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Mean curvature flow of pinched submanifolds of $\mathbb{CP}^n$

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We consider the evolution by mean curvature flow of a closed submanifold of the complex projective space. We show that, if the submanifold has small codimension and satisfies a suitable pinching condition on the second fundamental form, then the evolution has two possible behaviors: either the submanifold shrinks to a round point in finite time, or it converges smoothly to a totally geodesic limit in infinite time. The latter behavior is only possible if the dimension is even. These results generalize previous works by Huisken and Baker on the mean curvature flow of submanifolds of the sphere.

1. Introduction

Let $F_0 : \mathcal{M} \to \mathbb{CP}^n$ be a smooth immersion of a closed connected manifold in the complex projective space. We denote by A the second fundamental form and by H the mean curvature vector associated with the immersion. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one–parameter family of immersions $F : \mathcal{M} \times [0, T_{max}[\to \mathbb{CP}^n$ satisfying

$$\begin{cases} \dfrac{\partial}{\partial t} F(p,t) = H, \quad p \in \mathcal{M},\, t \geq 0, \\ F(\cdot, 0) = F_0. \end{cases} \tag{1.1}$$

We denote by $\mathcal{M}_t = F(\mathcal{M}, t)$ the evolution of $\mathcal{M}_0$ at time t. It is well known that this problem has a unique smooth solution up to some maximal time $T_{max} \leq \infty$. Moreover, if T_{max} is finite the curvature of $\mathcal{M}_t$ necessarily becomes unbounded as $t \to T_{max}$ and we say that the flow develops a singularity in finite time. The main theorem proved in this work is the following.

Theorem 1.1. *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$ of real dimension m and codimension $k = 2n - m$. Suppose either $n \geq 3$ and $k = 1$, or $n \geq 7$*

and $2 \le k < \frac{2n-3}{5}$ *(equivalently,* $2 \le k < \frac{m-3}{4}$*). If at every point of* $\mathcal{M}_0$ *the inequality*

$$(1.2) \qquad |A|^2 < \begin{cases} \dfrac{1}{m-1}|H|^2 + 2 & \text{if } k = 1, \\ \dfrac{1}{m-1}|H|^2 + \dfrac{m-3-4k}{m} & \text{if } k \ge 2, \end{cases}$$

is satisfied, then the same holds on $\mathcal{M}_t$ *for all* $0 < t < T_{max}$*. Moreover, one of the two following properties holds:*

1) $T_{max} < \infty$*, and* $\mathcal{M}_t$ *contracts to a point as* $t \to T_{max}$*,*
2) $T_{max} = \infty$*, and* $\mathcal{M}_t$ *converges to a smooth totally geodesic submanifold as* $t \to T_{max}$*.*

Case 2) can only occur if m *is even, and the limit submanifold is isometric to* $\mathbb{CP}^{\frac{m}{2}}$*.*

An inequality of the form (1.2) above is usually called a *pinching condition* on the second fundamental form. For instance, in the case $k = 1$ it gives a bound on how much each principal curvature of the submanifold can differ from the others.

The above statement says in particular that in odd dimension a submanifold satisfying our assumptions necessarily shrinks to a point under mean curvature flow. We remark that this property is not proved directly: we show that the only alternative to a round point is the behavior in 2), but such a behavior is excluded for odd dimension because the only totally geodesic submanifolds of $\mathbb{CP}^n$ with small codimension as in our hypotheses are isometric to a complex projective space.

When $\mathcal{M}_t$ shrinks to a point in finite time as in case 1) above, one can show that, after an appropriate rescaling, it converges to an m-dimensional sphere, a behavior which is usually described as "convergence to a round point", see e.g. [H1, §9-10], [LXZ, §6]. As a consequence, we obtain the following classification result.

Corollary 1.2. *Let* $\mathcal{M}_0$ *satisfy the hypotheses of Theorem 1.1. Then, if* m *is odd,* $\mathcal{M}_0$ *is diffeomorphic to an* $\mathbb{S}^m$*, while if* k *is even,* $\mathcal{M}_0$ *is diffeomorphic either to an* $\mathbb{S}^m$ *or to a* $\mathbb{CP}^{\frac{m}{2}}$*. In every case* $\mathcal{M}_0$ *is simply connected.*

The behavior of submanifolds evolving by mean curvature flow has been studied by several authors in the last decades, especially in the case of codimension one. The first fundamental result was obtained by Huisken [H1], who showed that any closed convex hypersurface in Euclidean space shrinks to a round point in finite time. He later proved [H2] that the same holds for hypersurfaces in general Riemannian manifolds satisfying a stronger convexity condition which takes into account the geometry of the ambient space. A similar analysis has then been carried out by several authors for flows driven by speeds different from the mean curvature, and many convergence results to a round point are known for hypersurfaces satisfy suitable convexity requirements, see [AMZ, AM] and the references therein. More recently, Andrews and Baker [AB] have considered the mean curvature flow in the case of higher codimension, and proved the convergence to a round point for submanifolds of arbitrary codimension of the Euclidean space satisfying a suitable pinching condition. Similar results have then been obtained by Liu, Xu,Ye and Zhao for submanifolds of hyperbolic spaces [LXYZ] and of general Riemannian manifolds [LXZ].

By contrast, very few authors have considered cases where the mean curvature flow converges to a stationary limit. In the context of weak solutions, there is a quite general result by White [W, Theorem 11.1], asserting that a mean convex solution either disappears in finite time or converges to a finite collection of stable minimal submanifolds. For classical solutions, results of this kind are known only in special cases. For the curve shortening flow, Grayson [G] showed that an embedded curve in a Riemannian surface either shrinks to a round point or converges smoothly to a geodesic. When the dimension of the evolving submanifold is larger than one, other kinds of singularities can occur and an analogous statement can only be expected under suitable restrictions. Until now, a higher dimensional analogue of the results of [G] has only been obtained for submanifolds of the sphere, which have been studied by Huisken [H3] for codimension one and by Baker [Ba] for arbitrary codimension. The results in the two cases can be stated together as follows.

Theorem 1.3. [H3, Ba] *Let $\mathcal{M}_0$ be a closed n dimensional submanifold of $\mathbb{S}^{n+k}$, with $n \geq 2$, and suppose that we have on $\mathcal{M}_0$*

$$(1.3)\qquad \begin{aligned} &|A|^2 < \tfrac{1}{n-1}|H|^2 + 2, && \text{if } n \geq 4, \text{ or } n = 3 \text{ and } k = 1,\\ &|A|^2 < \tfrac{3}{4}|H|^2 + \tfrac{4}{3}, && \text{if } n = 2 \text{ and } k = 1,\\ &|A|^2 < \tfrac{4}{3n}|H|^2 + \tfrac{2(n-1)}{3}, && \text{if } n = 2, 3 \text{ and } k > 1. \end{aligned}$$

Then one of the following holds:

1) *T_{max} is finite and the $\mathcal{M}_t$'s converge to a round point as $t \to T_{max}$,*

2) *T_{max} is infinite and the $\mathcal{M}_t$'s converge to a smooth totally geodesic hypersurface $\mathcal{M}_\infty$, isometric to $\mathbb{S}^n$.*

As underlined in the above statements, a key ingredient in all these results is the invariance under mean curvature flow of a pinching condition of the form $|A|^2 < a|H|^2 + b$, for some $a > 0$ and $b \in \mathbb{R}$. The values of a, b such that the invariance holds depend on the properties of the ambient manifold. If the ambient manifold is flat [H1, AB], or hyperbolic [LXYZ], or general [LXZ], the invariance can only be obtained for suitable values of $b \leq 0$, so that the condition rules out the possibility of a stationary limit. In the case of the sphere, it is possible instead to have invariance with some $b > 0$, which allows the two possible behaviors of the above statements. In addition, a pinching condition with $b > 0$ is substantially weaker: for example, in the case of codimension one it allows for some nonconvex hypersurfaces. Although the special structure of the sphere is used in an essential way in [H3, Ba], it is natural to expect that similar results should hold for more general ambient spaces of positive curvature.

The results of this paper confirm this expectation in the case of the complex projective space, showing that suitably pinched submanifolds evolving by mean curvature flow exhibit similar properties to the ones of the sphere. The complex projective space is a natural ambient space to consider beside the sphere, since it is a symmetric Einstein manifold with positive, but no longer constant, sectional curvature. The different structure of the Riemann curvature tensor complicates the study of the evolution of the curvature quantities with respect to [H3, Ba], and forces us to restrict our analysis to submanifolds with suitably small codimension.

The paper is organized as follows. After recalling some notation and preliminary results, we prove in Section 3 the invariance of the pinching condition. In this part, in order to efficiently estimate the reaction terms in the evolution equations, it is crucial to choose normal and tangent frames strongly linked with the geometry of $\mathbb{CP}^n$. In Section 4 we study the behavior of the norm of the traceless part of the second fundamental form, which is used to measure the improvement of pinching as the maximal time is approached. Since our estimates have additional lower order terms compared with [H3, Ba], to prove our main theorem we have to treat separately the cases of T_{max} finite and T_{max} infinite, which we do in Sections 5 and 6 respectively. The former case is more technically involved, and the convergence is obtained by integral estimates as in the previous papers, while for T_{max} infinite the result follows from a more direct argument. Finally, in Section 7 we show that in the case of hypersurfaces our main result also holds for the mean curvature flow in quaternionic projective spaces.

2. Preliminaries

The ambient manifold $\mathbb{CP}^n$ is a Kähler manifold of complex dimension n with complex structure J. It can be regarded as a real Riemannian manifold of dimension $2n$ endowed with the Fubini-Study metric g_{FS}. We denote the curvature tensor and the Levi-Civita connection of $(\mathbb{CP}^n, g_{FS})$ with $\bar{R}$ and $\bar{\nabla}$ respectively. Then $\bar{R}$ has the explicit form, for all tangent vector fields X, Y, Z, W,

$$\begin{aligned}\bar{R}(X,Y,Z,W) = {} & g_{FS}(X,Z)g_{FS}(Y,W) - g_{FS}(X,W)g_{FS}(Y,Z) \\ & + g_{FS}(X,JZ)g_{FS}(Y,JW) \\ & - g_{FS}(X,JW)g_{FS}(Y,JZ) \\ & + 2g_{FS}(X,JY)g_{FS}(Z,JW).\end{aligned} \tag{2.1}$$

In particular, the sectional curvature of a tangent plane spanned by two orthonormal vector fields X and Y is

$$\bar{K}(X,Y) = 1 + 3g_{FS}(X,JY)^2, \tag{2.2}$$

therefore $1 \le \bar{K} \le 4$ and $\bar{K} = 1$ (resp. $\bar{K} = 4$) if and only if JY is orthogonal (resp. tangent) to X. Moreover $(\mathbb{CP}^n, g_{FS})$ is a symmetric space, so $\bar{\nabla}\bar{R} = 0$, and is an Einstein manifold with Einstein constant $2(n+1)$.

Let now $\mathcal{M}$ be a closed submanifold of $\mathbb{CP}^n$, with induced metric g, curvature tensor R and connection ∇. The tangent and normal space to $\mathcal{M}$ at a point p are denoted by $T_p\mathcal{M}$ and $N_p\mathcal{M}$ respectively. Throughout the paper we denote by m the dimension of $\mathcal{M}$ and by $k = 2n - m$ its codimension. Unless specified otherwise, Latin letters $i, j, l, \ldots$ run from 1 to m, Greek letters $\alpha, \beta, \gamma, \ldots$ run from $m+1$ to $m+k$.

Let $e_1, \ldots, e_{m+k}$ be an orthonormal frame tangent to $\mathbb{CP}^n$ at a point of $\mathcal{M}$, such that the first m vectors are tangent to $\mathcal{M}$ and the other ones are normal. With respect to this frame, the second fundamental form can be written

$$A = \sum_\alpha h^\alpha \otimes e_\alpha,$$

where the $h^\alpha = \left(h^\alpha_{ij}\right)$ are symmetric 2-tensors. The trace of the second fundamental form with respect to the metric g is the mean curvature vector H:

$$H = \sum_\alpha \operatorname{tr} h^\alpha e_\alpha = \sum_\alpha \sum_{ij} g^{ij} h^\alpha_{ij} e_\alpha.$$

The traceless part of the second fundamental form is defined as $\mathring{A} = A - \frac{1}{m} H \otimes g$, and its components are $\mathring{h}^\alpha_{ij} = h^\alpha_{ij} - \frac{H}{m}^\alpha g_{ij}$, where $H^\alpha = \sum_{rs} g^{rs} h^\alpha_{rs}$. In particular, the squared length satisfies $|\mathring{A}|^2 = |A|^2 - \frac{1}{m}|H|^2$.

If $\mathcal{M}$ is a hypersurface, then the mean curvature vector is a multiple of the unit normal vector ν and satisfies

$$H = -(\lambda_1 + \cdots + \lambda_m)\nu,$$

where $\lambda_1 \leq \cdots \leq \lambda_m$ are the principal curvatures. In addition, we have $|A|^2 = \lambda_1^2 + \cdots + \lambda_m^2$ and

$$|\mathring{A}|^2 = |A|^2 - \frac{1}{m}|H|^2 = \frac{1}{m}\sum_{i<j}(\lambda_i - \lambda_j)^2, \tag{2.3}$$

so that smallness of $|\mathring{A}|^2$ implies that the curvatures are close to each other.

The evolution equations of the main curvature quantities of a submanifold evolving by mean curvature flow in a general Riemannian space have been computed in [AB] and [Ba]. In our case, they take a simpler form because the ambient manifold is symmetric. We recall here the equations satisfied by $|H|^2, |A|^2$ and by the volume form $d\mu_t$ associated with the immersion at time t.

Lemma 2.1. *On a submanifold evolving by mean curvature flow in a symmetric ambient space we have*

1) $$\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2\sum_{i,j}\left(\sum_\alpha H^\alpha h^\alpha_{ij}\right)^2 + 2\sum_{l,\alpha,\beta}\bar{R}_{l\alpha l\beta}H^\alpha H^\beta,$$

$$
\begin{aligned}
2)\quad \frac{\partial}{\partial t}|A|^2 = {} & \Delta |A|^2 - 2|\nabla A|^2 + 2\sum_{\alpha,\beta}\left(\sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta}\right)^2 \\
& + 2\sum_{i,j,\alpha,\beta}\left[\sum_p h_{ip}^{\alpha} h_{jp}^{\beta} - h_{ip}^{\beta} h_{jp}^{\alpha}\right]^2 \\
& + 4\sum_{i,j,p,q}\bar{R}_{ipjq}\left(\sum_{\alpha} h_{pq}^{\alpha} h_{ij}^{\alpha}\right) - 4\sum_{j,l,p}\bar{R}_{ljlp}\left(\sum_{i,\alpha} h_{pi}^{\alpha} h_{ij}^{\alpha}\right) \\
& + 2\sum_{l,\alpha,\beta}\bar{R}_{l\alpha l\beta}\left(\sum_{ij} h_{ij}^{\alpha} h_{ij}^{\beta}\right) - 8\sum_{j,p,\alpha,\beta}\bar{R}_{jp\alpha\beta}\left(\sum_i h_{ip}^{\alpha} h_{ij}^{\beta}\right),
\end{aligned}
$$

$$
3)\quad \frac{\partial}{\partial t} d\mu_t = -|H|^2\, d\mu_t.
$$

When the codimension is one these equations have a simpler form.

Lemma 2.2. *On a hypersurface evolving by mean curvature flow in a symmetric ambient space we have*

$$
\begin{aligned}
1)\quad & \frac{\partial}{\partial t}|H|^2 = \Delta |H|^2 - 2|\nabla H|^2 + 2|H|^2\left(|A|^2 + \bar{R}ic(\nu,\nu)\right), \\
2)\quad & \frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2\left(|A|^2 + \bar{R}ic(\nu,\nu)\right) \\
& \qquad - 4\sum_{i,j,p,l}\left(h_{ij} h_j^{\ p} \bar{R}_{pli}^{\ \ \ l} - h^{ij} h^{lp} \bar{R}_{pilj}\right),
\end{aligned}
$$

where $\bar{R}ic$ is the Ricci tensor of the ambient manifold.

3. Invariance of pinching

In this section we prove that the pinching condition (1.2) is invariant under the flow. To obtain the desired estimates, it is important to perform the computations using special tangent frames with suitable properties, which we now describe.

A first kind of frames, which was also considered in [AB, LXZ], can be defined at any point where $H \neq 0$ in the following way. We choose a privileged normal direction setting

$$e_{m+1} = \frac{H}{|H|}. \tag{3.1}$$

Then we can choose $e_{m+2}, \ldots, e_{m+k}$ such that $\{e_{m+1}, \ldots, e_{m+k}\}$ is an orthonormal basis of $N_p\mathcal{M}_t$ and choose any orthonormal basis $\{e_1, \ldots, e_m\}$ of $T_p\mathcal{M}_t$. Any tangent frame obtained in this way will be called of kind **(B1)**.

With such a choice of tangent frame, the second fundamental form and its traceless part satisfy

$$\begin{cases} \operatorname{tr} h^{m+1} = |H|\,, & \\ \operatorname{tr} h^{\alpha} = 0, & \alpha \geq m+2 \end{cases}$$

and

$$\begin{cases} \mathring{h}^{m+1} = h^{m+1} - \dfrac{|H|}{m} g, & \\ \mathring{h}^{\alpha} = h^{\alpha}, & \alpha \geq m+2. \end{cases}$$

When using a basis of kind (B1), we adopt the following notation:

$$\begin{aligned} |h_1|^2 &:= \left|h^{m+1}\right|^2, \qquad |\mathring{h}_1|^2 := \left|\mathring{h}^{m+1}\right|^2, \\ |h_-|^2 &= |\mathring{h}_-|^2 := \sum_{\alpha=m+2}^{2n} |\mathring{h}^{\alpha}|^2. \end{aligned} \tag{3.2}$$

A second kind of frames, more linked with the geometry of $\mathbb{CP}^n$, is useful when we have to compute explicitly the components of the Riemann curvature tensor of the ambient manifold. The properties required in this case are described in the following lemma.

Lemma 3.1. *Let $\mathcal{M}$ be a submanifold of $\mathbb{CP}^n$ of dimension m and codimension k. If $k \leq m$, then for every point $p \in \mathcal{M}$ there exist $\{e_1, \ldots, e_m\}$ an orthonormal basis of $T_p\mathcal{M}$ and $\{e_{m+1}, \ldots, e_{m+k}\}$ an orthonormal basis of $N_p\mathcal{M}$ such that:*

1) *for every $r \leq \frac{k}{2}$ we have*

$$\begin{cases} Je_{m+2r-1} = \tau_r e_{2r-1} + \nu_r e_{m+2r}, \\ Je_{m+2r} = \tau_r e_{2r} - \nu_r e_{m+2r-1}, \end{cases} \tag{3.3}$$

with $\tau_r, \nu_r \in [0,1]$ and $\tau_r^2 + \nu_r^2 = 1$.

2) *If k is odd then $Je_{m+k} = e_k$.*

3) *The remaining vectors satisfy*

$$Je_{k+1} = e_{k+2}, Je_{k+3} = e_{k+4}, \dots, Je_{m-1} = e_m. \tag{3.4}$$

Proof. For every point $p \in \mathcal{M}$ the function

$$\begin{array}{rccl} \varphi : & N_p\mathcal{M} \times N_p\mathcal{M} & \to & \mathbb{R} \\ & (X, Y) & \mapsto & \varphi(X, Y) := g(JX, Y) \end{array}$$

is a skew-symmetric bilinear form. It is a well-known fact that there is an orthonormal basis $\{e_{m+1}, \dots, e_{m+k}\}$ of $N_p\mathcal{M}$ such that φ is represented by the matrix

$$M_\varphi = \begin{pmatrix} \begin{matrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix} 0 & \nu_2 \\ -\nu_2 & 0 \end{matrix} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & \nu_s \\ -\nu_s & 0 \end{matrix} \end{pmatrix} \quad \text{if } k = 2s,$$

$$M_\varphi = \begin{pmatrix} \begin{matrix} 0 & \nu_1 \\ -\nu_1 & 0 \end{matrix} & 0 & \cdots & 0 & 0 \\ 0 & \begin{matrix} 0 & \nu_2 \\ -\nu_2 & 0 \end{matrix} & & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \begin{matrix} 0 & \nu_s \\ -\nu_s & 0 \end{matrix} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \quad \text{if } k = 2s+1.$$

Using the property that $|\varphi(X,Y)| \le |X||Y|$, we find that $|\nu_r| \le 1$ for any r, and after possibly reversing signs we can have $\nu_r \in [0,1]$.

Observe first that if k is odd statement 2 follows easily. When we consider the other vectors of the basis, the above construction implies that, for every $r \le \frac{k}{2}$, the normal component of Je_{m+2r-1} is given by $\nu_r e_{m+2r}$, while the normal component of Je_{m+2r} is given by $-\nu_r e_{m+2r-1}$. Now let us distinguish the cases $\nu_r < 1$ and $\nu_r = 1$. In the first case, we have

$$\begin{cases} Je_{m+2r-1} = \tau_r T_{2r-1} + \nu_r e_{m+2r}, \\ Je_{m+2r} = \hat{\tau}_r T_{2r} - \nu_r e_{m+2r-1}, \end{cases} \tag{3.5}$$

where the T_i are unit vectors of $T_p\mathcal{M}$ and $\tau_r, \hat{\tau}_r \in \mathbb{R}$. The above relations imply

$$\tau_r^2 + \nu_r^2 = 1 = \hat{\tau}_r^2 + \nu_r^2,$$

so, up to changing the sign of T_{2r-1} and T_{2r}, we can obtain $\tau_r = \hat{\tau}_r \in (0,1]$.

If instead $\nu_r = 1$, this means that Je_{m+2r-1} coincides with e_{m+2r}. In this case, we choose T_{2r-1} to be any unit tangent vector which is orthogonal to $T_1, \ldots, T_{2r-2}$ and which is also orthogonal to $Je_{m+1}, \ldots, Je_{m+k}$. It is easy to see that such a vector exists because of the assumption $k \leq m$. We then define $T_{2r} = JT_{2r-1}$. By construction, T_{2r} is a tangent unit vector orthogonal to $T_1, \ldots, T_{2r-1}$. Observe that equations (3.5) hold also in this case, with $\tau_r = \hat{\tau}_r = 0$.

In general, since $\{e_{m+1}, \ldots, e_{m+k}\}$ is an orthonormal basis, from equations (3.5), we have for any $i \neq j$

$$g(T_i, T_j) = 0.$$

Then we define $e_i = T_i$ for $i = 1, \ldots, k$, and we complete the basis of $T_p\mathcal{M}$ in an orthonormal way by choosing $e_{k+1}, \ldots, e_m$ in such a way that requirement 3 is satisfied. □

Any basis satisfying the properties of the previous lemma will be called of kind **(B2)**. Since $J^2 = -id$, from (3.3) it follows easily that such a basis also satisfies

$$\begin{cases} Je_{2r-1} = -\nu_r e_{2r} - \tau_r e_{m+2r-1}, \\ Je_{2r} = \nu_r e_{2r-1} - \tau_r e_{m+2r}. \end{cases} \tag{3.6}$$

If k is odd, it is convenient to define $\tau_r = 1$, $\nu_r = 0$ for $r = \frac{k+1}{2}$. In this way, the first equations in (3.3) and in (3.6) hold also for this value of r.

In general, the requirements for (B1) and (B2) are incompatible and the two kinds of bases are different. Thus when we use frames of type B2, we have $H = \sum_\alpha H^\alpha e_\alpha$, with H^α not necessarily zero for $\alpha > m+1$.

Observe that when $k = 1$ these constructions are trivial: there is an unique (up to sign) normal unit vector e_{2n}, H is a multiple of such vector and $e_1 = Je_{2n}$ is a tangent vector. Then for a hypersurface we can choose a basis that is at the same time of type (B1) and (B2).

When $k \geq 2$, we introduce the following notation taken from [AB]

$$R_1 := \sum_{\alpha,\beta} \left(\sum_{i,j} h_{ij}^\alpha h_{ij}^\beta \right)^2 + \sum_{i,j,\alpha,\beta} \left[\sum_p h_{ip}^\alpha h_{jp}^\beta - h_{ip}^\beta h_{jp}^\alpha \right]^2,$$

$$R_2 := \sum_{i,j} \left(\sum_\alpha H^\alpha h_{ij}^\alpha \right)^2.$$

If we use a frame of kind (B1), it is easily checked that

$$R_2 = \begin{cases} |\mathring{h}_1|^2|H|^2 + \dfrac{1}{m}|H|^4 & \text{if } H \neq 0 \\ 0 & \text{if } H = 0. \end{cases} \tag{3.7}$$

The following result, proved in [AB, §3] and in [Ba, §5.2], is useful in the estimation of the reaction terms occurring in the evolution equations of Lemma 2.1. It only uses the algebraic properties of R_1 and R_2 and is independent on the flow.

Lemma 3.2. *At a point where $H \neq 0$ we have, for any $a \in \mathbb{R}$*

$$2R_1 - 2aR_2 \leq 2|\mathring{h}_1|^4 - 2\left(a - \frac{2}{m}\right)|\mathring{h}_1|^2|H|^2 - \frac{2}{m}\left(a - \frac{1}{m}\right)|H|^4 + 8|\mathring{h}_1|^2|\mathring{h}_-|^2 + 3|\mathring{h}_-|^4.$$

In addition, if $a > 1/m$ and if $b \in \mathbb{R}$ is such that $|A|^2 = a|H|^2 + b$, we have

$$2R_1 - 2aR_2 \leq \left(6 - \frac{2}{ma-1}\right)|\mathring{A}|^2|\mathring{h}_-|^2 - 3|\mathring{h}_-|^4 + \frac{2mab}{ma-1}|\mathring{h}_1|^2 + \frac{4b}{ma-1}|\mathring{h}_-|^2 - \frac{2b^2}{ma-1}.$$

We now derive a sharp estimate on the gradient terms appearing in the evolution equations for $|A|^2$ and $|H|^2$ which will be used many times in the rest of the paper. Observe that the results are independent of the flow. Our starting point is the following inequality, first proved in Lemma 2.2 of [H2] in the case of hypersurfaces, and then extended to general codimension in Lemma 3.2 of [LXZ].

Lemma 3.3. *Let $\overline{\mathcal{M}}$ an Riemannian manifold and $\mathcal{M}$ a submanifold of $\overline{\mathcal{M}}$ of dimension m and arbitrary codimension. Then*

$$|\nabla A|^2 \geq \left(\frac{3}{m+2} - \eta\right)|\nabla H|^2 - \frac{2}{m+2}\left(\frac{2}{m+2}\eta^{-1} - \frac{m}{m-1}\right)|\omega|^2, \tag{3.8}$$

holds for any $\eta > 0$. Here $\omega = \sum_{ij\alpha} \bar{R}_{\alpha jij} e_i \otimes \omega_\alpha$, where ω_α is the dual frame to e_α.

Note that if the ambient space is Einstein, like $\mathbb{CP}^n$, and if $\mathcal{M}$ is a hypersurface, then $\omega = 0$. So we can let $\eta \to 0$ in inequality (3.8) and find

$$|\nabla A|^2 \geq \frac{3}{m+2}|\nabla H|^2. \tag{3.9}$$

For submanifolds of higher codimension, ω is in general nonzero. However, using the special properties of $\mathbb{CP}^n$, we can prove the following estimate.

Lemma 3.4. *Let $\mathcal{M}$ be a submanifold of $\mathbb{CP}^n$ of dimension m and codimension $k \leq m$. Then we have, at any point of $\mathcal{M}$,*

$$|\nabla A|^2 \geq \frac{2}{9}(m+1)|\omega|^2.$$

Proof. We first compute explicitly $|\omega|^2$ using a basis of type (B2). The relations (3.6) and the expression of $\bar{R}$ give

$$\begin{aligned}\bar{R}_{\alpha jij} &= 3g_{FS}(e_\alpha, Je_j)g_{FS}(e_i, Je_j)\\ &= \begin{cases} 3\tau_r\nu_r & \text{if } \alpha = m+2r-1,\ i = 2r,\ j = 2r-1,\\ -3\tau_r\nu_r & \text{if } \alpha = m+2r,\ i = 2r-1,\ j = 2r,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We recall that if k is odd then $\nu_r = 0$ for $r = \frac{k+1}{2}$. Thus we have, for a general k,

$$|\omega|^2 = 18\sum_{r \leq \frac{k}{2}} \tau_r^2 \nu_r^2. \tag{3.10}$$

Next we recall a lower bound on $|\nabla A|$ for general submanifolds $\mathcal{M}$ of $\mathbb{CP}^n$ which was proved in [Ko]. Following the notation of that paper, for any vector field X tangent to $\mathcal{M}$, we write $JX = PX + FX$, where PX and FX are the tangent and normal component of JX respectively. Similarly, for a normal vector field V we write $JV = tV + fV$ where tV is tangent to $\mathcal{M}$ and fV is normal. Then Lemma 3.6 of [Ko] asserts that, at any point of $\mathcal{M}$, we have

$$|\nabla A|^2 \geq 2\left(|P|^2\,|t|^2 + |FP|^2\right). \tag{3.11}$$

In a given orthonormal basis, the above norms are

$$|P|^2 = \sum_{i=1}^{m} |Pe_i|^2, \quad |t|^2 = \sum_{\alpha=m+1}^{2n} |te_\alpha|^2 \text{ and } |FP|^2 = \sum_{i=1}^{m} |FPe_i|^2.$$

We choose again a basis of type (B2) and estimate the above expressions in the cases k even and k odd separately, using the relations (3.3), (3.6). If $k = 2d$ we have

$$|P|^2 = (m-k) + \left(2\sum_{r\leq d} \nu_r^2\right) = m - \left(2\sum_{r\leq d} \tau_r^2\right),$$
$$|t|^2 = 2\sum_{r\leq d} \tau_r^2.$$

Therefore, using the property $\nu_r^2 + \tau_r^2 = 1$ and the assumption $m \geq k$, we find

$$\begin{aligned} |P|^2|t|^2 &= 2m\sum_{r\leq d} \tau_r^2 - 4\sum_{r,s\leq d} \tau_r^2\tau_s^2 \\ &\geq 2m\sum_{r\leq d} \tau_r^2 - 2\sum_{r,s\leq d} (\tau_r^4 + \tau_s^4) \\ &= 2m\sum_{r\leq d} \tau_r^2 - 2k\sum_{r\leq d} \tau_r^4 \\ &\geq 2m\sum_{r\leq d} (\tau_r^2 - \tau_r^4) = \frac{m}{9}|\omega|^2. \end{aligned} \tag{3.12}$$

If instead $k = 2d+1$ we find

$$|P|^2 = (m-k) + \left(2\sum_{r\leq d} \nu_r^2\right) = m - 1 - \left(2\sum_{r\leq d} \tau_r^2\right),$$
$$|t|^2 = 1 + 2\sum_{r\leq d} \tau_r^2.$$

Therefore,

$$\begin{aligned}|P|^2|t|^2 &\geq m-1+2(m-2)\sum_{r\leq d}\tau_r^2-2(k-1)\sum_{r\leq d}\tau_r^4\\ &\geq m-1+2(m-2)\sum_{r\leq d}\tau_r^2\nu_r^2.\end{aligned}$$

Since for every r we have $\nu_r^2+\tau_r^2=1$, we deduce that $\nu_r^2\tau_r^2\leq\frac{1}{4}$. Therefore, using that $m-1\geq k-1=2d$, we find

$$\begin{aligned}|P|^2|t|^2 &\geq 2d+2(m-2)\sum_{r\leq d}\tau_r^2\nu_r^2\\ &\geq 2(m+2)\sum_{r\leq d}\tau_r^2\nu_r^2=\frac{m+2}{9}|\omega|^2.\end{aligned}\tag{3.13}$$

Finally, we have for any k

$$|FP|^2=2\sum_{r\leq\frac{k}{2}}\tau_r^2\nu_r^2=\frac{|\omega|^2}{9}.\tag{3.14}$$

Putting together inequalities (3.11), (3.12), (3.13) and (3.14) the conclusion follows. □

The previous result allows us to obtain an estimate similar to (3.9) for general codimension.

Lemma 3.5. *For any submanifold $\mathcal{M}$ of $\mathbb{CP}^n$ with dimension satisfying the assumptions of Theorem 1.1, we have*

$$|\nabla A|^2\geq\frac{16}{9(m+2)}|\nabla H|^2.$$

Proof. If the codimension is 1, then the result follows directly from (3.9). In the case of higher codimension, the trick is to combine the estimates from Lemma 3.3 and Lemma 3.4 as follows:

$$\begin{aligned}3|\nabla A|^2 &= 2|\nabla A|^2+|\nabla A|^2\\ &\geq 2\left(\frac{3}{m+2}-\eta\right)|\nabla H|^2\\ &\quad+\left[\frac{2}{9}(m+1)-\frac{4}{m+2}\left(\frac{2}{m+2}\eta^{-1}-\frac{m}{m-1}\right)\right]|\omega|^2.\end{aligned}$$

Now we choose $\eta = 1/3(m+2)$ to obtain

$$3|\nabla A|^2 \geq \frac{16}{3}\frac{1}{m+2}|\nabla H|^2 + \left[\frac{2}{9}(m+1) - \frac{24}{m+2}\right]|\omega|^2,$$

and the term inside square brackets is positive for m as in our hypotheses. □

We are now ready to prove the invariance of the pinching condition of Theorem 1.1. We treat separately the case of hypersurfaces, where the analysis is simpler, and the case of higher codimension, where the two kinds of bases introduced before are essential. However, the strategy of proof is the same in the two cases: we consider the function

$$Q = |A|^2 - a|H|^2 - b$$

for suitable a, b, and we analyze its evolution equation showing that, if $Q(x,t) = 0$ at some point $(x,t) \in \mathcal{M} \times [0, T_{max}[$, then $\left(\frac{\partial}{\partial t} - \Delta\right) Q \leq 0$ at this point. By the maximum principle, the result will follow.

Proposition 3.6. *Let $\mathcal{M}_0$ be a closed hypersurface of $\mathbb{CP}^n$, with $n \geq 3$. Then the pinching condition*

$$|A|^2 \leq \frac{1}{m-1+\varepsilon}|H|^2 + 2(1-\varepsilon) \tag{3.15}$$

is preserved by the mean curvature flow for any $\varepsilon \in [0,1)$.

Proof. Let us set $Q = |A|^2 - a|H|^2 - b$ with $a = (m-1+\varepsilon)^{-1}$ and $b = 2(1-\varepsilon)$. Lemma 2.2 gives

$$\begin{aligned} \frac{\partial}{\partial t}Q &= \Delta Q - 2\left(|\nabla A|^2 - a|\nabla H|^2\right) + 2\left(|A|^2 - a|H|^2\right)\left(|A|^2 + \bar{r}\right) \\ &\quad - 4\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right) \\ &= \Delta Q - 2\left(|\nabla A|^2 - a|\nabla H|^2\right) + 2Q\left(|A|^2 + \bar{r}\right) + 2b\left(|A|^2 + \bar{r}\right) \\ &\quad - 4\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right), \end{aligned} \tag{3.16}$$

where we have set

$$\bar{r} = \bar{R}ic(\nu,\nu) = 2(n+1). \tag{3.17}$$

By Lemma 3.5 the gradient terms in equation (3.16) are non-positive and it suffices to consider the contribution of the reaction terms. Fix an orthonormal basis tangent to $\mathcal{M}_t$ that diagonalizes the second fundamental form and call $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_m$ its eigenvalues. Recalling that any sectional curvature $\bar{K}_{ij}$ satisfies $\bar{K}_{ij} \ge 1$, we find

$$\begin{aligned} -4\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right) &= -4\left(\lambda_j^2\delta_{ij}\delta_{jp}\bar{R}_{plil} - \lambda_j\lambda_l\delta_{ij}\delta_{lp}\bar{R}_{pilj}\right) \\ &= -4\sum_{j,l}\left(\lambda_j^2 - \lambda_j\lambda_l\right)\bar{R}_{jljl} \\ &= -2\sum_{j,l}\left(\lambda_j - \lambda_l\right)^2\bar{K}_{jl} \\ &\le -2\sum_{j,l}\left(\lambda_j - \lambda_l\right)^2 = -4m|\mathring{A}|^2. \end{aligned} \tag{3.18}$$

Since $2/a \ge 2m - 2 \ge m + 3 = \bar{r}$, we have

$$2b\left(|A|^2 + \bar{r}\right) - 4m\left(|A|^2 - \frac{1}{m}|H|^2\right) = -\frac{4}{a}\left(|A|^2 - a|H|^2 - \frac{a}{2}b\bar{r}\right) \le -\frac{4}{a}Q.$$

By the maximum principle, the assertion follows. □

Proposition 3.7. *Let $\mathcal{M}_0$ be a closed submanifold of $\mathbb{CP}^n$ of dimension m and codimension $2 \le k < \dfrac{2n-3}{5}$. Then the pinching condition*

$$|A|^2 \le \frac{1}{m-1+\varepsilon}|H|^2 + \frac{m-3-4k}{m}(1-\varepsilon) \tag{3.19}$$

is preserved by the flow for any $\varepsilon \in [0,1)$.

Proof. Again, let us set $Q = |A|^2 - a|H|^2 - b$, where

$$a = \frac{1}{m-1+\varepsilon}, \qquad b = \frac{m-3-4k}{m}(1-\varepsilon).$$

By Lemma 2.1 we have

$$\frac{\partial}{\partial t}Q = \Delta Q - 2(|\nabla A|^2 - a|\nabla H|^2) + 2R_1 - 2aR_2 + P_a, \tag{3.20}$$

where $P_a = I + II + III$, with

$$I = 4\sum_{i,j,p,q} \bar{R}_{ipjq}\left(\sum_{\alpha} h^{\alpha}_{pq}h^{\alpha}_{ij}\right) - 4\sum_{j,s,p} \bar{R}_{sjsp}\left(\sum_{i,\alpha} h^{\alpha}_{pi}h^{\alpha}_{ij}\right),$$

$$II = 2\sum_{s,\alpha,\beta} \bar{R}_{s\alpha s\beta}\left(\sum_{ij} h^{\alpha}_{ij}h^{\beta}_{ij}\right) - 2a\sum_{s,\alpha,\beta} \bar{R}_{s\alpha s\beta}H^{\alpha}H^{\beta},$$

$$III = -8\sum_{j,p,\alpha,\beta} \bar{R}_{jp\alpha\beta}\left(\sum_{i} h^{\alpha}_{ip}h^{\beta}_{ij}\right).$$

By Lemma 3.5 the gradient terms in equation (3.20) are non-positive and it suffices to consider the contribution of the reaction terms. Let us divide the analysis into two cases: $H = 0$ and $H \neq 0$. Consider first a point where $Q = 0$ and $H \neq 0$. To estimate I, we fix α and choose a tangent basis $\{\widetilde{e}_1, \dots, \widetilde{e}_m\}$, not necessarily of kind (B1) or (B2), that diagonalizes h^{α}, i.e. $h^{\alpha}_{ij} = \lambda^{\alpha}_i \delta_{ij}$. Like in estimate (3.18), we have

$$\begin{aligned} &4\sum_{i,j,p,q} \bar{R}_{ipjq}h^{\alpha}_{pq}h^{\alpha}_{ij} - 4\sum_{j,s,p} \bar{R}_{sjsp}\left(\sum_{i} h^{\alpha}_{pi}h^{\alpha}_{ij}\right) \\ &= 4\sum_{i,p} \bar{R}_{ipip}(\lambda^{\alpha}_i\lambda^{\alpha}_p - (\lambda^{\alpha}_i)^2) \\ &= -2\sum_{i,p} \bar{K}_{ip}(\lambda^{\alpha}_i - \lambda^{\alpha}_p)^2 \leq -4m|\mathring{h}^{\alpha}|^2. \end{aligned}$$

Hence we obtain

$$I \leq -4m|\mathring{A}|^2. \tag{3.21}$$

A basis of type (B2) is useful for estimating the terms II and III. We recall that the curvature tensor of the Fubini-Study metric, for every X, Y, Z and W tangent vector fields of $\mathbb{CP}^n$, is

$$\begin{aligned} \bar{R}(X,Y,Z,W) = {} & g_{FS}(X,Z)g_{FS}(Y,W) - g_{FS}(X,W)g_{FS}(Y,Z) \\ & + g_{FS}(X,JZ)g_{FS}(Y,JW) \\ & - g_{FS}(X,JW)g_{FS}(Y,JZ) \\ & + 2g_{FS}(X,JY)g_{FS}(Z,JW). \end{aligned} \tag{3.22}$$

In order to study the term II, note that, with our choice of the basis, we have that $\bar{R}_{s\alpha s\beta} = 0$ for any s if $\alpha \neq \beta$. Otherwise we have

$$\bar{R}_{s\alpha s\alpha} = \bar{K}_{s\alpha} = 1 + 3g_{FS}(e_s, Je_\alpha)^2,$$

which implies that $1 \leq \bar{K}_{s\alpha} \leq 1 + 3\delta_{s,\alpha-m}$. Therefore, since $a \geq \frac{1}{m}$, we have

$$\begin{aligned} II &= 2\sum_{s,\alpha} \bar{K}_{s\alpha} \left(|h^\alpha|^2 - a\,|H^\alpha|^2 \right) \\ &= 2\sum_{s,\alpha} \bar{K}_{s\alpha} \left(\left|\mathring{h}^\alpha\right|^2 - \left(a - \frac{1}{m}\right) |H^\alpha|^2 \right) \\ &\leq 2\sum_{s,\alpha} (1 + 3\delta_{s,\alpha-m}) \left|\mathring{h}^\alpha\right|^2 \\ &= 2(m+3)|\mathring{A}|^2. \end{aligned} \tag{3.23}$$

The most difficult term is III. Since $\bar{R}_{jp\alpha\beta}$ is anti-symmetric in j, p, while h^α_{jp} is symmetric, we have

$$III = -8 \sum_{j,p,\alpha,\beta} \bar{R}_{jp\alpha\beta} \left(\sum_i h^\alpha_{ip} h^\beta_{ij} \right) = -8 \sum_{j,p,\alpha,\beta} \bar{R}_{jp\alpha\beta} \left(\sum_i \mathring{h}^\alpha_{ip} \mathring{h}^\beta_{ij} \right).$$

We now analyze the possible values of $\bar{R}_{jp\alpha\beta}$. First fix α and β coupled by (3.3), meaning that $\min\{\alpha, \beta\} = m + 2r - 1$ and $\max\{\alpha, \beta\} = m + 2r$ for some $r \leq k/2$. By symmetry, it suffices to consider the case where $\alpha < \beta$. We find

$$\bar{R}_{jp\alpha\beta} = \tau_r^2 \left(\delta_{j,2r-1}\delta_{p,2r} - \delta_{j,2r}\delta_{p,2r-1} \right) - 2\nu_r g_{FS}(e_j, Je_p),$$

and

$$g_{FS}(e_j, Je_p) = \begin{cases} -\nu_s & \text{if } j = 2s, \quad p = 2s-1, \quad s \leq \frac{k}{2}; \\ \nu_s & \text{if } j = 2s-1, \quad p = 2s, \quad s \leq \frac{k}{2}; \\ 1 & \text{if } j = k+2s, \quad p = k+2s-1, \quad s \leq \frac{m-k}{2}; \\ -1 & \text{if } j = k+2s-1, \quad p = k+2s, \quad s \leq \frac{m-k}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

If α and β are not coupled by (3.3), there are two indices $r \neq s$ such that α is (or is coupled with) e_{m+2r-1} and β is (or is coupled with) e_{m+2s-1}. In this case we have

$$\bar{R}_{jp\alpha\beta} = \tau_r\tau_s\left(\delta_{j,\alpha-m}\delta_{p,\beta-m} - \delta_{j,\beta-m}\delta_{p,\alpha-m}\right).$$

Using what we have just found and summing all similar terms we have

$$\begin{aligned}
III = {} & 16\sum_r \left(2\nu_r^2 - \tau_r^2\right)\sum_i \left(\mathring{h}_{i\,2r}^{m+2r-1}\mathring{h}_{i\,2r-1}^{m+2r} - \mathring{h}_{i\,2r-1}^{m+2r-1}\mathring{h}_{i\,2r}^{m+2r}\right)\\
& - 8\sum_{r\neq s\leq\frac{k}{2}} \tau_r\tau_s\sum_i\left(\mathring{h}_{i\,2s}^{m+2r}\mathring{h}_{i\,2r}^{m+2s} - \mathring{h}_{i\,2r}^{m+2r}\mathring{h}_{i\,2s}^{m+2s}\right)\\
& - 16\sum_{r\neq s, r\leq\frac{k}{2}, s\leq\frac{k+1}{2}} \tau_r\tau_s\sum_i\left(\mathring{h}_{i\,2s-1}^{m+2r}\mathring{h}_{i\,2r}^{m+2s-1} - \mathring{h}_{i\,2r}^{m+2r}\mathring{h}_{i\,2s-1}^{m+2s-1}\right)\\
& - 8\sum_{r\neq s\leq\frac{k+1}{2}} \tau_r\tau_s\sum_i\left(\mathring{h}_{i\,2s-1}^{m+2r-1}\mathring{h}_{i\,2r-1}^{m+2s-1} - \mathring{h}_{i\,2r-1}^{m+2r-1}\mathring{h}_{i\,2s-1}^{m+2s-1}\right)\\
& + 32\sum_{r\neq s\leq\frac{k}{2}} \nu_r\nu_s\sum_i\left(\mathring{h}_{i\,2s}^{m+2r-1}\mathring{h}_{i\,2s-1}^{m+2r} - \mathring{h}_{i\,2s-1}^{m+2r-1}\mathring{h}_{i\,2s}^{m+2r}\right)\\
& + 32\sum_{r\leq\frac{k}{2}}\nu_r\sum_{s\leq\frac{m-k}{2}}\sum_i\left(\mathring{h}_{i\,k+2s-1}^{m+2r-1}\mathring{h}_{i\,k+2s}^{m+2r} - \mathring{h}_{i\,k+2s}^{m+2r-1}\mathring{h}_{i\,k+2s-1}^{m+2r}\right).
\end{aligned}$$

Obviously $III \leq |III|$. Using repeatedly the triangle inequality and Young's inequality, and taking into account that for any r and s

$$\begin{cases}
\left|2\nu_r^2 - \tau_r^2\right| \leq 2,\\
|\tau_r\tau_s| \leq 1,\\
|\nu_r\nu_s| \leq 1,\\
|\nu_r| \leq 1,
\end{cases}$$

we have

$$\begin{aligned}
III \leq {} & 16\sum_{r\leq\frac{k}{2}}\sum_i\left(\left|\mathring{h}_{i\,2r}^{m+2r-1}\right|^2 + \left|\mathring{h}_{i\,2r-1}^{m+2r}\right|^2 + \left|\mathring{h}_{i\,2r-1}^{m+2r-1}\right|^2 + \left|\mathring{h}_{i\,2r}^{m+2r}\right|^2\right)\\
& + 4\sum_{r\neq s\leq\frac{k}{2}}\sum_i\left(\left|\mathring{h}_{i\,2s}^{m+2r}\right|^2 + \left|\mathring{h}_{i\,2r}^{m+2s}\right|^2 + \left|\mathring{h}_{i\,2r}^{m+2r}\right|^2 + \left|\mathring{h}_{i\,2s}^{m+2s}\right|^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+8\sum_{r\neq s, r\leq\frac{k}{2}, s\leq\frac{k+1}{2}}\sum_i\left(\left|\mathring{h}_{i\ 2s-1}^{m+2r}\right|^2+\left|\mathring{h}_{i\ 2r}^{m+2s-1}\right|^2+\left|\mathring{h}_{i\ 2r}^{m+2r}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2s-1}\right|^2\right)\\
&+4\sum_{r\neq s\leq\frac{k+1}{2}}\sum_i\left(\left|\mathring{h}_{i\ 2s-1}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ 2r-1}^{m+2s-1}\right|^2+\left|\mathring{h}_{i\ 2r-1}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2s-1}\right|^2\right)\\
&+16\sum_{r\neq s\leq\frac{k}{2}}\sum_i\left(\left|\mathring{h}_{i\ 2s}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2r}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ 2s}^{m+2r}\right|^2\right)\\
&+16\sum_{r\leq\frac{k}{2}, s\leq\frac{m-k}{2}}\sum_i\left(\left|\mathring{h}_{i\ k+2s-1}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ k+2s}^{m+2r}\right|^2+\left|\mathring{h}_{i\ k+2s}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ k+2s-1}^{m+2r}\right|^2\right).
\end{aligned}
$$

Note that, if $k=2$, there are no indices $r\neq s\leq\frac{k+1}{2}$. Then, some of the sums in the expressions above are empty and we easily find that

$$III\leq 16|\mathring{A}|^2.$$

If $k>2$, by collecting similar terms we find

$$
\begin{aligned}
III\leq&\sum_{i,r}\left(16\left|\mathring{h}_{i\ 2r}^{m+2r-1}\right|^2+16\left|\mathring{h}_{i\ 2r-1}^{m+2r}\right|^2+8k\left|\mathring{h}_{i\ 2r}^{m+2r}\right|^2+8k\left|\mathring{h}_{i\ 2r-1}^{m+2r-1}\right|^2\right)\\
&+24\sum_{i,r\neq s\leq\frac{k}{2}}\left(\left|\mathring{h}_{i\ 2s}^{m+2r}\right|^2+\left|\mathring{h}_{i\ 2s}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2r}\right|^2+\left|\mathring{h}_{i\ 2s-1}^{m+2r-1}\right|^2\right)\\
&+16\sum_{i,r,s\leq\frac{m-k}{2}}\left(\left|\mathring{h}_{i\ k+2s-1}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ k+2s}^{m+2r}\right|^2+\left|\mathring{h}_{i\ k+2s}^{m+2r-1}\right|^2+\left|\mathring{h}_{i\ k+2s-1}^{m+2r}\right|^2\right)\\
\leq& 8k|\mathring{A}|^2.
\end{aligned}
$$

So we can say that in any case

$$III\leq 8k|\mathring{A}|^2. \tag{3.24}$$

By (3.21), (3.23) and (3.24), we conclude that

$$P_a=I+II+III\leq -2(m-3-4k)|\mathring{A}|^2.$$

Now let $R = 2R_1 - 2aR_2 + P_a$. If we again consider a frame of type (B1), Lemma 3.2 says that at any point with $Q = 0$ we have

$$R \leq \left(6 - \frac{2}{ma-1}\right) |\mathring{A}|^2 |\mathring{h}_-|^2 + \left(\frac{2mab}{ma-1} - 2(m-3-4k)\right) |\mathring{h}_1|^2 - 3|\mathring{h}_-|^4 + \left(\frac{4b}{ma-1} - 2(m-3-4k)\right) |\mathring{h}_-|^2 - \frac{2b^2}{ma-1}.$$

Observe that, for our choice of a and b, the coefficient of $|\mathring{A}|^2|\mathring{h}_-|^2$ is negative, while the one multiplying $|\mathring{h}_1|^2$ is zero. In addition, the assumptions $Q = 0$ and $a > 1/m$ imply that $|\mathring{A}|^2 \geq b$. Using this, we obtain

$$\begin{aligned} R &\leq -3|\mathring{h}_-|^4 + \left[\left(6 - \frac{2}{ma-1}\right) b + \frac{4b}{ma-1} - 2(m-3-4k)\right] |\mathring{h}_-|^2 \\ &\quad - \frac{2b^2}{ma-1} \\ &= -3|\mathring{h}_-|^4 + 4b|\mathring{h}_-|^2 + 2b(b - m + 3 + 4k). \end{aligned}$$

Using $4b|\mathring{h}_-|^2 \leq 3|\mathring{h}_-|^4 + \frac{4}{3}b^2$, we deduce

$$R \leq 2b\left(\frac{5}{3}b - m + 3 + 4k\right).$$

Our choice of b then implies that $R < 0$.

Finally, let us consider the case of a point where $Q = |H|^2 = 0$. Then we have $|A|^2 = |\mathring{A}|^2 = b$, $R_2 = 0$. Moreover, using Theorem 1 from [LL], we find that $2R_1 \leq 3\,|A|^4 = 3b^2$. As before, we obtain that $P_a \leq -2(m-3-4k)|\mathring{A}|^2 = -2(m-3-4k)b$. Therefore,

$$R \leq 3b^2 - 2(m-3-4k)b,$$

which is negative for our choice of b. By the maximum principle, the assertion follows. □

4. The traceless second fundamental form

Following an approach which goes back to [Ha, H1], the description of the asymptotic behavior of $\mathcal{M}_t$ will be obtained analyzing the traceless part of the second fundamental form and showing that it becomes small in a suitable sense if the curvature becomes unbounded.

Since our initial manifold $\mathcal{M}_0$ satisfies the assumption (1.2), it also satisfies inequality (3.15), respectively (3.19), for some $\varepsilon > 0$. We know from the results of the previous section that these inequalitis are preserved by the flow for all $t > 0$.

As in [H3, Ba], we introduce the functions

$$W := \alpha |H|^2 + \beta, \qquad f_\sigma := \frac{|\mathring{A}|^2}{W^{1-\sigma}}.$$

Here σ is a suitably small non-negative constant, while α, β are defined by

$$(4.1) \qquad \begin{aligned} \alpha &= \begin{cases} \dfrac{2}{(m-1+\varepsilon)(2+\bar{r}-2\varepsilon)} & \text{if } k = 1 \\ \dfrac{m-10}{3m^2} & \text{if } k \geq 2 \end{cases} \\ \beta &= \begin{cases} 2 & \text{if } k = 1 \\ \dfrac{m-3-4k}{m} & \text{if } k \geq 2. \end{cases} \end{aligned}$$

The main result of this section is the next proposition, which gives a differential inequality satisfied by f_σ.

Proposition 4.1. *Under the assumptions of Theorem 1.1 there is a σ_1 depending only on $\mathcal{M}_0$ that for all $0 \leq \sigma \leq \sigma_1$*

$$(4.2) \qquad \begin{aligned} \frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma &+ \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle \\ &- 2C_1 W^{\sigma-1} |\nabla H|^2 + 2\sigma |A|^2 f_\sigma - 2C_2 f_\sigma, \end{aligned}$$

for some constants $C_1 > 0$ and $C_2 > 0$ depending only on m and the initial data.

Proof. Let us analyze the evolution equation for f_σ. A straightforward computation gives

$$(4.3) \qquad \begin{aligned} \Delta f_\sigma = W^{\sigma-1}\Delta|\mathring{A}|^2 &- \alpha(1-\sigma)\frac{f_\sigma}{W}\Delta |H|^2 \\ &- \frac{2\alpha(1-\sigma)}{W} \left\langle \nabla f_\sigma, \nabla |H|^2 \right\rangle \\ &+ \alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2} \left| \nabla |H|^2 \right|^2. \end{aligned}$$

Therefore

$$\frac{\partial}{\partial t} f_\sigma - \Delta f_\sigma = W^{\sigma-1}\left(\frac{\partial}{\partial t}|\mathring{A}|^2 - \Delta|\mathring{A}|^2\right) - \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\frac{\partial}{\partial t}|H|^2 - \Delta|H|^2\right) + \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_\sigma, \nabla |H|^2\right\rangle - \alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2}\left|\nabla |H|^2\right|^2.$$

Let us first consider the case of hypersurfaces $k = 1$. Using Lemma 2.2, and neglecting the negative $\left|\nabla |H|^2\right|^2$ term, we have

$$\begin{aligned}\frac{\partial}{\partial t} f_\sigma \le \Delta f_\sigma &+ \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_\sigma, \nabla |H|^2\right\rangle - 2W^{\sigma-1}|\nabla A|^2 \\ &+ 2W^{\sigma-1}\left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]|\nabla H|^2 \\ &+ 2\beta\frac{(1-\sigma)}{W} f_\sigma\left(|A|^2 + \bar{r}\right) + 2\sigma f_\sigma\left(|A|^2 + \bar{r}\right) \\ &- 4W^{\sigma-1}\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right).\end{aligned} \tag{4.4}$$

Our choice of α and β gives $0 \le f_0 < 1$. Hence, by Lemma 3.5,

$$\begin{aligned}&-|\nabla A|^2 + \left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]|\nabla H|^2 \\ &\le \left(\frac{1}{m} + \alpha\right)|\nabla H|^2 - |\nabla A|^2 = -C_1|\nabla H|^2,\end{aligned} \tag{4.5}$$

where $C_1 = \frac{3}{m+2} - \frac{1}{m} - \alpha$ is positive for our choice of α and $m \ge 5$. It remains to estimate the reaction terms. Let us set

$$\begin{aligned}R := 2\beta\frac{(1-\sigma)}{W} f_\sigma\left(|A|^2 + \bar{r}\right) &+ 2\sigma f_\sigma\left(|A|^2 + \bar{r}\right) \\ &- 4W^{\sigma-1}\left(h_{ij}h_j^{\ p}\bar{R}_{pli}^{\ \ \ l} - h^{ij}h^{lp}\bar{R}_{pilj}\right).\end{aligned}$$

Using inequality (3.18) we have

$$R \le 2f_\sigma\left[\beta(1-\sigma)\frac{|A|^2 + \bar{r}}{W} + \sigma(|A|^2 + \bar{r}) - 2m\right].$$

From (3.15) and the definitions (3.17), (4.1) of $\bar{r}$, α and β, we obtain

$$|A|^2 + \bar{r} \le \frac{1}{m-1+\varepsilon}|H|^2 + 2(1-\varepsilon) + \bar{r} = \frac{2 + \bar{r} - 2\varepsilon}{\beta}W.$$

Since $m \geq 5$ and ε is small, we have

$$\begin{aligned} R &\leq 2f_\sigma \left[(1-\sigma)(\beta + \bar{r} - 2\varepsilon) + \bar{r}\sigma - 2m\right] + 2\sigma f_\sigma \left|A\right|^2 \\ &= 2f_\sigma \left[5 - m - 2\varepsilon + \sigma(2\varepsilon - 2)\right] + 2\sigma f_\sigma \left|A\right|^2 \leq -4\varepsilon f_\sigma + 2\sigma f_\sigma \left|A\right|^2. \end{aligned}$$

This inequality, together with (4.4) and (4.5), implies the assertion for the case of hypersurfaces, with $C_2 = 2\varepsilon$.

Let us now turn to the case $k \geq 2$. From Lemma 2.1 and the properties of the curvature tensor $\bar{R}$, arguing as in the estimation of term II in the proof of Proposition 3.7, we find

$$\begin{aligned} \frac{\partial}{\partial t}\left|H\right|^2 &= \Delta\left|H\right|^2 - 2\left|\nabla H\right|^2 + 2R_2 + 2\sum_{s,\alpha}\bar{K}_{s\alpha}\left|H^\alpha\right|^2 \\ &\geq \Delta\left|H\right|^2 - 2\left|\nabla H\right|^2 + 2R_2 + 2m\left|H\right|^2. \end{aligned} \tag{4.6}$$

Moreover, by Lemma 2.1, we have

$$\frac{\partial}{\partial t}|\mathring{A}|^2 = \Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{m}\left|\nabla H\right|^2\right) + 2\left(R_1 - \frac{1}{m}R_2\right) + P_{\frac{1}{m}},$$

where, like in the proof of Proposition 3.7,

$$P_{\frac{1}{m}} \leq -2(m-3-4k)|\mathring{A}|^2.$$

Then

$$\begin{aligned} \frac{\partial}{\partial t}f_\sigma \leq\ & W^{\sigma-1}\left(\Delta|\mathring{A}|^2 - 2\left(|\nabla A|^2 - \frac{1}{m}\left|\nabla H\right|^2\right)\right) \\ &+ W^{\sigma-1}\left(2\left(R_1 - \frac{1}{m}R_2\right) - 2(m-3-4k)|\mathring{A}|^2\right) \\ &- \alpha(1-\sigma)\frac{f_\sigma}{W}\left(\Delta\left|H\right|^2 - 2\left|\nabla H\right|^2 + 2R_2 + 2m\left|H\right|^2\right). \end{aligned}$$

Using the expression found previously for Δf_σ, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}f_\sigma \leq\ & \Delta f_\sigma + \frac{2\alpha(1-\sigma)}{W}\left\langle\nabla f_\sigma, \nabla\left|H\right|^2\right\rangle - 2W^{\sigma-1}\left|\nabla A\right|^2 \\ &+ 2W^{\sigma-1}\left[\frac{1}{m} + f_0(1-\sigma)\alpha\right]\left|\nabla H\right|^2 \\ &+ 2W^{\sigma-1}\left(R_1 - \frac{1}{m}R_2\right) - 2\alpha(1-\sigma)\frac{f_\sigma}{W}R_2 \\ &- 2m\alpha(1-\sigma)\frac{f_\sigma}{W}\left|H\right|^2 - 2(m-3-4k)W^{\sigma-1}|\mathring{A}|^2. \end{aligned} \tag{4.7}$$

To estimate the gradient terms, we use Lemma 3.5. Let us set

$$C_1 = \frac{16}{9(m+2)} - \frac{4m-10}{3m^2},$$

which is positive for all $m \geq 0$. Then we have, using again $0 \leq f_0 < 1$,

$$\begin{aligned}\left[\frac{1}{m} + f_0(1-\sigma)\alpha\right] |\nabla H|^2 &\leq \left(\frac{1}{m} + \alpha\right) |\nabla H|^2 = \frac{4m-10}{3m^2} |\nabla H|^2 \\ &= \left(\frac{16}{9(m+2)} - C_1\right) |\nabla H|^2 \leq |\nabla A|^2 - C_1 |\nabla H|^2,\end{aligned}$$

which yields the desired estimate. Let us now analyze the reaction terms. We can write them as

$$R = 2W^{\sigma-2}R' + 2\alpha\sigma\frac{f_\sigma}{W}R_2$$

where

$$R' = \left(R_1 - \frac{1}{m}R_2\right) W - \alpha|\mathring{A}|^2 R_2 - \alpha m(1-\sigma)|\mathring{A}|^2 |H|^2 - (m-3-4k)|\mathring{A}|^2 W.$$

We first estimate

$$\begin{aligned}2\alpha\sigma\frac{f_\sigma}{W}R_2 &\leq 2\alpha\sigma\frac{f_\sigma}{W} |A|^2 |H|^2 \\ &= 2\sigma f_\sigma |A|^2 - 2\sigma\beta\frac{f_\sigma}{W} |H|^2 \leq 2\sigma f_\sigma |A|^2.\end{aligned} \tag{4.8}$$

The rest of the proof is devoted to the estimation of R'. By Lemma 3.2

$$R_1 - \frac{1}{m}R_2 \leq |\mathring{h}_1|^4 + \frac{1}{m}|\mathring{h}_1|^2 |H|^2 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4.$$

Moreover $|\mathring{A}|^2 = |\mathring{h}_1|^2 + |\mathring{h}_-|^2$ and $R_2 = |\mathring{h}_1|^2 |H|^2 + \frac{1}{m} |H|^4$, so

$$\begin{aligned}R' \leq\ & 3\alpha|\mathring{h}_1|^2|\mathring{h}_-|^2 |H|^2 + \frac{3}{2}\alpha|\mathring{h}_-|^4 |H|^2 - \frac{\alpha}{m}|\mathring{h}_-|^2 |H|^4 \\ &+ \beta|\mathring{h}_1|^4 + 4\beta|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}\beta|\mathring{h}_-|^4 \\ &+ \left(\frac{\beta}{m} - m\alpha(1-\sigma) - \alpha(m-3-4k)\right) |\mathring{h}_1|^2 |H|^2 \\ &- \alpha\left(m(1-\sigma) + m - 3 - 4k\right) |\mathring{h}_-|^2 |H|^2 \\ &- \beta(m-3-4k)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right).\end{aligned} \tag{4.9}$$

Since the pinching condition (1.2) holds, we have that

$$\left(\frac{1}{m-1}-\frac{1}{m}\right)|H|^2 \geq \left(|\mathring{h}_1|^2+|\mathring{h}_-|^2-\beta\right).$$

Then we have

$$\begin{aligned}
R' &= R' + \frac{3\alpha}{m(m-1)}|\mathring{h}_-|^2\,|H|^4 + \frac{2\beta}{m(m-1)}\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right)|H|^2 \\
&\quad - \frac{3\alpha}{m(m-1)}|\mathring{h}_-|^2\,|H|^4 - \frac{2\beta}{m(m-1)}\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right)|H|^2 \\
&\leq R' + \frac{3\alpha}{m(m-1)}|\mathring{h}_-|^2\,|H|^4 + \frac{2\beta}{m(m-1)}\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right)|H|^2 \\
&\quad - 3\alpha(|\mathring{h}_1|^2+|\mathring{h}_-|^2-\beta)|\mathring{h}_-|^2\,|H|^2 \\
&\quad - 2\beta\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right)\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2-\beta\right) \\
&\leq -\alpha\frac{m-4}{m(m-1)}|\mathring{h}_-|^2\,|H|^4 \\
&\quad + \left[\beta\left(\frac{2}{m-1}-\frac{1}{m}\right)-\alpha(m(2-\sigma)-3-4k)\right]|\mathring{h}_1|^2\,|H|^2 \\
&\quad + \left[\beta\left(\frac{2}{m-1}-\frac{2}{m}\right)-\alpha(m(2-\sigma)-3-4k-3\beta)\right]|\mathring{h}_-|^2\,|H|^2 \\
&\quad + \beta\,(2\beta-m+3+4k)\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right).
\end{aligned}$$

Our hypotheses give $\beta \leq \frac{1}{4}(m-3-4k)$. We can further assume that σ is small, say $\sigma < \frac{1}{4}$. Using these inequalities, the condition $m > 4k+3$ and the inequalities

$$\frac{2\beta}{m(m-1)} < \frac{\beta(m+1)}{m(m-1)} < \frac{m-3-4k}{4m},$$

we obtain

$$\begin{aligned}
R' &\leq \left[\beta\frac{m+1}{m(m-1)}-\frac{\alpha}{4}(7m-12-16k)\right]|\mathring{h}_1|^2\,|H|^2 \\
&\quad + \left[\frac{2\beta}{m(m-1)}-\alpha\left(\frac{7}{4}m-3-4k-3\beta\right)\right]|\mathring{h}_-|^2\,|H|^2 \\
&\quad + \beta\,(2\beta-m+3+4k)\left(|\mathring{h}_1|^2+|\mathring{h}_-|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\frac{m-3-4k}{4m} - \frac{\alpha}{4}(7m-12-16k)\right] |\mathring{h}_1|^2 |H|^2 \\
&\quad + \left[\frac{m-3-4k}{4m} - \frac{\alpha}{4}(4m-3-4k)\right] |\mathring{h}_-|^2 |H|^2 \\
&\quad - 2\beta^2 \left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right) \\
&\leq \left[\frac{m-11}{4m} - \frac{3m}{4}\alpha\right] (|\mathring{h}_1|^2 + |\mathring{h}_-|^2) |H|^2 \\
&\quad - 2\beta^2 \left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right) \\
&\leq -C_2 |\mathring{A}|^2 W,
\end{aligned}$$

for some positive constant C_2 depending only on m. Together with (4.8), this implies that

$$R \leq 2\sigma f_\sigma |A|^2 + 2W^{\sigma-2}R' \leq 2\sigma f_\sigma |A|^2 - 2C_2 f_\sigma,$$

which concludes our proof. □

We now prove some other estimates which will be needed in the following.

Lemma 4.2. *We have the estimates:*

1) $\dfrac{\partial}{\partial t}|\mathring{A}|^2 \leq \Delta|\mathring{A}|^2 - 2C_3 |\nabla A|^2 + 4|A|^2 |\mathring{A}|^2$, *for some* $C_3 > 0$ *only depending on* m,
2) $\dfrac{\partial}{\partial t}|H|^4 \geq \Delta |H|^4 - 12 |H|^2 |\nabla H|^2 + \dfrac{4}{m} |H|^6$.

Proof. In the case of hypersurfaces, inequality 1) follows easily from Lemma 2.2, inequality (3.9) and estimate (3.18). For higher codimension we use Lemma 2.1:

$$\begin{aligned}
\frac{\partial}{\partial t}|\mathring{A}|^2 \leq \Delta|\mathring{A}|^2 &- 2\left(|\nabla A|^2 - \frac{1}{m}|\nabla H|^2\right) \\
&+ 2\left(R_1 - \frac{1}{m}R_2\right) + P_{\frac{1}{m}}.
\end{aligned}$$

By Lemma 3.5, we have that $-2\left(|\nabla A|^2 - \frac{1}{m}|\nabla H|^2\right) \leq -2C_3 |\nabla A|^2$ for some positive constant C_3. Moreover, using Lemma 3.2,

$$\begin{aligned} R_1 - \frac{1}{m} R_2 &\leq |\mathring{h}_1|^4 + 4|\mathring{h}_1|^2|\mathring{h}_-|^2 + \frac{3}{2}|\mathring{h}_-|^4 + \frac{1}{m}|\mathring{h}_1|^2\,|H|^2 \\ &\leq 2\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)^2 + \frac{2}{m}\,|H|^2\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right) = 2|\mathring{A}|^2\,|A|^2 . \end{aligned}$$

Finally, like in the proof of Proposition 3.7,

$$P_{\frac{1}{m}} \leq -2(m-3-4k)|\mathring{A}|^2 \leq 0.$$

This proves inequality 1). To prove the second part, we use again Lemma 2.2 and (2.1). For hypersurfaces we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\,|H|^4 &= \Delta\,|H|^4 - 2\left|\nabla\,|H|^2\right|^2 - 4\,|H|^2\,|\nabla H|^2 + 4\,|H|^4\,(|A|^2 + \bar{r}) \\ &\geq \Delta\,|H|^4 - 12\,|H|^2\,|\nabla H|^2 + \frac{4}{m}\,|H|^6 . \end{aligned}$$

For higher codimension we use the inequality

$$2R_2 = 2\,|H|^2\left(|\mathring{h}_1|^2 + \frac{1}{m}\,|H|^2\right) \geq \frac{2}{m}\,|H|^4$$

and we find

$$\begin{aligned} \frac{\partial}{\partial t}\,|H|^4 &= \Delta\,|H|^4 - 2\left|\nabla\,|H|^2\right|^2 - 4\,|H|^2\,|\nabla H|^2 \\ &\quad + 2\,|H|^2\left(2R_2 + 2\sum_{s,\alpha}\bar{K}_{s\alpha}\,|H^\alpha|^2\right) \\ &\geq \Delta\,|H|^4 - 12\,|H|^2\,|\nabla H|^2 + \frac{4}{m}\,|H|^6 . \end{aligned}$$

□

Finally, we consider the evolution equation for $|\nabla H|^2$. With the same proof of Corollary 5.10 in [Ba], we have the following result.

Proposition 4.3. *There exists a constant C_4 depending only on $\mathcal{M}_0$ such that*

$$\frac{\partial}{\partial t}\,|\nabla H|^2 \leq \Delta\,|\nabla H|^2 + C_4(|H|^2 + 1)\,|\nabla A|^2 .$$

5. Finite maximal time

In this section we consider the case that our flow develops a singularity in finite time and prove convergence to a round point as stated in Theorem 1.1.

Since $\mathcal{M}_0$ is compact, there is also an $\varepsilon > 0$ small enough such that

$$|A|^2 \leq a\,|H|^2 + b, \tag{5.1}$$

where

$$a = \frac{1}{m-1+\varepsilon}, \qquad b = \begin{cases} 2(1-\varepsilon) & \text{if } k = 1 \\ \dfrac{m-3-4k}{m}(1-\varepsilon) & \text{if } k \geq 2. \end{cases} \tag{5.2}$$

We know that inequality (5.1) with the above choice of constants remains preserved during the flow. As in the previous section, we let $W = \alpha\,|H|^2 + \beta$, where α, β are chosen according to (4.1). We observe that

$$\begin{aligned} 2mW &\geq 2m\left(a - \frac{1}{m}\right)|H|^2 + 2mb \\ &= 2\frac{1-\varepsilon}{m-1+\varepsilon}\,|H|^2 + 2mb > a\,|H|^2 + b \geq |A|^2\,. \end{aligned} \tag{5.3}$$

Theorem 5.1. *Let the assumptions of Theorem 1.1 hold. If T_{max} is finite, there are constants $C_0 < \infty$ and $\sigma_0 > 0$ depending only on the initial manifold $\mathcal{M}_0$ such that for all $0 \leq t < T_{max}$ we have*

$$|\mathring{A}|^2 \leq C_0(|H|^2 + 1)^{1-\sigma_0}.$$

To prove this result we will bound from above the function f_σ introduced in the previous section. For $\sigma > 0$, the positive term $2\sigma f_\sigma\,|A|^2$ in (4.1) prevents us from using the maximum principle. Therefore, as in Huisken [H3] and Baker [Ba], we will obtain integral estimates on f_σ exploiting the good negative $|\nabla H|^2$ term by the divergence theorem. These estimates allow to deduce the desired sup-estimate through a standard iteration procedure.

The starting point of our proof is the contracted Simons identity computed in [AB], formula (23). Using this we easily obtain

$$\Delta|\mathring{A}|^2 \geq 2|\nabla\mathring{A}|^2 + 2\left\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\right\rangle + 2Z - c|A|^2,$$

where $c > 0$ is a suitable constant only depending on m, k and

$$Z = \sum_{i,j,p,\alpha,\beta} H^\alpha h^\alpha_{ip} h^\beta_{pj} h^\beta_{ij} - \sum_{\alpha,\beta}\left(\sum_{i,j} h^\alpha_{ij} h^\beta_{ij}\right)^2 - \sum_{i,j,\alpha,\beta}\left(\sum_p \left(h^\alpha_{ip} h^\beta_{pj} - h^\alpha_{jp} h^\beta_{ip}\right)\right)^2.$$

Using our pinching assumption we also deduce

$$\Delta|\mathring{A}|^2 \geq 2|\nabla \mathring{A}|^2 + 2\left\langle \mathring{h}_{ij}, \nabla_i \nabla_j H\right\rangle + 2Z - \gamma W, \tag{5.4}$$

where γ only depends on m, k.

To understand the properties of Z in the case of hypersurfaces, it is interesting to relate the pinching condition (1.2) to the positivity of the intrinsic sectional curvature of the submanifold $\mathcal{M}_t$.

Proposition 5.2. *There exists a constant $c = c(m)$ such that if $k = 1$ the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies at any point*

$$K > \varepsilon c W > 0.$$

Proof. Let $e_1, \ldots, e_m$ be a orthonormal tangent basis that diagonalizes the second fundamental form. For any $i \neq j$ the Gauss equation gives

$$K_{ij} = \bar{K}_{ij} + \lambda_i \lambda_j.$$

Like in [H3], we can use the following algebraic property: for any $i \neq j$

$$\begin{aligned} |A|^2 - \frac{1}{m-1}|H|^2 &= -2\lambda_i\lambda_j + \left(\lambda_i + \lambda_j - \frac{|H|}{m-1}\right)^2 \\ &\quad + \sum_{l \neq i,j}\left(\lambda_l - \frac{|H|}{m-1}\right)^2 \\ &\geq -2\lambda_i\lambda_j. \end{aligned} \tag{5.5}$$

Then we have

$$\begin{aligned}
2K_{ij} &\geq 2 - |A|^2 + \frac{1}{m-1}|H|^2 \\
&\geq \left(\frac{1}{m-1} - a\right)|H|^2 + 2 - b \\
&= \varepsilon\left(\frac{1}{(m-1)(m-1+\varepsilon)}|H|^2 + 2\right) \\
&\geq \varepsilon c\left(\alpha|H|^2 + \beta\right) > 0,
\end{aligned}$$

for a suitable $c = c(m)$. □

We cannot use the same argument in higher codimension because we cannot diagonalize simultaneously the tensors h^α, for $\alpha = m+1, \ldots, 2n$. However, as a consequence of our other estimates, we will prove at the end of this section that also in this case the sectional curvature of the evolving submanifold becomes positive for time large enough.

Lemma 5.3. *There exists $\rho > 0$ depending only on m, k such that Z satisfies*

$$Z + 2mb|\mathring{A}|^2 \geq \rho\varepsilon|\mathring{A}|^2 W.$$

Proof. Let us first consider the case of hypersurfaces. Choosing a basis that diagonalizes the second fundamental form, using Gauss equations, Proposition 5.2 and $\bar{K} \leq 4$, we have

$$\begin{aligned}
Z &= \left(\sum_i \lambda_i\right)\left(\sum_i \lambda_i^3\right) - \left(\sum_i \lambda_i^2\right)^2 \\
&= \sum_{i<j} \lambda_i\lambda_j\left(\lambda_i - \lambda_j\right)^2 \\
&= \sum_{i<j} K_{ij}\left(\lambda_i - \lambda_j\right)^2 - \sum_{i<j} \bar{K}_{ij}\left(\lambda_i - \lambda_j\right)^2 \\
&\geq \varepsilon c(m) W|\mathring{A}|^2 - 4m|\mathring{A}|^2 = \varepsilon c(m) W|\mathring{A}|^2 - 2bm|\mathring{A}|^2.
\end{aligned}$$

For $k \geq 2$ we need to distinguish the cases $H = 0$ and $H \neq 0$. Let us examine first the case $H \neq 0$. We use an estimate proved by Andrews and Baker, see page 384 in [AB], which gives

$$\begin{aligned} Z \geq & -\frac{m}{2}|\mathring{h}_1|^4 - \frac{3}{2}|\mathring{h}_-|^4 - \frac{m+2}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2 \\ & + \frac{1}{2(m-1)}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)|H|^2 . \end{aligned}$$

Since (5.1) and (5.2) hold, we have $|H|^2 \geq \frac{m(m-1+\varepsilon)}{1-\varepsilon}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b\right)$. Then

$$\begin{aligned} Z \geq & -\frac{m}{2}|\mathring{h}_1|^4 - \frac{3}{2}|\mathring{h}_-|^4 - \frac{m+2}{2}|\mathring{h}_1|^2|\mathring{h}_-|^2 \\ & + \frac{m}{2(1-\varepsilon)}\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2\right)\left(|\mathring{h}_1|^2 + |\mathring{h}_-|^2 - b\right) \\ = & \frac{\varepsilon m}{2(1-\varepsilon)}|\mathring{h}_1|^4 + \frac{m-3+3\varepsilon}{2(1-\varepsilon)}|\mathring{h}_-|^4 \\ & + \frac{m-2+\varepsilon(m+2)}{2(1-\varepsilon)}|\mathring{h}_1|^2|\mathring{h}_-|^2 - \frac{m}{2(1-\varepsilon)}b|\mathring{A}|^2. \end{aligned}$$

We may assume that $\varepsilon > 0$ is small enough in order to have $2m > \frac{m}{2(1-\varepsilon)}$. Then the above estimate shows that there exists $\rho_1 = \rho_1(m) > 0$ such that

$$Z + 2mb|\mathring{A}|^2 \geq \varepsilon\rho_1|\mathring{A}|^4.$$

On the other hand, using the definition of Z and estimating various terms with Peter-Paul's inequality, we find

$$Z \geq \rho_2|\mathring{A}|^2\,|H|^2 - \rho_3|\mathring{A}|^4,$$

for ρ_2 and ρ_3 depending on m. Combining these two inequalities we obtain for any $0 \leq c \leq 1$

$$Z + 2mb|\mathring{A}|^2 \geq c\left(\rho_2|\mathring{A}|^2\,|H|^2 - \rho_3|\mathring{A}|^4 + 2mb|\mathring{A}|^2\right) + (1-c)\left(\varepsilon\rho_1|\mathring{A}|^4\right).$$

Choosing $\bar{c} = \frac{\varepsilon\rho_1}{\varepsilon\rho_1+\rho_3}$ we have

$$Z + 2mb|\mathring{A}|^2 \geq \bar{c}\left(\rho_2\,|H|^2 + 2mb\right)|\mathring{A}|^2.$$

The assertion follows for ρ small enough.

When $H = 0$ we have $|A|^2 = |\mathring{A}|^2 \le b$ and $W = \beta = b$. Using Theorem 1 in [LL] we find

$$Z \ge -\frac{3}{2}|A|^4 \ge -\frac{3}{2}b|\mathring{A}|^2.$$

Hence we have

$$Z + 2mb|\mathring{A}|^2 \ge \left(2m - \frac{3}{2}\right) b|\mathring{A}|^2 = \left(2m - \frac{3}{2}\right)|\mathring{A}|^2 W \ge \varepsilon\rho|\mathring{A}|^2 W,$$

provided $\rho > 0$ is small enough. □

Next we derive a Poincaré-type inequality on f_σ.

Proposition 5.4. *There exists a constant C_5 depending only on m, k and $\mathcal{M}_0$ such that, for any $p \ge 2$, $0 < \sigma < 1/4$ and $\eta > 0$, we have*

$$\begin{aligned}\varepsilon\rho\int_{\mathcal{M}_t} f_\sigma^p W\,d\mu \le (\eta(p+1)+5)&\int_{\mathcal{M}_t} W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2\,d\mu \\ &+ \frac{p+1}{\eta}\int_{\mathcal{M}_t} f_\sigma^{p-2}|\nabla f_\sigma|^2\,d\mu \\ &+ 4mb\int_{\mathcal{M}_t} f_\sigma^p d\mu + \frac{1}{p}C_5^p.\end{aligned}$$

Proof. Plugging equation (5.4) into (4.3), we find

$$\begin{aligned}\Delta f_\sigma \ge\ & 2W^{\sigma-1}|\nabla\mathring{A}|^2 + 2W^{\sigma-1}\left\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\right\rangle \\ &+ 2W^{\sigma-1}Z - \gamma W^\sigma - \alpha(1-\sigma)\frac{f_\sigma}{W}\Delta|H|^2 \\ &- \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla_i f_\sigma, \nabla_i|H|^2\right\rangle + \alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2}\left|\nabla|H|^2\right|^2.\end{aligned}$$

The terms $2W^{\sigma-1}|\nabla\mathring{A}|^2$ and $\alpha^2\sigma(1-\sigma)\frac{f_\sigma}{W^2}\left|\nabla|H|^2\right|^2$ are positive, so we can omit them. Thanks to Lemma 5.3, we have

$$\begin{aligned}\Delta f_\sigma \ge\ & 2W^{\sigma-1}\left\langle \mathring{h}_{ij}, \nabla_i\nabla_j H\right\rangle \\ &- \alpha(1-\sigma)\frac{f_\sigma}{W}\Delta|H|^2 - \frac{2\alpha(1-\sigma)}{W}\left\langle \nabla f_\sigma, \nabla|H|^2\right\rangle \\ &+ 2\varepsilon\rho W^\sigma|\mathring{A}|^2 - 4mbf_\sigma - \gamma W^\sigma.\end{aligned}$$

We multiply the above inequality by f_σ^{p-1} and integrate on $\mathcal{M}_t$. All terms, except the last two negative ones, can be estimated as in Lemma 2.4

in [H3] and Proposition 5.5 in [Ba]. In this way we obtain, for any $\eta > 0$,

$$\begin{aligned}2\varepsilon\rho\int_{\mathcal{M}_t} f_\sigma^p W d\mu \le (\eta(p+1)+5)\int_{\mathcal{M}_t} W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2 d\mu \\ + \frac{p+1}{\eta}\int_{\mathcal{M}_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu \\ + 4mb\int_{\mathcal{M}_t} f_\sigma^p d\mu + \gamma\int_{\mathcal{M}_t} W^\sigma f_\sigma^{p-1} d\mu.\end{aligned}$$

In order to estimate the last term we use Young's inequality:

$$\gamma W^\sigma f_\sigma^{p-1} \le \gamma W\left(\frac{r^p}{p}W^{(\sigma-1)p} + \frac{p-1}{p}r^{-\frac{p}{p-1}} f_\sigma^p\right), \quad \forall r > 0.$$

Choose r such that $\frac{p-1}{p}\gamma r^{-\frac{p}{p-1}} = \varepsilon\rho$. Observe that r is uniformly bounded from above for large p. Moreover $(\sigma-1)p+1 < 0$ and $W \ge \beta > 0$. Then $W^{(\sigma-1)p+1} \le \beta^{(\sigma-1)p+1}$ and we have

$$\begin{aligned}\frac{1}{p}\gamma r^p\int_{\mathcal{M}_t} W^{(\sigma-1)p+1} d\mu &\le \frac{1}{p}\gamma r^p \beta^{(\sigma-1)p+1}\mathrm{vol}(\mathcal{M}_t) \\ &\le \frac{1}{p}\gamma r^p \beta^{(\sigma-1)p+1}\mathrm{vol}(\mathcal{M}_0) \le \frac{1}{p}C_5^p\end{aligned}$$

for a suitable $C_5 > 0$ depending on $\mathcal{M}_0$. □

We can now bound high L^p-norms of f_σ, provided σ is of order $p^{-\frac{1}{2}}$. This is the step where the hypothesis $T_{max} < \infty$ is used in an essential way.

Proposition 5.5. *If $T_{max} < +\infty$, there is a constant C_6 depending only on $m, k, \mathcal{M}_0, T_{max}$ such that for all*

$$p \ge \frac{16}{C_1\varepsilon} + 2 \qquad \sigma \le \frac{\sqrt{C_1}\rho}{2^7 m\sqrt{p}}\varepsilon^2$$

we have the inequality

$$\left(\int_{\mathcal{M}_t} f_\sigma^p d\mu\right)^{\frac{1}{p}} \le C_6, \qquad \textit{for all } t < T_{max}.$$

Proof. We multiply inequality (4.2) by pf_σ^{p-1}, integrate and obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{M}_t} f_\sigma^p d\mu + p(p-1)\int_{\mathcal{M}_t} f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu \\
&+ 2C_1 p\int_{\mathcal{M}_t}\left|\nabla H\right|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\
\leq{}& 4p\alpha\int_{\mathcal{M}_t}\left|H\right| W^{-1}\left|\nabla H\right|\left|\nabla f_\sigma\right| f_\sigma^{p-1} d\mu \\
&+ 2\sigma p\int_{\mathcal{M}_t}\left|A\right|^2 f_\sigma^p d\mu - 2C_2 p\int_{\mathcal{M}_t} f_\sigma^p d\mu.
\end{aligned}$$

Using that $\alpha\left|H\right| \leq W^{\frac{1}{2}}$ and $f_\sigma \leq W^\sigma$, we have

$$\begin{aligned}
&4p\alpha\int_{\mathcal{M}_t}\left|H\right| W^{-1}\left|\nabla H\right|\left|\nabla f_\sigma\right| f_\sigma^{p-1} d\mu \\
\leq{}& \frac{p(p-1)}{2}\int_{\mathcal{M}_t}\left(\frac{\alpha\left|H\right|}{W} f_\sigma\right) f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 W^{\frac{1}{2}-\sigma} d\mu \\
&+ \frac{8p}{p-1}\int_{\mathcal{M}_t}\frac{\alpha\left|H\right|}{W} W^{\sigma-\frac{1}{2}} f_\sigma^{p-1}\left|\nabla H\right|^2 d\mu \\
\leq{}& \frac{p(p-1)}{2}\int_{\mathcal{M}_t} f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu \\
&+ \frac{8p}{p-1}\int_{\mathcal{M}_t} W^{\sigma-1} f_\sigma^{p-1}\left|\nabla H\right|^2 d\mu.
\end{aligned}$$

With our choice of p, we have $C_1 p \leq 2C_1 p - \frac{8p}{p-1}$. In addition, (5.3) shows that $\left|A\right|^2 \leq 2mW$. Therefore

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{M}_t} f_\sigma^p d\mu + \frac{p(p-1)}{2}\int_{\mathcal{M}_t} f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu \\
&+ C_1 p\int_{\mathcal{M}_t}\left|\nabla H\right|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\
\leq{}& 2\sigma p\int_{\mathcal{M}_t}\left|A\right|^2 f_\sigma^p d\mu - 2C_2 p\int_{\mathcal{M}_t} f_\sigma^p d\mu \\
\leq{}& 4\sigma pm\int_{\mathcal{M}_t} W f_\sigma^p d\mu - 2C_2 p\int_{\mathcal{M}_t} f_\sigma^p d\mu.
\end{aligned}$$

Thanks to Lemma 5.4, we obtain for any $\eta > 0$

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathcal{M}_t} f_\sigma^p d\mu + \frac{p(p-1)}{2}\int_{\mathcal{M}_t} f_\sigma^{p-2}\left|\nabla f_\sigma\right|^2 d\mu \\
&+ C_1 p \int_{\mathcal{M}_t} |\nabla H|^2 W^{\sigma-1} f_\sigma^{p-1} d\mu \\
\leq\ & \frac{4\sigma p m}{\varepsilon\rho}\left[(\eta(p+1)+5)\int_{\mathcal{M}_t} W^{\sigma-1} f_\sigma^{p-1}|\nabla H|^2 d\mu \right.\\
&\qquad \left. + \frac{p+1}{\eta}\int_{\mathcal{M}_t} f_\sigma^{p-2}|\nabla f_\sigma|^2 d\mu + 4mb\int_{\mathcal{M}_t} f_\sigma^p d\mu + \frac{1}{p} C_5^p\right] \\
&- 2C_2 p \int_{\mathcal{M}_t} f_\sigma^p d\mu.
\end{aligned}
$$

Choosing $\eta = \frac{\sqrt{C_1}\varepsilon}{4\sqrt{p}}$ and using our assumptions on m, p and σ, we have

$$
\frac{4\sigma p m}{\varepsilon\rho}(\eta(p+1)+5) \leq C_1 p, \qquad \frac{4\sigma p(p+1)}{\varepsilon\rho\eta} \leq \frac{p(p-1)}{2}.
$$

Then

$$
\frac{d}{dt}\int_{\mathcal{M}_t} f_\sigma^p d\mu \leq \bar{C}_2 \int_{\mathcal{M}_t} f_\sigma^p d\mu + \bar{C}_5,
$$

where

$$
\bar{C}_2 = \frac{32m^2 pb\sigma}{\rho} - 2C_2 p, \qquad \bar{C}_5 = \frac{8\sigma m}{\rho} C_5^p.
$$

Since T_{max} is finite, we obtain the assertion for a constant C_6 independent of p. □

To prove Theorem 5.1, we can now proceed as in [H3] via a Stampacchia iteration procedure to uniformly bound the function f_σ when $T_{max} < \infty$.

Next we establish a gradient estimate for the mean curvature flow. This estimate is required to compare the mean curvature at different points of the submanifold. First we need some technical inequalities. As before, we denote by C_i constants only depending on m, k and the initial data.

Lemma 5.6.

$$
\begin{aligned}
\frac{\partial}{\partial t}|H|^2|\mathring{A}|^2 \leq\ & \Delta(|H|^2|\mathring{A}|^2) - C_3|H|^2|\nabla A|^2 + C_7|\nabla A|^2 \\
& + 2|H|^2|\mathring{A}|^2(3|A|^2 + 4m)
\end{aligned}
$$

for some constant $C_7 > 0$.

Proof. By Lemma 2.1 and 4.2,

$$\frac{\partial}{\partial t}|H|^2|\mathring{A}|^2 \leq \Delta(|H|^2|\mathring{A}|^2) - 2\left\langle \nabla|H|^2, \nabla|\mathring{A}|^2\right\rangle - 2C_3|H|^2|\nabla A|^2 - 2|\mathring{A}|^2|\nabla H|^2 + 2|\mathring{A}|^2|H|^2(3|A|^2+4m).$$

Furthermore we have

$$\begin{aligned}-2\left\langle \nabla|H|^2, \nabla|\mathring{A}|^2\right\rangle &\leq 4|H|\left\langle|\nabla H|, \nabla|\mathring{A}|^2\right\rangle \\ &\leq 8|H||\nabla H||\mathring{A}|^2|\nabla A| \\ &\leq 6|H|\sqrt{m+2}|\nabla A|^2|\mathring{A}|^2.\end{aligned}$$

We can estimate the last term using Theorem 5.1 and Young's inequality, to find that there exists a constant $C_7 > 0$ such that

$$\begin{aligned}6|H|\sqrt{m+2}|\nabla A|^2|\mathring{A}|^2 &\leq 6|H|\sqrt{m+2}|\nabla A|^2\sqrt{C_0}\left(|H|^2+1\right)^{\frac{1-\sigma}{2}} \\ &\leq C_3|H|^2|\nabla A|^2 + C_7|\nabla A|^2.\end{aligned}$$

□

Now we consider the function

$$g = |H|^2|\mathring{A}|^2 + \left(\frac{C_7}{C_3}+1\right)|\mathring{A}|^2. \tag{5.6}$$

Using Lemma 4.2, Lemma 5.6 and $|H|^2 \leq m|A|^2$ we obtain

$$\begin{aligned}\frac{\partial}{\partial t}g &\leq \Delta g - C_3|H|^2|\nabla A|^2 + C_7|\nabla A|^2 + 2|\mathring{A}|^2|H|^2(3|A|^2+4m) \\ &\quad \left(\frac{C_7}{C_3}+1\right)\left(-2C_3|\nabla A|^2 + 4|A|^2|\mathring{A}|^2\right) \\ &\leq \Delta g - C_3|H|^2|\nabla A|^2 - 4C_3|\nabla A|^2 + 2|\mathring{A}|^2|H|^2(3|A|^2+4m) \\ &\quad + 4\left(\frac{C_7}{C_3}+1\right)|\mathring{A}|^2|A|^2 \\ &\leq \Delta g - C_3(|H|^2+1)|\nabla A|^2 + 2|\mathring{A}|^2|A|^2(3m|A|^2+C_8),\end{aligned} \tag{5.7}$$

where $C_8 = 4m^2 + 2\frac{C_7}{C_3} + 2$.

Proposition 5.7. *If $T_{max} < \infty$, for every $\eta > 0$ small enough there exists a constant $C_\eta > 0$ depending only on η such that the inequality*

$$|\nabla H|^2 \leq \eta |H|^4 + C_\eta$$

holds for all times.

Proof. Let $f = |\nabla H|^2 + \frac{1}{C_3}(C_4+1)g - \eta |H|^4$ with $\eta > 0$. By Lemma 4.2, Proposition 4.3 and inequality (5.7) we have

$$\begin{aligned}\frac{\partial}{\partial t} f &\leq \Delta f + C_4(|H|^2+1)\,|\nabla A|^2 - (C_4+1)(|H|^2+1)\,|\nabla A|^2 \\ &\quad + \frac{2}{C_3}(C_4+1)|\mathring{A}|^2\,|A|^2\,(3m\,|A|^2 + C_8) \\ &\quad - \eta\left(\frac{4}{m}\,|H|^6 - 12\,|H|^2\,|\nabla H|^2\right).\end{aligned}$$

We can use Lemma 3.5 to find

$$-\left(|H|^2+1\right)|\nabla A|^2 + 12\eta\,|H|^2\,|\nabla H|^2 \leq \left(-|H|^2 - 1 + \frac{27}{4}(m+2)\eta\right)|\nabla A|^2,$$

and therefore the gradient terms are non-positive for η sufficiently small. The remaining terms are

$$R := \frac{2}{C_3}(C_4+1)|\mathring{A}|^2\,|A|^2\,(3m\,|A|^2 + C_8) - \frac{4\eta}{m}\,|H|^6.$$

Using the pinching condition (5.1) we have

$$R \leq \frac{2}{C_3}(C_4+1)|\mathring{A}|^2\left(a\,|H|^2 + b\right)\left(3ma\,|H|^2 + C_9\right) - \frac{4\eta}{m}\,|H|^6,$$

where $C_9 = 3mb + C_8$. Hence, thanks to Theorem 5.1, we obtain

$$\begin{aligned}R &\leq \frac{2}{C_3}(C_4+1)C_0\left(|H|^2+1\right)^{1-\sigma}\left(a\,|H|^2+b\right)\left(3ma\,|H|^2 + C_9\right) - \frac{4\eta}{m}\,|H|^6 \\ &\leq \frac{2}{C_3}(C_4+1)C_0\left(\mu(1-\sigma)\left(|H|^2+1\right) + \sigma\mu^{\frac{\sigma-1}{\sigma}}\right) \\ &\quad \times \left(a\,|H|^2+b\right)\left(3ma\,|H|^2+C_9\right) - \frac{4\eta}{m}\,|H|^6 \\ &\leq C_{10},\end{aligned}$$

for some constant C_{10} if μ is small enough. Putting these estimates together, we have $\frac{\partial}{\partial t} f \leq \Delta f + C_{10}$. Since $T_{max} < \infty$, we conclude that there exists a constant C_η depending only on η such that $f \leq C_\eta$. Then, from the definition of f, we have

$$|\nabla H|^2 \leq |\nabla H|^2 + \frac{1}{C_3}(C_4+1)g \leq \eta\,|H|^4 + C_\eta.$$

□

As we have mentioned at the beginning of this section, when the codimension is greater than one we cannot repeat the proof of Proposition 5.2. However, using Theorem 5.1 and Proposition 5.7 we can prove that, if time is large enough, the sectional curvature of the evolving submanifold becomes positive.

Proposition 5.8. *There are constants $\mu > 0$ and $\vartheta > 0$ such that, for any time $\vartheta < t < T_{max} < \infty$, the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies*

$$K > \mu W > 0.$$

Proof. From Gauss equation we have that

$$2K_{ij} = 2\bar{K}_{ij} + 2\sum_{\alpha=m+1}^{2n}\left(h_{ii}^{\alpha}h_{jj}^{\alpha} - \left(h_{ij}^{\alpha}\right)^2\right), \tag{5.8}$$

where K_{ij} is the sectional curvature of $\mathcal{M}_t$ of the plane spanned by two orthonormal vectors e_i, e_j, and $\bar{K}_{ij}$ is the sectional curvature of the same plane, but in $\mathbb{CP}^n$. The idea is to use (5.5) restricted to the normal direction parallel to H. To this purpose, we fix an orthonormal basis of type (B1) with the additional requirement that $e_1, \ldots, e_m$ diagonalize h^{m+1}, and let $\lambda_1^{m+1} \leq \cdots \leq \lambda_m^{m+1}$ be the eigenvalues of h^{m+1}. Recalling that $\bar{K} \geq 1$, (5.8) becomes

$$\begin{aligned} 2K_{ij} &\geq 2 + 2\lambda_i^{m+1}\lambda_j^{m+1} + 2\sum_{\alpha=m+2}^{2n}\left(\mathring{h}_{ii}^{\alpha}\mathring{h}_{jj}^{\alpha} - \left(\mathring{h}_{ij}^{\alpha}\right)^2\right) \\ &\geq 2 + \frac{1}{m-1}\,|H|^2 - |h_1|^2 - 2|\mathring{h}_-|^2 \\ &= 2 + \frac{1}{m(m-1)}\,|H|^2 - |\mathring{h}_1|^2 - 2|\mathring{h}_-|^2 \\ &\geq 2 + \frac{1}{m(m-1)}\,|H|^2 - 2|\mathring{A}|^2. \end{aligned} \tag{5.9}$$

By Theorem 5.1 we have

$$2K_{ij} \geq 2 + \frac{1}{m(m-1)} |H|^2 - 2C_0 \left(|H|^2 + 1 \right)^{1-\sigma}. \tag{5.10}$$

Fix some $0 < \mu < \min\{\frac{1}{2\alpha m(m-1)}, \frac{1}{\beta}\}$. Then there exists H^* such that, if $|H| \geq H^*$, then

$$2 + \frac{1}{m(m-1)} |H|^2 - 2C_0 \left(|H|^2 + 1 \right)^{1-\sigma} \geq 2\mu W = 2\mu(\alpha |H|^2 + \beta). \tag{5.11}$$

Let $\bar{H}(t) = \max_{\mathcal{M}_t} |H|$. Since $T_{max} < +\infty$, we know that $\bar{H}(t) \to +\infty$ as $t \to T_{max}$, and so there exists ϑ such that $\bar{H}(t) \geq \bar{H}$ for all $\vartheta \leq t < T_{max}$. Fix some $0 < \eta < \frac{1}{2}$. By Theorem 5.7, there is a constant C_η with $|\nabla H| \leq \frac{1}{2}\eta^2 |H|^2 + C_\eta$. By choosing larger H^* and ϑ if necessary, we can assume that $C_\eta \leq \frac{1}{2}\eta^2 (H^*)^2$ and so $|\nabla H| \leq \eta^2 \bar{H}(t)^2$ on $\mathcal{M}_t$ for $t > \vartheta$. Now fix any $t \in]\vartheta, T_{max}[$ and let x be a point on $\mathcal{M}_t$ where $|H|$ assumes its maximum. Along any geodesic starting from x of length at most $r = [\eta \bar{H}(t)]^{-1}$, we have $|H| \geq (1-\eta)\bar{H}(t) > \frac{1}{2}\bar{H}(t)$. By inequalities (5.10) and (5.11) we find that

$$K > \mu W > \mu\alpha |H|^2 \geq \mu\alpha \frac{\bar{H}(t)^2}{4} > 0$$

holds in all $B_r(x)$, with μ independent on the choice of η. Then in $B_r(x)$ we have $Ric_{ij} \geq (m-1)\frac{\mu\alpha}{4}\bar{H}(t)^2 g_{ij}$. Applying Myers' theorem to geodesics in $B_r(x)$ we have that, if such a geodesic has length at least $2\pi(\bar{H}(t)\sqrt{\mu\alpha})^{-1}$, then it has a conjugate point. So if η is small, precisely such that

$$\frac{2\pi}{\bar{H}(t)\sqrt{\mu\alpha}} < r = \frac{1}{\eta \bar{H}(t)}$$

then $B_r(x)$ covers all $\mathcal{M}_t$. □

To conclude the proof of the convergence of $\mathcal{M}_t$ to a round point we use the main result of [LXZ], which states the following: given any Riemannian manifold with bounded geometry (in particular, the complex projective space), there is a constant $b_0 > 0$ such that if a submanifold of dimension m satisfies

$$|A|^2 < \frac{1}{m-1} |H|^2 - b_0, \tag{5.12}$$

then the mean curvature flow of this submanifold contracts to a round point in finite time. Our pinching condition (1.2) on $\mathcal{M}_0$ is weaker than (5.12), but our analysis implies that (5.12) holds on $\mathcal{M}_t$ for t sufficiently close to T_{max}, as the next result shows.

Proposition 5.9. *For every $b_0 > 0$, there exists a time $0 < \vartheta < T_{max}$ such that inequality (5.12) holds on $\mathcal{M}_t$ for all $\vartheta < t < T_{max}$.*

Proof. By Theorem 5.1 we have

$$\begin{aligned} |A|^2 - \frac{1}{m-1}|H|^2 + b_0 &= |\mathring{A}|^2 - \frac{1}{m(m-1)}|H|^2 + b_0 \\ &\leq C_0\left(|H|^2+1\right)^{1-\sigma} - \frac{1}{m(m-1)}|H|^2 + b_0, \end{aligned}$$

which is negative at the points (x,t) where $|H|^2(x,t)$ is big enough. Using Myers' theorem as in the the proof of Proposition 5.8 we obtain the assertion. □

6. Infinite maximal time

Throughout this section we assume $T_{max} = \infty$. In this case, the argument is simpler than in the case of finite maximal time, because the improvement of pinching can be obtained directly from the maximum principle, as shown in the next result.

Proposition 6.1. *There are positive constants C_0 and δ_0 depending only on the initial manifold $\mathcal{M}_0$ such that*

$$|\mathring{A}|^2 \leq C_0\left(|H|^2+1\right)e^{-\delta_0 t}$$

holds for any time $0 \leq t < T_{max} = \infty$.

Proof. Using Proposition 4.1 with $\sigma = 0$ and the maximum principle, we have that

$$f_0 \leq C_0' e^{-\delta_0 t},$$

for some positive constants C_0' and δ_0 that depend only on the initial data. Recalling that

$$f_0 = \frac{|\mathring{A}|^2}{\alpha|H|^2+\beta},$$

we obtain the assertion for an appropriate constant C_0. □

Note that the above result is trivial for small values of t, while it becomes significant when t is arbitrarily large. As a first consequence of this estimate, we can prove that the intrinsic sectional curvature of the evolving submanifold becomes positive for time large enough, similarly to the case of finite maximal time.

Proposition 6.2. *There are constants $\mu > 0$ and $\vartheta > 0$ such that, for any time $\vartheta < t < T_{max} = \infty$, the intrinsic sectional curvature of $\mathcal{M}_t$ satisfies*

$$K > \mu W > 0.$$

Proof. As in the proof of Proposition 5.8, we have $2K_{ij} \geq 2 + \frac{1}{m(m-1)} |H|^2 - 2|\mathring{A}|^2$. By the exponential decay of $|\mathring{A}|^2$ proved in Proposition 6.1, we have

$$2K_{ij} \geq 2 + \frac{1}{m(m-1)} |H|^2 - 2C_0 \left(|H|^2 + 1\right) e^{-\delta_0 t} \geq 2\mu W > 0,$$

for $\mu > 0$ small enough and t sufficiently big. □

Now we can follow a procedure similar to the previous section.

Lemma 6.3. *There exists $C_7 > 0$ such that*

$$\begin{aligned}\frac{\partial}{\partial t} |H|^2 |\mathring{A}|^2 \leq \Delta(|H|^2 |\mathring{A}|^2) - C_3 |H|^2 |\nabla A|^2 + C_7 |\nabla A|^2 \\ + 2 |H|^2 |\mathring{A}|^2 (3 |A|^2 + 4m).\end{aligned}$$

Proof. We proceed like in the proof of Lemma 5.6, but this time we use Proposition 6.1, to find that

$$\begin{aligned}6 |H| \sqrt{m+2} |\nabla A|^2 |\mathring{A}|^2 &\leq 6 |H| \sqrt{m+2} |\nabla A|^2 \sqrt{C_0(|H|^2+1)} e^{-\delta_0 t/2} \\ &\leq C_3 |H|^2 |\nabla A|^2 + C_7 |\nabla A|^2\end{aligned}$$

if C_7 is chosen large enough. In fact, for t large enough this follows from the exponential decay of $|\mathring{A}|^2$, while if t varies on any compact interval of $[0, \infty[$ it follows from the boundedness of $|H|^2$. □

Now we consider the function g defined in (5.6). Using Lemma 4.2 and Lemma 6.3, we can repeat the computations of the previous sections to conclude that inequality (5.7) holds also in this case. We can now prove a gradient estimate for the curvature.

Theorem 6.4. *For every $\eta > 0$ small enough there exists a constant $C_\eta > 0$ depending only on η such that for all times we have the estimate*

$$|\nabla H|^2 \leq \left(\eta |H|^4 + C_\eta\right) e^{-\delta_0 t/2}.$$

Proof. The proof is similar to Proposition 5.7. Let us define

$$f = e^{\delta_0 t/2}\left(|\nabla H|^2 + \frac{1}{C_3}(C_4 + \delta_0 m)g\right) - \eta |H|^4 .$$

By Proposition 4.3, Lemma 4.2 and inequality (5.7) we have

$$\begin{aligned}\frac{\partial}{\partial t} f \leq \Delta f + &\left[\frac{\delta_0}{2}|\nabla H|^2 + \frac{\delta_0}{2C_3}(C_4+\delta_0 m)\left(|H|^2|\mathring{A}|^2 + 2(C_7+1)|\mathring{A}|^2\right)\right] e^{\delta_0 t/2} \\ + &\Big[-\delta_0 m(|H|^2+1)|\nabla A|^2 \\ &\quad + \frac{2}{C_3}(C_4+\delta_0 m)|\mathring{A}|^2|A|^2\left(3m|A|^2 + C_8\right)\Big] e^{\delta_0 t/2} \\ - &\eta\left(\frac{2}{m}|H|^6 - 12|H|^2|\nabla H|^2\right).\end{aligned}$$

By Lemma 3.5, the gradient terms satisfy

$$\begin{aligned}&\left[\frac{\delta_0}{2}|\nabla H|^2 - \delta_0 m\left(|H|^2+1\right)|\nabla A|^2\right] e^{\delta_0 t/2} + 12\eta|H|^2|\nabla H|^2 \\ \leq &\left[\frac{\delta_0}{2} - \frac{16\delta_0 m}{9(m+2)}(|H|^2+1) + 12\eta|H|^2\right]|\nabla H|^2 e^{\delta_0 t/2}\end{aligned}$$

and therefore they are non-positive for η sufficiently small. We call R the remaining terms. Using condition (1.2) and Theorem 6.1, we can find a constant Λ such that

$$\begin{aligned}R &\leq C_0\Lambda\left(|H|^2+1\right)\left(|H|^4+1\right)e^{-\delta_0 t/2} - \frac{2\eta}{m}|H|^6 \\ &\leq \left[C_0\Lambda\left(|H|^2+1\right)\left(|H|^4+1\right)e^{-\delta_0 t/4} - \frac{2\eta}{m}|H|^6\right] e^{-\delta_0 t/4} \\ &\leq C_{10}e^{-\delta_0 t/4},\end{aligned}$$

for a suitably large constant C_{10}. Note that this is true, because $e^{-\delta_0 t/4}$ is small, for t big enough, and because $|H|^2$ is bounded, for t small. Then there exists a constant C_η such that $f \leq C_\eta$. Recalling the definition of f we conclude the proof. □

We now show that, if $T_{max} = \infty$, the curvature is uniformly bounded.

Lemma 6.5. *If $T_{max} = \infty$, then $|H|^2$ is bounded uniformly for all t.*

Proof. Let b_0 the constant which appears in the main theorem in [LXZ]. From Proposition 6.1 we have

$$\begin{aligned} |A|^2 - \frac{1}{m-1}|H|^2 + b_0 &= |\mathring{A}|^2 - \frac{1}{m(m-1)}|H|^2 + b_0 \\ &\leq C_0\left(|H|^2+1\right)e^{-\delta_0 t} - \frac{1}{m(m-1)}|H|^2 + b_0. \end{aligned}$$

Observe that the right-hand side is negative if t and $|H|^2$ are big enough. Using Proposition 6.2 and Theorem 6.4, we can apply Myers' theorem like in the proof of Proposition 5.9 to show that, if the curvature is sufficiently large at some point, then it is large everywhere. Therefore, if $|H|^2$ becomes arbitrarily large as $t \to \infty$ we obtain that, for t big enough,

$$|A|^2 - \frac{1}{m-1}|H|^2 + b_0 < 0$$

everywhere on $\mathcal{M}_t$. Then the main theorem in [LXZ] implies that the mean curvature flow with initial value $\mathcal{M}_t$ shrinks to a point in finite time, giving a contradiction. □

Now we have all the ingredients to prove the convergence in the case $T_{max} = \infty$. Since $|H|^2$ stays bounded, Proposition 6.1 and Theorem 6.4 give that there is a constant C such that

$$|\mathring{A}|^2 \leq Ce^{-\delta_0 t}, \qquad |\nabla H|^2 \leq Ce^{-\delta_0 t/2}.$$

Applying once again Myers' theorem, the diameter of $\mathcal{M}_t$ is uniformly bounded and so $|H|^2_{max} - |H|^2_{min} \leq Ce^{-\delta_0 t/2}$. Moreover $|H|^2_{min} = 0$ otherwise the evolution equation for $|H|^2$ from Lemma 2.1, together with (3.7), would imply by a standard comparison argument the finite time blow up of $|H|^2$, in contradiction with the assumption $T_{max} = +\infty$. Then $|H|^2$ decays exponentially fast and

$$|A|^2 = |\mathring{A}|^2 + \frac{1}{m}|H|^2 \leq Ce^{-\delta_0 t/2},$$

for some $C > 0$. We deduce

$$\int_0^\infty \left|\frac{\partial}{\partial t}g_{ij}\right| dt = \int_0^\infty |H|\,|A|\,dt \leq \sqrt{m}\int_0^\infty |A|^2\,dt \leq \sqrt{m}C\int_0^\infty e^{-\delta_0 t/2} \leq \bar{C},$$

for some $\bar{C} > 0$. So we can apply a result by Hamilton [Ha, Lemma 14.2] to obtain that there is a continuous limit metric $g_{ij}(\infty)$. By the same method used in [H1, §10], we can show that the exponential decay for $|A|^2$ gives the exponential decay for all derivatives $\nabla^k A$ by means of interpolation inequalities. This finally gives C^∞-convergence to a smooth totally geodesic submanifold $\mathcal{M}_\infty$. By our smallness assumption on the codimension k, the only possibility is that $\mathcal{M}_\infty = \mathbb{CP}^{n'}$ for some $n' < n$ as implied by Theorem 3.25 of [Be1]. Therefore, if k is odd this possibility cannot happen and we can only have a singularity in finite time. This concludes the proof of Theorem 1.1.

7. Extensions to quaternionic projective spaces

In this last section we show that in the case of hypersurfaces our main result, Theorem 1.1, can be easily extended to the flow in a quaternionic projective space. Let $\mathbb{K}$ be either the field $\mathbb{C}$ of complex numbers or the associative algebra $\mathbb{H}$ of quaternions, and let c be a positive constant. We denote by $\mathbb{KP}^n(4c)$ the projective space over $\mathbb{K}$ with sectional curvature $c \le \bar{K} \le 4c$, and we consider the mean curvature flow of a real hypersurface of $\mathbb{KP}^n(4c)$.

Theorem 7.1. *Let $n \ge 3$, $c > 0$, and let $\mathcal{M}_0$ be a closed real hypersurface of $\mathbb{KP}^n(4c)$. Let m be the real dimension of $\mathcal{M}_0$ and suppose that $\mathcal{M}_0$ satisfies*

$$|A|^2 < \frac{1}{m-1}|H|^2 + 2c. \tag{7.1}$$

Then the mean curvature flow with initial condition $\mathcal{M}_0$ has a smooth solution $\mathcal{M}_t$ on a finite time interval $0 \le t < T_{max} < \infty$ and the flow converges to a round point as t goes to T_{max}.

The proof is the same exposed in the previous sections for the case of hypersurfaces of $\mathbb{CP}^n = \mathbb{CP}^n(4)$. The constants used are

$$m = \begin{cases} 2n-1 & \text{if } \mathbb{K} = \mathbb{C}, \\ 4n-1 & \text{if } \mathbb{K} = \mathbb{H}, \end{cases} \qquad \text{and} \qquad \bar{r} = \begin{cases} 2(n+1)c & \text{if } \mathbb{K} = \mathbb{C}, \\ 4(n+2)c & \text{if } \mathbb{K} = \mathbb{H}. \end{cases}$$

As we have observed in the complex case, the proof that the flow develops a singularity in finite time is in some sense indirect and is related to the global structure of the projective spaces we are considering. Namely, we show that a solution defined for all times would converge to a totally geodesic hypersurface, but this is excluded because in $\mathbb{KP}^n(4c)$ there are no such hypersurfaces.

Theorem 7.1 implies the following classification result.

Corollary 7.2. *Let $n \geq 3$ and $c > 0$.*

1) *If $\mathcal{M}_0$ is a closed real hypersurface of $\mathbb{KP}^n(4c)$ satisfying the pinching condition* (7.1)*, then $\mathcal{M}_0$ is diffeomorphic to a sphere.*
2) *For any minimal closed real hypersurface of $\mathbb{KP}^n(4c)$, $|A|^2 \geq 2c$ holds.*

Theorem 7.1 is the generalization of the main theorem of [H3] about pinched hypersurfaces of the sphere to all CROSSes (compact rank-one symmetric spaces) with sufficiently large dimension. Unfortunately, these techniques do not allow to find an analogous result for the Cayley plane $\mathbb{C}a\mathbb{P}^2$.

The next example shows that Theorem 7.1 is not a trivial consequence of the general result in [H2], because there are non-convex hypersurfaces in the class considered.

Example 7.3. Consider for simplicity $c = 1$ and let $\mathcal{M}_0$ be a geodesic sphere in $\mathbb{CP}^n$. In [NR] it is proved that $\mathcal{M}_0$ has two distinct principal curvatures: $\lambda_1 = 2\cot(2u)$ with multiplicity 1 and $\lambda_2 = \cot(u)$ with multiplicity $2(n-1)$, for some $0 < u < \frac{\pi}{2}$. For any $u > \frac{\pi}{4}$, we have $\lambda_1 < 0$ and $\lambda_2 > 0$, so $\mathcal{M}_0$ is not convex. Moreover, it is easy to compute that in this case condition (7.1) is equivalent to

$$2(2n-3)\cot^2(2u) - 2(n-1)\cot^2(u) < 0.$$

Hence, there are non-convex examples in our class for every n. In the same way, a geodesic sphere in $\mathbb{HP}^n$ has principal curvatures $\lambda_1 = 2\cot(2u)$ with multiplicity 3 and $\lambda_2 = \cot(u)$ with multiplicity $4(n-1)$, for some $0 < u < \frac{\pi}{2}$ (see for example [MP]). Condition (7.1) in this case becomes

$$3(4n-5)\cot^2(2u) - 4(n-1)\cot^2(u) + 4n - 5 < 0,$$

so we have non-convex examples in our class for $\mathbb{K} = \mathbb{H}$ too. We remark that, even if the initial hypersurface is not convex, it becomes convex approaching the maximal time, as a consequence of the convergence to a round point.

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