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G_2–instantons, associative submanifolds and Fueter sections

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We give sufficient conditions for a family of G_2–instantons to be "spontaneously" be born out of a Fueter section of a bundle of moduli spaces of ASD instantons over an associative submanifold. This phenomenon is one of the key difficulties in defining the conjectural G_2 *Casson invariant* proposed by Donaldson–Thomas in [DT98].

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1. Introduction

Fix a compact 7–manifold Y together with a positive 3–form ϕ satisfying a certain non-linear partial differential equation; see (2.3) and the discussion preceding it. The 3–form ϕ canonically equips Y with a metric (and orientation) such that the holonomy group $\mathrm{Hol}(g)$ is contained in the exceptional Lie group G_2; hence, (Y, ϕ) is commonly called a G_2–manifold and ϕ is called a torsion-free G_2–structure.

Given a G–bundle E over Y, Donaldson and Thomas [DT98] noted that there is a Chern–Simons type functional on $\mathscr{B}(E)$, the space of gauge equivalence classes of connections, whose critical points $[A]$ satisfy

$$*\,(F_A \wedge \phi) = -F_A. \tag{1.1}$$

Solutions of (1.1) are called G_2–instantons. These are the central objects in gauge theory on G_2–manifolds. The moduli space of G_2–instantons

$$\mathscr{M}(E, \phi) := \{[A] \in \mathscr{B}(E) : *(F_A \wedge \phi) = -F_A\}$$

can, in general, be a very complicated space. However, after gauge fixing, (1.1) has an elliptic deformation theory of index zero, i.e., $\mathscr{M}(E, \phi)$ has virtual dimension zero. Thus one can try to "count" $\mathscr{M}(E, \phi)$, say, by a suitable perturbation scheme or via virtual cycle techniques and arrive at a number

$$n(E, \phi) := \#\mathscr{M}(E, \phi).$$

How does $n(E, \phi)$ depend on ϕ? Since the deformation theory of G_2–instantons is very well-behaved, the key question one needs to understand is: how can G_2–instantons degenerate as ϕ varies? Consider a family of G_2–instantons $(A_t)_{t\in(0,T]}$ over a family of G_2–manifolds $(Y, \phi_t)_{t\in(0,T]}$ and assume that ϕ_t converges to a torsion-free G_2–structure ϕ_0 as $t \to 0$. From classical results due to Uhlenbeck [Uhl82a], Price [Pri83] and Nakajima [Nak88] and more recent progress by Tian [Tia00] and Tao and Tian [TT04] one can conclude the following:

- There is a closed subset P of Y of finite 3–dimensional Hausdorff measure and a G_2–instanton B over $(Y \setminus P, \phi_0)$ such that up to gauge transformations a subsequence of (A_t) converges to B in C^∞_{loc} on $Y \setminus P$ as $t \to 0$.
- B can be extended to the complement of a closed set $\mathrm{sing}(B)$ of vanishing 3–dimensional Hausdorff measure. However, $\mathrm{sing}(B)$ might very

well be non-empty, that is: one might encounter non-removable singularities.

- P supports an integral current calibrated by ϕ_0, or more informally: P is a, possibly wildly singular, associative submanifold in (Y, ϕ_0). At almost every point $x \in P$, the degeneration of (A_t) is modelled on (a bubbling tree of) ASD instantons bubbling off in the direction transverse to P.

This, of course, represents the worse case scenario. One would expect that a generic deformation is less wild. In this article we only consider the case when B extends to all of Y and P is smooth. Moreover, we form a bundle $\mathfrak{M}$ over P whose fibres are moduli spaces of ASD instantons, as explained in Section 4, and assume that the ASD instantons bubbling off transverse to P give rise to a section $\mathfrak{I} \in \Gamma(\mathfrak{M})$. Since the ASD instantons bubbling off do not have a canonical scale, $\mathfrak{I}$ is unique only up to the action of $C^\infty(P, \mathbf{R}_{>0})$. Donaldson and Segal [DS11] noticed that $\mathfrak{I}$ cannot be arbitrary but should satisfy a non-linear p.d.e. called the Fueter equation, provided scalings are chosen appropriately; see Section 4 for more details. This equation is elliptic of index zero; however, since $\mathfrak{M}$ is a bundle of cones, one only expects solutions to appear only at isolated values in 1–parameter families. In particular, for a generic torsion-free G_2–structure ϕ, no Fueter section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ ought to exists. Hence, the bubbling phenomenon for G_2–instantons should only occur in codimension one.

The main result of this article is to prove that given the data $(B, P, \mathfrak{I})$ and assuming certain "acyclicity/unobstructedness conditions" (which are expounded in Definitions 2.9, 2.16 and 4.9), we can produce a family of G_2–instantons yielding $(B, P, \mathfrak{I})$ in the limit.

Theorem 1.1. *Let Y be a compact 7–manifold equipped with a family of torsion-free G_2–structures $(\phi_t)_{t\in(-T,T)}$. Suppose we are given:*

- *an acyclic G_2–instanton B on a G–bundle E_0 over (Y, ϕ_t),*
- *an unobstructed associative submanifold P in (Y, ϕ_0) and*
- *a Fueter section $\mathfrak{I}$ of an instanton moduli bundle $\mathfrak{M}$ over P associated $E_0|_P$ which is unobstructed with respect to (ϕ_t).*

Then there is a constant $\Lambda > 0$, a G–bundle E together with a family of connections $(A_\lambda)_{\lambda\in(0,\Lambda]}$ and a continuous function $t : [0, \Lambda] \to (-T, T)$ with $t(0) = 0$ such that:

- *A_λ is a G_2–instanton on E over $(Y, \phi_{t(\lambda)})$ for all $\lambda \in (0, \Lambda]$.*
- *A_λ converges to B on the complement of P and at each point $x \in P$ an ASD instanton in the equivalence class given by $\mathfrak{I}(x)$ bubbles off transversely as $\lambda \to 0$.*

As was already pointed out by Donaldson and Segal [DS11], an immediate consequence is that $n(E, \phi)$ has no reason to be invariant under (large) deformations of ϕ. They suggest that one should try to construct a counter term, say $m(E, \phi)$, as a weighted count of associative submanifolds and G_2–instantons on bundles of "smaller" topological type than E, so that the sum $n(E, \phi) + m(E, \phi)$ is invariant under deformations. The crucial point is to find out what these weights should be. A candidate for the definition of these weights in the "low energy" SU(2)–theory, which was hinted at by Donaldson–Segal [DS11], is explained in more detail in the author's PhD thesis [Wal13a, Chapter 6]. A more systematic approach, based on generalised Seiberg–Witten equations and the ADHM construction, is currently being developed by Haydys and the author; see [HW14] for a first step.

Remark 1.2. It would be interesting to see a concrete example of the input required by Theorem 1.1. Unfortunately, no such example is known currently. The main difficulty with constructing such examples is to ensure that $\mathfrak{I}$ is unobstructed with respect to (ϕ_t). It should be pointed out, however, that a construction closely related to Theorem 1.1 has been used by the author to construct Spin(7)–instantons from a Fueter section of a bundle of moduli space of ASD instantons over a Cayley submanifold [Wal14].

Remark 1.3. The proof of Theorem 1.1 is based on a gluing construction and the analysis involved is an extension of that required for the construction of G_2–instantons on generalised Kummer constructions in [Wal13b]. As such there are some similarities with Lewis' construction of Spin(7)–instantons [Lew98], unpublished work by Brendle on Spin(7)–instantons [Bre03] and Pacard–Ritoré's work on the Allen–Cahn equation [PR03].

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2. Review of geometry on G_2–manifolds

We begin with a terse review of the basic notions of G_2–geometry. This is mainly to fix notation and conventions, and also recall a few results which we will make use of later. The reader who is interested in a more detailed exposition is referred to Joyce's book [Joy00], which is the standard reference for most of the material in this section.

Definition 2.1. A 3–form ϕ on a 7–dimensional vector space is called *positive* if for each non-zero vector $v \in V$ the 2–form $i(v)\phi$ on $V/\langle v\rangle$ is symplectic.

Example 2.2. The 3–form $\phi_0 \in \Omega^3(\mathbf{R}^7)$ defined by

$$\phi_0 := \mathrm{d}x^{123} - \mathrm{d}x^{145} - \mathrm{d}x^{167} - \mathrm{d}x^{246} + \mathrm{d}x^{257} - \mathrm{d}x^{347} - \mathrm{d}x^{356} \tag{2.1}$$

is positive.

This example is representative in the sense that for any positive 3–form ϕ on V there exists a basis of V with respect to which ϕ is given by ϕ_0; see, e.g., [SW10, Theorem 3.2]. Hence, the space of positive 3–forms on V is a $\mathrm{GL}(V)$–orbit. The stabiliser of a fixed positive 3–form is isomorphic to the exceptional Lie group G_2. The choice of a positive 3–form ϕ equips V with a canonical metric g and orientation on V such that

$$i(v_1)\phi \wedge i(v_2)\phi \wedge \phi = 6g(v_1, v_2)\mathrm{vol}. \tag{2.2}$$

In particular, if $\mathscr{P}(V)$ denotes the space of positive 3–forms on V, then there is a *non-linear* map $\Theta : \mathscr{P}(V) \to \Lambda^4 V^*$ defined by

$$\Theta(\phi) := *_\phi \phi.$$

Definition 2.3. A *G_2–structure* on a 7–manifold Y is a positive 3–form $\phi \in \Gamma(\mathscr{P}(TY)) \subset \Omega^3(Y)$. It is called *torsion-free* if

$$\mathrm{d}\phi = 0 \quad \text{and} \quad \mathrm{d}\Theta(\phi) = 0. \tag{2.3}$$

A 7–manifold Y equipped with a torsion-free G_2–structure ϕ is called a *G_2–manifold.*

Remark 2.4. From the above discussion is clear that a G_2–structure is equivalent to a reduction of the structure group of the tangent bundle from

$\mathrm{GL}(7)$ to G_2. A theorem of Fernández and Gray [FG82, Theorem 5.2] asserts that (2.3) is equivalent to $\nabla_g \phi = 0$; hence, for a torsion-free G_2–structure, the holonomy group $\mathrm{Hol}(g)$ is contained in G_2.

Examples of G_2–manifold with $\mathrm{Hol}(g)$ strictly contained in G_2 are easy to come by. For our purposes the following very trivial example will play an important rôle.

Example 2.5. Choose coordinates $(x^1, x^2, x^3, y^1, \dots, y^4)$ on $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ and set

$$\omega_1 = \mathrm{d}y^{12} + \mathrm{d}y^{34}, \quad \omega_2 = \mathrm{d}y^{13} - \mathrm{d}y^{24} \quad \text{and} \quad \omega_3 = \mathrm{d}y^{14} + \mathrm{d}y^{23}.$$

Then

$$\phi = \mathrm{d}x^{123} - \mathrm{d}x^1 \wedge \omega_1 - \mathrm{d}x^2 \wedge \omega_2 - \mathrm{d}x^3 \wedge \omega_3$$

is a torsion-free G_2–structure on $\mathbf{R}^7$.

There is by now a plethora of examples of G_2–manifolds due to Bryant [Bry87], Bryant and Salamon [BS89], Joyce [Joy96], Kovalev [Kov03], Kovalev and Lee [KL11], and Corti, Haskins, Nordström and Pacini [CHNP15]. The construction techniques (especially in the latter cases, which yield compact examples) are quite involved and we will not go into any detail.

2.1. Gauge theory on G_2–manifolds

Let (Y, ϕ) be a *compact* G_2–manifold and let E be a G–bundle over Y where G is a compact Lie group, say $G = \mathrm{SO}(3)$ or $G = \mathrm{SU}(2)$. Denote by $\mathscr{A}(E)$ the space of connections on E.

Definition 2.6. A connection $A \in \mathscr{A}(E)$ on E is called a G_2*–instanton* on (Y, ϕ) if it satisfies (1.1), i.e.,

$$*(F_A \wedge \phi) = -F_A.$$

Since ϕ is closed, it follows from the Bianchi identity that G_2–instantons are Yang–Mills connections. In fact, there is an energy identity which shows that G_2–instantons are absolute minima of the Yang–Mills functional.

Example 2.7. The pullback of an ASD instanton over $\mathbf{R}^4$ to $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$, as in Example 2.5, is a G_2–instanton.

The first non-trivial examples of G_2–instantons (with structure group $G = \mathrm{SO}(3)$) where recently constructed by the author in [Wal13b]. Those live on manifolds arising from Joyce's generalised Kummer construction. A method to produce G_2–instantons on G_2–manifolds arising from the twisted connected sum construction was presented by Sá Earp and the author in [SEW15] and used to produce concrete examples by the author in [Wal15].

From an analytical point of view equation (1.1) is slightly inconvenient to work with, because its linearisation supplemented with the Coulomb gauge is not elliptic. However, we can make use of the following result whose proof can be found, e.g., in [Wal13b, Proposition 3.7].

Proposition 2.8. *Set $\psi := \Theta(\phi)$. Let $A \in \mathscr{A}(E)$ be a connection on E. Then the following are equivalent.*

1) *A is G_2–instanton.*
2) *A satisfies $F_A \wedge \psi = 0$.*
3) *There is a $\xi \in \Omega^0(Y, \mathfrak{g}_E)$ such that*

$$F_A \wedge \psi + *\mathrm{d}_A \xi = 0. \tag{2.4}$$

From Proposition 2.8 one can see that G_2–instantons are in many ways similar to flat connections on 3–manifolds. In particular, if $A_0 \in \mathscr{A}(E)$ is a G_2–instanton, then there is a G_2–Chern–Simons functional

$$CS^\psi(A_0 + a) := \int_Y \left\langle a \wedge \mathrm{d}_{A_0} a + \frac{1}{3} a \wedge [a \wedge a] \right\rangle \wedge \psi$$

whose critical points are precisely the G_2–instantons on E. Very roughly speaking the conjectural G_2 Casson invariant, suggested by Donaldson and Thomas [DT98], should be a signed count the critical points of CS^ψ on a suitable completion of $\mathscr{A}(E)/\mathscr{G}(E)$. Here $\mathscr{G}(E)$ denotes the group of gauge transformations of E.

The infinitesimal deformation theory of G_2–instantons around $A \in \mathscr{A}(E)$ is governed by the self-dual elliptic complex

$$\Omega^0(Y, \mathfrak{g}_E) \xrightarrow{\mathrm{d}_A} \Omega^1(Y, \mathfrak{g}_E) \xrightarrow{\psi \wedge \mathrm{d}_A} \Omega^6(Y, \mathfrak{g}_E) \xrightarrow{\mathrm{d}_A} \Omega^7(Y, \mathfrak{g}_E). \tag{2.5}$$

Definition 2.9. A G_2–instanton $A \in \mathscr{A}(E)$ is called *irreducible*, *rigid* or *unobstructed* if (2.5) has vanishing cohomology in degree zero, one or two respectively. It is called *acyclic* if the cohomology of (2.5) vanishes completely.

Remark 2.10. Since (2.5) is self-dual, rigid and unobstructed are the same thing; in particular, A is acyclic if and only if it is irreducible and rigid/unobstructed.

For any $A \in \mathscr{A}(E)$ we define $L_A = L_{A,\phi} : \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E) \to \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E)$ by

$$L_{A,\phi} := \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix} \tag{2.6}$$

where $\psi := \Theta(\phi)$. This is a self-adjoint elliptic operator. It appears as the linearisation of equation (2.4) supplemented with the Coulomb gauge and therefore controls the infinitesimal deformation theory of G_2–instantons. Alternatively, L_A is obtained by folding the complex (2.5).

As an immediate consequence of the implicit function theorem we have the following result.

Proposition 2.11. *Let Y be a compact 7–manifold and let $(\phi_t)_{t\in(-T,T)}$ be a family of torsion-free G_2–structures on Y. Suppose that $A \in \mathscr{A}(E)$ is an unobstructed G_2–instanton on a G–bundle E over (Y, ϕ_0). Then there is a constant $T' \in (0, T]$ and a unique family of G_2–instantons $(A_t)_{t\in(-T',T')}$ on E over (Y, ϕ_t) with $A_0 = A$.*

2.2. Associative submanifolds in G_2–manifolds

Let (Y, ϕ) be a compact G_2–manifold. The 3–form ϕ is a *calibration* in the sense of Harvey–Lawson [HL82], meaning that ϕ is closed and that for each oriented 3–dimensional subspace P of T_xY the following inequality holds

$$\mathrm{vol}_P \le \phi|_P.$$

Definition 2.12. An oriented submanifold P of Y is called an *associative submanifold* in (Y, ϕ) if it is calibrated by ϕ, that is, for each $x \in P$ we have

$$\mathrm{vol}_{T_xP} = \phi|_{T_xP}.$$

Example 2.13. $\mathbf{R}^3 \times \{0\} \subset \mathbf{R}^3 \oplus \mathbf{R}^4$, as in Example 2.5, is an associative submanifold.

Associative submanifolds also arise as 3–dimensional fixed point sets of orientation reversing involutions of Y mapping ϕ to $-\phi$. For concrete examples we refer the reader to Joyce [Joy96, Part II, Section 4.2]. The recent work of Corti, Haskins, Nordström and Pacini [CHNP15] gives a number of concrete examples of associative submanifolds in twisted connected sums.

The importance of associative submanifolds in the study of gauge theory on G_2–manifolds is due to the following fact: Consider $(\mathbf{R}^7, \phi_0)$ and *any* orthogonal decomposition $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$. Let I be a connection on a bundle over $\mathbf{R}^4$. Then the pullback of I to $\mathbf{R}^7$ is a G_2–instanton if and only if there is an orientation on $\mathbf{R}^3$ with respect to which it is calibrated by ϕ_0 and I is an ASD instanton on $\mathbf{R}^4$. This is the underlying reason why the bubbling locus of a sequence of G_2–instantons is associative and why the connections bubbling off transversely are ASD instantons.

In the following we will discuss some results due to McLean [McL98] concerning the deformation theory of associative submanifolds. If P is an associative submanifold, then there is a natural identification

$$TP \cong \Lambda^+ N^* P : v \mapsto -i(v)\phi \tag{2.7}$$

given by (the negative of) inserting tangent vectors to P into ϕ (and restricting to NP). Thinking of $\Lambda^+ N^* P$ as a sub-bundle of $\mathfrak{so}(NP)$ this yields a Clifford multiplication $\gamma : TP \to \mathrm{End}(NP)$. Denote by $\bar{\nabla}$ the connection on NP induced by the Levi–Civita connection on Y.

Definition 2.14. The *Fueter operator* $F_P = F_{P,\phi} : \Gamma(NP) \to \Gamma(NP)$ associated with P is defined by

$$F_{P,\phi}(n) := \sum_{i=1}^{3} \gamma(e_i)\bar{\nabla}_i n \tag{2.8}$$

with (e_i) a local orthonormal frame on P.

Remark 2.15. The Fueter operator F_P can be identified with a twisted Dirac operator as follows. Pick a spin structure $\mathfrak{s}$ on P. Because of the identification (2.7) there is a unique SU(2)–bundle $\mathfrak{u}$ over P such that $\mathfrak{s} \times \mathfrak{u}$ is a spin structure on NP. The bundle $\mathfrak{u}$ also comes with a connection, such that the resulting connection on $\mathfrak{s} \times \mathfrak{u}$ is a spin connection. If $\not{S}$ and U denote the quaternionic line bundles corresponding to $\mathfrak{s}$ and $\mathfrak{u}$, then $\not{S} \otimes_{\mathbf{C}} U$ has a natural real structure and its real part can be identified with NP. With respect to this identification F_P becomes the twisted Dirac operator $\not{D} : \Gamma(\mathrm{Re}(\not{S} \otimes_{\mathbf{C}} U)) \to \Gamma(\mathrm{Re}(\not{S} \otimes_{\mathbf{C}} U))$.

The importance of F_P is that it controls the infinitesimal deformation theory of the associative submanifold P. In particular, the moduli space of associative submanifolds near P is modelled on the zero set of a smooth map from a neighbourhood of zero in the kernel of F_P to its cokernel.

Definition 2.16. An associative submanifold P is called *rigid* (*unobstructed*) if F_P is injective (surjective).

Remark 2.17. Since F_P is self-adjoint, unobstructed and rigid are the same thing. So unobstructed associative submanifolds are also rigid.

Using McLean's setup for the deformation theory of associative submanifolds developed in [McL98] the following is a simple consequence of the implicit function theorem.

Proposition 2.18. *Let Y be a compact 7–manifold and let $(\phi_t)_{t\in(-T,T)}$ be a family of torsion-free G_2–structures on Y. Suppose that P is an unobstructed associative submanifold in (Y,ϕ_0). Then there is a constant $T'\in(0,T]$ and a unique family of associative submanifolds $(P_t)_{t\in(-T',T')}$ in (Y,ϕ_t) with $P_0=P$.*

3. Moduli spaces of ASD instantons over $\mathbf{R}^4$

In the next section we will explain the construction of the bundle $\mathfrak{M}$ of moduli spaces of ASD instantons, the Fueter equation and provide more detail for the discussion preceding Theorem 1.1. As a preparation we quickly recall some basic facts about moduli spaces of ASD instantons over $\mathbf{R}^4$.

Fix a G–bundle E over $S^4=\mathbf{R}^4\cup\{\infty\}$. Denote by M the moduli space of ASD instantons on E framed over the point at infinity. These moduli spaces are smooth manifolds, because ASD instantons over S^4 are always unobstructed as a consequence of the Weitzenböck formula; see, e.g., [Tau82, Proposition 2.2]. By Uhlenbeck's removable singularities theorem [Uhl82b, Theorem 4.1] we can think of M as a moduli space of framed finite energy ASD instantons on $\mathbf{R}^4$. In a suitable functional analytic setup incorporating decay conditions at infinity, see, e.g., [Tau83] or [Nak90], the infinitesimal deformation theory of a framed ASD instanton I over $\mathbf{R}^4$ is governed by the linear operator $\delta_I:\Omega^1(\mathbf{R}^4,\mathfrak{g}_E)\to\Omega^0(\mathbf{R}^4,\mathfrak{g}_E)\oplus\Omega^+(\mathbf{R}^4,\mathfrak{g}_E)$ defined by

$$\delta_I a := (\mathrm{d}_I^* a, \mathrm{d}_I^+ a). \tag{3.1}$$

From the work of Taubes [Tau83] it is known that δ_I is always surjective and that its kernel lies in L^2. More precisely, we have the following result whose proof can be found, e.g., in [Wal13b, Proposition 5.10].

Proposition 3.1. *Let E be a G–bundle over $\mathbf{R}^4$ and let $I \in \mathscr{A}(E)$ be a finite energy ASD instanton on E. Then the following holds.*

1) *If $a \in \ker \delta_I$ decays to zero at infinity, that is to say*

$$\lim_{r\to\infty} \sup_{\partial B_r(0)} |a| = 0,$$

then $|\nabla^k a| = O(r^{-3-k})$ for $k \geq 0$. Here $r : \mathbf{R}^4 \to [0,\infty)$ denotes the radius function $r(x) := |x|$.

2) *If $(\xi,\omega) \in \ker \delta_I^*$ decays to zero at infinity, then $(\xi,\omega) = 0$.*

In particular, this implies (once more) that M is a smooth manifold and that it can be equipped with an L^2 metric arising from the standard metric on $\mathbf{R}^4$. Clearly, $\Lambda^+ := \Lambda^+(\mathbf{R}^4)^* \cong \mathfrak{so}(4)$ acts SO(4)–equivariantly on $\mathbf{R}^4$ and on $\mathbf{R} \oplus \Lambda^+$. It is a straight-forward computation to verify that the corresponding actions of Λ^+ on $\Omega^1(\mathbf{R}^4, \mathfrak{g}_E)$ and on $\Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ commute with δ_I. Hence, we obtain an SO(4)–equivariant action of Λ^+ on TM.

Remark 3.2. If we fix an identification $\mathbf{R}^4 = \mathbf{H}$ and correspondingly $\Lambda^+ = \operatorname{Im}\mathbf{H}$, then the above defines a hyperkähler structure on TM. However, for our purpose it is more natural not to fix such an identification.

M has carries an action of $\mathbf{R}^4 \rtimes \mathbf{R}^+$ where $\mathbf{R}^4$ acts by translation and $\mathbf{R}^+$ acts by dilation, i.e., by pullback via $s_\lambda : \mathbf{R}^4 \to \mathbf{R}^4$ where

$$s_\lambda(x) := \lambda x$$

for $\lambda \in \mathbf{R}^+$. Since the centre of mass of the measure $|F_I|^2\mathrm{dvol}$ is equivariant with respect to the $\mathbf{R}^4$–action, we can write

$$M = \mathring{M} \times \mathbf{R}^4$$

where $\mathring{M}$ is the space of instantons centred at zero. The action of Λ^+ preserves this product structure and Λ^+ acts on the factor $\mathbf{R}^4$ in the usual way.

Example 3.3. If E is the unique SU(2)–bundle over S^4 with $c_2(E) = 1$, then E carries a single ASD instanton I, commonly called "the one-instanton", unique up to scaling, translation and changing the framing at infinity. We can naturally write the corresponding moduli space as $M = \mathring{M} \times \mathbf{R}^4 = (\not{S}^+ \setminus \{0\})/\mathbf{Z}_2 \times \mathbf{R}^4$. Here S^+ is the positive spin representation associated with $\mathbf{R}^4$.

Example 3.4. In general, if E is an $\mathrm{SU}(r)$–bundle over S^4, then M can be understood rather explicitly in terms the ADHM construction [DK90, Section 3.3].

4. Fueter sections of instanton moduli bundles

Let (Y, ϕ) be a G_2–manifold and let P be an associative submanifold in Y. Fix a moduli space M of framed finite energy ASD instantons on $\mathbf{R}^4$, as in Section 3, and let E_∞ be a G–bundle over P together with a connection A_∞. In the context of Theorem 1.1 we take $E_\infty := E_0|_P$ and $A_\infty := B|_P$.

Definition 4.1. The *instanton moduli bundle* $\mathfrak{M}$ over P associated with E_∞ and M is defined by

$$\mathfrak{M} := (\mathrm{Fr}(NP) \times E_\infty) \times_{\mathrm{SO}(4)\times G} M.$$

Similarly, we define $\mathring{\mathfrak{M}}$ with $\mathring{M}$ instead of M.

Example 4.2. Let $M = (\not{S}^+ \setminus \{0\})/\mathbf{Z}_2 \times \mathbf{R}^4$ be the moduli space of framed ASD instantons from Example 3.3. If we pick $\mathfrak{s}$ and $\mathfrak{u}$ as in Remark 2.15, then

$$\mathfrak{M} = (\mathfrak{s} \times \mathfrak{u} \times E_\infty) \times_{\mathrm{Spin}(4)\times\mathrm{SU}(2)} M = (\mathrm{Re}(\not{S} \otimes E_\infty) \setminus \{0\})/\mathbf{Z}_2 \times NP.$$

Here we used the fact that the $\mathrm{SO}(4)$ action on M lifts to an action of $\mathrm{Spin}(4)$.

Denote by $N_\infty P := \mathrm{Fr}(NP) \times_{\mathrm{SO}(4)} S^4$ the sphere-bundle obtained from NP by adjoining a section at infinity.

Theorem 4.3 (Donaldson–Segal [DS11] and Haydys [Hay12]). *To each section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ we can assign a G–bundle $E = E(\mathfrak{I})$ over $N_\infty P$ together with a connection $I = I(\mathfrak{I})$ and a framing $\Phi : E|_\infty \to E_\infty$ such that:*

- *For each $x \in P$ the restriction of I to $N_x P$ represents $\mathfrak{I}(x)$.*
- *The framing Φ identifies the restriction of I to the section at infinity with A_∞.*

The actions of $\mathbf{R}^+$ on $\mathbf{R}^4$ and M lift to fibre-wise actions on NP and $\mathfrak{M}$. The construction in Theorem 4.3 is equivariant with respect to this action. In particular, $I(s_\lambda^*\mathfrak{I}) = s_\lambda^* I(\mathfrak{I})$. It will be convenient to use the shorthand notations

$$I_\lambda := I(s_{1/\lambda}^*\mathfrak{I}) \quad \text{and} \quad \mathfrak{I}_\lambda := s_{1/\lambda}^*\mathfrak{I}.$$

If a section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ does arise from a sequence of G_2–instantons bubbling along P, then it is reasonable to expect that in the limit as $\lambda \to 0$ the connection I_λ is "close to being a G_2–instanton". To make sense of that notion we define the 4–form ψ_0 on NP to be the zeroth order Taylor expansion of $\psi := \Theta(\phi)$ off P. More explicitly, we can write ψ_0 as

$$\psi := \mathrm{vol}_{NP} - e^1 \wedge e^2 \wedge \omega_{e_3} - e^2 \wedge e^3 \wedge \omega_{e_1} - e^3 \wedge e^1 \wedge \omega_{e_2}. \tag{4.1}$$

Here (e_i) is a local positive orthonormal frame on P, (e^i) is its dual frame, vol_{NP} is the fibre-wise volume form on NP and $v \in TP \mapsto \omega_v \in \Lambda^+ N^*P$ is given by the identification (2.7). With this notation set up the natural requirement is that

$$\lim_{\lambda \to 0} \lambda^{-2} s_\lambda^*(F_{I_\lambda} \wedge \psi_0) = F_I \wedge (\psi_0 - \mathrm{vol}_{NP}) = 0. \tag{4.2}$$

If we introduce a bi-grading on k–forms on NP according to the splitting $TNP = \pi_1^*TP \oplus \pi_2^*NP$ corresponding to the connection on NP with $\pi_1 : TP \to P$ and $\pi_2 : NP \to P$ denoting the canonical projections, then it is easy to see that equation (4.2) splits into two parts. The first one is simply the condition that the anti-self-dual part of $F_I^{0,2}$ must vanish, while the second part is given by

$$F_I^{1,1} \wedge \psi_0 = 0.$$

This condition can be understood as a partial differential equation on $\mathfrak{I}$ as follows. Define the vertical tangent bundle $V\mathfrak{M}$ to $\mathfrak{M}$ by

$$V\mathfrak{M} := (\mathrm{Fr}(NP) \times E_\infty) \times_{\mathrm{SO}(4)\times G} TM.$$

If $\mathfrak{I}$ is a section of $\mathfrak{M}$, then the action of Λ^+ on M induces a Clifford multiplication $\gamma : TP \to \mathrm{End}(\mathfrak{I}^*V\mathfrak{M})$ in view of the identification (2.7). Moreover, the connections on NP and E_∞ induce a connection ∇ on $\mathfrak{M}$ assigning to each section $\mathfrak{I}$ its covariant derivative $\nabla\mathfrak{I} \in \Omega^1(\mathfrak{I}^*V\mathfrak{M})$.

Definition 4.4. The *Fueter operator* $\mathfrak{F}$ associated with $\mathfrak{M}$ is defined by

$$\mathfrak{I} \in \Gamma(\mathfrak{M}) \mapsto \mathfrak{F}\mathfrak{I} := \sum_{i=1}^{3} \gamma(e_i)\nabla_i \mathfrak{I} \in \Gamma(\mathfrak{I}^* V\mathfrak{M})$$

with (e_i) a local orthonormal frame on P. A section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is called a *Fueter section* if it satisfies $\mathfrak{F}\mathfrak{I} = 0$.

Example 4.5. If M is as in Theorem 3.3, then the Fueter operator $\mathfrak{F}$ lifts to the twisted Dirac operator $\not{D} : \Gamma(\mathrm{Re}(S \otimes_{\mathbf{C}} (E_\infty \oplus U)) \to \Gamma(\mathrm{Re}(S \otimes_{\mathbf{C}} (E_\infty \oplus U))$, cf. Remark 2.15.

The Fueter operator $\mathfrak{F}$ is compatible with the product structure on

$$\mathfrak{M} = \mathring{\mathfrak{M}} \times NP$$

corresponding to $M = \mathring{M} \times \mathbf{R}^4$. Its restriction to the second factor is given by the Fueter operator F_P associated with P.

Theorem 4.6 (Donaldson–Segal [DS11] and Haydys [Hay12]). *If* $\mathfrak{I} \in \Gamma(\mathfrak{M})$*, then we can identify* $\Gamma(\mathfrak{I}^* V\mathfrak{M})$ *with a subspace of* $\Omega^1\left(NP, \mathfrak{g}_{E(\mathfrak{I})}\right)$*. With respect to this identification we have the identity*

$$\mathfrak{F}\mathfrak{I} = *_0(F^{1,1}_{I(\mathfrak{I})} \wedge \psi_0)$$

where $*_0$ *is the Hodge–$*$–operator on* NP*. In particular,* $I(\mathfrak{I})$ *satisfies equation* (4.2) *if and only if* $\mathfrak{I}$ *is a Fueter section.*

Definition 4.7. The *linearised Fueter operator* $F_{\mathfrak{I}} = F_{\mathfrak{I},\phi} : \Gamma(\mathfrak{I}^* V\mathfrak{M}) \to \Gamma(\mathfrak{I}^* V\mathfrak{M})$ for $\mathfrak{I} \in \Gamma(\mathfrak{M})$ is defined by

$$F_{\mathfrak{I},\phi}(\hat{\mathfrak{I}}) := \sum_i \gamma(e_i)\nabla_i \hat{\mathfrak{I}} \tag{4.3}$$

with (e_i) a local orthonormal frame on P.

Example 4.8. If M is as in Example 3.3, then the linearised Fueter operator $F_{\mathfrak{I}}$ lifts to the twisted Dirac operator $\not{D} : \Gamma(\mathrm{Re}(\not{S} \otimes_{\mathbf{C}} (E_\infty \oplus U)) \to \Gamma(\mathrm{Re}(\not{S} \otimes_{\mathbf{C}} (E_\infty \oplus U))$. In particular, it only depends on the spin structure $\mathfrak{s}$ and not on $\mathfrak{I}$.

The operator $F_{\mathfrak{I}}$ is self-adjoint and elliptic; however, it can never be invertible if $\mathfrak{I}$ is a Fueter section $\mathfrak{I}$. This is because Fueter sections come in 1–parameter families $(\mathfrak{I}_\lambda)_{\lambda\in\mathbf{R}^+}$. In particular, taking the derivative at $\lambda = 1$ yields an element in the kernel of $F_{\mathfrak{I}}$. If $\hat{v} \in \Gamma(V\mathfrak{M})$ denotes the vector field generating the action of $\mathbf{R}^+$ on $\mathfrak{M}$, then we can succinctly write this element of the kernel as $\hat{v} \circ \mathfrak{I}$.

Let $(\phi_t)_{t\in(-T,T)}$ be a family of torsion-free G_2–structures on Y, let B_0 be an unobstructed G_2–instanton on a G–bundle E over (Y, ϕ_0) and let P_0 be an unobstructed associative submanifold in (Y, ϕ_0). Then by Proposition 2.11 and Proposition 2.18 we obtain a family of G_2–instantons $(B_t)_{t\in(-T',T')}$ over (Y, ϕ_t) and a family of associative submanifolds $(P_t)_{t\in(-T',T')}$ in (Y, ϕ_t) for some $T' \in (0, T]$. Now, carry out the above construction with $P = P_t$, $E_\infty = E|_{P_t}$, $A_\infty = B_t|_{P_t}$ and a fixed moduli space M of framed finite energy ASD instantons to obtain a family of instanton moduli bundles $(\mathfrak{M}_t)_{t\in(-T',T')}$ along with a family of Fueter operators $(\mathfrak{F}_t)_{t\in(-T',T')}$. If $\mathfrak{I}_0$ is a Fueter section of $\mathfrak{M}_0$ with

$$\dim\ker F_{\mathfrak{I}_0} = 1,$$

then, using the implicit function theorem, we obtain a family $(\mathfrak{I}_t)_{t\in(-T',T')}$ of sections of $\mathfrak{M}_t$ satisfying

$$\mathfrak{F}_t\mathfrak{I}_t + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_t = 0 \tag{4.4}$$

where $\mu : (-T', T') \to \mathbf{R}$ is a smooth function vanishing at zero.

Definition 4.9. In the above situation we say that $\mathfrak{I}_0$ is *unobstructed* with respect to (ϕ_t) if

$$\left.\frac{\partial\mu}{\partial t}\right|_{t=0} \neq 0.$$

Remark 4.10. One can work with a slightly weaker notion of unobstructedness where one only requires that μ is strictly monotone near $t = 0$. A slight variation of Theorem 1.1 still holds in this case. We will pick up this thread again in Section 10.

Example 4.11. If M is as in Example 3.3, then equation (4.4) can be viewed as the spectral flow of a family of twisted Dirac operators and $\mathfrak{I}_0$ is unobstructed if and only if this spectral flow has a regular crossing at $\mathfrak{I}_0$.

5. Pregluing construction

In this section we begin the proof of Theorem 1.1 in earnest. Suppose that Y, $(\phi_t)_{t\in(-T,T)}$, B, P and $\mathfrak{I} \in \Gamma(\mathfrak{M})$ are as in the hypothesis of Theorem 1.1.

Convention 5.1. We fix constants $T' \in (0,T]$ and $\Lambda > 0$ such that all of the statements of the kind "if $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$, then ..." appearing in the following are valid. This is possible since there is only a finite number of these statements and each one of them is valid provided T' and Λ are sufficiently small. By $c > 0$ we will denote a generic constant whose value depends neither on $t \in (-T',T')$ nor on $\lambda \in (0,\Lambda]$ but may change from one occurrence to the next.

As discussed at the end of Section 4, $B_0 := B$ and $P_0 := P$ give rise to:

- a family $(B_t)_{t\in(-T',T')}$ of G_2–instantons on E_0 over (Y,ϕ_t),
- a family of associative submanifolds $(P_t)_{t\in(-T',T')}$ in (Y,ϕ_t) and, hence,
- a family of instanton moduli bundles $(\mathfrak{M}_t)_{t\in(-T',T')}$ with $\mathfrak{M}_0 = \mathfrak{M}$ and Fueter operators $(\mathfrak{F}_t)_{t\in(-T',T')}$ together with sections $(\mathfrak{I}_t)_{t\in(-T',T')}$ satisfying $\mathfrak{I}_0 = \mathfrak{I}$ and

$$\mathfrak{F}_t\mathfrak{I}_t + \mu(t)\hat{v} \circ \mathfrak{I}_t = 0 \tag{5.1}$$

where $\mu : (-T',T') \to \mathbf{R}$ is a smooth function vanishing at zero with

$$\left.\frac{\partial \mu}{\partial t}\right|_{t=0} \neq 0.$$

The proof of Theorem 1.1 proceeds via a gluing construction. As a first step we explain how to construct approximate solutions.

Proposition 5.2. *For each $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$ we can explicitly construct a G–bundle $E_{t,\lambda}$ together with a connection $A_{t,\lambda} = A_t\#_\lambda\mathfrak{I}_t$ from E_0, $A_t \in \mathscr{A}(E_0)$ and $\mathfrak{I}_t$. The bundles $E_{t,\lambda}$ are pairwise isomorphic.*

Before we embark on the proof, let us set up some notation. Fix a constant $\sigma > 0$ such that for all $t \in (-T',T')$ the exponential map identifies a tubular neighbourhood of width 8σ of P_t in Y with a neighbourhood of the zero section in NP_t. For $I \subset \mathbf{R}$ we set

$$U_{I,t} := \{v \in NP_t : |v| \in I\} \quad \text{and} \quad V_{I,t} := \{x \in Y : r_t(x) \in I\}.$$

Here $r_t := d(\cdot, P_t) : Y \to [0, \infty)$ denotes the distance from P_t. Fix a smooth-cut off function $\chi : [0, \infty) \to [0, 1]$ which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we define $\chi^-_{t,\lambda} : Y \to [0, 1]$ and $\chi^+_t : Y \to [0, 1]$ by

$$\chi^-_{t,\lambda}(x) := \chi(r_t(x)/2\lambda) \quad \text{and} \quad \chi^+_t(x) := 1 - \chi(r_t(x)/2\sigma),$$

respectively.

Proof of Proposition 5.2. Via radial parallel transport we can identify $E(\mathfrak{I}_t)$ over $U_{(R,\infty),t}$ for some $R > 0$ with the pullback of $E(\mathfrak{I}_t)|_\infty$ to said region and similarly we can identify E_0 over $V_{[0,\sigma),t}$ with the pullback of $E_0|_{P_t}$. Hence, via the framing Φ we can identify $s^*_{1/\lambda}E(\mathfrak{I}_t)$ with E_0 on the overlap $V_{(\lambda,\sigma),t}$ for $\lambda \in (0, \Lambda]$. Patching both bundles via this identification yields $E_{t,\lambda}$.

To construct a connection on $E_{t,\lambda}$ note that on the overlap $I_{t,\lambda} := s^*_{1/\lambda}I(\mathfrak{I}_t)$ and B_t can be written as

$$I_{t,\lambda} = B_t|_{P_t} + i_{t,\lambda} \quad \text{and} \quad B_t = B_t|_{P_t} + b_t.$$

Here and in the following, by a slight abuse of notation, we denote by $B_t|_{P_t}$ the pullback of $B_t|_{P_t}$ to the overlap. We define $A_{t,\lambda}$ by interpolating between I_λ and B_t on the overlap as follows

$$A_{t,\lambda} := B_t|_{P_t} + \chi^-_{t,\lambda} b_t + \chi^+_t i_{t,\lambda}. \tag{5.2}$$

□

Now, in view of Proposition 2.8, the task at hand is to solve the equation

$$*_{\phi_t}\left(F_{A_{t,\lambda}+a} \wedge \psi_t\right) + \mathrm{d}_{A_{t,\lambda}+a}\xi = 0 \tag{5.3}$$

where $\psi_t := \Theta(\phi_t) = *_{\phi_t}\phi_t$, $t = t(\lambda)$, $a = a(\lambda)$ and $\xi = \xi(\lambda)$. If we could find an appropriate analytic setup in which $(a, \xi) = 0$ becomes closer and closer to being a solution of equation (5.3) while at the same time the linearisations $L_{t,\lambda} := L_{A_{t,\lambda},\phi_t}$, as defined in (2.6), possess right inverses that can be controlled *uniformly* in t and λ, then it would not be too difficult to solve equation (5.3) for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$. Since the properties of $L_{t,\lambda}$ are closely linked, among other things, to those of $F_{\mathfrak{I}_t}$ and since $F_{\mathfrak{I}_0}$

has a one-dimensional cokernel, however, we will only be able to solve equation (5.3) "modulo the cokernel of $F_{\mathfrak{I}_0}$". More precisely, when $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we will be able to solve the equation

$$L_{t,\lambda}\underline{a} + \eta \cdot \iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t + Q_{t,\lambda}(\underline{a}) + e_{t,\lambda} = 0 \tag{5.4}$$

for $\underline{a} = (\xi, a) \in \Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ and $\eta \in \mathbf{R}$ with $Q_{t,\lambda}$ and $e_{t,\lambda}$ defined by

$$Q_{t,\lambda}(\underline{a}) := \frac{1}{2} * ([a \wedge a] \wedge \psi_t) + [\xi, a]. \tag{5.5}$$

and

$$e_{t,\lambda} := *(F_{A_{t,\lambda}} \wedge \psi_t) + \mu(t) \cdot \iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t,$$

respectively. Here the map $\iota_{t,\lambda} : \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t) \to \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ is defined by

$$\iota_{t,\lambda}\hat{\mathfrak{I}} := \chi_t^+ s_{1/\lambda}^* \hat{\mathfrak{I}}$$

where we first identify $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ with an element of $\Omega^1(NP, E(\mathfrak{I}_t))$, then view the restriction of its pullback via s_λ^{-1} to $U_{[0,\sigma),t}$ as lying in $\Omega^1(V_{[0,\sigma),t}, \mathfrak{g}_{E_{t,\lambda}})$ and finally extended it to all of Y by multiplication with χ_t^+. After solving (5.4) we are left with the residual scalar equation

$$\mu(t) + \eta(t, \lambda) = 0.$$

It will turn out that η and $\partial_t \eta$ go to zero as $\lambda \to 0$. Since $\partial_t \mu(0) \neq 0$, finding $t = t(\lambda)$ such that equation (5.3) is satisfied is then a simple consequence of an implicit function theorem.

Let us now discuss some aspects of the analysis. First of all we will introduce appropriate weighted Hölder spaces in Section 6. One should think of these weighted spaces as a convenient framework to deal with different local scales simultaneously. In our case they are constructed to counteract the fact that the curvature of the connection $A_{t,\lambda}$ around P_t becomes larger and larger as $\lambda \to 0$. We will see in Section 7 that the amount by which our approximate solutions $A_{t,\lambda}$ fail to be solutions of equation (5.3) "modulo the cokernel of $F_{\mathfrak{I}_0}$" measured in our weighted Hölder norms goes to zero at a certain rate as $\lambda \to 0$. The key difficulty then lies in analysing the linearisation $L_{t,\lambda}$. As is the case in most adiabatic limit constructions, the linearisation $L_{t,\lambda}$ is rather badly behaved on an infinite dimensional space: For every $\hat{\mathfrak{I}} \in \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ the appropriate norm of $\iota_{t,\lambda}\hat{\mathfrak{I}}$ is essentially independent of λ, while the appropriate norm of $L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}}$ tends to zero as $\lambda \to 0$. To overcome this issue it is convenient to split the problem at hand into a part coming from $\Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ and the part orthogonal to it. We define $\pi_{t,\lambda} : \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) \to \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ by

$$(\pi_{t,\lambda} a)(x) := \sum_{\kappa} \int_{N_x P} \langle a, \iota_{t,\lambda}\kappa \rangle \, \kappa$$

for $x \in P_t$. Here κ runs through an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}(x)}$ with respect to the inner product $\langle \iota_{t,\lambda}\cdot, \iota_{t,\lambda}\cdot \rangle$. Clearly, $\pi_{t,\lambda}\iota_{t,\lambda} = \mathrm{id}$; hence, $\bar{\pi}_{t,\lambda} := \iota_{t,\lambda}\pi_{t,\lambda}$ is a projection. We denote the complementary projection by $\rho_{t,\lambda} := \mathrm{id} - \bar{\pi}_{t,\lambda}$. If we define

$$\mathfrak{A}_{t,\lambda} := \Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \ker \pi_{t,\lambda},$$

then we can write

$$\Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) = \mathfrak{A}_{t,\lambda} \oplus \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$$

and decompose $L_{t,\lambda}$ accordingly into a 2–by–2 matrix of operators. We will see in Section 8 that the diagonal entries can be controlled in terms of certain models on $\mathbf{R}^7$, L_{A_0} and the linearised Fueter operator $F_{\mathfrak{I}_0}$, while the off-diagonal terms are negligibly small. In Section 9 we discuss how to control the non-linearity $Q_{t,\lambda}$ in equation (5.4). The completion of the proof of Theorem 1.1 in Section 10 will then be rather straight-forward.

6. Weighted Hölder norms

For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we define a family of weight functions $w_{\ell,\delta;t,\lambda}$ on Y depending on two additional parameters $\ell, \delta \in \mathbf{R}$ as follows

$$w_{\ell,\delta;t,\lambda}(x) := \begin{cases} \lambda^\delta(\lambda + r_t(x))^{-\ell-\delta} & \text{if } r_t(x) \leq \sqrt{\lambda} \\ r_t(x)^{-\ell+\delta} & \text{if } r_t(x) > \sqrt{\lambda} \end{cases}$$

and set $w_{\ell,\delta;t,\lambda}(x,y) := \min\{w_{\ell,\delta;t,\lambda}(x), w_{\ell,\delta;t,\lambda}(x)\}$. For a Hölder exponent $\alpha \in (0,1)$ and $\ell, \delta \in \mathbf{R}$ we define (semi-)norms

$$\begin{aligned}
\|f\|_{L^\infty_{\ell,\delta;t,\lambda}(U)} &:= \|w_{\ell,\delta;t,\lambda} f\|_{L^\infty(U)}, \\
[f]_{C^{0,\alpha}_{\ell,\delta;t,\lambda}(U)} &:= \sup_{\substack{x \neq y \in U: \\ d(x,y) \leq \lambda + \min\{r_t(x), r_t(y)\}}} w_{\ell-\alpha,\delta;t,\lambda}(x,y) \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \quad \text{and} \\
\|f\|_{C^{k,\alpha}_{\ell,\delta;t,\lambda}(U)} &:= \sum_{j=0}^{k} \|\nabla^k f\|_{L^\infty_{\ell-j,\delta;t,\lambda}(U)} + [\nabla^k f]_{C^{0,\alpha}_{\ell-j,\delta;t,\lambda}}.
\end{aligned}$$

Here f is a section of a vector bundle over $U \subset Y$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = Y$. We will primarily use this norm for $\mathfrak{g}_{E_{t,\lambda}}$–valued tensor fields.

Remark 6.1. The reader may find the following heuristic useful. Let f be a k–form on Y. Fix a small ball centred at a point $x \in P_t$, identify it with a small ball in $T_xY = T_xP_t \oplus N_xP_t$ and rescale this ball by a factor $1/\lambda$. Upon pulling everything back to this rescaled ball the weight function $w_{-k,\delta,t,\lambda}$ becomes essentially $\lambda^k(1+|y|)^{k-\delta}$, where y denotes the N_xP_t–coordinate. Thus as λ goes to zero a uniform bound $\|f_\lambda\|_{L^\infty_{-k,\delta,t,\lambda}}$ on a family (f_λ) of k–forms ensures that the pullbacks of f_λ decay like $|y|^{-k+\delta}$ in the direction of N_xP_t. At the same time it forces f_λ not to blowup at a rate faster than $r_t^{-k-\delta}$ along P_t. The "discrepancy" in the exponents can be seen to be rather natural by considering the action of the inversion $y \mapsto \lambda y/|y|^2$.

Proposition 6.2. *If $(f,g) \mapsto f \cdot g$ is a bilinear form satisfying $|f \cdot g| \leq |f||g|$, then*

$$\|f \cdot g\|_{C^{k,\alpha}_{\ell_1+\ell_2,\delta_1+\delta_2;t,\lambda}} \leq \|f\|_{C^{k,\alpha}_{\ell_1,\delta_1;t,\lambda}} \|g\|_{C^{k,\alpha}_{\ell_2,\delta_2;t,\lambda}}.$$

Proof. This follows immediately from the above definition. □

Corollary 6.3. *If $\delta < 0$, then there is a constant $c > 0$ which is independent of $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ such that*

$$\|f\|_{C^{k,\alpha}_{\ell,\delta;t,\lambda}} \leq c\lambda^{\delta/2}\|f\|_{C^{k,\alpha}_{\ell,0;t,\lambda}} \quad \text{and} \quad \|f\|_{C^{k,\alpha}_{\ell,0;t,\lambda}} \leq c\|f\|_{C^{k,\alpha}_{\ell,\delta;t,\lambda}}$$

Proof. Use $\|1\|_{C^{k,\alpha}_{0,\delta;t,\lambda}} \leq c\lambda^{\delta/2}$ and $\|1\|_{C^{k,\alpha}_{0,-\delta;t,\lambda}} \leq c$ for $\delta < 0$. □

Proposition 6.4. *For $\ell \leq -1$ and $\delta \in \mathbf{R}$ such that $\ell - \alpha + \delta > -3$ and $\ell + \delta < -1$ there is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|\iota_{t,\lambda}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{\ell,\delta;t,\lambda}} &\leq c\lambda^{-1-\ell}\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \quad \text{and} \\ \|\pi_{t,\lambda}a\|_{C^{0,\alpha}} &\leq c\lambda^{1+\ell-\alpha}\|a\|_{C^{0,\alpha}_{\ell,\delta;t,\lambda}(V_{[0,\sigma);t})}. \end{aligned}$$

In particular, $\bar{\pi}_{t,\lambda} = \iota_{t,\lambda}\pi_{t,\lambda}$ and $\rho_{t,\lambda}$ are bounded by $c\lambda^{-\alpha}$ with respect to the $C^{0,\alpha}_{\ell,\delta;t,\lambda}$–norms.

Proof. From Proposition 3.1 it follows at once that

$$\|s_{1/\lambda}^*\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{[0,\sigma),t})} \le c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{0,\alpha}}.$$

The first inequality thus is a consequence of Prop. 6.2 since $\|\chi_t^+\|_{C^{0,\alpha}_{3+\ell,\delta;t,\lambda}} \le c\lambda^{-3-\ell}$ for $\ell+\delta > -3$.

To prove the second inequality, note that by Proposition 3.1 for $\kappa \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ we have $|s_{1/\lambda}^*\kappa|(x) \le c\lambda^2/(\lambda+|x|)^3\|\kappa\|_{L^2}$ and thus

$$\begin{aligned}\int_{N_xP}\left\langle a, \chi_t^+ s_{1/\lambda}^*\kappa\right\rangle &\le c\int_0^{\sqrt{\lambda}} \lambda^{2-\delta}(\lambda+r)^{\ell+\delta-3}r^3\mathrm{d}r\cdot\|a\|_{L^\infty_{\ell,\delta;t,\lambda}}\|\kappa\|_{L^2} \\ &\quad + c\int_{\sqrt{\lambda}}^{\sigma}\lambda^2 r^{\ell-\delta}(\lambda+r)^{-3}r^3\mathrm{d}r\cdot\|a\|_{L^\infty_{\ell,\delta;t,\lambda}}\|\kappa\|_{L^2} \\ &\le c\lambda^{3+\ell}\|a\|_{L^\infty_{\ell,\delta;t,\lambda}}\|\kappa\|_{L^2}\end{aligned}$$

since $\ell \le -1$ and $\ell+\delta < -1$. If κ is an element of an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ with respect to $\langle \iota_{t,\lambda}\cdot, \iota_{t,\lambda}\cdot\rangle$, then $\|\kappa\|_{L^2} \le c/\lambda$ since for $\kappa_1,\kappa_2 \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$

$$\lambda^2\langle\kappa_1,\kappa_2\rangle_{L^2} \sim \left\langle \chi_t^+ s_{1/\lambda}^*\kappa_1, \chi_t^+ s_{1/\lambda}^*\kappa_2\right\rangle_{L^2}$$

where $\sim$ means comparable uniformly in t and λ. Therefore,

$$\|\pi_{t,\lambda}a\|_{L^\infty} \le c\lambda^{1+\ell}\|a\|_{L^\infty_{\ell,\delta;t,\lambda}}.$$

The estimates on the Hölder norms follow by the same kind of argument. □

7. Pregluing estimate

In the following we will need to differentiate various tensors over Y and NP_t depending on $t \in (-T,T)$. For tensors over Y we could simply differentiate using ∂_t; however, ∂_t does not drag P_t along in a parallel fashion, which causes some additional error terms to show up, and also it is preferable to differentiate tensors over Y and NP_t in way that is consistent with the identification $V_{I,t} = U_{I,t}$ for $I \subset [0,2\sigma)$. Therefore we use a fixed set of connections constructed as follows: For each $t \in (-T',T')$ we can write $P_t = \{\exp_p(v_t) : p \in P_0\}$ for some unique normal vector field $v_t \in \Gamma(P_0, NP_0)$; hence, the bundle $\coprod_{t\in(-T',T')} P_t \to (-T',T')$ comes with a canonical connection. Pick a connection on $\mathscr{N} := \coprod_{t\in(-T',T')} NP_t \to (-T',T')$ such that for each parallel path $t \mapsto p_t \in P_t$ its lift to the zero section $t \mapsto 0_{p_t} \in NP_t$ is also parallel.

Moreover, we pick a connection on $Y \times (-T', T') \to (-T', T')$ which agrees with the connection on $\mathscr{N}$ on $\bigcup_{t\in(-T',T')} V_{[0,2\sigma),t} = \bigcup_{t\in(-T',T')} U_{[0,2\sigma),t}$ and with ∂_t on $Y \setminus \bigcup_{t\in(-T',T')} V_{[0,4\sigma),t}$. These connections induce various connections on bundles of tensors over Y and NP_t; we denote the associated covariant derivatives by ∇_t.

Proposition 7.1. *There is a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|e^{1,0}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda \quad \text{and} \quad \|\nabla_t e^{1,0}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda.$$

and

$$\|e^{0,1}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda^2 \quad \text{and} \quad \|\nabla_t e^{0,1}_{t,\lambda}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda^2.$$

Here we use the bi-grading introduced after (4.2)*.*

Remark 7.2. We will later improve this error term further by performing one adjustment "by hand", see Proposition 10.1.

The proof of this result requires some preparation.

Proposition 7.3. *In the tubular neighbourhood $V_{[0,\sigma);t}$ of P_t we can write $\psi_t := \Theta(\phi_t) = *_{\phi_t}\phi_t$ as*

$$\psi_t = \psi_{0;t} + \psi_{1;t} + \psi_{\ge 2;t}$$

where $\psi_{0;t}$ is defined as in equation (4.1)*, $\psi_{1;t}$ takes values in $\Lambda^2 T^*P_t \otimes \Lambda^+ N^*P_t$. Moreover, $\psi_{0;t}$, $\psi_{1;t}$ and $\psi_{\ge 2;t}$ depend continuously differentiably on t, and there is a constant $c > 0$ such that for all $t \in (-T', T')$ we have*

$$\|\psi_{0;t}\|_{C^{0,\alpha}_{0,0;t,\lambda}(V_{[0,\sigma);t})} + \|\psi_{1;t}\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} + \|\psi_{\ge 2;t}\|_{C^{0,\alpha}_{2,0;t,\lambda}(V_{[0,\sigma);t})} \le c$$

and

$$\|\nabla_t\psi_{0;t}\|_{C^{0,\alpha}_{0,0;t,\lambda}(V_{[0,\sigma);t})} + \|\nabla_t\psi_{1;t}\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} + \|\nabla_t\psi_{\ge 2;t}\|_{C^{0,\alpha}_{2,0;t,\lambda}(V_{[0,\sigma);t})} \le c.$$

Proof of Proposition 7.3. Let $P = P_t$ and $\psi = \psi_t$. If we pull the identity map of a tubular neighbourhood of P back to a tubular neighbourhood of the zero section of NP via the exponential map, then the Taylor expansion of its derivative around P can be expressed in the splitting $TNP = \pi_1^*TP \oplus \pi_2^*NP$ as

$$(x, y) \mapsto (x, y) + (\mathrm{II}_y(x), y) + O\left(|y|^2\right)$$

where II is the second fundamental form of P in Y which we think of as a map from NP to $\mathrm{End}(TP)$. This immediately yields the desired expansion of ψ near P, with ψ_1 taking values in $\Lambda^2T^*P \otimes \Lambda^+N^*P$, since we know that ψ is given by ψ_0 along P. Moreover, we have $\nabla^k\psi_1 = O\left(|y|^{1-k}\right)$ and $\nabla^k\psi_{\geq 2} = O\left(|y|^{2-k}\right)$ for $k = 0, 1$ which implies the first estimate and, since everything depends smoothly on t, also the second estimate. $\square$

The same reasoning also proves the following result.

Proposition 7.4. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\| *_0 - * \|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} + \|\nabla_t(*_0 - *)\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} \leq c.$$

Proposition 7.5. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \left\|F^{2,0}_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right\|_{C^{0,\alpha}_{-2,0;t,\lambda}(V_{[0,\sigma);t})} + \left\|\nabla_t\left(F^{2,0}_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)\right\|_{C^{0,\alpha}_{-2,0;t,\lambda}(V_{[0,\sigma);t})} &\leq c\lambda, \\ \left\|F^{1,1}_{I_{t,\lambda}}\right\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{[0,\sigma);t})} + \left\|\nabla_t F^{1,1}_{I_{t,\lambda}}\right\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{[0,\sigma);t})} &\leq c\lambda^2 \\ \textit{and} \quad \left\|F^{0,2}_{I_{t,\lambda}}\right\|_{C^{0,\alpha}_{-4,0;t,\lambda}(V_{[0,\sigma);t})} + \left\|\nabla_t F^{0,2}_{I_{t,\lambda}}\right\|_{C^{0,\alpha}_{-4,0;t,\lambda}(V_{[0,\sigma);t})} &\leq c\lambda^2. \end{aligned}$$

Proof. Theorem 4.3 asserts that the restriction of $I_t = I(\mathfrak{I}_t)$ to the section at infinity agrees with $B_t|_{P_t}$. Hence, $F^{2,0}_{I_t} - F_{B_t|_{P_t}}$ vanishes along the section at infinity which when viewed from the zero section in NP_t means that

$$\left|F^{2,0}_{I_t} - F_{B_t|_{P_t}}\right| \leq \frac{c}{1 + |y|}.$$

The first estimate now follows from a simple scaling consideration and by realising that the above reasoning also applies to $\nabla_t\left(F^{2,0}_{I_t} - F_{B_t|_{P_t}}\right)$.

The last two estimates follow from Theorem 4.3, Proposition 3.1, the fact that the curvature of a finite energy ASD instanton decays at least like $|y|^{-4}$ and simple scaling considerations. $\square$

Proposition 7.6. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned}\|i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{(\lambda,\sigma);t})} + \|\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda}\|_{C^{0,\alpha}_{-4,0;t,\lambda}(V_{(\lambda,\sigma);t})} &\le c\lambda^2 \quad \textit{and} \\ \|\nabla_t i_{t,\lambda}\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{(\lambda,\sigma);t})} + \|\nabla_t(\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda})\|_{C^{0,\alpha}_{-4,0;t,\lambda}(V_{(\lambda,\sigma);t})} &\le c\lambda^2\end{aligned}$$

as well as

$$\begin{aligned}\|b_t\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} + \|\mathrm{d}_{B_t|_{P_t}} b_t\|_{C^{0,\alpha}_{0,0;t,\lambda}(V_{[0,\sigma);t})} &\le c \quad \textit{and} \\ \|\nabla_t b_t\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} + \|\nabla_t(\mathrm{d}_{B_t|_{P_t}} b_t)\|_{C^{0,\alpha}_{0,0;t,\lambda}(V_{[0,\sigma);t})} &\le c.\end{aligned}$$

Proof. The first two estimates follow from the fact that we put I_t into radial gauge from the section at infinity in $N_\infty P_t$ and a simple scaling consideration, while the last two estimates follow from the fact that we put B_t into radial gauge from the zero section in NP_t. $\square$

Proof of Proposition 7.1. We proceed in four steps. First we estimate $\tilde{e}_{t,\lambda}$, an approximation of $e_{t,\lambda}$. Then we estimate the difference $e_{t,\lambda} - \tilde{e}_{t,\lambda}$ separately in the three subsets $V_{[0,\lambda);t}$, $V_{[\lambda,\sigma/2);t}$ and $V_{[\sigma/2,\sigma);t}$ constituting $V_{[0,\sigma);t}$ which contains the support of $e_{t,\lambda}$.

It will be convenient to use the following shorthand notation

$$\|f\|_{\ell,U} := \|f\|_{C^{0,\alpha}_{\ell,0;t,\lambda}(U)} + \|\nabla_t f\|_{C^{0,\alpha}_{\ell,0;t,\lambda}(U)}.$$

Note that if $(f, g) \mapsto f \cdot g$ is a bilinear map satisfying $|f \cdot g| \le |f||g|$ and the Leibniz rule with respect to ∇_t, then it follows from Proposition 6.2 that $\|f \cdot g\|_{\ell_1+\ell_2,U} \le \|f\|_{\ell_1,U} \cdot \|g\|_{\ell_2,U}$.

Step 1. *The term*

$$\tilde{e}_{t,\lambda} := *\left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right) \wedge \psi_t\right] + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_{t,\lambda}$$

satisfies $\|\tilde{e}^{1,0}_{t,\lambda}\|_{-2,V_{[0,\sigma);t}} + \|\tilde{e}^{0,1}_{t,\lambda}\|_{-2,V_{[0,\sigma);t}} \le c\lambda.$

Because of Theorem 4.6, the fact that $F^{0,2}_{I_{t,\lambda}}$ is anti-self-dual and Proposition 7.3 we can write $\tilde{e}_{t,\lambda}$ on $V_{[0,\sigma);t}$ as

$$\begin{aligned}\tilde{e}_{t,\lambda} = {}& *\left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \wedge \psi_t\right] + *\left[F^{1,1}_{I_{t,\lambda}} \wedge (\psi_{1;t} + \psi_{\ge 2;t})\right] \\ &+ *\left(F^{0,2}_{I_{t,\lambda}} \wedge \psi_{\ge 2;t}\right) + (* - *_0)\left(F^{1,1}_{I_{t,\lambda}} \wedge \psi_{0;t}\right).\end{aligned}$$

Using Proposition 7.3 and Proposition 7.5 as well as $\|1\|_{-1,V_{[0,\sigma);t}} \leq c$ we estimate $\|\tilde{e}_{t,\lambda}\|_{-2,V_{[0,\sigma);t}}$ by

$$\begin{aligned}
&\left\| \left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \right\|_{-2,V_{[0,\sigma);t}} \cdot \|\psi_t\|_{0,V_{[0,\sigma);t}} \\
&+ \left\| F^{1,1}_{I_{t,\lambda}} \right\|_{-3,V_{[0,\sigma);t}} \cdot \left(\|\psi_{1;t}\|_{1,V_{[0,\sigma);t}} + \|1\|_{-1,V_{[0,\sigma);t}} \cdot \|\psi_{\geq 2;t}\|_{2,V_{[0,\sigma);t}} \right) \\
&+ \left\| F^{0,2}_{I_{t,\lambda}} \right\|_{-4,V_{[0,\sigma);t}} \cdot \|\psi_{\geq 2;t}\|_{2,V_{[0,\sigma);t}} \\
&+ \| * - *_0 \|_{1,V_{[0,\sigma);t}} \cdot \left\| F^{1,1}_{I_{t,\lambda}} \right\|_{-3,V_{[0,\sigma);t}} \cdot \|\psi_{0;t}\|_{0,V_{[0,\sigma);t}} \leq c\lambda.
\end{aligned}$$

This proves the first part of the assertion. To prove the second part, note that all but the first term can be bounded by λ^2. The first part arises from

$$\begin{aligned}
* \left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \wedge \psi_t \right] = {} & *_0 \left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \wedge \psi_{0;t} \right] \\
& + *_0 \left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \wedge (\psi_{1;t} + \psi_{\geq 2;t}) \right] \\
& + (* - *_0) \left[\left(F_{I_{t,\lambda}} - F_{B_t|_{P_t}}\right)^{2,0} \wedge \psi_t) \right].
\end{aligned}$$

Again, all but the first term can be bounded by λ^2 with respect to $\| \cdot \|_{-2,V_{[0,\sigma);t}}$. The first term, however, is of type $(1,0)$.

Step 2. *We prove that* $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{[0,\lambda);t}} \leq c\lambda^2$.

Since

$$\left\| F_{B_t|_{P_t}} \wedge \psi_t \right\|_{-2,V_{[0,\lambda);t}} \leq \|1\|_{-2,V_{[0,\lambda);t}} \cdot \left\| F_{B_t|_{P_t}} \wedge \psi_t \right\|_{0,V_{[0,\lambda);t}} \leq c\lambda^2,$$

it suffices to estimate $F_{A_{t,\lambda}} - F_{I_{t,\lambda}}$ in $V_{[0,\lambda);t}$. Now, in $V_{[0,\lambda);t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{I_{t,\lambda}} + \chi^-_{t,\lambda} \mathrm{d}_{I_{t,\lambda}} b_t + \frac{1}{2} (\chi^-_{t,\lambda})^2 [b_t \wedge b_t] + \mathrm{d}\chi^-_{t,\lambda} \wedge b_t.$$

Using Proposition 7.6 and the fact that the cut-off functions $\chi^-_{t,\lambda}$ where constructed so that $\|\chi^-_{t,\lambda}\|_{0,V_{[0,\sigma)}} + \|\mathrm{d}\chi^-_{t,\lambda}\|_{-1,V_{[0,\sigma)}} \leq c$ we obtain

$$\begin{aligned}
&\|F_{A_{t,\lambda}} - F_{I_{t,\lambda}}\|_{-2,V_{[0,\lambda);t}} \\
&\leq \|1\|_{-2,V_{[0,\lambda);t}} \cdot \|\chi^-_{t,\lambda}\|_{0,V_{[0,\lambda);t}} \cdot \|\mathrm{d}_{B_t|_{P_t}} b_t\|_{0,V_{[0,\lambda);t}} \\
&\quad + \|\chi^-_{t,\lambda}\|_{0,V_{[0,\lambda);t}} \cdot \|i_{t,\lambda}\|_{-3,V_{[\lambda/2,\sigma);t}} \cdot \|b_t\|_{1,V_{[0,\lambda);t}} \\
&\quad + \frac{1}{2}\|1\|_{-4,V_{[0,\lambda);t}} \cdot \|\chi^-_{t,\lambda}\|^2_{0,V_{[0,\lambda);t}} \cdot \|b_t\|^2_{1,V_{[0,\lambda);t}} \\
&\quad + \|1\|_{-2,V_{[0,\lambda);t}} \cdot \|\mathrm{d}\chi^-_{t,\lambda}\|_{-1,V_{[0,\lambda);t}} \cdot \|b_t\|_{1,V_{[0,\lambda);t}} \leq c\lambda^2.
\end{aligned}$$

Step 3. *We prove that* $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{(\lambda,\sigma/2);t}} \leq c\lambda^2$.

This is an immediate consequence of $F_{B_t} \wedge \psi_t = 0$ and Proposition 7.6 since in $V_{[\lambda,\sigma/2);t}$ the curvature of $A_{t,\lambda}$ is given by $F_{A_{t,\lambda}} = F_{B_t} + [i_{t,\lambda} \wedge b_t] + F_{I_{t,\lambda}} - F_{B_t|_{P_t}}$.

Step 4. *We prove that* $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{[\sigma/2,\sigma);t}} \leq c\lambda^2$.

In $V_{[\sigma/2,\sigma);t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{B_t} + \chi^+_t \mathrm{d}_{B_t} i_{t,\lambda} + \frac{1}{2}(\chi^+_t)^2[i_{t,\lambda} \wedge i_{t,\lambda}] + \mathrm{d}\chi^+_t \wedge i_{t,\lambda}.$$

Since $\|\chi^+_t\|_{\ell,V_{[\sigma/2,\sigma);t}} + \|\mathrm{d}\chi^+_t\|_{\ell,V_{[\sigma/2,\sigma);t}} \leq c$, it follows that

$$\begin{aligned}
&\|F_{A_{t,\lambda}} - F_{B_t}\|_{-2,V_{[\sigma/2,\sigma);t}} \\
&\leq \|\chi^+_t\|_{2,V_{[\sigma/2,\sigma);t}} \cdot \|\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda}\|_{-4,V_{[\sigma/2,\sigma);t}} \\
&\quad + \|\chi^+_t\|_{0,V_{[\sigma/2,\sigma);t}} \cdot \|b_t\|_{1,V_{[\sigma/2,\sigma);t}} \cdot \|i_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma);t}} \\
&\quad + \frac{1}{2}\|\chi^+_t\|^2_{2,V_{[\sigma/2,\sigma);t}} \cdot \|i_{t,\lambda}\|^2_{-3,V_{[\sigma/2,\sigma);t}} \\
&\quad + \|\mathrm{d}\chi^+_t\|_{1,V_{[\sigma/2,\sigma);t}} \cdot \|i_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma);t}} \leq c\lambda^2.
\end{aligned}$$

We are thus left with estimating

$$\|\iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t - \hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{-2,V_{[\sigma/2,\sigma);t}} \leq c\|\chi^+_t - 1\|_{1,V_{[\sigma/2,\sigma);t}} \cdot \|\hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma);t}}.$$

To conclude the proof we observe that $\|\chi^+_t - 1\|_{1,V_{[\sigma/2,\sigma);t}} \leq c$ and that

$$\|\hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-3,0;t,\lambda}(V_{[0,\sigma);t})} \leq c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{k,\alpha}} \tag{7.1}$$

as a consequence of Proposition 3.1 and a simple scaling consideration. □

8. Linear estimates

We denote by $\mathfrak{X}_{t,\lambda}$ and $\mathfrak{Y}_{t,\lambda}$ the Banach spaces $C^{1,\alpha} \oplus \mathbf{R}$ and $C^{0,\alpha} \oplus \mathbf{R}$ equipped with the norms

$$\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}} := \lambda^{-\delta/2}\|\rho_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}} + \lambda\|\pi_{t,\lambda}\underline{a}\|_{C^{1,\alpha}} + \lambda|\eta| \quad \text{and}$$
$$\|(\underline{a},\eta)\|_{\mathfrak{Y}_{t,\lambda}} := \lambda^{-\delta/2}\|\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \lambda\|\pi_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} + \lambda|\eta|,$$

respectively. Here we fixed $\delta \in (-1,0)$ and $0 < \alpha \ll |\delta|$. For concreteness one may take $\delta = -\frac{1}{2}$ and $\alpha = \frac{1}{256}$. It will become apparent in the course of this section that the choice of the relative weights between terms involving $\rho_{t,\lambda}\underline{a}$ and those involving $\pi_{t,\lambda}\underline{a}$ is not completely unnatural. We consider the linear operator $\mathbf{L}_{t,\lambda} : \mathfrak{X}_{t,\lambda} \to \mathfrak{Y}_{t,\lambda}$ defined by

$$\mathbf{L}_{t,\lambda}(\underline{a},\eta) := \left(L_{t,\lambda} + \eta \cdot \iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t, \langle \pi_{t,\lambda}\underline{a}, \hat{v} \circ \mathfrak{I}_t\rangle\right).$$

Proposition 8.1. *For all $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$ the linear operator $\mathbf{L}_{t,\lambda}$ is invertible, $\mathbf{L}_{t,\lambda}^{-1}$ depends continuously differentiably on t and continuously on λ and, moreover, there exists a constant $c > 0$, which is independent of $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$, such that*

$$\|\mathbf{L}_{t,\lambda}^{-1}(\underline{b},\zeta)\|_{\mathfrak{X}_{t,\lambda}} \leq c\|(\underline{b},\zeta)\|_{\mathfrak{Y}_{t,\lambda}} \quad \textit{and} \tag{8.1}$$
$$\|\nabla_t\mathbf{L}_{t,\lambda}^{-1}(\underline{b},\zeta)\|_{\mathfrak{X}_{t,\lambda}} \leq c\|(\underline{b},\zeta)\|_{\mathfrak{Y}_{t,\lambda}}. \tag{8.2}$$

The key to this proposition is the following estimate which we will prove in the course of this section.

Proposition 8.2. *There exists a constant $c > 0$, which is independent of $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$, such that*

$$\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}} \leq c\|\mathbf{L}_{t,\lambda}(\underline{a},\eta)\|_{\mathfrak{Y}_{t,\lambda}}. \tag{8.3}$$

Proposition 8.3. *The family of operators $\mathbf{L}_{t,\lambda} : \mathfrak{X}_{t,\lambda} \to \mathfrak{Y}_{t,\lambda}$ depends continuously differentiably on t and continuously on λ and there exists a constant $c > 0$ such that for all $t \in (-T',T')$ and $\lambda \in (0,\Lambda]$ we have*

$$\|\mathbf{L}_{t,\lambda}(\underline{a},\eta)\|_{\mathfrak{Y}_{t,\lambda}} \leq c\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}} \quad \textit{and}$$
$$\|\nabla_t\mathbf{L}_{t,\lambda}(\underline{a},\eta)\|_{\mathfrak{Y}_{t,\lambda}} \leq c\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}}.$$

Proof of Proposition 8.1. By Proposition 8.2 the operator $\mathbf{L}_{t,\lambda}$ is injective and has closed range. Hence, we can identify its cokernel with the kernel of

$\mathbf{L}_{t,\lambda}^*$. Since $\mathbf{L}_{t,\lambda}$ is formally self-adjoint, it follows from elliptic regularity that the kernel of $\mathbf{L}_{t,\lambda}^*$ agrees with the kernel of $\mathbf{L}_{t,\lambda}^*$ and thus is trivial. Therefore, $\mathbf{L}_{t,\lambda}$ is invertible. Now, (8.1) follows at once from (8.3). Since $\mathbf{L}_{t,\lambda}$ depends continuously differentiably on t and continuously on λ, so does $\mathbf{L}_{t,\lambda}$. Since $\nabla_t\mathbf{L}_{t,\lambda}^{-1} = -\mathbf{L}_{t,\lambda}^{-1}\nabla_t\mathbf{L}_{t,\lambda}\mathbf{L}_{t,\lambda}^{-1}$, (8.2) follows from (8.1) and Proposition 8.3. □

8.1. The model operator on $\mathbf{R}^7$

Let I be a finite energy ASD instanton on a G–bundle E over $\mathbf{R}^4$. By a slight abuse of notation we denote the pullbacks of I and E to $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ by I and E as well. We define $L_I : \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ by

$$\begin{pmatrix} 0 & \mathrm{d}_I^* \\ \mathrm{d}_I & *(\psi_0 \wedge \mathrm{d}_I) \end{pmatrix}$$

where

$$\psi_0 := \frac{1}{2}\omega_1 \wedge \omega_1 - \mathrm{d}x^{23} \wedge \omega_1 - \mathrm{d}x^{31} \wedge \omega_2 - \mathrm{d}x^{12} \wedge \omega_3 \quad \text{and}$$
$$\omega_1 := \mathrm{d}x^{45} + \mathrm{d}x^{67}, \quad \omega_2 := \mathrm{d}x^{46} - \mathrm{d}x^{47} \quad \text{and} \quad \omega_3 := \mathrm{d}x^{47} + \mathrm{d}x^{56}.$$

Denote by $\pi_{\mathbf{R}^4} : \mathbf{R}^3 \oplus \mathbf{R}^4 \to \mathbf{R}^4$ the projection onto the second summand and define weight functions

$$w(x) := 1 + |\pi_{\mathbf{R}^4}(x)| \quad \text{and} \quad w(x,y) := \min\{w(x), w(y)\}. \tag{8.4}$$

For a Hölder exponent $\alpha \in (0,1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$[f]_{C^{0,\alpha}_\beta(U)} := \sup_{d(x,y)\le w(x,y)} w(x,y)^{\alpha-\beta}\frac{|f(x)-f(y)|}{d(x,y)^\alpha},$$
$$\|f\|_{L^\infty_\beta(U)} := \|w^{-\beta}f\|_{L^\infty(U)} \quad \text{and}$$
$$\|f\|_{C^{k,\alpha}_\beta(U)} := \sum_{j=0}^k \|\nabla^j f\|_{L^\infty_{\beta-j}(U)} + [\nabla^j f]_{C^{0,\alpha}_{\beta-j}(U)}.$$

Here f is a section of a vector bundle over $U \subset \mathbf{R}^7$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = \mathbf{R}^7$. We denote by $C^{k,\alpha}_\beta$ the subspace of elements f of the Banach space $C^{k,\alpha}$ with $\|f\|_{C^{k,\alpha}_\beta} < \infty$ equipped with the norm $\|\cdot\|_{C^{k,\alpha}_\beta}$.

The linear operators L_I can serve as a model for $L_{t,\lambda}$ in the following sense: Fix $t \in (0, T']$ and $x \in P_t$. Set $I := I(\mathfrak{I})|_{N_xP_t}$ and $E := E(\mathfrak{I}_t)|_{N_xP_t}$.

Identify $T_xY = T_xP_t \oplus N_xP_t$ with $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ in such a way that the summands are preserved and $\psi_t|_{T_xY}$ is identified with ψ_0. For $\epsilon_1, \epsilon_2 > 0$ we define $V_{\epsilon_1,\epsilon_2;t}$ to be the open set which under the exponential map based at x is identified with $\tilde{U}_{\epsilon_1,\epsilon_2} := B_{\epsilon_1}(0) \times B_{\epsilon_2}(0) \subset \mathbf{R}^3 \oplus \mathbf{R}^4$. With respect to this identification a $\mathfrak{g}_{E_{t,\lambda}}$–valued tensor field f on $V_{\epsilon_1,\epsilon_2;t}$ is identified with a $s^*_{1/\lambda}\mathfrak{g}_E$–valued tensor field $\tilde{f}$ on $\tilde{U}_{\epsilon_1,\epsilon_2;\lambda}$, and if $k \in \mathbf{N}$ is a scaling parameter, then with f we can associate a $\mathfrak{g}_E$–valued tensor field $s_{k,\lambda}f$ on $U_{\epsilon_1,\epsilon_2;\lambda} := \lambda^{-1}\tilde{U}_{\epsilon_1,\epsilon_2}$ defined by

$$(s_{k,\lambda}f)(x,y) := \lambda^k \tilde{f}(\lambda x, \lambda y).$$

Proposition 8.4. *There is are constants $c, \epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$, $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\frac{1}{c}\|s_{k,\lambda}f\|_{L^\infty_{\ell+\delta}(U_{\epsilon,\sqrt{\lambda};\lambda_i})} \leq \lambda^{k+\ell}\|f\|_{L^\infty_{\ell,\delta;t,\lambda}(V_{\epsilon,\sqrt{\lambda};t})} \leq c\|s_{k,\lambda}f\|_{L^\infty_{\ell+\delta}(U_{\epsilon,\sqrt{\lambda};\lambda_i})},$$
$$\frac{1}{c}\|s_{k,\lambda}f\|_{C^{k,\alpha}_{\ell+\delta}(U_{\epsilon,\sqrt{\lambda};\lambda_i})} \leq \lambda^{k+\ell}\|f\|_{C^{k,\alpha}_{\ell,\delta;t,\lambda}(V_{\epsilon,\sqrt{\lambda};t})} \leq c\|s_{k,\lambda}f\|_{C^{k,\alpha}_{\ell+\delta}(U_{\epsilon,\sqrt{\lambda};\lambda_i})}$$

and

$$\left\|L_{t,\lambda}\underline{a} - s^{-1}_{2,\lambda}L_I s_{1,\lambda}\underline{a}\right\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}(V_{\epsilon,\sqrt{\lambda};t})} \leq c(\epsilon + \sqrt{\lambda})\,\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}(V_{\epsilon,\sqrt{\lambda};t})}.$$

To better understand L_I it is useful to rewrite it as follows.

Proposition 8.5. *If we identify $T^*\mathbf{R}^3$ with Λ^+ via $\mathrm{d}x^i \mapsto -\omega_i$ and accordingly $\Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ with $\Omega^0(\mathbf{R}^7, \left(\mathbf{R} \oplus T^*\mathbf{R}^3 \oplus T^*\mathbf{R}^4\right) \otimes \mathfrak{g}_E)$, then the linear operator L_I can be written as $L_I = F + D_I$ where*

$$F(\xi, \omega, a) := \sum_{i=1}^{3}\left(-\langle\partial_i\omega, \omega_i\rangle, \partial_i\xi \cdot \omega_i, *_4(\partial_i a \wedge \omega_i)\right),$$
$$D_I := \begin{pmatrix} 0 & \delta_I \\ \delta_I^* & 0 \end{pmatrix} \quad \textit{and}$$

$\delta_I : \Omega^1(\mathbf{R}^4, \mathfrak{g}_E) \to \Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ is as defined in (3.1)*. Moreover,*

$$L_I^* L_I = \Delta_{\mathbf{R}^3} + \begin{pmatrix} \delta_I\delta_I^* & \\ & \delta_I^*\delta_I \end{pmatrix}.$$

Proof. This is easy to verify by a straight-forward computation. □

Let us now recall a key result from [Wal13b, Appendix A].

Definition 8.6. A Riemannian manifold X is said to be of *bounded geometry* if it is complete, its Riemann curvature tensor is bounded from above and its injectivity radius is bounded from below. A vector bundle over X is said to be of *bounded geometry* if it has trivialisations over balls of a fixed radius such that the transitions functions and all of their derivatives are uniformly bounded. We say that a complete oriented Riemannian manifold X has *subexponential volume growth* if for each $x \in X$ the function $r \mapsto \mathrm{vol}\,(B_r(x))$ grows subexponentially as $r \to \infty$.

Lemma 8.7. *Let E be a vector bundle of bounded geometry over a Riemannian manifold X of bounded geometry with subexponential volume growth and suppose that $D : C^\infty(X, E) \to C^\infty(X, E)$ is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, i.e., $\langle Da, a\rangle \geq 0$ for all $a \in W^{2,2}(X, E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

and $\|a\|_{L^\infty}$ is finite, then a is constant in the $\mathbf{R}^n$–direction, that is $a(x, y) = a(y)$. Here, by slight abuse of notation, we denote the pullback of E to $\mathbf{R}^n \times X$ by E as well.

Remark 8.8. The statement in [Wal13b, Appendix A] also requires $\|\nabla a\|_{L^\infty}$ to be finite. This, however, can be deduced from $\|a\|_{L^\infty} < \infty$, elliptic estimates and the equation $(\Delta_{\mathbf{R}^n} + D)a = 0$.

Corollary 8.9. *If $\underline{a} \in \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ satisfies $L_I = 0$ and $\|\underline{a}\|_{L^\infty}$ is finite, then $\underline{a}$ is the pullback of an element in the kernel of δ_I.*

Proposition 8.10. *For $\beta \in \mathbf{R}$ there is a constant $c = c(I) > 0$ depending continuously on I such that the following estimate holds*

$$\|\underline{a}\|_{C^{1,\alpha}_\beta} \leq c \left(\|L_I \underline{a}\|_{C^{0,\alpha}_{\beta-1}} + \|\underline{a}\|_{L^\infty_\beta} \right).$$

Proof. This is a standard result. The argument we use goes back to work of Nirenberg–Walker [NW73, Theorem 3.1].

The desired estimate is local in the sense that is enough to prove estimates of the form

$$\|\underline{a}\|_{C^{1,\alpha}_\beta(U_i)} \leq c\big(\|L_I\underline{a}\|_{C^{0,\alpha}_{\beta-1}} + \|\underline{a}\|_{L^\infty_\beta}\big)$$

with $c > 0$ independent of i, where $\{U_i\}$ is a suitable open cover of $\mathbf{R}^3 \times X$.

Fix $R > 0$ suitably large and set $U_0 := \{(x, y) \in \mathbf{R}^3 \times X : |\pi_{\mathbf{R}^4}(x)| \leq R\}$. Then there clearly is a constant $c > 0$ such that the above estimate holds for $U_i = U_0$. Pick a sequence $(x_i, y_i) \in \mathbf{R}^3 \times X$ such that $r_i := |\pi_{\mathbf{R}^4}(y_i)| \geq R$ and the balls $U_i := B_{r_i/8}(x_i, y_i)$ cover the complement of U_0. On U_i, we have a Schauder estimate of the form

$$\begin{aligned}\|\underline{a}\|_{L^\infty(U_i)} + r_i^\alpha[\underline{a}]_{C^{0,\alpha}(U_i)} + r_i\|\nabla_I\underline{a}\|_{L^\infty(U_i)} + r_i^{1+\alpha}[\nabla_I\underline{a}]_{C^{0,\alpha}(U_i)}\\ \leq c\left(r_i\|L_I\underline{a}\|_{L^\infty(V_i)} + r_i^{1+\alpha}[L_I\underline{a}]_{C^{0,\alpha}(V_i)} + \|\underline{a}\|_{L^\infty(V_i)}\right)\end{aligned}$$

where $V_i = B_{r_i/4}(x_i, y_i)$ and $\underline{a} = (\xi, a)$. By rescaling the balls V_i to a ball of fixed radius one can see that the constant $c > 0$ can be chosen to work for all i simultaneously. Since on V_i we have $\frac{1}{2}r_i \leq w \leq 2r_i$, multiplying the above Schauder estimate by $r_i^{-\beta}$ yields the desired local estimate. □

8.2. Schauder estimate

Proposition 8.11. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following estimate holds*

$$\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}} \leq c\left(\|L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \|\underline{a}\|_{L^\infty_{-1,\delta;t,\lambda}}\right). \tag{8.5}$$

Proof. It suffices to show that there is a constant $c > 0$ such that for all $t \in (-T', T')$, $\lambda \in (0, \Lambda]$ and $x \in Y$ there exist open sets U and V such that

$$\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}(U)} \leq c\left(\|L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}(V)} + \|\underline{a}\|_{L^\infty_{-1,\delta;t,\lambda}(V)}\right).$$

For $x \in Y$ with $r_t(x) \leq \sqrt{\lambda}$ such an estimate follows from Proposition 8.4 and Proposition 8.10. For $x \in Y$ with $r_t(x) > \sqrt{\lambda}$ one can take $U = B_{r_t(x)/8}(x)$ and $V = B_{r_t(x)/4}(x)$ and argue as in the proof of Proposition 8.10. □

8.3. Estimate of $\|\rho_{t,\lambda}\underline{a}\|_{L^\infty_{-1,\delta;t,\lambda}}$

Proposition 8.12. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following estimate holds*

$$\|\underline{a}\|_{L^\infty_{-1,\delta;t,\lambda}} \le c\left(\|L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \|\bar{\pi}_{t,\lambda}\underline{a}\|_{L^\infty_{-1,\delta;t,\lambda}}\right). \tag{8.6}$$

Proof. If not, then there exist sequences (t_i), (λ_i) and $(\underline{a}_i)$ such that $\lim_{i\to\infty}\lambda_i = 0$,

$$\begin{aligned} \|\underline{a}_i\|_{L^\infty_{-1,\delta;t_i,\lambda_i}} &= 1, \\ \lim_{i\to\infty}\|L_{t_i,\lambda_i}\underline{a}_i\|_{C^{0,\alpha}_{-2,\delta;t_i,\lambda_i}} &= 0 \quad \text{and} \\ \lim_{i\to\infty}\|\bar{\pi}_{t_i,\lambda_i}\underline{a}_i\|_{L^\infty_{-1,\delta;t_i,\lambda_i}} &= 0. \end{aligned} \tag{8.7}$$

After passing to a subsequence we can assume that (t_i) converges to a limit t. From Proposition 8.11 it follows that

$$\|\underline{a}_i\|_{C^{1,\alpha}_{-1,\delta;t_i,\lambda_i}} \le c. \tag{8.8}$$

Pick a sequence (x_i) of points in Y such that

$$w_{-1,\delta;t_i,\lambda_i}(x_i)|\underline{a}_i|(x_i) = 1.$$

After passing to a subsequence we can assume that one of the following cases occurs. We will show that each of them leads to a contradiction, thus proving the proposition.

Case 1. *The sequence (x_i) accumulates away from P_t: $\lim_{i\to\infty} r_{t_i}(x_i) > 0$.*

By (8.8) the sequence $(\underline{a}_i)$ is uniformly bounded in $C^{1,\alpha}$ on each compact subset of $Y \setminus P_t$. Arzelà–Ascoli and a diagonal sequence argument thus yield a subsequence of $(\underline{a}_i)$ which converges to a limit $\underline{a}$ on $Y \setminus P_t$ in $C^{1,\alpha/2}_{\mathrm{loc}}$. Since we can also arrange that the corresponding subsequence of (x_i) converges to a limit $x \in Y \setminus P_t$ for which $r_t(x)^{1+\delta}|\underline{a}|(x) = 1$, it follows that $\underline{a}$ cannot vanish identically. However, $\underline{a}$ also satisfies

$$\|r_t^{1+\delta}\underline{a}\|_{L^\infty} \le 1 \quad \text{and} \tag{8.9}$$

$$L_{B_t,\phi_t}\underline{a} = 0 \quad \text{on } Y \setminus P_t. \tag{8.10}$$

Since $\delta < 2$, it follows from (8.9) that $\underline{a}$ satisfies (8.10) on all of Y in the sense of distribution and hence is smooth by elliptic regularity. This contradicts the hypothesis that A_0 and hence B_t is acyclic, i.e., that L_{B_t,ϕ_t} has trivial kernel.

Case 2. *The sequence (x_i) quickly accumulates near P_t:* $\lim_{i\to\infty} r_{t_i}(x_i)/\lambda_i < \infty$.

After passing to a subsequence we can assume that (x_i) converges to a point $x \in P_t$. With the notation of the paragraph preceding Proposition 8.4, set

$$\underline{b}_i := s_{1,\lambda_i}\left(\underline{a}_i|_{V_{\sqrt{\lambda_i},\sqrt{\lambda_i};t}}\right).$$

This sequence satisfies

$$\|\underline{b}_i\|_{C^{1,\alpha}_{-1+\delta}(U_{\sqrt{\lambda_i},\sqrt{\lambda_i};\lambda_i})} \leq c \quad \text{and} \quad \lim_{i\to\infty} \|L_I\underline{b}_i\|_{C^{0,\alpha}_{-2+\delta}} = 0,$$

and if (y_i) denotes the sequence of points in $U_{\sqrt{\lambda_i},\sqrt{\lambda_i};\lambda_i}$ corresponding to the sequence (x_i), then

$$w(y_i)^{1-\delta}|\underline{b}_i|(y_i) \geq \frac{1}{2}.$$

where $w = 1 + |\pi_{\mathbf{R}^4}|$ as in (8.4). Since the sequence of subsets $U_{\sqrt{\lambda_i},\sqrt{\lambda_i};\lambda_i} \subset \mathbf{R}^7$ is exhaustive, Arzelà–Ascoli and a diagonal sequence argument yield a subsequence of $(\underline{b}_i)$ which converges to a limit $\underline{b}$ on $\mathbf{R}^7$ in $C^{1,\alpha/2}_{\mathrm{loc}}$. By translation we can arrange that the $\mathbf{R}^3$–component of y_i is zero and thus $|y_i|$ is bounded. After passing to a further subsequence (y_i) converges to a limit $y \in \mathbf{R}^7$. At this point we must have $w(y)^{1-\delta}|\underline{b}|(y) \geq 1/2$ and thus $\underline{b}$ cannot vanish identically. It follows from Proposition 8.4 that $\underline{b}$ satisfies

$$\|w^{1-\delta}\underline{b}\|_{L^\infty} \leq 2 \quad \text{and} \quad L_I\underline{b} = 0.$$

Moreover, using (8.7) and arguing as in the proof of Proposition 6.4, making use of the hypothesis $\delta < 0$, one can show that each restriction of $\underline{b} = 0$ to a slice $\{x\} \times \mathbf{R}^4$ is L^2–orthogonal to $\ker \delta_I$. This, however, contradicts Corollary 8.9.

Case 3. *The sequence (x_i) slowly accumulates near P_t:* $\lim_{i\to\infty} r_{t_i}(x_i)/\lambda_i = \infty$.

In a similar manner as in the previous case we set

$$\underline{b}_i := s_{1,\lambda_i}\left(\underline{a}_i|_{V_{\sqrt{\lambda}_i,\sigma;t}}\right)$$

and denote by (y_i) the sequence of points in $U_{\sqrt{\lambda}_i,\sigma;\lambda_i}$. Again, we can assume that the $\mathbf{R}^3$–component of y_i is zero. After passing to a subsequence we can assume that one of the following two cases occurs.

Case 3.1. *We have* $|y_i| \leq 1/\sqrt{\lambda}_i$ *for all* $i \in \mathbf{N}$.

Set

$$\tilde{\underline{b}}_i := |y_i|^{1-\delta}\underline{b}_i(|y_i|\cdot -) \quad \text{and} \quad \tilde{y}_i := y_i/|y_i|.$$

Again, Arzelà–Ascoli and a diagonal sequence argument yield a subsequence of $(\tilde{\underline{b}}_i)$ converging to a limit $\tilde{\underline{b}}$ on $\mathbf{R}^7 \setminus (\mathbf{R}^3 \times \{0\})$ which cannot vanish identically, since $|\tilde{y}|^{1-\delta}|\tilde{\underline{a}}|(\tilde{y}) \geq 1/4$ with $\tilde{y} := \lim_{t\to\infty}\tilde{y}_i$. However, $\tilde{\underline{b}}$ also satisfies

$$\|\tilde{w}^{1-\delta}\tilde{\underline{b}}\|_{L^\infty} \leq 4 \quad \text{and} \tag{8.11}$$

$$L\tilde{\underline{b}} = 0 \quad \text{on } \mathbf{R}^7 \setminus (\mathbf{R}^3 \times \{0\}). \tag{8.12}$$

Here $\tilde{w} := |\pi_{\mathbf{R}^4}|$ and L is defined by

$$L\underline{a} := (\mathrm{d}^*a, \mathrm{d}\xi + *(\psi \wedge \mathrm{d}a)). \tag{8.13}$$

Since $\delta > -2$, it follows from (8.11) that $\tilde{\underline{b}}$ solves (8.12) on all of $\mathbf{R}^7$ in the sense of distributions and hence is smooth by elliptic regularity. Moreover, using standard elliptic estimates one can show that $\tilde{\underline{b}}$ is uniformly bounded near $\mathbf{R}^3$ and therefore by (8.11) on all of $\mathbf{R}^7$ since $\delta \leq 1$. Because $L^*L = \Delta_{\mathbf{R}^3} + \Delta_{\mathbf{R}^4}$, we can now apply Lemma 8.7 to conclude that $\tilde{\underline{b}}$ is invariant under translations in the $\mathbf{R}^4$–direction. We can thus think of $\tilde{\underline{b}}$ as a vector of harmonic functions on $\mathbf{R}^4$. Since $\delta < 1$ it follows that the components of $\tilde{\underline{b}}$ decay to zero at infinity and thus vanish by the maximum principle. This, however, contradicts the fact that $\tilde{\underline{b}}$ cannot vanish identically.

Case 3.2. *We have* $|y_i| > 1/\sqrt{\lambda}_i$ *for all* $i \in \mathbf{N}$.

If we set

$$\tilde{\underline{b}}_i := \lambda_i^\delta|y_i|^{1+\delta}\underline{b}_i(|y_i|\cdot -) \quad \text{and} \quad \tilde{y}_i := y_i/|y_i|,$$

then we obtain the desired contradiction by arguing, mutatis mutandis, as the previous case. The relevant constraint on δ is easily seen to be that $\delta \in (-1, 2)$. □

8.4. Comparison with $F_{\mathfrak{I}_t}$

The connection on $\coprod_{t\in(-T,T)} NP_t$ induces connections on the bundles over $(-T,T)$ whose fibres are $C^{0,\alpha}(\mathfrak{I}^*_{t,\lambda}V\mathfrak{M}_t)$ and $C^{1,\alpha}(\mathfrak{I}^*_{t,\lambda}V\mathfrak{M}_t)$, respectively. We denote the corresponding covariant derivatives by ∇_t.

Proposition 8.13. *There is a constant $c>0$ such that for all $t\in(-T',T')$ and $\lambda\in(0,\Lambda]$ the following estimate holds*

$$\|L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \quad \textit{and}$$
$$\|(\nabla_t L_{t,\lambda})\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}(\nabla_t F_{\mathfrak{I}_t})\hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \le c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

Proof. Consider the operator

$$\tilde{L}_t : \Omega^0(NP_t,\mathfrak{g}_{E_t}) \oplus \Omega^1(NP_t,\mathfrak{g}_{E_t}) \to \Omega^0(NP_t,\mathfrak{g}_{E_t}) \oplus \Omega^1(NP_t,\mathfrak{g}_{E_t})$$

defined by

$$\tilde{L}_{t,\lambda}\underline{a} := (\mathrm{d}^*_{I_{t,\lambda}}a, \mathrm{d}_{I_{t,\lambda}}\xi + *_0(\psi_{0;t}\wedge \mathrm{d}_{I_{t,\lambda}}a)).$$

If we identify $\hat{\mathfrak{I}}\in\Gamma(\mathfrak{I}^*_t V\mathfrak{M}_t)$ with an element of $\Omega^1(NP_t,\mathfrak{g}_{E_t})$, then since $\delta_{\mathfrak{I}_t(x)}(\hat{\mathfrak{I}}|_{N_xP_t})=0$ we have

$$\tilde{L}_{t,\lambda}\hat{\mathfrak{I}} = \left(0, *_0\left[\psi_{0;t}\wedge(\mathrm{d}_{I_{t,\lambda}}\hat{\mathfrak{I}})^{1,1}\right]\right)$$

which is the same as $s^*_{1/\lambda}\circ F_{\mathfrak{I}_t}\circ s^*_\lambda(\hat{\mathfrak{I}})$.

In order to prove the first estimate it thus suffices to control the following terms

$$\begin{aligned} L_{t,\lambda}\iota_{t,\lambda}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}} &= L_{t,\lambda}(\iota_{t,\lambda}\hat{\mathfrak{I}} - \hat{\mathfrak{I}}_\lambda) + (L_{t,\lambda}-\tilde{L}_{t,\lambda})\hat{\mathfrak{I}}_\lambda \\ &\quad + (s^*_{1/\lambda}\circ F_{\mathfrak{I}_t}\hat{\mathfrak{I}} - \iota_{t,\lambda}F_{\mathfrak{I}_t}\hat{\mathfrak{I}}) \\ &=: \mathrm{I}+\mathrm{II}+\mathrm{III} \end{aligned}$$

on $V_{[0,\sigma);t}$. It is easy to see that

$$\|\mathrm{I}\|_{C^{0,\alpha}_{-2,0;t,\lambda}(V_{[0,\sigma);t})} + \|\mathrm{III}\|_{C^{0,\alpha}_{-2,0;t,\lambda}(V_{[0,\sigma);t})} \le c\lambda^2\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}$$

by using that fact that I and III are supported in $V_{[\sigma/2,\sigma);t}$ and the estimates

$$\|L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,0;t,\lambda}(V_{[0,\sigma);t})} \le c\|\underline{a}\|_{C^{1,\alpha}_{-1,0;t,\lambda}(V_{[0,\sigma);t})} \quad \text{and} \quad \|F_{\mathfrak{I}_t}\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \le c\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.$$

as well as

$$
\begin{aligned}
&\|\iota_{t,\lambda}\hat{\mathfrak{I}} - \hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-\ell,0;t,\lambda}(V_{[\sigma/2,\sigma);t})} \\
&\leq \|\chi_t^+ - 1\|_{C^{k,\alpha}_{\ell+3,0;t,\lambda}(V_{[\sigma/2,\sigma);t})} \cdot \|\hat{\mathfrak{I}}_\lambda\|_{C^{k,\alpha}_{-3,0;t,\lambda}(V_{[\sigma/2,\sigma);t})} \\
&\leq c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{k,\alpha}}.
\end{aligned}
$$

To estimate II we expand it as

$$
\begin{aligned}
\mathrm{II} = &* \Big(\psi_t \wedge (A_{t,\lambda} - I_{t,\lambda}) \wedge \hat{\mathfrak{I}}_\lambda\Big) + * \Big((\psi_{1;t} + \psi_{\geq 2;t}) \wedge \mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda\Big) \\
&+ (* - *_0)(\psi_{0;t} \wedge \mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda) \\
=&: \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.
\end{aligned}
$$

It follows from Proposition 7.6 that

$$
\|A_{t,\lambda} - I_{t,\lambda}\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} = \|\chi^-_{t,\lambda} b_t + (\chi_t^+ - 1) i_{t,\lambda}\|_{C^{0,\alpha}_{1,0;t,\lambda}(V_{[0,\sigma);t})} \leq c \tag{8.14}
$$

which in conjunction with (7.1) yields

$$
\|\mathrm{II}_1\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \leq c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.
$$

From Proposition 3.1 and simple scaling considerations it follows that

$$
\|(\mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda)^{1,1}\|_{C^{0,\alpha}_{-3,0;t,\lambda}(V_{[0,\sigma);t})} + \|(\mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda)^{0,2}\|_{C^{0,\alpha}_{-4,0;t,\lambda}(V_{[0,\sigma);t})} \leq c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.
$$

Since $\delta_{\mathfrak{I}_t(x)}(\hat{\mathfrak{I}}|_{N_x P_t}) = 0$, we have

$$
\psi_{0;t} \wedge (\mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda)^{0,2} = \psi_{1;t} \wedge (\mathrm{d}_{I_{t,\lambda}} \hat{\mathfrak{I}}_\lambda)^{0,2} = 0.
$$

These facts together with Proposition 7.3 imply that

$$
\|\mathrm{II}_2\|_{C^{0,\alpha}_{-2,0;t,\lambda}} + \|\mathrm{II}_3\|_{C^{0,\alpha}_{-2,0;t,\lambda}} \leq c\lambda^2 \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}}.
$$

This finishes the proof of the first estimate. To prove the second estimate note that the individual terms of

$$
\begin{aligned}
(\nabla_t L_{t,\lambda}\iota_{t,\lambda})\hat{\mathfrak{I}} - \nabla_t F_{\mathfrak{I}_t}\hat{\mathfrak{I}} = &(\nabla_t L_{t,\lambda})(\iota_{t,\lambda}\hat{\mathfrak{I}} - \hat{\mathfrak{I}}_\lambda) + \nabla_t(L_{t,\lambda} - \tilde{L}_{t,\lambda})\hat{\mathfrak{I}}_\lambda \\
&+ (s^*_{1/\lambda} \circ \nabla_t F_{\mathfrak{I}_t}\hat{\mathfrak{I}} - \iota_{t,\lambda}\nabla_t F_{\mathfrak{I}_t}\hat{\mathfrak{I}})
\end{aligned}
$$

can be estimated just as above. $\square$

Proposition 8.14. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \leq c \left(\|\pi_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathfrak{I}}\|_{C^{0,\alpha}} + \left| \langle \hat{\mathfrak{I}}, \hat{v} \circ \mathfrak{I}_t \rangle \right| \right).$$

Proof. By hypothesis we have

$$\|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \leq c \left(\|F_{\mathfrak{I}_t} \hat{\mathfrak{I}}\|_{C^{0,\alpha}} + \left| \langle \hat{\mathfrak{I}}, \hat{v} \circ \mathfrak{I}_t \rangle \right| \right)$$

for $t = 0$ and thus also for $t \in (-T', T')$. Together with

$$\|F_{\mathfrak{I}_t} \hat{\mathfrak{I}}\|_{C^{0,\alpha}} \leq c \left(\|\pi_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathfrak{I}}\|_{C^{0,\alpha}} + \lambda^{1-\alpha} \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \right),$$

which is an immediate corollary of Proposition 6.4 and Proposition 8.13, this immediately implies the asserted estimate by rearranging. □

8.5. Cross-term estimates

Proposition 8.15. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|\rho_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} &\leq c\lambda^{2-\alpha} \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \quad \textit{and} \\ \|\rho_{t,\lambda} (\nabla_t L_{t,\lambda}) \iota_{t,\lambda} \hat{\mathfrak{I}}\|_{C^{0,\alpha}_{-2,0;t,\lambda}} &\leq c\lambda^{2-\alpha} \|\hat{\mathfrak{I}}\|_{C^{1,\alpha}} \end{aligned}$$

as well as

$$\begin{aligned} \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} &\leq c\lambda^{-\alpha} \|\rho_{t,\lambda} \underline{a}\|_{C^{1,\alpha}_{-1,0;t,\lambda}} \quad \textit{and} \\ \|\pi_{t,\lambda} (\nabla_t L_{t,\lambda}) \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} &\leq c\lambda^{-\alpha} \|\rho_{t,\lambda} \underline{a}\|_{C^{1,\alpha}_{-1,0;t,\lambda}}. \end{aligned}$$

Proof. The first two estimates are immediate consequences of Proposition 6.4 and Proposition 8.13 because

$$\rho_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathfrak{I}} = \rho_{t,\lambda} (L_{t,\lambda} \iota_{t,\lambda} \hat{\mathfrak{I}} - \iota_{t,\lambda} F_{\mathfrak{I}_t} \hat{\mathfrak{I}})$$

and similarly for $\nabla_t L_{t,\lambda}$.

To prove the last two estimates first note that we can assume without loss of generality that $\underline{a}$ is supported in $V_{[0,\sigma)}$ and that $\underline{a} = \rho_{t,\lambda} \underline{a}$. Define

$\tilde{\pi}_{t,\lambda} : \Omega^1(NP_t, \mathfrak{g}_{E_t}) \to \Gamma(\mathfrak{I}_t^* V\mathfrak{M}_t)$ by

$$(\tilde{\pi}_{t,\lambda} a)(x) := \sum_\kappa \int_{N_x P_t} \left\langle a, s_{1/\lambda}^* \kappa \right\rangle s_{1/\lambda}^* \kappa$$

where, at each point $x \in P_t$, κ runs through an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ with respect to $\left\langle s_{1/\lambda}^* \cdot, s_{1/\lambda}^* \cdot \right\rangle$ and set $\tilde{\rho}_{t,\lambda} := \mathrm{id} - \tilde{\pi}_{t,\lambda}$. One can check that $\tilde{\pi}_{t,\lambda} a = 0$ implies that $\tilde{\pi}_{t,\lambda} \tilde{L}_{t,\lambda} a = 0$ where $\tilde{L}_{t,\lambda}$ is as defined in the proof of Proposition 8.13. Therefore

$$\begin{aligned} \pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a} &= \pi_{t,\lambda}(L_{t,\lambda} - \tilde{L}_{t,\lambda}) \rho_{t,\lambda} \underline{a} + (\pi_{t,\lambda} - \tilde{\pi}_{t,\lambda}) \tilde{L}_{t,\lambda} \rho_{t,\lambda} \underline{a} \\ &\quad + \tilde{\pi}_{t,\lambda} \tilde{L}_{t,\lambda} (\rho_{t,\lambda} - \tilde{\rho}_{t,\lambda}) \underline{a} \\ &=: \pi_{t,\lambda} \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Define $\tilde{\tilde{\pi}}_{t,\lambda}$ like $\tilde{\pi}_{t,\lambda}$ but take the inner product with $\iota_{t,\lambda}\kappa$ instead of $s_{1/\lambda}^* \kappa$ and let κ run through an orthonormal basis with respect to $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle_{L^2}$. If $\tilde{\mathrm{II}}$ and $\tilde{\mathrm{III}}$ denote the same expressions as II and III but with $\tilde{\tilde{\pi}}_{t,\lambda}$ in place of $\tilde{\pi}_{t,\lambda}$ and $\mathrm{id} - \tilde{\tilde{\pi}}_{t,\lambda}$ in place of $\tilde{\rho}_{t,\lambda}$, then $\tilde{\mathrm{II}}$ and $\tilde{\mathrm{III}}$ are supported in $V_{[\sigma/2,\sigma)}$ and one can argue as in the proof of Proposition 8.13 to show

$$\|\tilde{\mathrm{II}}\|_{C^{0,\alpha}} + \|\tilde{\mathrm{III}}\|_{C^{0,\alpha}} \le c\lambda^{-\alpha} \|\underline{a}\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}}.$$

The eigenvalues of the quadratic form $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle$ with respect to $\left\langle s_{1/\lambda}^* \cdot, s_{1/\lambda}^* \cdot \right\rangle$ differ from one by $O(\lambda^4)$; hence, the differences between II and $\tilde{\mathrm{II}}$ as well as between III and $\tilde{\mathrm{III}}$ are negligibly small.

To estimate I we write it as

$$\begin{aligned} \mathrm{I} = &* (\psi_t \wedge (A_{t,\lambda} - I_{t,\lambda}) \wedge \rho_{t,\lambda} \underline{a}) + * \left((\psi_{1;t} + \psi_{\ge 2;t}) \wedge \mathrm{d}_{I_{t,\lambda}} \rho_{t,\lambda} \underline{a} \right) \\ &+ (* - *_0) \left(\psi_0 \wedge \mathrm{d}_{I_{t,\lambda}} \rho_{t,\lambda} \underline{a} \right). \end{aligned}$$

Using Propositions 6.4, 7.3 and 7.4 as well as and (8.14) it follows that

$$\|\pi_{t,\lambda} \mathrm{I}\|_{C^{0,\alpha}} \le c\lambda^{-\alpha} \|\mathrm{I}\|_{C^{0,\alpha}_{-1,0;t,\lambda}(V_{[0,\sigma);t})} \le c\lambda^{-\alpha} \|\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}}.$$

This finishes the proof of the third estimate. The last estimate is proved along the same lines. □

8.6. Proof of Proposition 8.2

Applying Proposition 8.11 and Proposition 8.12 to $\rho_{t,\lambda}\underline{a}$ and using Proposition 6.4 yields

$$\begin{aligned}\|\rho_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}} &\leq c\|L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} \\ &\leq c\big(\|\rho_{t,\lambda}L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \|\rho_{t,\lambda}L_{t,\lambda}\bar{\pi}_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} \\ &\qquad + \lambda^{1-\alpha}\|\pi_{t,\lambda}L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}}\big).\end{aligned}$$

By Proposition 8.14

$$\|\pi_{t,\lambda}\underline{a}\|_{C^{1,\alpha}} \leq c\left(\|\pi_{t,\lambda}L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} + |\langle\pi_{t,\lambda}\underline{a}, \hat{v}\circ\mathfrak{I}_t\rangle| + \|\pi_{t,\lambda}L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}}\right).$$

Recalling the definitions of $\|\cdot\|_{\mathfrak{X}_{t,\lambda}}$, $\|\cdot\|_{\mathfrak{Y}_{t,\lambda}}$ and $\mathbf{L}_{t,\lambda}$, and using Proposition 8.15 it follows that

$$\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}} \leq c\left(\|\mathbf{L}_{t,\lambda}(\underline{a},\eta)\|_{\mathfrak{Y}_{t,\lambda}} + \lambda^{1-\alpha}\|(\underline{a},\eta)\|_{\mathfrak{X}_{t,\lambda}}\right)$$

which yields (8.3) by rearranging. □

8.7. Proof of Proposition 8.3

It is clear that $L_{t,\lambda}$ depends continuously differentiably on t and continuously on λ. By Proposition 8.13 and Proposition 8.15 as well as Corollary 6.3 and Proposition 6.4 we have

$$\begin{aligned}\|\rho_{t,\lambda}L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} &\leq c\Big(\|L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \lambda^{1-\alpha}\|\pi_{t,\lambda}L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} \\ &\qquad + \|\rho_{t,\lambda}L_{t,\lambda}\bar{\pi}_{t,\lambda}\underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}}\Big) \\ &\leq c\Big(\|\rho_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}} + \lambda^{2+\delta/2-\alpha}\|\pi_{t,\lambda}\underline{a}\|_{C^{1,\alpha}}\Big)\end{aligned}$$

and

$$\begin{aligned}\|\pi_{t,\lambda}L_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} &\leq \|F_{\mathfrak{I}_t}\pi_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} + \|\pi_{t,\lambda}L_{t,\lambda}\rho_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} \\ &\qquad + c\lambda^{1-\alpha}\|\pi_{t,\lambda}\underline{a}\|_{C^{0,\alpha}} \\ &\leq c\Big(\|\pi_{t,\lambda}\underline{a}\|_{C^{1,\alpha}} + \lambda^{-\alpha}\|\rho_{t,\lambda}\underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}}\Big).\end{aligned}$$

This yields $\|\mathbf{L}_{t,\lambda}\underline{a}\|_{\mathfrak{Y}_{t,\lambda}} \leq c\|\underline{a}\|_{\mathfrak{X}_{t,\lambda}}$. In a similar way one shows that

$$\|\nabla_t\mathbf{L}_{t,\lambda}\underline{a}\|_{\mathfrak{Y}_{t,\lambda}} \leq c\|\underline{a}\|_{\mathfrak{X}_{t,\lambda}}.$$

This completes the proof. □

9. Quadratic estimate

By a slight abuse of notation we denote by $Q_{t,\lambda}$ the quadratic form defined in (5.5) as well as the associated bilinear form.

Proposition 9.1. *The bilinear form $Q_{t,\lambda}$ depends continuously differentiably on t and continuously on λ and there exists a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned}
&\|\rho_{t,\lambda} Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} \\
&\leq c\lambda^{-\alpha} \Big(\|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\|\rho_{t,\lambda} \nabla_t Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} \\
&\leq c\lambda^{-\alpha} \Big(\|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \Big)
\end{aligned}$$

as well as

$$\begin{aligned}
&\lambda\|\pi_{t,\lambda} Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}} \\
&\leq c\lambda^{-\alpha} \Big(\|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \lambda\|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \Big)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda\|\pi_{t,\lambda} \nabla_t Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}} \\
&\leq c\lambda^{-\alpha} \Big(\|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \|\rho_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \\
&\qquad + \|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} + \lambda\|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}} \Big).
\end{aligned}$$

Proof. The first two estimates are immediate consequences of Proposition 6.2 and Proposition 6.4. For the last two estimates we only have to explain why we get a factor λ (instead of one) in front of $\|\pi_{t,\lambda}\underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda}\underline{a}_2\|_{C^{0,\alpha}}$. Note that

$$\left[*_0 \left(\iota_{t,\lambda}\hat{\mathfrak{I}}_1 \wedge \iota_{t,\lambda}\hat{\mathfrak{I}}_2 \wedge \psi_{0;t}\right)\right]^{0,1} = 0$$

on grounds of simple bi-degree considerations. Therefore, using Propositions 6.4, 7.3 and 7.4,

$$\begin{aligned}
&\|\pi_{t,\lambda}Q_{t,\lambda}(\iota_{t,\lambda}\hat{\mathfrak{I}}_1, \iota_{t,\lambda}\hat{\mathfrak{I}}_2)\|_{C^{0,\alpha}} \\
&\quad \leq c\lambda^{-\alpha}\Big(\|(* - *_0)(\iota_{t,\lambda}\hat{\mathfrak{I}}_1 \wedge \iota_{t,\lambda}\hat{\mathfrak{I}}_2 \wedge \psi)\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}} \\
&\qquad\qquad + \| *_0 \left[\iota_{t,\lambda}\hat{\mathfrak{I}}_1 \wedge \iota_{t,\lambda}\hat{\mathfrak{I}}_2 \wedge (\psi_{1;t} + \psi_{\geq 2;t})\right]\|_{C^{0,\alpha}_{-1,\delta;t,\lambda}}\Big) \\
&\quad \leq c\lambda^{-\alpha}\|\hat{\mathfrak{I}}_1\|_{C^{0,\alpha}} \cdot \|\hat{\mathfrak{I}}_2\|_{C^{0,\alpha}}.
\end{aligned}$$

□

10. Conclusion of the proof of Theorem 1.1

As promised earlier we first explain how to improve the pregluing error.

Proposition 10.1. *For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ there exists $\underline{a}_0 \in \ker \pi_{t,\lambda}$ depending continuously differentiably on t and continuously on λ such that*

$$\rho_{t,\lambda}L_{\tilde{A}_{t,\lambda}}\underline{a}_0 + \rho_{t,\lambda}e_{t,\lambda} = 0$$

and $\|\underline{a}\|_{C^{1,\alpha}_{-1,\delta/1024;t,\lambda}} \leq c\lambda$. The improved pregluing error

$$e'_{t,\lambda} := L_{\tilde{A}_{t,\lambda}}\underline{a}_0 + Q_{t,\lambda}(\underline{a}_0) + e_{t,\lambda}$$

satisfies

$$\|\rho_{t,\lambda}e'_{t,\lambda}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} \leq \lambda^{2+\delta} \quad \textit{and} \quad \|\pi_{t,\lambda}e'_{t,\lambda}\|_{C^{0,\alpha}} \leq \lambda^{1+\delta/1024-\alpha}$$

and so does $\nabla_t e'_{t,\lambda}$.

Proof. It follows from Propositions 8.12, 8.15 and 6.4 that $\rho_{t,\lambda}L_{t,\lambda}|_{\ker \pi_{t,\lambda}} : C^{1,\alpha}_{-1,\delta/1024;t,\lambda} \to C^{0,\alpha}_{-2,\delta/1024;t,\lambda}$ is invertible and its inverse is uniformly bounded. From this the existence of $\underline{a}$ with the asserted bound follows since $\|\rho_{t,\lambda}e_{t,\lambda}\|_{C^{0,\alpha}_{-2,\delta/1024;t,\lambda}} \leq c\lambda^{1+\delta/1024}$ by Proposition 7.1 and Corollary 6.3

The first part of the estimate follows from Proposition 9.1 and Corollary 6.3. The second part also makes use of Propositions 8.15, 6.4, 9.1 and 7.1. □

Now instead of working with $\tilde{A}_{t,\lambda}$ we can work with $\tilde{A}_{t,\lambda} + \underline{a}_0$. The initial error term is then $e'_{t,\lambda}$. In order to not make the notation even more cumbersome, at this point, we replace $\tilde{A}_{t,\lambda}$ with $\tilde{A}_{t,\lambda} + \underline{a}_0$ and correspondingly $e_{t,\lambda}$ with $e'_{t,\lambda}$. In this new notation

$$\|e_{t,\lambda}\|_{\mathfrak{Y}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha}$$

and similarly for $\nabla_t e'_{t,\lambda}$.

Proposition 10.2. *There is a constant $c > 0$ and for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ there are $\underline{a}(t,\lambda) \in C^{1,\alpha}\left(Y, (\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g}_{E_{t,\lambda}}\right)$ and $\eta(t,\lambda) \in \mathbf{R}$ depending continuously differentiably on t and continuously on λ such that the connection $\tilde{A}_{t,\lambda} := A_{t,\lambda} + a(t,\lambda)$ satisfies*

$$* \left(F_{\tilde{A}_{t,\lambda}} \wedge \psi_t\right) + \mathrm{d}_{\tilde{A}_{t,\lambda}} \xi(t,\lambda) + (\mu(t) + \eta(t,\lambda)) \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t = 0 \tag{10.1}$$

and

$$\|\underline{a}(t,\lambda)\|_{\mathfrak{X}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha} \quad \text{and} \quad |\eta(t,\lambda)| + |\partial_t \eta(t,\lambda)| \leq c\lambda^{1+\delta/1024-\alpha}.$$

The proof relies on the preceding analysis and the following simple consequence of Banach's fixed point theorem, cf. [DK90, Lemma 7.2.23].

Lemma 10.3. *Let X be a Banach space and let $T : X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \leq c\,(\|x\| + \|y\|)\,\|x - y\|.$$

Then if $y \in X$ satisfies $\|y\| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $\|x\| \leq \frac{1}{5c}$ solving

$$x + Tx = y.$$

The unique solution satisfies $\|x\| \leq 2\|y\|$. Moreover, if T and y depend continuously or continuously differentiably on a parameter in an open subset of $\mathbf{R}^n$, then so does the solution x.

Proof of Proposition 10.2. We solve (10.1) with the additional constraints

$$\mathrm{d}^*_{A_{t,\lambda}} a = 0 \quad \text{and} \quad \langle \pi_{t,\lambda}\underline{a}, \hat{v} \circ \mathfrak{I}_t \rangle = 0.$$

This can be written as

$$\mathbf{L}_{t,\lambda}(\underline{a}, \eta) + Q_{t,\lambda}(\underline{a}) + e_{t,\lambda} = 0.$$

With $(\underline{a}, \eta) = \mathbf{L}_{t,\lambda}^{-1}(\underline{b}, \zeta)$ this becomes

$$(\underline{b}, \zeta) + \tilde{Q}_{t,\lambda}(\underline{b}, \zeta) + e_{t,\lambda} = 0 \tag{10.2}$$

where $\tilde{Q}_{t,\lambda} := Q_{t,\lambda} \circ \mathbf{L}_{t,\lambda}^{-1}$. It follows from Proposition 8.1 and Proposition 9.1 that

$$\begin{aligned} &\|\tilde{Q}_{t,\lambda}(\underline{b}_1, \zeta_1) - \tilde{Q}_{t,\lambda}(\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t,\lambda}} \\ &\leq c\lambda^{-2-\delta/2-\alpha} \left(\|(\underline{b}_1, \zeta_1)\|_{\mathfrak{Y}_{t,\lambda}} + \|(\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t,\lambda}} \right) \|(\underline{b}_1, \zeta_1) - (\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t,\lambda}}. \end{aligned}$$

Since

$$\|e_{t,\lambda}\|_{\mathfrak{Y}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha},$$

we can solve (10.2) using Lemma 10.3 since $\delta \in (-1, 0)$ and $0 < \alpha \ll |\delta|$. The solution satisfies $\|(\underline{b}, \zeta)\|_{\mathfrak{Y}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha}$ and $(\nabla_t \underline{b}, \partial_t \zeta)$ solves the equation

$$(\nabla_t \underline{b}, \partial_t \zeta) + 2\tilde{Q}_{t,\lambda}((\underline{b}, \zeta), (\nabla_t \underline{b}, \partial_t \zeta)) + (\nabla_t \tilde{Q}_{t,\lambda})(\underline{b}, \zeta) + \nabla_t e_{t,\lambda} = 0. \tag{10.3}$$

Since

$$\begin{gathered} \|2\tilde{Q}_{t,\lambda}(\underline{b}, \cdot)\|_{\mathfrak{Y}_{t,\lambda}} \leq \frac{1}{2}\| \cdot \|_{\mathfrak{Y}_{t,\lambda}} \quad \text{and} \\ \|(\nabla_t \tilde{Q}_{t,\lambda})(\underline{b}, \zeta) + \nabla_t e_{t,\lambda}\|_{\mathfrak{Y}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha}, \end{gathered}$$

it follows that $\|(\nabla_t \underline{b}, \partial_t \zeta)\|_{\mathfrak{Y}_{t,\lambda}} \leq c\lambda^{2+\delta/1024-\alpha}$. This implies the desired estimates on $(\underline{a}, \eta) = \mathbf{L}_{t,\lambda}^{-1}(\underline{b}, \zeta)$ and its derivative by Proposition 8.1. □

The problem of finding A_λ is now reduced to constructing a continuous function $t : [0, \Lambda] \to (-T', T')$ such that $t(0) = 0$ and

$$\mu(t(\lambda)) + \eta(t(\lambda), \lambda) = 0$$

for $t \in (0, \Lambda]$. Since $\mu(0) = 0$ and $\partial_t \mu(0) \neq 0$, we can invert μ locally around $t = 0$ and rewrite this equation as

$$\tilde{\mu}(\lambda) + \eta(\mu^{-1} \circ \tilde{\mu}(\lambda), \lambda) = 0 \tag{10.4}$$

with $\tilde{\mu} = \mu \circ t$. Because $|\eta| + |\partial_t \eta| \leq c\lambda^{1+\delta/1024-\alpha}$, this equation on the other hand can immediately be solved for $\tilde{\mu}$ and thus $t = \mu^{-1} \circ \tilde{\mu}$ by appealing to Lemma 10.3.

Remark 10.4. If we assume the situation of Remark 4.10, that is, μ is just monotone (but possibly $\partial_t \mu(0) = 0$), then one can still find a continuous inverse μ^{-1} find solutions of (10.4) using Brouwer's fixed point theorem. However, these solutions might not be described by the graph of a function; e.g., if $\mu(t) = t^3$ and $\eta(t, \lambda) = -t\lambda^2$, then the set of solutions of $\mu(t) + \eta(t, \lambda) = 0$ is a union of three graphs: $t = 0$ and $t = \pm\lambda$.

The resulting connection $A_\lambda := \tilde{A}_{t(\lambda),\lambda}$ will be smooth by elliptic regularity. That A_λ converges to B_0 on the complement of P_0 and that at each point $x \in P_0$ an ASD instanton in the equivalence class of $\mathfrak{I}(x)$ bubbles off transversely is clear, since we constructed $A_{t,\lambda}$ accordingly and $\bar{A}_\lambda$ is a small perturbation of $A_{t,\lambda}$. This concludes the proof of Theorem 1.1. □

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