Phi_C(\mathrm{Graph}\, w)$. We denote by π_Σ the projection of C onto Σ, which induces a linear isomorphism $\pi_\Sigma^* : H^1(\Sigma; \mathbb{R}) \to H^1(C \setminus \overline{B_R}; \mathbb{R})$, so that we may write $w = \pi_\Sigma^* \eta_W + dh_W$ for some 1-form η_W on Σ and some $h_W : C \setminus \overline{B_R} \to \mathbb{R}$. By results of Joyce [12, Equations (7.7) and (7.8)] we have:

Lemma 4.5. *If $\alpha < 2$ and $h_W = O(r^\alpha)$ then we have $\Delta_C h_W = O(r^{2(\alpha - 2)})$.*

We can extend a result of Joyce [12, Theorem 7.11] as follows:

Lemma 4.6. *Let $W \in \mathcal{W}_\lambda$, $\lambda' < \lambda < 2$ and $[\lambda', \lambda) \cap \Lambda = \emptyset$. Then we have $W \in \mathcal{W}_{\lambda'}$.*

Proof. For each integer $n \geq 0$ we put $\lambda(n) = 2^n(\lambda - 2) + 2$. We can take a unique integer ν such that $\lambda(\nu + 1) < \lambda' \leq \lambda(\nu)$. By an induction on $n = 0, 1, \ldots, \nu$, we shall prove $h_W = O(r^{\lambda(n)})$ for every $n = 0, 1, \ldots, \nu$.

By the property of h_W we have $h_W = O(r^\lambda) = O(r^{\lambda(0)})$. If $\nu = 0$ we can then complete the induction automatically. We suppose therefore $\nu > 0$. Suppose also that we have $h_W = O(r^{\lambda(n)})$ for some $n = 0, 1, \ldots, \nu - 1$.

By Lemma 4.5 we have $\Delta h_W = O(r^{2(\lambda(n)-2)}) = O(r^{\lambda(n+1)-2})$. Since $n < \nu$ we get $\lambda(n+1) \geq \lambda'$ and so $[\lambda(n+1), \lambda) \cap \Lambda = \emptyset$. Applying Corollary 4.3 to $h_W, \lambda, \lambda(n+1)$ in place of u, p, q respectively, we get $h_W = O(r^{\lambda(n+1)})$, completing the induction.

We have thus proved $h_W = O(r^{\lambda(n)})$ for every $n = 0, 1, \ldots, \nu$. Putting $n = \nu$ we get $h_W = O(r^{\lambda(\nu)})$. By Lemma 4.5 we have $\Delta_C h_W = O(r^{\lambda(\nu+1)-2})$. By the definition of ν we have $\lambda(\nu+1) \leq \lambda'$ and so $\Delta_C h_W = O(r^{\lambda'-2})$. Applying Corollary 4.3 to h_W, λ, λ' in place of u, p, q respectively, we get $h_W = O(r^{\lambda'})$, completing the proof. $\qquad\square$

Joyce [13, Definition 3.6] defines the stability of C. We have:

Theorem 4.7. *Let C be stable in the sense of Joyce. Then there exists $b \in \mathbb{R}^{2m}$ such that $W - b \in \mathcal{W}_0$.*

Proof. By the stability of C we have $\Lambda \cap (1, 2) = \emptyset$ and $E_1 = \{b \cdot x : b \in \mathbb{R}^{2m}\}$. We take $\lambda \in (1, 1 + \epsilon)$. Let h_W be as above. Then by Lemma 4.6 we have $h_W = O(r^\lambda)$. By Lemma 4.5, therefore, we have $\Delta_C h_W = O(r^{2(\lambda-2)})$. Applying Corollary 4.3 to $h_W, \lambda, 2(\lambda - 2)$ in place of u, p, q we get $h_W = b \cdot x|_C + O(r^{2(\lambda-2)+2})$.

We may suppose that for each $t \in [0, 1]$ there exist a compact subset K_t of $\mathbb{R}^{2m}$ and a 1-form w_t on $C \setminus \overline{B_R}$ such that $(\text{Spt} \|W\| - tb) \setminus K_t = \Phi_C(\text{Graph } w_t)$. We put $\beta = (b \cdot x) \circ \Phi_C$. We have then a function $\beta : UC \to \mathbb{R}$. Notice that $\Phi_C^{-1}(W - b)$ is the image of the time-one map of the flow generated by $d\beta \lrcorner \omega_C$. Then we have $\partial w_t/\partial t = -w_t^* d\beta$ and so $w_1 = w_0 - d \int_0^1 w_t^* \beta dt = \pi_\Sigma^* \eta_W + d(h_W - \int_0^1 w_t^* \beta dt)$. Hence we get

$$h_{W-b} = h_W - \int_0^1 w_t^* \beta dt = h_W - \beta|_C - \int_0^1 (w_t^* \beta - \beta|_C) dt$$
$$= O(r^{2\lambda-2}) + O(r^{\lambda-1}) = O(r^{2\lambda-2}).$$

By results of Joyce [12, Equations (7.7) and (7.8)] we have $\Delta_C h_{W-b} = O(r^{4\lambda-8})$. Applying Corollary 4.3 to $h_{W-b}, 2\lambda - 2, 4\lambda - 8$ in place of u, p, q we get $c \in \mathbb{R}$ such that $h_{W-b} = c + O(r^{4\lambda-6}) = O(r^0)$ as we may suppose $4\lambda - 6 < 0$. This completes the proof. $\qquad\square$

Let $\lambda < 2$ and $W \in \mathcal{W}_\lambda$. Take a compact subset K_W of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \to \text{Spt} \|W\| \setminus K_W$ with $f_W - i_C = O(r^{\lambda-1})$. Then we have a Riemannian metric $f_W^* g_0$ over $C \setminus \overline{B_R}$. With respect to $f_W^* g_0$ we can define the Laplacian $\Delta_W : C^\infty(C \setminus \overline{B_R}; \mathbb{R}) \to C^\infty(C \setminus \overline{B_R}; \mathbb{R})$. We have then:

Proposition. *If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$ and $u = O(r^\alpha)$ then we have $\Delta_W u = \Delta_C u + O(r^{\alpha+\lambda-4})$.*

Proof. Since $f_W - i_C = O(r^{\lambda-1})$ we get $f_W^* g_0 = i_C^* g_0 + O(r^{\lambda-2})$. We denote by ∇_W and ∇_C the Levi-Civita connexions over $C \setminus \overline{B_R}$ with respect to $f_W^* g_0$ and $i_C^* g_0$ respectively. We have then $\nabla_W = \nabla_C + O(r^{\lambda-3})$ and so $\Delta_W u = \Delta_C u + O(r^{\alpha+\lambda-4})$ as we want. $\qquad\square$

We give a corollary to this:

Corollary 4.8. *If $u \in C^\infty(C \setminus \overline{B_R}; \mathbb{R})$, $\Delta_W u = 0$ and $u = O(r^0)$ then we have $u = c + O(r^{2-m})$ for some $c \in \mathbb{R}$.*

Proof. By the proposition above we have $\Delta_C u = O(r^{0+\lambda-4}) = O(r^{\lambda-4})$. If $\lambda - 2 < 2 - m$ then we can complete the proof by applying Corollary 4.3 to $0, \lambda - 2$ in place of p, q respectively.

We suppose therefore $\lambda - 2 \geq 2 - m$. We take $\lambda' \in (\lambda, 2)$ such that $\lambda' - 2 > 2 - m$. Applying Corollary 4.3 to $0, \lambda' - 2$ in place of p, q respectively, we get $u = c + O(r^{\lambda'-2})$ for some $c > 0$. We have thus improved the decay order estimate for u, and so we can complete the proof in a way similar to the proof of Lemma 4.6. $\qquad\square$

We suppose now

$$C = \{(z_1, \ldots, z_m) \in \mathbb{C}^m \setminus \{0\} : |z_1| = \cdots = |z_m|, z_1 \cdots z_m \in (0, \infty)\}.$$

This is an extension of (1.1) to dimension m. Harvey and Lawson [8, Chapter III.3.A, Theorem 3.1] prove that C is a special Lagrangian submanifold of $(\mathbb{R}^{2m} \setminus \{0\}, \omega_0, J_0, \Omega_0)$.

We define a T^{m-1}-action on $\mathbb{C}^m$ as follows. We write $T^{m-1} = S^1 \times \cdots \times S^1$ and $S^1 = \{t \in \mathbb{C} : |t| = 1\}$. For each $j \in \{2, \ldots, m\}$ we define the j-th S^1-action on $\mathbb{C}^m$ by setting $t \cdot (z_1, \ldots, z_j, \ldots, z_m) = (tz_1, \ldots, t^{-1}z_j, \ldots, z_m)$ for each $(z_1, \ldots, z_m) \in \mathbb{C}^m$. This action preserves the subset $C \subset \mathbb{C}^m$.

We also define a map $\mu_j : \mathbb{C}^m \to \mathbb{R}$ by $2\mu_j(z_1, \ldots, z_m) = |z_1|^2 - |z_j|^2$. We shall identify $\mathbb{R}$ with the Lie algebra of S^1 so that μ_j will be the moment map on $(\mathbb{C}^m, \omega_0)$ with respect to the S^1-action. One important property is that $\mu_j|_C \equiv 0$.

We prove:

Theorem. *Let $m > 2$. If $W \in \mathcal{W}_0$ and $j \in \{2, \ldots, m\}$ then we have $\mathrm{grad}_{TW}\, \mu_j = 0$ almost everywhere on $\mathbb{R}^{2m}$ with respect to $\|W\|$.*

Proof. We can take a compact subset K_W of $\mathbb{R}^{2m}$, an $R > 0$ and a diffeomorphism $f_W : C \setminus \overline{B_R} \to \text{Spt} \, \|W\| \setminus K_W$ with $f_W - i_C = O(r^{-1})$. Since $\mu_j = O(r^2)$ and $\mu_j \circ i_C = 0$ we get $f_W^* \mu_j = \mu_j \circ f_W = \mu_j \circ f_W - \mu_j \circ i_C = \int_0^1 d\mu_j|_{(1-t)f_W + ti_C} (f_W - i_C) dt = O(r^{2-1} r^{-1}) = O(r^0)$ and so $f_W^* \mu_j = O(r^0)$. By a result of Joyce [13, Lemma 3.4] we have $\Delta_W f_W^* \mu_j = 0$ for each $j = 2, \ldots, m$. By Corollary 4.8 we can find $c_j \in \mathbb{R}$ such that $f_W^* \mu_j - c_j = O(r^{2-m})$. Putting $\mu_j' = \mu_j - c_j$ we get $f_W^* \mu_j' = O(r^{2-m})$. We have clearly $\text{grad} \, \mu_j' = \text{grad} \, \mu_j$.

Take a smooth function $\chi : \mathbb{R} \to [0, 1]$ with $\chi = 1$ on B_1 and $\chi = 0$ on $\mathbb{R}^6 \setminus B_2$. Let $R > 0$, and define a function $\chi_R : \mathbb{R}^{2m} \to [0, 1]$ by setting $\chi_R(x) = \chi(|x|/R)$. Since W has first variation 0 in B_{2R} we get then

$$\int_{B_{2R}} \text{div}_{TW} (\chi_R \mu_j' \, \text{grad} \, \mu_j) d\|W\| = 0.$$

Also by a result of Joyce [13, Lemma 3.4] we have $\text{div}_{TW} \, \text{grad} \, \mu_j = 0$ and so

$$(4.1) \qquad \int_{B_{2R}} \mu_j' (\text{grad} \, \chi_R, \text{grad}_{TW} \, \mu_j) + \chi_R |\text{grad}_{TW} \, \mu_j|^2 d\|W\| = 0.$$

Notice that $(\text{grad} \, \chi_R, \text{grad}_{TW} \, \mu_j) = (d\chi_R, d\mu_j|_{\text{Spt} \, \|W\| \setminus K_W})$ on $\text{Spt} \, \|W\| \setminus K_W$. Take R sufficiently large so that $K_W \subset B_R$. Then we have $\text{grad} \, \chi_R = 0$ on K_W and so $(\text{grad} \, \chi_R, \text{grad}_{TW} \, \mu_j) = (d\chi_R, d\mu_j|_{\text{Spt} \, \|W\| \setminus K_W})$ on $\text{Spt} \, \|W\|$. We have therefore

$$\int_{B_{2R}} -\mu_j' (\text{grad} \, \chi_R, \text{grad}_{TW} \, \mu_j) d\|W\| \le \sup_{\text{Spt} \, \|W\| \setminus K_W} |\mu_j' (d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|$$

and so by (4.1) we have

$$(4.2) \qquad \int_{B_{2R}} \chi_R |\text{grad}_{TW} \, \mu_j|^2 d\|W\| \le \sup_{\text{Spt} \, \|W\| \setminus K_W} |\mu_j' (d\chi_R, d\mu_j)| \int_{B_{2R}} d\|W\|.$$

Since $f_W : C \setminus \overline{B_R} \to \text{Spt} \, \|W\| \setminus K_W$ is a diffeomorphism we get

$$(4.3) \qquad \sup_{\text{Spt} \, \|W\| \setminus K_W} |\mu_j' (d\chi_R, d\mu_j)| = \sup_{C \setminus \overline{B_R}} |f_W^* \mu_j' (df_W^* \chi_R, df_W^* \mu_j)|.$$

Since $\chi_R = \chi(r/R)$ we get $|d\chi_R| \le kR^{-1}$ for some $k > 0$ independent of R. Since $f_W^* \mu_j' = O(r^{2-m})$ we get

$$(4.4) \qquad \sup_{C \setminus \overline{B_R}} |f_W^* \mu_j' (d\chi_R, df_W^* \mu_j)| \le kR^{2-m} R^{-1} R^{1-m} = kR^{2-2m}$$

for some $k > 0$ independent of R. By (4.2)–(4.4) we have

$$\int_{B_{2R}} \chi_R |\text{grad}_{TW}\, \mu_j|^2 d\|W\| \le kR^{2-2m} \int_{B_{2R}} d\|W\|.$$

On the other hand, by the monotonicity formula, we have $\int_{B_{2R}} d\|W\| \le kR^m$ for some $k > 0$ depending only on m and $T_\infty W$. We have therefore

$$\int_{B_{2R}} \chi_R |\text{grad}_{TW}\, \mu_j|^2 d\|W\| \le kR^{2-m}, k > 0 \text{ independent of } R.$$

Letting $R \to \infty$ we get $\int_{\mathbb{R}^{2m}} |\text{grad}_{TW}\, \mu_j|^2 d\|W\| = 0$ because $m > 2$, completing the proof. $\qquad\square$

We give a corollary to the theorem above. We define a map $f : \mathbb{C}^m \to \mathbb{C}$ by setting $f(z_1, \ldots, z_m) = i^{m+1} z_1 \cdots z_m$ for each $(z_1, \ldots, z_m) \in \mathbb{C}^m$. We put $\text{Im}\, f = (f - \bar{f})/2i : \mathbb{C}^m \to \mathbb{R}$ and $F = (\mu_2, \ldots, \mu_m, \text{Im}\, f) : \mathbb{C}^m \to \mathbb{R}^m$. We have then:

Corollary 4.9. *If $W \in \mathcal{W}_0$ then we have $T_y W = \text{Ker}\, dF|_y$ for $\|W\|$-almost every $y \in \mathbb{R}^{2m}$.*

This follows readily from the proof of Harvey and Lawson [8, Chapter III.3.A, Theorem 3.1].

We have moreover:

Corollary 4.10. *For every $W \in \mathcal{W}_0$ there exists $c \in \mathbb{R}^m$ such that $F = c$ on $\text{Spt}\, \|W\|$.*

Proof. By Corollary 4.9 we have $dF|_{\text{Spt}\,\|W\| \setminus K_W} = 0$, and $F|_{\text{Spt}\,\|W\| \setminus K_W}$ is therefore locally constant. Since $\text{Spt}\, \|W\| \setminus K_W \cong C \setminus \overline{B_R} \cong (R, \infty) \times T^{m-1}$ we see that $\text{Spt}\, \|W\| \setminus K_W$ is connected, and $F|_{\text{Spt}\,\|W\| \setminus K_W}$ is therefore constant; i.e. we have $F|_{\text{Spt}\,\|W\| \setminus K_W} = c$ for some $c \in \mathbb{R}^m$.

Put $\phi = |F - c|^2$. Then we have $\phi = 0$ on $\text{Spt}\, \|W\| \setminus K_W$ and so $\text{Spt}\, \phi \cap \text{Spt}\, \|W\| \subset K_W$. By Corollary 4.9 we have $\text{grad}_{TW}\, \phi = 0$ almost everywhere on $\mathbb{R}^{2m}$ with respect to $\|W\|$.

We shall now use a result of Michael and Simon [19, Theorem 2.1], who prove a Poincaré–Sobolev inequality for varifolds; we refer also to Simon [22, Theorem 18.6], who uses varifolds more explicitly. We are going to use the following version:

Lemma. *Let W be a stationary integral varifold of dimension m in $(\mathbb{R}^n, g_0)$. Suppose that we have a smooth function $\phi : \mathbb{R}^n \to [0, \infty)$ with $\text{Spt}\, \phi\, \cap$*

Spt $\|W\|$ *compact and* $\mathrm{grad}_{TW}\,\phi = 0$ *almost everywhere on* $\mathbb{R}^n$ *with respect to* $\|W\|$. *Then we have* $\phi = 0$ *on* Spt $\|W\|$.

We give a proof for the sake of clarity:

Proof. We define $r : \mathbb{R}^n \to [0, \infty)$ by setting $r(y) = |y|$. We can then define $\partial_r = \partial/\partial r$ as a smooth vector field on $\mathbb{R}^n \setminus \{0\}$. It is easy to see that $r\partial_r$ extends smoothly to $\mathbb{R}^n$. Let ϕ be as above, and let χ be a compactly-supported smooth function on $\mathbb{R}^n$ with $\chi = 1$ on Spt $\phi \cap$ Spt $\|W\|$. Then we can define $\phi r\partial_r$ as a smooth vector field on $\mathbb{R}^n$. Since W has first variation 0 we get

$$\int_{\mathbb{R}^n} \mathrm{div}_{TW}\,\chi\phi r\partial_r d\|V\| = 0.$$

Since $\chi = 1$ on Spt $\phi \cap$ Spt $\|W\|$ we get $\int_{\mathrm{Spt}\,\phi \cap \mathrm{Spt}\,\|W\|} \mathrm{div}_{TW}\,\phi r\partial_r d\|W\| = 0$. Since $\mathrm{grad}_{TW}\,\phi = 0$ on Spt $\|W\|$ we get $\int_{\mathrm{Spt}\,\phi \cap \mathrm{Spt}\,\|W\|} \phi\,\mathrm{div}_{TW}\,r\partial_r d\|W\| = 0$. Since $\mathrm{div}_{TW}\,r\partial_r = m$ we get $\int_{\mathrm{Spt}\,\phi \cap \mathrm{Spt}\,\|W\|} m\phi d\|W\| = 0$. Since $\phi \geq 0$ we get $\phi = 0$ on Spt $\phi \cap$ Spt $\|V\|$, completing the proof. $\square$

Hence we get $\phi = |F - c|^2 = 0$ on Spt $\|W\|$, completing the proof of Corollary 4.10. $\square$

We suppose now $m = 3$. For the fibres of $F : \mathbb{C}^3 \to \mathbb{R}^3$ we have an explicit description of the topology and asymptotic behaviour at infinity, which we shall use next. The behaviour of $F : \mathbb{C}^m \to \mathbb{R}^m$ is rather complicated if $m > 3$, which we shall not discuss.

We put

$$Y = \{(a, 0, 0) \in \mathbb{R}^3 : a \geq 0\}$$
$$\cup \{(0, a, 0) \in \mathbb{R}^3 : a \geq 0\} \cup \{(-a, -a, 0) \in \mathbb{R}^3 : a \geq 0\}.$$

We note that if $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ has no fixed point with respect to the T^2-action

$$(e^{i\theta}, e^{i\phi}) \cdot (z_1, z_2, z_3) = (e^{i\theta}z_1, e^{i\phi}z_2, e^{-i\theta-i\phi}z_3).$$

We have:

Proposition 4.11. *Let* $W \in \mathcal{W}$ *and suppose* Spt $\|W\| \subset F^{-1}(c)$ *for some* $c \in \mathbb{R}^3$. *Then we have* $c \in Y$.

For the proof we shall use:

Lemma 4.12. *If $c \in \mathbb{R}^3 \setminus Y$ then $F^{-1}(c)$ is a submanifold of $\mathbb{R}^6$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$ and asymptotic to $C \cup -C$ with multiplicity 1 at infinity.*

Proof. We put $c = (c_1, c_2, c_3)$. We may suppose $c_1 \geq c_2 \geq 0$ without loss of generality. Since $(c_1, c_2, c_3) \in \mathbb{R}^3 \setminus Y$ we get $c_3 \neq 0$ or $c_2 > 0$. For each $t \in \mathbb{R}$ we can find a unique $\phi_c(t) \in [0, \infty)$ such that

$$(\phi_c(t) + c_1)(\phi_c(t) + c_2)\phi_c(t) = |t + ic_3|^2.$$

It is easy to see that $\phi_c(t)$ depends smoothly on t. We put

$$\psi_c(t) = \sqrt{\phi_c(t) + c_1}\sqrt{\phi_c(t) + c_2}.$$

Since $c_3 \neq 0$ or $c_2 > 0$ we get $\psi_c(t) > 0$ for every $t \in \mathbb{R}$, and so we can define $\left(\psi_c(t)\right)^{-1} > 0$ for every $t \in \mathbb{R}$. Define a smooth map $\Phi_c : \mathbb{R} \times S^1 \times S^1 \to F^{-1}(c)$ by setting

$$\Phi_c(t, u, v) = \left(\sqrt{\phi_c(t) + c_1}\,u,\ \sqrt{\phi_c(t) + c_2}\,v,\ \frac{t + ic_3}{\psi_c(t)uv}\right),$$

and a smooth map $\Psi_c(t) : F^{-1}(c) \to \mathbb{R} \times S^1 \times S^1$ by setting

$$\Psi_c(z_1, z_2, z_3) = \left(\mathrm{Re}\, z_1 z_2 z_3,\ \frac{z_1}{(\phi_c(\mathrm{Re}\, z_1 z_2 z_3) + c_1)^{1/2}},\ \frac{z_2}{(\phi_c(\mathrm{Re}\, z_1 z_2 z_3) + c_2)^{1/2}}\right).$$

Then $\Psi_c \circ \Phi_c$ is clearly the identity map of $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $\Phi_c \circ \Psi_c$ is the identity map of $F^{-1}(c)$, and so $F^{-1}(c)$ is a submanifold of $\mathbb{C}^3$ diffeomorphic to $\mathbb{R} \times S^1 \times S^1$. It is also easy to see that $F^{-1}(c)$ is asymptotic to $C \cup -C$ with multiplicity 1 at infinity. $\square$

Proof of Proposition 4.11. We give a proof by contradiction, and so suppose $c \notin Y$. By Lemma 4.12, then, $F^{-1}(c)$ will be a connected submanifold of $\mathbb{R}^6$. By a constancy theorem of Allard [1, Theorem 4.6(3)] or Simon [22, Theorem 41.1], therefore, the multiplicity function Θ_W will be constant on $F^{-1}(c)$ (we recall that Θ_W is characterized by the condition $\Theta_W \mathcal{H}^3 = \|W\|$ where $\mathcal{H}^3$ denotes the 3-dimensional Hausdorff measure on $\mathbb{R}^6$). On the other hand we have $\Theta_W = 1$ near infinity on $\mathrm{Spt}\,\|W\|$ and so we shall have $\Theta_W = 1$ on $F^{-1}(c)$, which implies $\|W\| = |F^{-1}(c)|$. By Lemma 4.12, however, $F^{-1}(c)$ is asymptotic at infinity to $C \cup -C$ with multiplicity 1, which contradicts that W is asymptotic at infinity to C with multiplicity 1. This completes the proof. $\square$

Suppose now that $c \in Y \subset \mathbb{R}^2 \times \{0\}$. We prove:

Proposition 4.13. *Let $W \in \mathcal{W}$ and suppose $\operatorname{Spt} \|W\| \subset F^{-1}(c)$ for some $c \in Y \subset \mathbb{R}^2 \times \{0\}$. Then we have $\|W\| = |sL|$ for some $s > 0$ and $L \in \{C, L_1, L_2, L_3\}$.*

Proof. We treat the two cases $c = \mathbf{0}$, $c \neq \mathbf{0}$ individually but in both cases the main tool is the constancy theorem (which we have already used in the proof of Proposition 4.11).

If $c = \mathbf{0} \in Y \subset \mathbb{R}^3$ then we have $F^{-1}(\mathbf{0}) = C \cup \{\mathbf{0}\} \cup -C$. Let $U_+ := \{\operatorname{Re} z_1 z_2 z_3 > 0\}$ and $U_- := \{\operatorname{Re} z_1 z_2 z_3 < 0\}$, which are open subsets of $\mathbb{C}^3$. Then we have $F^{-1}(\mathbf{0}) \cap U_+ = C$ and $F^{-1}(\mathbf{0}) \cap U_- = -C$, which are sub-manifolds of U_+ and U_- respectively. Hence by the constancy theorem we find some integers $n_\pm \geqslant 0$ such that $\|W\| \llcorner U_\pm = n_\pm |F^{-1}(0) \cap U_\pm|$ where $\|W\| \llcorner U_\pm$ denotes the Radon measure on $\mathbb{R}^6$ with $\|W\| \llcorner U_\pm(A) = \|W\|(U_\pm \cap A)$ for $A \subset \mathbb{R}^6$. On the other hand W is asymptotic at infinity to C, and so $n_+ = 1$ and $n_- = 0$, which implies $\|W\| = |C|$ as we want.

We turn now to the case $c \neq \mathbf{0}$. One easily sees that $F^{-1}(c)$ may be written as the union of two $S^1 \times \mathbb{R}^2$ intersecting at $S^1 \times \{\mathbf{0}\}$ and $F^{-1}(c) \cap U_+$, $F^{-1}(c) \cap U_-$ are diffeomorphic to $S^1 \times (\mathbb{R}^2 \setminus \{\mathbf{0}\})$. Thus $F^{-1}(c)$ is topologically different from $F^{-1}(\mathbf{0})$, but otherwise the treatment above is valid with $F^{-1}(c)$ in place of $F^{-1}(\mathbf{0})$, which implies $\|W\| = |F^{-1}(c) \cap U_+|$. Finally from (1.2)–(1.4), the definition of L_1, L_2, L_3, it follows readily that the closure of $F^{-1}(c) \cap U_+$ in $\mathbb{R}^6$ is equal to sL for some $s > 0$ and $L \in \{L_1, L_2, L_3\}$, which completes the proof of Proposition 4.13. $\qquad\square$

We shall now complete the proof of Theorem 4.1. We suppose that W is as in Theorem 4.1. By Theorem 4.7, then, we can find $b \in \mathbb{R}^6$ such that $W - b \in \mathcal{W}_0$. By Corollary 4.10, therefore, we can find $c \in \mathbb{R}^3$ such that $\operatorname{Spt} \|W - b\| \subset F^{-1}(c)$. By Proposition 4.11, therefore, we have $c \in Y$. By Proposition 4.13, therefore, we have $\|W - b\| = |sL|$ for some $s > 0$ and $L \in \{C, L_1, L_2, L_3\}$ as we want.

5. Combining results of §§3–4

As in §1 the proof of Theorems 1.4 and 1.5, the main theorems of the present paper, is based on the results of §§3–4 above. In the present section therefore we combine those results into a convenient form (Theorem 5.2 below). We begin by recalling some basic notation from §1.

Let (M, ω, J, Ω) be an almost Calabi–Yau manifold of complex dimension
3. Let $\mathcal{V}$ denote the space of all compactly-supported special Lagrangian
integral varifolds with no boundary in M.

Let $\mathcal{X}$ denote the subspace of $\mathcal{V}$ consisting of compact special Lagrangian
3-folds in M with only one singular point modelled on C with multiplicity
1 where C denotes the T^2-cone in $\mathbb{C}^3$ given by (1.1).

Let L_1, L_2, L_3 be as in (1.2)–(1.4), which are non-compact special La-
grangian submanifolds properly-embedded in $\mathbb{C}^3$ and asymptotic at infin-
ity to C with multiplicity 1. For each $s > 0$ and $L \in \{L_1, L_2, L_3\}$ we write
$sL := \{sz \in \mathbb{C}^3 : z \in L\}$.

In the notation above the main result of the present section may be
stated briefly as follows:

Theorem 5.1. *Let $X \in \mathcal{X}$ and let x denote the unique singular point of
X. Then there exist a neighbourhood B_x of x in X and a neighbourhood $\mathcal{U}$
of X in $\mathcal{V}$ such that if $V \in \mathcal{U} \setminus \mathcal{X}$ then the following two statements hold:*

- **(I)** *V is a multiplicity-one non-singular varifold and V restricted to $M \setminus
  B_x$ is C^1-close to X;*

- **(II)** *there exists an open set $B_{x,V} \subset B_x$ with the following two properties:
  (i) V restricted to $B_{x,V}$ is C^1-close to sL for some $s > 0$ and $L \in
  \{L_1, L_2, L_3\}$; **(ii)** V restricted to $B_x \setminus B_{x,V}$ is C^1-close to the tangent
  cone to X at x.*

Proof. By Allard's regularity theorem we can take a neighbourhood $\mathcal{U}$ of X
in $\mathcal{V}$ and a neighbourhood B_x of x such that if $V \in \mathcal{U} \setminus \mathcal{X}$ then V will satisfy
the latter part of (I) above.

Near x we can apply Theorem 3.7 so that as V approaches X we can
take its re-scaled limit W with positive energy; more precisely the proof of
Theorem 3.7 implies that we can take $y \in B_x$, $\delta > 0$ and a special Lagrangian
integral varifold W with no boundary in $\mathbb{C}^3$ satisfying the following two
properties:

- (a) $\mathcal{E}\big(V \llcorner (B_x \setminus B_\delta(y))\big) \leqslant \epsilon/2$ where $B_\delta(y)$ denotes a δ-ball about y in M
  and ϵ denotes a constant given by Theorem 3.3;

- (b) if we identify B_x with an open set in $\mathbb{C}^3$, translate V by $y \in \mathbb{C}^3$ and
  dilate it by δ^{-1} then the resulting varifold $\delta^{-1}(V - y)$ is close to W in
  the varifold topology.

From (b) and Theorem 4.1 it follows that W is a multiplicity-one non-
singular varifold represented by some $L \in \{L_1, L_2, L_3\}$ up to dilation and

translation in $\mathbb{R}^6 \cong T_x M$. This combined with Allard's regularity theorem implies (i) above.

We have not seen yet what happens to V in the annular region $B_x \setminus B_\delta(y)$ but in that region V has little energy as in (a) above. This combined with Theorem 3.3 implies (ii) above with $B_{x,V} = B_\delta(y)$.

Finally the former part of (I) follows from (i), (ii) and the latter part of (I). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In Theorem 5.1 above we have frequently used the expression 'C^1-close' for Lagrangian submanifolds, but in later sections we shall need to describe them in Weinstein neighbourhoods [25] (a class of tubular neighbourhoods of Lagrangian submanifolds). In what follows therefore we introduce some notation concerning Weinstein neighbourhoods and re-state Theorem 5.1 in that notation.

We take a linear isomorphism $\gamma : \mathbb{R}^6 \to T_x M$ with $\gamma^* g|_x = g_0$, $\gamma^* J|_x = J|_0$, $\gamma^* \Omega|_x = \Omega_0$ and $\gamma(C)$ a multiplicity 1 smooth tangent cone to X at x. We write $X' := X \setminus \{x\}$.

As in §2 we define $\psi : M \to (0, \infty)$ by (2.1). We have then $\omega_0 = \psi^2(x)\gamma^* \omega|_x$. By Darboux's theorem we can find a real number $\delta > 0$ and an embedding $\Gamma : B_\delta \to M$ with $\Gamma(0) = x$, $d\Gamma|_0 = \psi(x)\gamma$ and $\Gamma^* \omega|_x = \omega_0$.

We define T^*C, ω_C, U_C and Φ_C as in Lemma 4.4. Joyce [12, Theorem 4.4 and Lemma 4.5] proves that making $\delta > 0$ smaller if necessary we can take an embedding f_X of $C \cap B_\delta$ into X', a function $h_X : C \cap B_\delta \to \mathbb{R}$ and an $\alpha > 2$ with $h_X = O(r^\alpha)$ such that $f_X^{-1}(X') = \Phi_C(\mathrm{Graph}\, dh_X)$. We put $Z = X' \setminus f_X(C \cap \overline{B_\delta})$. It is clear that Z is an open subset of X' with boundary diffeomorphic to T^2.

We denote by T^*X' the cotangent bundle over X', by 0_C the zero-section of T^*X', and by ω_X the canonical symplectic form on T^*X'. Since f_X maps $C \cap B_\delta$ diffeomorphically onto $X' \setminus Z$ we get a vector-bundle isomorphism $T^*f_X : T^*(X' \setminus Z) \to T^*(C \cap B_\delta)$ covering $f_X^{-1} : X' \setminus Z \to C \cap B_\delta$. Joyce [12, Theorem 4.6] constructs a neighbourhood UX' of $\mathrm{Im}\, 0_X$ in T^*X' and a diffeomorphism Φ_X of UX' into M with $\Phi_X \circ 0_X = i_X$, $\Phi_X^* \omega = \omega_X$ and

$$\Phi_X|_{UX' \setminus T^*Z} = \Gamma \circ \Phi_C \circ (+dh_X) \circ T^*f_X$$

where $+dh_X$ denotes the fibrewise translation of T^*C by dh_X.

For each $y \in M$ we denote by $P|_y$ the set of all linear isomorphisms $\phi : \mathbb{R}^6 \to T_y M$ with $\phi^* g|_y = g_0$, $\phi^* J|_y = J_0$ and $\phi^* \Omega|_y = \Omega_0$. We put $P = \bigcup_{y \in M} P|_y$. It is clear that P is a principal bundle over M with structure group SU_3. We have $(x, \gamma) \in P$. By a result of Joyce [13, Theorem 5.2] we can take a neighbourhood $U_{x,\gamma}$ of (x, γ) in P such that for all $p = (y, \phi) \in U_{x,\gamma}$

we can construct embeddings Γ_p of B_δ into M depending smoothly on p with $\Gamma_p(0) = y$, $d\Gamma_p|_0 = \phi$ and $\psi^2(y)\Gamma_p^*\omega = \omega_0$, and embeddings Φ_p of UX' into M depending smoothly on p with $\Phi_p \circ 0_X = i_X$, $\Phi_p^*\omega = \omega_X$ and

$$\Phi_p|_{UX'\setminus T^*Z} = \Gamma_p \circ \Phi_C \circ (+dh_X) \circ T^*f_X.$$

We turn now to $L \in \{L_1, L_2, L_3\}$. We take an open subset K of L with boundary diffeomorphic to T^2, a real number $R > 0$ and a diffeomorphism f_L of $C \setminus \overline{B_R}$ onto $L \setminus \overline{K}$ with $f_L - i_C = O(r^{-1})$. Making $R > 0$ larger if necessary we can find a 1-form η_L on $C \setminus \overline{B_R}$ with $\Phi_C(\mathrm{Graph}\,\eta_L) = L \setminus K$.

We denote by T^*L the cotangent bundle over L, by 0_L the zero-section of T^*L, and by ω_L the canonical symplectic form on T^*L. Since f_L maps $C \setminus \overline{B_R}$ diffeomorphically onto $L \setminus K$ we get a vector-bundle isomorphism $T^*f_L : T^*(L \setminus K) \to T^*(C \setminus \overline{B_R})$ covering $f_L^{-1} : L \setminus K \to C \cap \overline{B_R}$. Joyce [12, Theorem 7.5] constructs a neighbourhood UL of $\mathrm{Im}\,0_L$ in T^*L and a diffeomorphism Φ_L of UL into $\mathbb{R}^6$ with $\Phi_L^*\omega_0 = \omega_L$, $\Phi_L \circ 0_L = i_L$ and

$$(5.1) \qquad\qquad \Phi_L|_{UL\setminus T^*K} = \Phi_C \circ (+\eta_L) \circ T^*f_L$$

where $+\eta_L$ denotes the fibrewise translation of T^*C by η_L.

We are ready now to refine the statement of Theorem 5.1:

Theorem 5.2. *For each $\epsilon > 0$ there exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that if $V \in \mathcal{U} \setminus \mathcal{X}$ then V will be a multiplicity-one non-singular varifold in M and we can find some $p \in P$, $s > 0$, $L \in \{L_1, L_2, L_3\}$, a closed 1-form β_C on $C \cap A_{sR,\delta}$ with $|\beta_C|_{[sR,\delta]}^{1,\mathrm{cyl}} < \epsilon$, a closed 1-form β_L on $\widehat{K}$ with $|\beta_L|_{C^1(\widehat{K})} < \epsilon$, and a closed 1-form β_X on $\widehat{Z}$ with $|\beta_X|_{C^1(\widehat{Z})} < \epsilon$ such that on $f_L^{-1}(\widehat{K}) \cap A_{sR,\delta}$ we have $s^2(f_L^*\beta_L + \eta_L) = \beta_C$, on $f_X^{-1}(\widehat{Z}) \cap A_{sR,\delta}$ we have $f_X^*\beta_X + dh_X = \beta_C$, and*

$$(5.2) \quad V = \Gamma_p\big(s\Phi_L(\mathrm{Graph}\,\beta_L)\big) \cup \Gamma_p \circ \Phi_C(\mathrm{Graph}\,\beta_C) \cup \Phi_X(\mathrm{Graph}\,\beta_X).$$

Remark. Here and subsequently if V is a multiplicity-one varifold in M then we shall treat V as a subset of M to simplify the notation as in (5.2).

The right-hand side of (5.2) defines a submanifold of M because of the two conditions $s^2(f_L^*\beta_L + \eta_L) = \beta_C$ on $f_L^{-1}(K) \cap A_{sR,\delta}$ and $f_X^*\beta_X + dh_X = \beta_C$ on $f_X^{-1}(Z) \cap A_{sR,\delta}$.

6. Remarks on Joyce's work

This section will be devoted to several remarks on Joyce's work [12–16].

We begin in §6.1 by recalling Joyce's topological condition [16, Theorem 10.4] which is necessary for his gluing construction and also for the precise statement of Theorems 1.4 and 1.5, the main theorems of the present paper.

In §6.2 we prove Theorem 1.4, which claims that if there is no $L \in \{L_1, L_2, L_3\}$ satisfying Joyce's topological condition then X is *unsmoothable*, which is a corollary to Theorem 5.2.

On the other hand Theorem 1.5 supposes that there is some $L \in \{L_1, L_2, L_3\}$ satisfying Joyce' topological condition and claims that there exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that the elements of $\mathcal{U} \setminus \mathcal{X}$ may be obtained by Joyce's gluing construction. Its proof will be given in §7 below after the preparation in §§6.3–6.5.

In §6.3 we prove that the varifold topology on $\mathcal{X}$ induced from $\mathcal{V}$ is equal to a stronger topology defined by Joyce [13, Definition 5.6]. This result may be of independent interest as an improvement of Joyce's work in the second paper [13].

In §6.4 we recall from Joyce's second paper [13, Corollary 6.11] that $\mathcal{X}$ is a *manifold* of finite dimension, which is a consequence of the fact that the T^2-cone C is *stable* in the sense of Joyce [13, §3.2]. Our presentation will be slightly different from that of Joyce, but it is superficial and we shall only re-phrase Joyce's statement [13, Theorem 6.10] so that we may use it directly in §7.

In §6.5 we give an explicit description of the gluing map $G : [0, \tau) \times \mathcal{Y} \to \mathcal{V}$ given in §1.

6.1. Joyce's topological condition

Our main results concern the topological condition given by Joyce [16, Theorem 10.4] which we therefore recall now. Let $f_X : C \cap B_\delta \to X'$ be as in §5 above, which induces a linear map

(6.1) $$f_X^* : H^1(X'; \mathbb{R}) \to H^1(C \cap B_\delta; \mathbb{R}) \cong H^1(T^2; \mathbb{R})$$

between cohomology groups. Its image will be denoted by $\operatorname{Im} f_X^*$, which is a linear subspace of $H^1(T^2; \mathbb{R})$. Now we use:

Lemma (Joyce [16, Lemma 10.1]). Let $X^\dagger$ be a compact orientable 3-dimensional manifold with boundary, and consider the natural restriction map $r^\dagger : H^1(X^\dagger; \mathbb{R}) \to H^1(\partial X^\dagger; \mathbb{R})$ and its image $\operatorname{Im} r^\dagger$ in $H^1(\partial X^\dagger; \mathbb{R})$. Then

we have

$$\dim_{\mathbb{R}} \operatorname{Im} r^{\dagger} = \tfrac{1}{2} \dim_{\mathbb{R}} H^1(\partial X^{\dagger}; \mathbb{R})$$

where $\dim_{\mathbb{R}} H^1(\partial X^{\dagger}; \mathbb{R})$ is always an even integer as $\partial X^{\dagger}$ is a compact orientable 2-dimensional manifold.

In our case X' may be retracted to some $X^{\dagger}$ with $\partial X^{\dagger} \cong T^2$ in the notation above, and so

Corollary. $\dim_{\mathbb{R}} \operatorname{Im} f_X^* = \tfrac{1}{2} \dim_{\mathbb{R}} H^1(T^2; \mathbb{R}) = 1$.

On the other hand let $L \in \{L_1, L_2, L_3\}$ and define a closed 1-form η_L as in §5.1. Then we can define its de Rham cohomology class

$$(6.2) \qquad Y(L) := [\eta_L] \in H^1(C \setminus \overline{B_R}; \mathbb{R}) \cong H^1(T^2; \mathbb{R})$$

which is compatible with the notation of Joyce [16, Definition 6.2 (see also Theorem 6.6)]. Joyce [16, Equation (77)] proves indeed:

Lemma 6.1. $Y(L) \neq 0$ for each $L \in \{L_1, L_2, L_3\}$ and moreover any two of $\{Y(L_1), Y(L_2), Y(L_3)\}$ are *linearly independent* as vectors in $H^1(T^2; \mathbb{R})$.

Now let $\langle Y(L) \rangle$ denote the 1-dimensional linear subspace of $H^1(T^2; \mathbb{R})$ generated by $Y(L)$. Then Joyce's topological condition [16, Theorem 10.4 (see also Theorem 7.3)] is equivalent to:

Condition 6.2. $\langle Y(L) \rangle = \operatorname{Im} f_X^* \subset H^1(T^2; \mathbb{R})$.

Thus there are four lines $\langle Y(L_1) \rangle, \langle Y(L_2) \rangle, \langle Y(L_3) \rangle$ and $\operatorname{Im} f_X^*$ in the plane $H^1(T^2; \mathbb{R}) \cong \mathbb{R}^2$, all passing thorough the origin $0 \in H^1(T^2; \mathbb{R})$. The three lines $\langle Y(L_1) \rangle, \langle Y(L_2) \rangle, \langle Y(L_3) \rangle$ are all distinct by Lemma 6.1, and Condition 6.2 is equivalent to the fourth line $\operatorname{Im} f_X^*$ overlapping one of those three lines.

It is now clear that there exists *at most* one $L \in \{L_1, L_2, L_3\}$ satisfying Condition 6.2, which we have mentioned in Lemma 1.3.

6.2. Proof of Theorem 1.4

We are ready now to prove Theorem 1.4 as a corollary to Theorem 5.2:

Corollary 6.3. *Suppose that for every neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ we have $\mathcal{U} \setminus \mathcal{X} \neq \emptyset$. Then there exists $L \in \{L_1, L_2, L_3\}$ with $\langle Y(L) \rangle = \operatorname{Im} f_X^*$.*

Remark. This statement is the contraposition of Theorem 1.4 and so they are equivalent.

Proof of Corollary 6.3. Theorem 5.2 and the hypothesis above imply that for each $\epsilon > 0$ there exist $L \in \{L_1, L_2, L_3\}$, a real number $s > 0$, a closed 1-form β_L with $|\beta_L|_{C^1(\widehat{K})} < \epsilon$, and a closed 1-form β_X on $\widehat{K}, \widehat{X}$ respectively such that

$$s^2[f_L^*\beta_L + \eta_L] = [f_X^*\beta_X + dh_X] \in H^1(T^2; \mathbb{R})$$

and so $[f_L^*\beta_L] + Y(L) \in \operatorname{Im} f_X^*$. Here L depends on ϵ but letting $\epsilon = 1/n$, $n = 1, 2, 3, \ldots$, and taking a subsequence we can make L independent of n and satisfying the following property: for infinitely many n there exist 1-forms $\beta_{L,n}$ on $\widehat{K}$ with $|\beta_{L,n}|_{C^1(\widehat{K})} < 1/n$ and $[f_L^*\beta_{L,n}] + Y(L) \in \operatorname{Im} f_X^*$. Consequently letting $n \to \infty$ we get $Y(L) \in \operatorname{Im} f_X^*$.

On the other hand $\langle Y(L) \rangle$ and $\operatorname{Im} f_X^*$ are both 1-dimensional as in §6.1, and so $\langle Y(L) \rangle = \operatorname{Im} f_X^*$, which completes the proof of Corollary 6.3. □

We can also strengthen Theorem 5.2 as follows:

Corollary 6.4. *If there exists $L \in \{L_1, L_2, L_3\}$ with $\langle Y(L) \rangle = \operatorname{Im} f_X^*$ then such an L is unique as proven in §6.1 and the statement of Theorem 5.2 holds for that unique L instead of assigning L_1, L_2 or L_3 to each $V \in \mathcal{U} \setminus \mathcal{X}$.*

Proof. In a way similar to the proof of Corollary 6.3 it follows that in the statement of Theorem 5.2 we may suppose that $\langle Y(L) \rangle$ is ϵ-close to $\operatorname{Im} f_X^*$ but such an L is unique as in §6.1, which completes the proof of Corollary 6.4. □

6.3. Topology on $\mathcal{X}$

We have used so far the varifold topology induced from $\mathcal{V}$ but in what follows we prove that it is actually equal to a stronger topology given by Joyce [13, Definition 5.6]. We begin with:

Lemma 6.5. *There exists a neighbourhood $\mathcal{Y}$ of X in $\mathcal{X}$ such that for each $Y \in \mathcal{Y}$ there exist unique $p(Y) = (y, \phi) \in P$ and a closed 1-form η_Y on X' with*

$$Y = \Phi_{p(Y)}(\operatorname{Graph} \eta_Y)$$

where $P, \Phi_{p(Y)}$ are as in §5; moreover near the singular point y in Y we may write $\eta_Y = dh_Y$ for some unique function $h_Y : X' \to \mathbb{R}$ decaying with any rate < 3.

Remark 6.6. If X were not modelled on the T^2-cone C then the rate of h_Y would be confined as in the definition of Joyce [13, Definition 3.7].

Proof of Lemma 6.5. By Allard's regularity theorem we may suppose that Y is C^1-close to X outside a neighbourhood of x in X. By Theorem 3.6 we may also suppose that the singular point y of Y is close to x and the tangent cone to Y at y is close to that to X at x. These two facts readily imply the existence of $p(Y) = (y, \phi), \eta_Y$ and the decay property of η_Y, as proven by Joyce [13, Theorem 5.3]. The uniqueness of $p(Y) = (y, \phi)$ follows from the definition of P, and the uniqueness of η_Y follows from the fact that $\Phi_{p(Y)}^{-1}(Y \setminus \{y\})$ is C^1-close to $X' := X \setminus \{x\}$, which completes the proof of Lemma 6.5. $\qquad\square$

Joyce [13, Definition 5.6] defines a topology on $\mathcal{X}$ dependent on $\mu \in (2, 3)$ in which $Y \in \mathcal{X}$ is close to X if $p(Y) = (y, \phi)$ is close to (x, γ) in P and η_Y is small in the weighted C^1-space $C_\mu^1(T^*X')$ where $p(Y) = (y, \phi)$ and η_Y are as in Lemma 6.5 above. We prove:

Theorem 6.7. *Joyce's topology is independent of μ and moreover equal to the varifold topology induced from $\mathcal{V}$.*

Proof. It is clear that if $X, Y \in \mathcal{X}$ are close in the μ-topology then so they are in the varifold topology. Conversely let $X, Y \in \mathcal{X}$ be close in the varifold topology. Allard's regularity theorem implies that Y is C^1-close to X outside a neighbourhood of x in X where x denotes the singular point of X. We may also write Y as in Lemma 6.5 and Theorem 3.6 implies that $p(Y) = (y, \phi)$ is close to (x, γ) in P. The last part of Lemma 6.5 implies that Y approaches X at x with any rate $\mu < 3$, which completes the proof of Theorem 6.7. $\quad\square$

Theorem 6.7 readily extends to compact special Lagrangian m-folds of X with isolated conical singularities in the sense of Joyce [12–16] where m is an arbitrary integer > 2 and the tangent cones to X need not be modelled on the stable T^2-cone C.

6.4. Consequence of stability of C

In §7 we use the fact that X is modelled on the *stable* cone C in the sense of Joyce [13, §3.2] who proves indeed (in Corollary 6.11 in the same paper) that a neighbourhood of X in $\mathcal{X}$ is a *manifold* of finite dimension and its tangent space $T_X\mathcal{X}$ is isomorphic to the compactly-supported de Rham cohomology group $H_c^1(X'; \mathbb{R}) \subset H^1(X'; \mathbb{R})$.

Here $H_c^1(X'; \mathbb{R})$ is embedded in $H^1(X'; \mathbb{R})$ because the map f_X^* : $H^0(X'; \mathbb{R}) \to H^0(T^2; \mathbb{R})$ is surjective. It is unnecessary for our purpose but may be worth remarking that Joyce [13] deals with a more general case where X has two or more singular points, and then $T_X \mathcal{X}$ is isomorphic to the image of the canonical map $H_c^1(X'; \mathbb{R}) \to H^1(X; \mathbb{R})$ which need not be injective.

We shall need in §7 a more detailed statement of Joyce's result above, which we therefore recall now. We take a neighbourhood $\mathcal{Y}$ of X in $\mathcal{X}$ and define a map $H : \mathcal{Y} \to H_c^1(X'; \mathbb{R})$ as follows. Let $\mathcal{Y}$ be as in Lemma 6.5 so that for each $Y \in \mathcal{Y}$ there exist unique $p(Y) = (y, \phi) \in P$ and a closed 1-form η_Y on X' with

$$Y = \Phi_{p(Y)}(\mathrm{Graph}\, \eta_Y).$$

Consider the de Rham cohomology class $[\eta_Y] \in H^1(X'; \mathbb{R})$, which maps to 0 under f_X^* because of the last property in Lemma 6.5. Consequently $[\eta_Y]$ lies in $H_c^1(X'; \mathbb{R})$, and we set $H(Y) = [\eta_Y]$.

With the notation above we can state the detailed version of Joyce's result [13, Theorem 6.10]:

Theorem 6.8. H maps a neighbourhood of X in $\mathcal{X}$ *homeomorphically* onto a neighbourhood of 0 in $H_c^1(X'; \mathbb{R})$.

6.5. Explicit description of gluing map

Finally we give an explicit description of $G : [0, \tau) \times \mathcal{Y} \to \mathcal{V}$. The basic notation is already given in §5 but we shall need some more notation.

We denote by η_L' the harmonic 1-form on T^2 with $[\eta_L'] = Y(L) \in H^1(T^2; \mathbb{R})$ and define a 1-form $\widetilde{\eta}_L$ on C by setting $\widetilde{\eta}_L := \pi^* \eta_L'$ where π denotes the projection of $C \cong (0, \infty) \times T^2$ onto T^2.

Since $\langle Y(L) \rangle = \mathrm{Im}\, f_X^*$ it is clear that there exists a closed 1-form ξ_L on X' with $f_X^*[\xi_L] = Y(L) \in H^1(T^2; \mathbb{R})$. Following the proof of Joyce (sketched in [16, Theorem 7.3] and completed in [15, Theorem 6.12]) we find that for a suitable choice of ξ_L the following statement holds:

Theorem 6.9. *For all $(t, Y) \in (0, \tau) \times \mathcal{Y}$ there exist three smooth functions $u_L : K \to \mathbb{R}$, $u_C : C \cap A_{tR, \delta} \to \mathbb{R}$, $u_X : Z \to \mathbb{R}$ such that letting $p(Y), \eta_Y$ be as in Lemma 6.5 we can construct a compact special Lagrangian submanifold*

of (M, ω, J, Ω) of the form

$$G(t, Y) := \Gamma_{p(Y)}\big(t\Phi_L(\text{Graph } du_L) \cup \Phi_C(\text{Graph } t^2\widetilde{\eta}_L + du_C)\big)$$
$$\cup \, \Phi_X(\text{Graph } t^2\xi_L + \eta_Y + du_X).$$

Moreover $|u_L|_{C^1(K)}$, $|u_C|_{[tR,\delta]}^{1,\text{cyl}}$, $|u_X|_{C^1(Z)}$ tend to 0 uniformly in Y as $t \to +0$.

7. Proof of Theorem 1.5

As in §6.5 above we suppose that there is some $L \in \{L_1, L_2, L_3\}$ satisfying Joyce's topological condition and define the gluing map $G : [0, \tau) \times \mathcal{Y} \to \mathcal{V}$. Theorem 1.5 claims that for τ and Y small enough G is a *homeomorphism* onto a neighbourhood of X in $\mathcal{V}$.

To prove it we shall construct an inverse map to G. We shall indeed take a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$, then define a map $F_1 : \mathcal{U} \to [0, \infty)$, then define a map $F_2 : \mathcal{U} \to \mathcal{X}$ and then set $F = (F_1, F_2)$, to get a map $F : \mathcal{U} \to [0, \infty) \times \mathcal{X}$ inverse to G.

In Donaldson's situation [5] (explained also by Freed and Uhlenbeck [7]) in Yang–Mills gauge theory there is a map similar to $F : \mathcal{U} \to [0, \infty) \times \mathcal{X}$ and the elements of $\mathcal{U}$ play the rôle of 'concentrated' instantons. There is a map similar to $F_2 : \mathcal{U} \to \mathcal{X}$ which assigns the 'centres' about which those instantons are concentrated, and there is also a map similar to $F_1 : \mathcal{U} \to [0, \infty)$ which measures how concentrated they are.

We return now to our situation. By Allard's regularity theorem there exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that for each $V \in \mathcal{U}$ we can take a unique closed 1-form β_X on Z such that

$$(7.1) \qquad\qquad V \cap \Phi_X(U X' \cap T^*Z) = \Phi_X(\text{Graph } \beta_X)$$

where by Theorem 5.2 we may suppose that V is a *multiplicity-one* varifold (with singularity at most one point) and so we may treat V as a subset of M.

Consider the de Rham cohomology class $[\beta_X] \in H^1(Z; \mathbb{R}) \cong H^1(X'; \mathbb{R})$. Then by Condition 6.2 we may write $f_X^*[\beta_X] = rY(L)$ for some unique $r \in \mathbb{R}$.

We claim that $r \geqslant 0$. If $V \in \mathcal{U} \cap \mathcal{X}$ then near x we may write V as the graph of an *exact* 1-form as in Lemma 6.5 and so $f_X^*[\beta_X] = 0$ in the notation above, which implies $r = 0$.

If $V \in \mathcal{U} \setminus \mathcal{X}$ then by Corollary 6.4 there exist a real number $s > 0$, a closed 1-form β_X on Z, and a closed 1-form β_L on K such that $[f_X^*\beta_X] = s^2\big([f_L^*\beta_L] + Y(L)\big) \in H^1(T^2; \mathbb{R})$. Hence recalling that $\text{Im } f_X^* = \langle Y(L) \rangle$ we

find $[f_L^* \beta_L] = cY(L)$ for some $c \in \mathbb{R}$. Making $\mathcal{U}$ smaller if necessary we may suppose that β_L is so small that $|c| < \frac{1}{100}$ and then $r = s^2(1 + c) > 0$ as we want.

Thus we have defined a map $F_1 : \mathcal{U} \to [0, \infty)$ and we turn now to the definition of $F_2 : \mathcal{U} \to \mathcal{X}$.

In Theorem 6.9 we have taken a closed 1-form ξ_L on X' with $f_X^*[\xi_L] = Y(L)$ and so $[\beta_X - r\xi_L] \in \mathrm{Ker}\, f_X^* = H_c^1(X'; \mathbb{R})$. We may suppose that $[\beta_X - r\xi_L]$ is so small that by Theorem 6.8 we can define $F_2(V) := H^{-1}[\beta_X - r\xi_L] \in \mathcal{Y}$.

Thus we have defined a map $F = (F_1, F_2) : \mathcal{U} \to [0, \infty) \times \mathcal{X}$, which readily satisfies the following two properties:

(i) $F(Y) = (0, Y)$ for $Y \in \mathcal{U} \cap \mathcal{X}$;

(ii) $F \circ G$ is the identity map of a neighbourhood of $(0, X)$ in $[0, \infty) \times \mathcal{X}$.

We can also prove:

Lemma 7.1. There exists a neighbourhood $\mathcal{U}$ of X in $\mathcal{V}$ such that if $V \in \mathcal{U} \setminus \mathcal{X}$ then V may be written as the graph of an *exact* 1-form α on $G \circ F(V)$ in a Weinstein neighbourhood of $G \circ F(V)$ in the symplectic manifold (M, ω).

Remark 7.2. The definitions of F and G readily imply that V and $G \circ F(V)$ are C^1-*close* to each other. On the other hand V and $G \circ F(V)$ are both *Lagrangian* submanifolds of (M, ω). Consequently V may be written as the graph of a *closed* 1-form α on $G \circ F(V)$. Thus the substantial part of Lemma 7.1 is the exactness of the 1-form α.

Proof of Lemma 7.1. We proceed in two steps: the first step introduces some notation to define α; and the second step proves that α is exact.

First Step. Let $\mathcal{U}$ be a neighbourhood of X in $\mathcal{V}$, let $V \in \mathcal{U} \setminus \mathcal{X}$ and write $F(V) = (t, Y) \in (0, \infty) \times \mathcal{X}$. Making $\mathcal{U}$ smaller if necessary we may apply Lemma 6.5 to Y and in particular we can define $p := p(Y) \in P$ as in Lemma 6.5.

Making $\mathcal{U}$ smaller if necessary we may also apply Corollary 6.4 to V. It is also easy to see that (5.2) holds with t in place of s and with p as above; i.e. there exist a closed 1-form β_X on Z, a closed 1-form β_C on $C \cap A_{tR, \delta}$, and a closed 1-form β_L on K such that

$$(7.2) \qquad V = \Gamma_p\big(t\Phi_L(\mathrm{Graph}\,\beta_L)\big) \cup t\Phi_C(\mathrm{Graph}\,\beta_C)\big) \cup \Phi_X(\mathrm{Graph}\,\beta_X).$$

We take now a Weinstein neighbourhood of V in (M, ω) as follows. Define a map $f_{LV} : K \to V$ by setting

$$f_{LV} := \Gamma_p \circ t\Phi_L \circ \beta_L$$

where β_L is regarded as a map K into $UL \cap T^*K$. Likewise we can define two maps $f_{CV} : C \cap A_{tR,\delta} \to V$ and $f_{XV} : Z \to V$ by setting

$$f_{CV} := \Gamma_p \circ t\Phi_C \circ \beta_C, \; f_{XV} := \Phi_X \circ \beta_X$$

respectively. Differentiating $f_{LV} : K \to V$ we get a vector-bundle homomorphism $f_{LV!} : T^*K \to T^*V$ covering $f_{LV} : K \to V$. Likewise we can also define the two maps $f_{CV!}$ and $f_{XV!}$. Let UV be a neighbourhood of the zero-section in T^*V such that we can define a map $\Phi_V : UV \to M$ by

$$\Phi_V \circ f_{LV!} = \Gamma_p \circ t\Phi_L \circ (+\beta_L), \quad \Phi_V \circ f_{CV!} = \Gamma_p \circ t\Phi_C \circ (+\beta_C)$$
$$\text{and} \quad \Phi_V \circ f_{XV!} = \Phi_L \circ (+\beta_X)$$

which defines a Weinstein neighbourhood of V in (M, ω).

By Theorem 6.9 we may write $G \circ F(V) = \Phi_V(\operatorname{Graph} \alpha)$ for some unique closed 1-form α on V.

Second Step. It remains to prove that $[\alpha] = 0$ in the de Rham cohomology group $H^1(V; \mathbb{R})$. The co-efficient field $\mathbb{R}$ will be omitted in what follows for the sake of brevity. We may topologically write V as a union of K and Z the intersection of which is diffeomorphic to $T^2 \times \mathbb{R}$. We consider the associated Mayer–Vietoris exact sequence. The map $H^0(K) \oplus H^0(Z) \to H^0(T^2)$ is clearly surjective, and so the map $H^1(V) \to H^1(K) \oplus H^1(Z)$ is injective.

It suffices therefore to prove that $[\alpha] \in H^1(V)$ maps to $0 \in H^1(K) \oplus H^1(Z)$. We begin by considering its image in $H^1(Z)$. By the definition of F we have $[\beta_X] = t^2 Y(L) + [\eta_{F_2(V)}] \in H^1(Z)$ where $\eta_{F_2(V)}$ is as in Lemma 6.5 with $F_2(V)$ in place of Y. On the other hand by the construction of $\Phi_V : UV \to M$ we have

$$\beta_X + f_{XV}^* \alpha = t^2 \xi_L + \eta_{F_2(V)} + du_L \text{ on } Z$$

in the notation of Theorem 6.9. Consequently $f_{XV}^*[\alpha] = 0 \in H^1(Z)$ as we want.

We also prove that $[\alpha]$ maps to 0 under the map $H^1(V) \to H^1(K)$, which is induced by f_{LV}. We can naturally compactify K into a manifold with

boundary diffeomorphic to T^2, which induces a map $H^1(K) \to H^1(T^2)$. Its kernel is isomorphic to the compactly-supported cohomology group $H^1_c(T^2)$. Since K is diffeomorphic to $S^1 \times \mathbb{R}^2$ it follows that $H^1_c(S^1 \times \mathbb{R}^2) \cong H^2(S^1 \times \mathbb{R}^2) = \{0\}$ and so the map $H^1(K) \to H^1(T^2)$ is injective.

It suffices therefore to prove that $[\alpha] \in H^1(V)$ maps to $0 \in H^1(T^2)$ under the composite map $H^1(V) \to H^1(K) \to H^1(T^2)$. Recall that it is contained in the Mayer–Vietoris sequence

$$H^1(V) \to H^1(K) \oplus H^1(Z) \to H^1(T^2)$$

and in particular that the two composite maps $H^1(V) \to H^1(K) \to H^1(T^2)$ and $H^1(V) \to H^1(Z) \to H^1(T^2)$ are equal. On the other hand we have already proved that $[\alpha] \in H^1(V)$ maps to $0 \in H^1(Z)$ and so to $0 \in H^1(T^2)$ as we want, which completes the proof of Lemma 7.1. $\qquad\square$

As a corollary to Lemma 7.1 we can prove:

Corollary 7.3. *$G \circ F$ is the identity map of a neighbourhood of X in $\mathcal{V}$.*

Proof. It suffices to prove that $V = G \circ F(V)$ in the situation of Lemma 7.1. Write the exact 1-form α as df for some smooth function $f : G \circ F(V) \to \mathbb{R}$. Since V and $G \circ F(V)$ are special Lagrangian it follows that f satisfies Hopf's maximum principle [10] and so f is constant as V is compact. Consequently $df = 0$ and so $V = G \circ F(V)$ as we want. $\qquad\square$

Theorem 1.5 is now an immediate consequence of Corollary 7.3 and the property (ii) stated before Lemma 7.1.

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