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ABP estimate and geometric inequalities

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In this paper, we present the proofs for several geometric inequalities using the Alexandrov-Bakelman-Pucci (ABP) estimate on Riemannian manifolds. First, we give new proofs of the Heintze-Karcher inequality for mean convex domain on manifolds with non-negative Ricci curvature and the classical Minkowski inequality for mixed volumes. Then, we prove the anisotropic Heintze-Karcher inequality. Along the new approach, we also establish an anisotropic version of ABP estimate which may be of independent interest.

1. Introduction

The classical Alexandrov-Bakelman-Pucci (ABP) (see [6]) estimate gives a point-wise bound for a function u by the L^n-norm of the Pucci extremal operator acting on u, the integration being over the contact set of u with its convex envelop. This estimate is essential in the regularity theory for second order elliptic equations [6]: the Krylov-Safonov Harnack inequality and the landmark works on fully non-linear equations by Caffarelli [7, 8]. On the other hand, it is remarkable that this PDE technique was also used, by Cabré [5], to give a neat proof for the classical isoperimetric inequality for smooth domains in Euclidean space (see also the recent survey paper on this topic [4]).

Given those important applications in both elliptic equations and geometry, it is interesting to extend the classical ABP estimate to Riemannian manifolds. However, ABP techniques are not directly applicable due to the fact that there is no corresponding notion of affine functions on general manifolds. To overcome this difficulty, Cabré [3] suggested to consider the square of distance functions instead of affine functions as the *touching functions* and obtained the Harnack inequalities for non-divergent elliptic equations on manifolds with non-negative sectional curvature. Based on Cabré's

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idea and a work of Savin [20], the second author and Wang [24] introduced a notion of *contact set* and established an explicit ABP type estimate on Riemannian manifolds with Ricci curvature bounded from below (see Section 2.1 for details). Similar to the classical ABP estimate in the Euclidean space, the upshot of the new estimate is that the integration is calculated only on the *contact set*. This allows one to establish the Krylov-Safanov Harnack inequalities on general manifolds. Moreover, it was also used to prove the Minkowski inequality on manifolds with Ricci curvatures bounded from below in [24].

In this paper, we continue to investigate the applications of ABP estimate established in [24] to geometric inequalities. In comparison with the Euclidean case, this ABP estimate allows one to do small variations of the boundary of the original domain, which brings the mean curvature of the boundary into play. Therefore, we are able to establish some inequalities relating volume of the domain, area of the boundary and the integral of the mean curvature.

In Section 2, we will first recall the definition of *contact set* defined in [24] and state the ABP estimate on general manifolds. Then, we present a proof of the classical Heintze-Karcher inequality by the ABP method.

Theorem 1.1. *Assume that (M^n, g) is a Riemannian manifold with $Ric \geq 0$. Let $\Omega \subset M$ be a connected subdomain with C^2 boundary $\partial\Omega$ with mean curvature $H > 0$. Then,*

$$|\Omega| \leq \frac{n-1}{n} \int_{\partial\Omega} \frac{1}{H} dA, \tag{1.1}$$

and equality holds if and only if Ω is isometric to an Euclidean ball.

In the second part of the paper, we will further investigate applications of the ABP method to geometric inequalities involving anisotropic curvature integrals. The anisotropy is an alternative way of talking about the relative geometry or the Minkowski geometry, which was initiated by Minkowski, Fenchel, etc., see e.g. [2, 14]. In particular, the mixed volumes about two convex bodies, when smooth and strictly convex, can be represented by the anisotropic curvature integrals.

In Section 3, we will briefly review the *anisotropic curvatures* and then use the ABP estimate in Section 2 to prove a Minkowski type inequality.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with convex and C^2 smooth boundary $\partial\Omega$. Then*

$$|\Omega| \int_{\partial\Omega} H_F dA \leq \frac{n-1}{n} |\partial\Omega|_F^2, \tag{1.2}$$

where H_F is the anisotropic mean curvature (see Section 3.1 for the definition) and $|\partial\Omega|_F = \int_{\partial\Omega} F(\nu)dA$ is the anisotropic area functional. Moreover, equality holds if and only if $\partial\Omega$ is a translation or rescaling of the Wulff shape $\mathcal{W}$ (uniquely determined by F).

We remark that inequality (1.2) is also called the *Minkowski inequality for mixed volumes*. Indeed, let K, L be two convex bodies in $\mathbb{R}^n$ and $V(K[i], L[n-i])$ be the mixed volumes for $i = 0, 1, \ldots, n$. Then, by approximations, (1.2) yields Minkowski's second inequality (see e.g. [21] Theorem 7.2.1):

$$V(K[n-1], L[1])^2 \geq V(K[n-2], L[2])V(K[n], L[0]).$$

In Section 4, we will establish an anisotropic analogue of the Heintze-Karcher inequality. However, the ABP estimate (Theorem 2.1) is not directly applicable and we will introduce a notion of *contact set* in the anisotropic setting and prove an anisotropic ABP estimate, Theorem 4.1. With this new version, we are able to prove the following inequality.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary $\partial\Omega$ satisfying $H_F > 0$. Then,*

$$|\Omega| \leq \frac{n-1}{n} \int_{\partial\Omega} \frac{F(\nu)}{H_F} dA, \tag{1.3}$$

Moreover, equality holds if and only if $\partial\Omega$ is a rescaling or translation of the Wulff shape $\mathcal{W}$ (uniquely determined by F).

This anisotropic Heintze-Karcher inequality was first proved in [11] using Heintze-Karcher's idea. More recently, Ma-Xiong [13] gave a new proof which follows Brendle's flow method [1].

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2. Heintze-Karcher inequality

In this section, we present a new proof for the classical Heintze-Karcher inequality by the ABP estimate obtained in [24].

2.1. ABP estimates on Riemannian manifolds

Let (M, g) be an n-dimensional Riemannian manifold with metric g and Ric be the Ricci curvature. Let $d_y(x)$ be the Riemannian distance between the points x, y. Given a connected domain $\Omega \subset M$ with smooth boundary $\partial\Omega$, we also denote dA and dA as the canonical measure of Ω and $\partial\Omega$, respectively. To simplify the notation, the volume and boundary area are denoted by $|\Omega| := \mu(\Omega)$ and $|\partial\Omega| := A(\partial\Omega)$, respectively. In [24], the following definition of *contact set* on (M, g) was introduced.

Definition 2.1. Let Ω be an open bounded subdomain of (M, g) and $u \in C(\Omega)$. For a given $a > 0$ and a compact set $E \subset M$, the contact set associated to u of opening a with vertex set E is defined by

$$A_a(E, \Omega, u) := \left\{x \in \overline{\Omega} : \exists\, y \in E \ \text{ s.t. } \inf_{z\in\overline{\Omega}} \left(u(z)+\frac{a}{2}d_y^2(z)\right) = u(x)+\frac{a}{2}d_y^2(x)\right\}.$$

Geometrically, $x \in A_a(E, \Omega, u)$ if and only if there exists a concave paraboloid of opening a and with vertex $y \in E$ that touches u in $\overline{\Omega}$ from below. Here, by a concave paraboloid, we mean a function of the form $P_{a,y}(\cdot) := -\frac{a}{2}d_y^2(\cdot) + c_y$ with $c_y \in \mathbb{R}, a > 0$.

Using this *contact set*, the following ABP estimate was established in [24].

Theorem 2.1 ([24]). *Let Ω be an open bounded subdomain in M with* $\mathrm{Ric} \geq 0$ *and $u \in C^2(\Omega)$.*

Given any compact set $E \subset M$ and a real number $a > 0$, if $A_a(E, \Omega, u) \subset \Omega$, then

$$|E| \leq \int_{A_a(E,\Omega,u)} \left(1 + \frac{\Delta u}{na}\right)^n dA. \tag{2.1}$$

The ABP estimate in [24] was proved for general Riemannian manifolds with Ricci curvature bounded from below. Here we only state the case that Ricci curvature is non-negative which is enough for our discussion in this paper.

2.2. The classical Heintze-Karcher inequality

Let (M, g) be a Riemannian manifold and $\Omega \subset M$ is a connected subdomain. Denote ν to be the unit outward normal of $\partial\Omega$ and H as the mean curvature with respect to ν. The classical Heintze-Karcher inequality is as follows:

Theorem 2.2. *Assume that (M^n, g) is a Riemannian manifold with $Ric \geq 0$. Let $\Omega \subset M$ be a connected subdomain with C^2 boundary $\partial\Omega$ satisfying $H > 0$. Then,*

$$|\Omega| \leq \frac{n-1}{n} \int_{\partial\Omega} \frac{1}{H} dA, \tag{2.2}$$

and equality holds if and only if Ω is isometric to an Euclidean ball.

The above Heintze-Karcher inequality is a sharp inequality for hypersurfaces of positive mean curvature inspired by a classical inequality due to Heintze and Karcher [10].

In 1987, Ros [19] provided a proof of the above theorem using the remarkable Reilly formula (see [17]), and applied it to show Alexandrov's rigidity theorem for high order mean curvatures. More recently, aiming to extend Alexandrov's rigidity theorem to more general geometric setting, Brendle [1] established inequality (1.1) in a large class of warped product spaces, including $\mathbb{S}^n, \mathbb{H}^n$ and the Schwarzschild manifolds. A geometric flow method, which is quite different from Ros' proof, was used in [1] and it allows one to weaken the assumption on the non-negativity of Ricci curvature. However, it requires the warped product structure of the manifolds. Motivated by Brendle's work, a new kind of Heintze-Karcher's inequality was established by the first author and Qiu [16] for compact manifolds with boundary and sectional curvature bounded below by -1.

In the following, we present a new proof for Theorem 2.2 via the ABP method. Heuristically, the key technique used to show Reilly's formula is integration by parts, which exploits the divergence structure of the elliptic operator. Meanwhile, ABP estimate is the crucial technique to study the regularity theory of non-divergent PDE. From this viewpoint, proof of

Theorem 2.2 given by Ros [19] is making use of the divergence structure of laplacian equation and our new proof plays with the non-divergence part.

Proof. In order to illustrate the idea more transparently, we first consider the case $\Omega \subset \mathbb{R}^n$. Later, we will point out how to modify the argument to general manifolds. Consider the Dirichlet problem

$$\begin{cases} \Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

It is clear that the above problem has a solution $u \in C^2(\bar{\Omega})$. Moreover,

$$f(x) := u_\nu(x) := \frac{\partial u}{\partial \nu}(x) > 0, \quad \text{for any } x \in \partial\Omega$$

by Hopf's boundary point lemma.

Next, for any fixed small $\varepsilon, \delta > 0$, we apply the ABP estimate (2.1) with $a = (\varepsilon + \delta)^{-1}$ and $E = \overline{\Omega}_\varepsilon$ which is enclosed by the following hypersurface

$$\partial\Omega_\varepsilon = \{y \in \mathbb{R}^n \mid y = x + \varepsilon f(x)\nu(x) \ \text{ for some } x \in \partial\Omega\}.$$

To apply the ABP estimate, we need to check that the contact set $A_a(E, \Omega, u) \subset \Omega$. Indeed, we can argue by contradiction and suppose there is $x_0 \in A_a(E, \Omega, u)$ and $x_0 \in \partial\Omega$. By the definition of contact set $A_a(E, \Omega, u)$, x_0 is a minimum point of the function

$$u(x) + \frac{a}{2}|y - x|^2 + c_y$$

for some $y \in \Omega_\varepsilon$. We first observe that $u(x) \equiv 0$ on $\partial\Omega$ implies $y - x_0$ is parallel to $\nu(x_0)$ since

$$a\langle y - x_0, \tau(x_0)\rangle = \langle \nabla u(x_0), \tau(x_0)\rangle = 0,$$

for any tangential vector $\tau(x_0)$ of $\partial\Omega$ at x_0. On the other hand, by the critical point condition of $x_0 \in \partial\Omega$, we know that

$$\langle y - x_0, \nu(x_0)\rangle \geq a^{-1}\langle \nabla u(x_0), \nu(x_0)\rangle > \varepsilon f(x_0).$$

But, this is impossible because $y \in \overline{\Omega}_\varepsilon$.

Now, we apply the ABP estimate and let $\delta \to 0$ to obtain

$$
\begin{aligned}
|\Omega_\varepsilon| &\le \int_{A_{\frac{1}{\varepsilon}}(E,\Omega,u)} \left(1+\frac{\varepsilon\Delta u}{n}\right)^n dA \le |\Omega|\left(1+\frac{\varepsilon}{n}\right)^n \\
&= |\Omega|\left(1+\varepsilon+\frac{\varepsilon^2}{2}\frac{n-1}{n}\right)+o(\varepsilon^2).
\end{aligned} \tag{2.4}
$$

To get the desired inequality, we need to approximate the volume $|\Omega_\varepsilon|$. Indeed, $\partial\Omega_\varepsilon$ can be viewed as the following evolving surface when $t=\varepsilon$:

$$
\partial_t X(x,t) = f(x)\nu(x), \quad x \in \partial\Omega. \tag{2.5}
$$

Note that the velocity of this flow is independent of t. To make use of the variational formulae for area and volume, we re-write the flow (2.5) into an equivalent form.

$$
\begin{aligned}
\partial_t X(x,t) &= f(x)\langle\nu(x),\nu(x,t)\rangle\nu(x,t) + f(x)\langle\nu(x),\tau(x,t)\rangle\tau(x,t) \\
&:= f^\perp(x,t)\nu(x,t) + f^\top(x,t)\tau(x,t).
\end{aligned} \tag{2.6}
$$

Here $\nu(x,t)$ and $\tau(x,t)$ denote the unit normal and tangential vectors of hypersurface $\partial\Omega_t$. The variational formula gives

$$
\begin{aligned}
\frac{d}{dt}|\Omega_t| &= \int_{\partial\Omega_t} f^\perp(x,t)dA_t; \\
\frac{d^2}{dt^2}\bigg|_{t=0}|\Omega_t| &= \int_{\partial\Omega}\frac{\partial}{\partial t}\bigg|_{t=0} f^\perp(x,t)dA + \int_{\partial\Omega}\left(f^\perp(x,t)\right)^2 H\bigg|_{t=0} dA.
\end{aligned}
$$

By standard computation, we have

$$
\begin{aligned}
f^\perp(x,t)\Big|_{t=0} &= \left(f(x)\langle\nu(x),\nu(x,t)\rangle\right)\Big|_{t=0} = f(x); \\
\frac{\partial}{\partial t}\bigg|_{t=0} f^\perp(x,t) &= \frac{\partial}{\partial t}\bigg|_{t=0}\left(f(x)\langle\nu(x),\nu(x,t)\rangle\right) \\
&= f(x)\langle\nu(x), \frac{\partial}{\partial t}\bigg|_{t=0}\nu(x,t)\rangle = 0.
\end{aligned}
$$

We used the fact that $\frac{\partial}{\partial t}\Big|_{t=0}\nu(x,t)$ is a tangential vector of $\partial\Omega$ at x to get the last equality. Therefore,

$$\begin{aligned}|\Omega_\varepsilon| &= |\Omega| + \varepsilon\frac{d}{dt}\Big|_{t=0}|\Omega_t| + \frac{\varepsilon^2}{2}\frac{d^2}{dt^2}\Big|_{t=0}|\Omega_t| + o\left(\varepsilon^2\right)\\ &= |\Omega| + \varepsilon\int_{\partial\Omega} f dA + \frac{\varepsilon^2}{2}\int_{\partial\Omega} f^2 H dA + o\left(\varepsilon^2\right).\end{aligned}$$

Compare this approximation with the right hand side of (2.4) and using the fact that $\int_{\partial\Omega} f = \int_{\partial\Omega} u_\nu = \int_\Omega \Delta u = |\Omega|$, we have

$$\int_{\partial\Omega} f^2 H dA = \int_{\partial\Omega} H u_\nu^2 dA \le \frac{n-1}{n}|\Omega|.$$

Finally, we have

$$|\Omega|^2 = \left(\int_\Omega \Delta u\right)^2 = \left(\int_{\partial\Omega} u_\nu\right)^2 \le \int_{\partial\Omega} H u_\nu^2 \int_{\partial\Omega}\frac{1}{H} \le \frac{n-1}{n}|\Omega|\int_{\partial\Omega}\frac{1}{H},$$

which gives us the desired inequality

$$|\Omega| \le \frac{n-1}{n}\int_{\partial\Omega}\frac{1}{H}dA.$$

To adapt this argument on Riemannian manifolds, we need to modify the definition of $\partial\Omega_\varepsilon$ as

$$\partial\Omega_\varepsilon = \{\, y\in M \mid y = \exp_x\left(\varepsilon f(x)\nu(x)\right) \quad \text{for some } x\in\partial\Omega\,\}.$$

And it satisfies the flow equation

$$\begin{aligned}\partial_t X(x,t) &= f(x) d\exp_x|_{t\nu(x)}\left(\nu(x)\right) := f(x) P_t\left(\nu(x)\right)\\ &= f(x)\langle P_t\left(\nu(x)\right), \nu(x,t)\rangle \nu(x,t)\\ &\quad + f(x)\langle P_t\left(\nu(x)\right), \tau(x,t)\rangle\tau(x,t)\\ &:= f^\perp(x,t)\nu(x,t) + f^\top(x,t)\tau(x,t),\ x\in\partial\Omega.\end{aligned} \tag{2.7}$$

Then, we can follow the same line as the Euclidean case to compute the variational formula along this flow by changing $\nu(x)$ to $P_t\left(\nu(x)\right)$. The only thing we need to verify is

$$f^{\perp}(x,t)\Big|_{t=0} = f(x), \quad \frac{\partial}{\partial t}\Big|_{t=0} f^{\perp}(x,t) = 0. \tag{2.8}$$

The first equality in (2.8) holds since $P_0\left(\nu(x)\right) = \nu(x,0) = \nu(x)$. For the second equality in (2.8),

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0} f^{\perp}(x,t) &= f(x)\frac{\partial}{\partial t}\Big|_{t=0} \langle P_t\left(\nu(x)\right), \nu(x,t)\rangle \\ &= f(x)\langle \partial_t\left(P_t\left(\nu(x)\right)\right), \nu(x,t)\rangle\Big|_{t=0} \\ &\quad + f(x)\langle P_t\left(\nu(x)\right), \partial_t \nu(x,t)\rangle\Big|_{t=0}. \end{aligned} \tag{2.9}$$

We know that, for small $t \geq 0$,

$$\langle P_t\left(\nu(x)\right), P_t\left(\nu(x)\right)\rangle = \langle \nu(x), \nu(x)\rangle = 1.$$

It follows that

$$\langle \partial_t\left(P_t\left(\nu(x)\right)\right), P_t\left(\nu(x)\right)\rangle = 0.$$

By evaluating at $t = 0$, and noting that $P_0\left(\nu(x)\right) = \nu(x)$, we find $\partial_t\left(P_t\left(\nu(x)\right)\right)|_{t=0}$ is a tangential vector of $\partial\Omega$ at x. Thus

$$\langle \partial_t\left(P_t\left(\nu(x)\right)\right), \nu(x,t)\rangle\Big|_{t=0} = 0.$$

It is clear that $\partial_t \nu(x,t)|_{t=0}$ is also a tangential vector of $\partial\Omega$ at x. Hence $\langle P_t\left(\nu(x)\right), \partial_t \nu(x,t)\rangle\Big|_{t=0} = 0$. Therefore we can verify the second equality in (2.8) from (2.9).

Let us now check the equality case in (2.2). If we achieve equality in (2.2), then each step in our argument must be equality. In particular, the ABP estimate (2.1) is also forced to be equality. By checking the proof for (2.1) in [24], p. 505, we note that the equality holds in

$$\ddot{\mathcal{J}}(t,x) \leq -\frac{1}{n} Ric(a^{-1}\nabla u, a^{-1}\nabla u)\mathcal{J}(t,x) \leq 0, \tag{2.10}$$

where $\mathcal{J}(t,x) := (\det J(t,x))^{\frac{1}{n}}$ and $J(t,x) = D\exp_x(a^{-1}t\nabla u)$ is the matrix representing the Jacobi fileds. Inequality (2.10) was derived by using the standard theory of Jacobi fields. In the proof of (2.10) (see e.g., Villani's book [23], page 367-370), the Cauchy-Schwarz inequality in the form

$$\mathrm{tr}(U^2) \geq \frac{1}{n}(\mathrm{tr}U)^2$$

is used, where $U = \dot{J}J^{-1}$, see [23], page 369. By the equality case for the Cauchy-Schwarz inequality, we know that $\dot{J}(0)$ is proportional to $J(0)$. Note that $\dot{J}(0) = a^{-1}\nabla^2 u$ and $J(0) = I_n$ (see [23], (14.8), note that in our situation $\xi = \nabla u$), we see $\nabla^2 u$ is proportional to g. Combining with $\Delta u = 1$, we deduce $\nabla^2 u = \frac{1}{n} g$. Thus, using Obata's theorem (see [18], Lemma 3), we see Ω is isometric to the Euclidean ball. $\square$

3. Minkowski inequality for mixed volume

In the next two sections, we move to study some geometric inequalities involving the so-called *anisotropic curvatures*.

The concept of anisotropy can date back to the time of Minkowski [14] and Wulff [26]. In 1901, Wulff initiated the study of the interface energy functional or the anisotropic area functional $\int_M F(\nu)dA$ in the theory of the physical models of crystal growth. The minimizer of the interface energy functional is the so-called *Wulff shape*. Such crystalline variational problems and models of crystal growth have been studied extensively by many mathematicians and theoretical physicists. On the other hand, the anisotropy also has its root in the relative or the Minkowski differential geometry, where the Wulff shape is named instead as "Eichkřoper", a fixed convex body, by Minkowski [14]. Especially, in the Euclidean geometry, the unit round sphere plays the role of the "Eichkřoper". The anisotropic curvatures we consider here have a great correspondence with the concept of mixed volume in the theory of convex bodies, see [2, 21].

3.1. Preliminaries on anisotropic curvature

Given a smooth closed strictly convex hypersurface $\mathcal{W} \subset \mathbb{R}^n$ containing the origin, the support function of $\mathcal{W}$, which is defined by

$$F(x) = \sup_{X \in \mathcal{W}} \langle x, X \rangle, \quad x \in \mathbb{S}^{n-1},$$

is a smooth positive function on $\mathbb{S}^{n-1}$. $\mathcal{W}$ can be represented by F as

$$\mathcal{W} = \{\psi(x) \in \mathbb{R}^n \mid \psi(x) = F(x)x + \nabla^{\mathbb{S}} F(x), x \in \mathbb{S}^{n-1}\}, \tag{3.1}$$

where $\nabla^{\mathbb{S}}$ denotes the covariant derivative on $\mathbb{S}^{n-1}$. Let $A_F : \mathbb{S}^{n-1} \to \Lambda^2 T^* \mathbb{S}^{n-1}$ be a 2-tensor defined by

$$A_F(x) = \nabla^{\mathbb{S}} \nabla^{\mathbb{S}} F(x) + F(x)\sigma \text{ for } x \in \mathbb{S}^{n-1},$$

and σ is the round metric on $\mathbb{S}^{n-1}$. The strictly convexity of $\mathcal{W}$ implies that A_F is positive definite. It is well-known that the eigenvalues of A_F are the principal radii of $\mathcal{W}$, i.e., the inverse of principal curvatures, see for example [21].

Let $X : M \to \mathbb{R}^n$ be a smooth embedding in $\mathbb{R}^n$ with induced metric g, and $\nu : M \to \mathbb{S}^{n-1}$ be its Gauss map. The *anisotropic Gauss map* of $X(M)$ with respect to the *Wulff shape* $\mathcal{W}$ is defined by

$$\begin{aligned} \nu_F : M &\longrightarrow \mathcal{W} \\ x &\to DF(\nu(x)) = F(\nu(x))\nu(x) + \nabla^{\mathbb{S}} F(\nu(x)). \end{aligned}$$

The *anisotropic principal curvature* κ_F of $X(M)$ is defined as the eigenvalues of

$$d\nu_F : T_x M \to T_{\nu_F(x)} \mathcal{W}.$$

In particular, the *anisotropic mean curvature* of $X(M)$ with respect to $\mathcal{W}$ is given by

$$H_F(x) := \mathrm{tr}_g(d\nu_F|_x) = \mathrm{tr}_g\left(A_F(\nu(x)) \circ d\nu|_x\right).$$

On the other hand, a variational characterization for the *anisotropic mean curvature* H_F also arises from the first variation of the parametric area functional

$$|X(M)|_F := \int_M F(\nu) dA.$$

More precisely, let $X(\cdot, t), t \in [0, \varepsilon)$ be a varation of X with variational vector field

$$\frac{\partial}{\partial t} X(x,t) = \psi(x,t)\nu_F(x,t) + \tau(x,t) = \psi(x,t)F(\nu(x,t))\nu(x,t) + \tilde{\tau}(x,t),$$

where $\psi \in C^\infty(M \times [0, \varepsilon))$ and $\tau, \tilde{\tau} \in T_x M$. Then, it follows that (see [27])

$$\frac{d}{dt} |X(M,t)|_F = \int_M \psi(x,t) H_F(x,t) F(\nu(x,t)) dA_t(x). \tag{3.2}$$

We extend $F \in C^\infty(\mathbb{S}^{n-1})$ to be a 1-homogeneous function $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ by

$$F(x) = |x| F\left(\frac{x}{|x|}\right), \quad x \in \mathbb{R}^n \setminus \{0\} \text{ and } F(0) = 0.$$

One can check easily that $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is in fact a *Minkowski norm* in $\mathbb{R}^n$ in the sense that

(i) F is a norm in $\mathbb{R}^n$, i.e., F is a convex, 1-homogeneous function satisfying $F(x) > 0$ when $x \neq 0$;

(ii) F satisfies a uniformly elliptic condition: $D^2\left(\frac{F^2}{2}\right)$ is positive definite in $\mathbb{R}^n \setminus \{0\}$.

Here D is the Euclidean gradient and D^2 is the Euclidean Hessian. In fact, (ii) is equivalent to the fact that $(\nabla^{\mathbb{S}}\nabla^{\mathbb{S}} F + F\sigma)$ is positive definite on $(\mathbb{S}^{n-1}, \sigma)$.

For a Minkowski norm $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$, its *dual norm* is defined as

$$F^0(\xi) := \sup_{x \neq 0} \frac{\langle x, \xi \rangle}{F(x)}, \quad \xi \in \mathbb{R}^n.$$

It follows from the definition of F^0 that F^0 is also a Minkowski norm and

$$\langle x, \xi \rangle \leq F(x) F^0(\xi), \quad \forall x, \xi \in \mathbb{R}^n. \tag{3.3}$$

This inequality allows one to verify that the *Wulff shape* $\mathcal{W} = \{\xi \in \mathbb{R}^n \mid F^0(\xi) = 1\}$ (see e.g. [11]).

Furthermore, using the norms given above, we can define *Legendre transforms* as

$$l(x) = D\left(\frac{1}{2}F^2\right)(x), \quad l^0(\xi) = D\left(\frac{1}{2}(F^0)^2\right)(\xi).$$

The following properties are easy consequences of the 1-homogeneity of F. We refer to [15], Lemma 1.1 for the proof.

Proposition 3.1. (i) *$F(DF^0(\xi)) = 1$, $DF(DF^0(\xi)) = \frac{\xi}{F^0(\xi)}$ for $\xi \neq 0$;*

$$F^0(DF(x)) = 1, \quad DF^0(DF(x)) = \frac{x}{F(x)} \text{ for } x \neq 0;$$

(ii) *The inverse map of l is l^0, i.e.,*

$$l(l^0(x)) = l^0(l(x)) = x;$$

(iii) *The inverse matrix of* $\left[D^2(\frac{1}{2}F^2)(x)\right]$ *is* $\left[D^2(\frac{1}{2}(F^0)^2(l(x))\right]$, *i.e.,*

$$\left[D^2(\frac{1}{2}F^2)(x)\right] \circ \left[D^2(\frac{1}{2}(F^0)^2(l(x))\right] = I_n, \text{ for } x \neq 0.$$

3.2. A Minkowski type inequality

In this subsection, we use the ABP estimate (2.1) to derive the Minkowski inequality.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded domain with convex and* C^2 *smooth boundary* $\partial\Omega$. *Then*

$$|\Omega| \int_{\partial\Omega} H_F dA \leq \frac{n-1}{n} |\partial\Omega|_F^2, \tag{3.4}$$

where $|\partial\Omega|_F = \int_{\partial\Omega} F(\nu) dA$ *is the anisotropic area functional. Moreover, equality holds if and only if* $\partial\Omega$ *is a translation or rescaling of the Wulff shape* $\mathcal{W}$.

Proof. We consider the Neumann problem

$$\begin{cases} \Delta u = \frac{|\partial\Omega|_F}{|\Omega|} & \text{in } \Omega, \\ u_\nu = F(\nu) & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

It is clear from the standard PDE theory that (3.5) admits a unique C^2 solution.

For any fixed $\varepsilon > 0$, let $E := \bar{\Omega}_\varepsilon$ be enclosed by the following hypersurface

$$\partial\Omega_\varepsilon := \{y \in \mathbb{R}^n \mid y = x + \varepsilon \nu_F(x) \text{ for some } x \in \partial\Omega\},$$

where $\nu_F(x) = DF(\nu(x))$ is the outward anisotropic normal vector at x.

We claim that for any $\delta > 0$ and $a = (\varepsilon + \delta)^{-1}$, $A_a(E, \Omega, u) \subset \Omega$. Suppose not, there exists some $x_0 \in A_a(E, \Omega, u) \cap \partial\Omega$. Let y_0 be the point in E such that

$$\inf_{\bar{\Omega}} \left(u(x) + \frac{a}{2}|y_0 - x|^2\right) = u(x_0) + \frac{a}{2}|y_0 - x_0|^2.$$

At x_0, we have

$$0 < F(\nu(x_0)) = u_\nu(x_0) \leq -a\langle x_0 - y_0, \nu(x_0)\rangle. \tag{3.6}$$

This implies $y_0 \notin \bar{\Omega}$. Let z_0 be the point on $\partial\Omega$ such that

$$y_0 = z_0 + \varepsilon_0 \nu_F(z_0) \text{ for some } 0 < \varepsilon_0 \leq \varepsilon.$$

and z_1 be the intersection point of the segment y_0z_0 and $x_0^\perp$, where $x_0^\perp$ denotes the tangent hyperplane of $\partial\Omega$ at x_0. It follows from the convexity of $\partial\Omega$ that z_1 lies on the segment y_0z_0. Moreover, it is easy to see from the definition of z_1 that

$$\langle y_0 - x_0, \nu(x_0)\rangle = \langle y_0 - z_1, \nu(x_0)\rangle. \tag{3.7}$$

By using (3.6), (3.7) and (3.3), we have

$$F(\nu(x_0)) \leq -a\langle x_0 - y_0, \nu\rangle = a\langle y_0 - z_1, \nu(x_0)\rangle \leq aF^0(y_0 - z_1)F(\nu(x_0)).$$

It follows that

$$F^0(y_0 - z_0) \geq F^0(y_0 - z_1) \geq \frac{1}{a} > \varepsilon.$$

On the other hand, by using Proposition 3.1 (i), we have

$$F^0(y_0 - z_0) = F^0(\varepsilon_0 \nu_F(z_0)) = \varepsilon_0 \leq \varepsilon.$$

A contradiction. Thus $x_0 \in \Omega$.

We can now apply the ABP estimate on $A_a(\bar{\Omega}_\varepsilon, \Omega, u)$ and let $\delta \to 0$ to deduce

$$\begin{aligned} |\Omega_\varepsilon| &\leq \int_\Omega \left(1 + \varepsilon \frac{\Delta u}{n}\right)^n dV \\ &= \int_\Omega 1 + \varepsilon \Delta u + \frac{n-1}{2n}(\Delta u)^2 \varepsilon^2 + o(\varepsilon^2) dV. \\ &= |\Omega| + \varepsilon |\partial\Omega|_F + \frac{n-1}{2n}\varepsilon^2 \int_\Omega (\Delta u)^2 dV + o(\varepsilon^2). \end{aligned} \tag{3.8}$$

On the other hand, $\bar{\Omega}_\varepsilon$ is the Minkowski sum of two convex bodies, $\bar{\Omega}$ and $\varepsilon\{y \in \mathbb{R}^n \mid F^0(y) \leq 1)\}$. Thus the variational formula (see e.g. [27]) gives

$$|\Omega_\varepsilon| = |\Omega| + \varepsilon|\partial\Omega|_F + \frac{1}{2}\varepsilon^2 \int_{\partial\Omega} H_F dA + o(\varepsilon^2). \tag{3.9}$$

By comparing the ε^2 term between (3.8) and (3.9), we obtain

$$\int_{\partial\Omega} H_F dA \leq \frac{n-1}{n} \frac{|\partial\Omega|_F^2}{|\Omega|}.$$

To characterize the equality, we first observe that the equality holds when $\partial\Omega = \mathcal{W}$. On the other hand, if the equality holds, we must have $D_iD_ju = c\delta_{ij}$ in Ω for $c > 0$ from the ABP estimate. Integrating $D_iD_ju = c\delta_{ij}$ from some point $x_0 \in \Omega$ gives

$$u(x) = \frac{c}{2}|x - x_0|^2 + \langle Du(x_0), x - x_0\rangle + u(x_0).$$

It follows that

$$F(\nu(x)) = \langle Du(x), \nu(x)\rangle = \langle c(x - x_0) + Du(x_0), \nu(x)\rangle \text{ for } x \in \partial\Omega. \tag{3.10}$$

The right hand side of (3.10) is in fact the support function for $c\partial\Omega$ with respect to the point $\xi_0 := cx_0 - Du(x_0)$. Thus (3.10) tells us, the support function for the hypersurface $c\partial\Omega - \xi_0$ with respect to the origin, viewed as a function on $\mathbb{S}^{n-1}$, is F. Note that F is the support function of $\mathcal{W}$ with respect to the origin. Since the support function uniquely determines a convex body or a convex hypersurface, we conclude that $\partial\Omega$ is a rescaling and translation of $\mathcal{W}$, precisely, $\partial\Omega = \frac{1}{c}(\mathcal{W} + \xi_0)$. □

4. Anisotropic Heintze-Karcher inequality

In this section, we prove an anisotropic analogue of the Heintze-Karcher inequality. We will utilize an anisotropic version of the ABP estimate.

4.1. Anisotropic ABP estimate

Before stating the result, we introduce a notion of *contact set* in the anisotropic setting.

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain and $E \subset \mathbb{R}^n$ be a compact set. Let $a > 0$ and $u \in C^1(\Omega)$. Set $N_u := \{x \in \Omega \mid Du = 0\}$. The F-contact set associated with u of opening a with vertex set E is defined by

$$\begin{aligned} A_a^F(E, \Omega, u) := \Big\{ & x \in \overline{\Omega} \;\Big|\; \exists\, y \in E \setminus N_u \text{ s.t.} \\ & \inf_{z \in \bar{\Omega}} \Big(u(z) + \frac{a}{2}(F^0(y - x))^2 \Big) = u(x) + \frac{a}{2}(F^0(y - x))^2 \Big\}. \end{aligned} \tag{4.1}$$

We remark that there are two differences between the F-contact set and the one given in Definition 2.1: one is that we replace the usual paraboloid

$\frac{1}{2}|y-z|^2+c_y$ by the F-paraboloid $\frac{1}{2}(F^0(y-z))^2+c_y$ and the other is that we remove the critical set N_u of u away from E.

To prove the anisotropic ABP, we also need to introduce the *anisotropic gradient* and *anisotropic Laplacian.* For $u \in C^1(\Omega)\cap C^2(\Omega\setminus N_u)$, we define

$$\nabla_F u := l(Du) = D\left(\frac{1}{2}F^2\right)(Du), \text{ for } x\in\Omega.$$

$$\Delta_F u(x) := \operatorname{div}(\nabla_F u) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_i}\left(\frac{1}{2}F^2\right)(Du)\right), \text{ for } x\in\Omega\setminus N_u.$$

It is worth to mention that the anisotropic Laplacian $\Delta_F u$ can only be defined pointwisely away from N_u since the function $\frac{1}{2}F^2$ may not be C^2 at the origin.The anisotropic Laplacian has been well studied in recent years, see for example [25] and the references therein.

Based on the idea in [24], we can prove the following anisotropic ABP estimate which links the F-contact set and the anisotropic Laplacian.

Theorem 4.1. *Let $\Omega\subset\mathbb{R}^n$ be an open bounded domain and $u\in C^1(\Omega)\cap C^2(\Omega\setminus N_u)$. Given any compact set $E\subset\mathbb{R}^n$ and any number $a>0$, if $A_a^F(E,\Omega,u)\subset\Omega$, then*

$$|E\setminus N_u| \le \int_{A_a^F(E,\Omega,u)\setminus N_u}\left(1+\frac{\Delta_F u}{na}\right)^n dx. \tag{4.2}$$

Proof. Consider the map

$$T_u:\mathbb{R}^n\to\mathbb{R}^n,\quad T_u(x)=x+a^{-1}\nabla_F u(x).$$

We claim that T_u is a C^1 surjective map from $A_a^F(E,\Omega,u)\setminus N_u$ to $E\setminus N_u$. First, the differentiability of T_u on $A_a^F(E,\Omega,u)\setminus N_u$ is obvious. Second, for $x\in A_a^F(E,\Omega,u)\setminus N_u\subset\Omega$, there exists some $y\in E\setminus N_u$ such that

$$\inf_{z\in\bar\Omega}\left(u(z)+\frac{a}{2}(F^0(y-z))^2\right)=u(x)+\frac{a}{2}(F^0(y-x))^2.$$

Then $Du(x)=al^0(y-x)$. Using Proposition 3.1 (ii) we have

$$\nabla_F u(x)=l(Du(x))=l\left(al^0(y-x)\right)=a(y-x).$$

Thus $T_u(x)=x+a^{-1}\nabla_F u(x)=y\in E\setminus N_u$. Third, for $y\in E\setminus N_u$, there exists $x\in\bar\Omega$ such that

$$\inf_{z\in\bar\Omega}\left(u(z)+\frac{a}{2}(F^0(y-z))^2\right)=u(x)+\frac{a}{2}(F^0(y-x))^2.$$

By the definition of $A_a^F(E,\Omega,u)$, we have $x \in A_a^F(E,\Omega,u)$. From the assumption that $A_a^F(E,\Omega,u) \subset \Omega$, we also have $x \in \Omega$. By the same argument as above, $\nabla_F u(x) = a(y-x)$. It follows that $x \notin N_u$ because otherwise $y = x \in N_u$. Therefore $x \in A_a^F(E,\Omega,u) \setminus N_u$ and $T_u(x) = y$. Thus T_u is surjective.

It follows from the area formula for Lipschitz map that

$$|E \setminus N_u| \le \int_{A_a^F(E,\Omega,u)\setminus N_u} |\det DT_u|\, dx. \tag{4.3}$$

Next, we show the matrix $(I_n + a^{-1}D(\nabla_F u))(x) \ge 0$ for $x \in A_a^F(E,\Omega,u) \setminus N_u$, where I_n is the identity matrix and

$$(D(\nabla_F u))_{ij} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_j}\frac{1}{2}F^2(Du)\right) = \sum_{k=1}^n \frac{\partial^2 \frac{1}{2}F^2}{\partial \xi_j \partial \xi_k}(Du) D_i D_k u.$$

By the fact that x is the minimum point of the function $u(z) + \frac{a}{2}(F^0(y-z))^2$ for some y, we know that

$$\begin{aligned} &Du(x) = a l^0(y-x) \quad \text{and} \\ &D_i D_k u(x) \ge -a D_i D_k \frac{1}{2}(F^0(y-x))^2 \quad \text{(as matrices)}. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{k=1}^n \frac{\partial^2(\frac{1}{2}F^2)}{\partial \xi_j \partial \xi_k}(Du(x)) D_i D_k u(x) \\ &\ge -a \sum_{k=1}^n \frac{\partial^2(\frac{1}{2}F^2)}{\partial \xi_j \partial \xi_k}\left(l^0(y-x)\right) \frac{\partial^2(\frac{1}{2}(F^0(y-x))^2)}{\partial x_i \partial x_k} \\ &= -a\delta_{ij}. \end{aligned}$$

The last equality follows from Proposition 3.1 (ii) and (iii).

Using the arithmetic geometric mean inequality in (4.3), we have

$$\begin{aligned} |E \setminus N_u| &\le \int_{A_a^F(E,\Omega,u)\setminus N_u} \det DT_u dx \\ &\le \int_{A_a^F(E,\Omega,u)\setminus N_u} \left[\frac{1}{n}\mathrm{tr}\left(I_n + a^{-1}D(\nabla_F u)(x)\right)\right]^n dx \\ &= \int_{A_a^F(E,\Omega,u)\setminus N_u} \left(1 + \frac{1}{na}\Delta_F u(x)\right)^n dx. \end{aligned} \tag{4.4}$$

□

4.2. Anisotropic Heintze-Karcher inequality

In this subsection we use Theorem 4.1 to derive the anisotropic Heintze-Karcher inequality.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with C^2 boundary $\partial\Omega$ satisfying $H_F > 0$. Then,*

$$|\Omega| \leq \frac{n-1}{n} \int_{\partial\Omega} \frac{F(\nu)}{H_F} dA, \tag{4.5}$$

and equality holds if and only if $\partial\Omega$ is a rescaling or translation of the Wulff shape $\mathcal{W}$.

Proof. Let u be the solution of the Dirichlet problem

$$\begin{cases} \Delta_F u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

in the weak sense that $u \in W_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega \sum_{i=1}^n D_{\xi_i} \left(\frac{1}{2} F^2\right) (Du) D_i v dx = \int_\Omega v dx, \text{ for any } v \in W_0^{1,2}(\Omega).$$

It is well-known that $u \in C^{1,\alpha}(\bar{\Omega}) \cap C^2(\bar{\Omega} \setminus N_u)$ for some $\alpha \in (0,1)$, see e.g. [22], Theorem 1 and [12], Theorem 1. The Hopf boundary point lemma still holds for (4.6), i.e., $u < 0$ in Ω and $u_\nu > 0$ on $\partial\Omega$, see [9], Theorem 2.1.

We claim that the critical set $N_u = \{Du = 0\}$ has Lebesgue measure zero. For $\min u \leq t \leq 0$, define $S_t = \{x \in \bar{\Omega} : u(x) = t\}$. It follows from the strong maximum principle that the domain $\Omega_t := \{x \in \bar{\Omega} : u(x) < t\}$ is connected and $\bar{\Omega}$ can be viewed as the union of set $\{S_t\}$ with $\min u \leq t \leq 0$. Therefore, for the point in each S_t with $\min u < t \leq 0$, we have $|\nabla u| \neq 0$. The points where $|\nabla u| = 0$ must then be a subset of $S_{\min u}$, and thus necessarily coincide with such set. On the other hand, it follows from the equation that $S_{\min u}$ cannot have interior point. Therefore, $S_{\min u}$, and in turn, N_u has Lebesgue measure zero.

Let $f(x) = u_{\nu_F}(x) = u_\nu(x) F(\nu(x)) > 0$ for $x \in \partial\Omega$ and $\varepsilon > 0$ be a small positive number. Let $E = \bar{\Omega}_\varepsilon$ be the enclosed domain by the hypersurface

$$\partial\Omega_\varepsilon = \{\, y \in \mathbb{R}^n \mid y = x + \varepsilon f(x) \nu_F(x),\, x \in \partial\Omega \,\}.$$

where $\nu_F = DF(\nu)$ is the outward anisotropic normal vector.

For any $\delta > 0$ and $a = (\varepsilon + \delta)^{-1}$, we claim that $A_a^F(E, \Omega, u) \subset \Omega$. Suppose not. There exists $x_0 \in A_a^F(E, \Omega, u) \cap \partial\Omega$. Let $y_0 \in E \setminus N_u$ be the point such that

$$\inf_{z\in\bar{\Omega}} u(z) + \frac{a}{2}(F^0)^2(y_0 - z) = u(x_0) + \frac{a}{2}(F^0)^2(y_0 - x_0). \tag{4.7}$$

Since $u=0$ on $\partial\Omega$, we see for any $\tau \in T_{x_0}(\partial\Omega)$, $\langle Du(x_0), \tau\rangle = 0$. From (4.7),

$$\langle l^0(y_0 - x_0), \tau\rangle = \left\langle D\frac{1}{2}(F^0)^2(y_0 - x_0), \tau \right\rangle = 0.$$

This implies $l^0(y_0 - x_0) = c\nu(x_0)$ and by using Proposition 3.1 (ii),

$$y_0 - x_0 = c \cdot (l^0)^{-1}\nu(x_0) = c \cdot l(\nu(x_0)) = c \cdot F(\nu(x_0))\nu_F(x_0)$$

for some constant c. On the other hand, from (4.7) we see that

$$u_{\nu_F}(x_0) - a\langle l^0(y_0 - x_0), \nu_F(x_0)\rangle \geq 0.$$

That is

$$\begin{aligned}
&u_{\nu_F}(x_0) - a\langle c\nu(x_0), \nu_F(x_0)\rangle \\
&= u_{\nu_F}(x_0) - ac \cdot F(\nu(x_0)) \\
&= u_{\nu_F}(x_0) - a\left\langle y_0 - x_0, \frac{\nu_F}{|\nu_F|^2}(x_0)\right\rangle \geq 0.
\end{aligned}$$

It follows that

$$\left\langle y_0 - x_0, \frac{\nu_F}{|\nu_F|^2}(x_0)\right\rangle \geq a u_{\nu_F}(x_0) > \varepsilon f(x_0).$$

This together with the fact that $y_0 - x_0$ is in the same direction as ν_F imply $y_0 \notin \overline{\Omega}_\varepsilon$, which gives the contradiction. Therefore, we obtain that $A_a^F(E, \Omega, u) \subset \Omega$.

We can apply the anisotropic ABP estimate on $A_a^F(E, \Omega, u)$ and let $\delta \to 0$,

$$\begin{aligned}
|\Omega_\varepsilon| = |\Omega_\varepsilon \setminus N_u| &\leq \int_{\Omega\setminus N_u} \left(1 + \varepsilon\frac{\Delta_F u}{n}\right)^n dV \\
&= \int_{\Omega\setminus N_u} 1 + \varepsilon\Delta_F u + \frac{n-1}{2n}(\Delta_F u)^2\varepsilon^2 + o(\varepsilon^2) dV.
\end{aligned} \tag{4.8}$$

We will compute the first and the second variation of $|\Omega_\varepsilon|$. By the same reason as in (2.6), we need to decompose the vector into two parts. For this purpose, we first recall a metric defined in [27]:

$$G_\xi(V,W) := \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \frac{1}{2}(F^0)^2(\xi)}{\partial\xi^\alpha \partial\xi^\beta} V^\alpha W^\beta,$$
$$Q_\xi(U,V,W) := \sum_{\alpha,\beta,\gamma=1}^{n} \frac{\partial^3 (\frac{1}{2}(F^0)^2(\xi)}{\partial\xi^\alpha \partial\xi^\beta \partial\xi^\gamma} U^\alpha V^\beta W^\gamma,$$

for $\xi \in \mathbb{R}^n \setminus \{0\}$, $U,V,W \in T_\xi\mathbb{R}^n$. It is easy to see

$$\begin{aligned} &G_{\nu_F}(\nu_F,\nu_F) = 1 \ \text{ and } \ G_{\nu_F}(\nu_F,\tau) = 0, \ \text{for tangential vector } \tau; \\ &Q_{\nu_F}(\nu_F,V,W) = 0, \qquad \text{for } V,W \in \mathbb{R}^n. \end{aligned} \tag{4.9}$$

Thus we can decompose any vector into an anisotropic part together with a tangential part by using G.

We continue the proof. As before, we view $\partial\Omega_t$, $t \in [0,\varepsilon]$ as a hypersurface flow

$$\partial_t X(x,t) = f(x,t)\nu_F(x,t) + \tau(x,t), \ t \in [0,\varepsilon]$$

where $f(x,t) := f(x) G_{\nu_F(x,t)}(\nu_F(x),\nu_F(x,t))$ and $\nu_F(x,t)$ is the outward anisotropic normal vector of $\partial\Omega_t$ and $\tau(x,t)$ is tangential to $\partial\Omega_t$.

It follows from the variational formula that

$$\begin{aligned} \frac{d}{dt}|\Omega_t| &= \int_{\partial\Omega_t} f(x,t)F(\nu(x,t))dA_t(x) \\ &= \int_{\partial\Omega_t} f(x)G_{\nu_F(x,t)}(\nu_F(x),\nu_F(x,t))\,F(\nu(x,t))dA_t(x) \end{aligned} \tag{4.10}$$

and in turn

$$\left.\frac{d}{dt}\right|_{t=0}|\Omega_t| = \int_{\partial\Omega} fF(\nu)dA = \int_{\partial\Omega} u_{\nu_F}F(\nu)dA.$$

For the second variation, we note that

$$\begin{aligned} &\left.\frac{\partial}{\partial t}\right|_{t=0} G_{\nu_F(x,t)}(\nu_F(x),\nu_F(x,t)) \\ &= G_{\nu_F(x,t)}(\nu_F(x),\partial_t\nu_F(x,t)) + Q_{\nu_F(x,t)}(\nu_F(x),\nu_F(x,t),\partial_t\nu_F(x,t)) \\ &= 0. \end{aligned} \tag{4.11}$$

The last equality follows from (4.9). Using (4.10), (4.11) and (3.2), we get

$$\frac{d^2}{dt^2}\bigg|_{t=0}|\Omega_t| = \int_{\partial\Omega} f(x)^2 H_F F(\nu)dA = \int_{\partial\Omega} u_{\nu_F}^2 H_F F(\nu)dA.$$

Thus, we have the expansion

$$|\Omega_\varepsilon| = |\Omega| + \varepsilon\int_{\partial\Omega} u_{\nu_F}F(\nu)dA + \frac{1}{2}\varepsilon^2\int_{\partial\Omega} u_{\nu_F}^2 H_F F(\nu)dA + o(\varepsilon^2). \tag{4.12}$$

Since $u = 0$ on $\partial\Omega$, $Du = u_\nu\nu$ on $\partial\Omega$. By integration by parts and $Du = 0$ on N_u,

$$\begin{aligned}\int_{\Omega\setminus N_u}\Delta_F u dV &= \int_{\partial\Omega}\left\langle D\frac{1}{2}F^2(Du),\nu\right\rangle dA\\ &= \int_{\partial\Omega} u_\nu\left\langle D\frac{1}{2}F^2(\nu),\nu\right\rangle dA = \int_{\partial\Omega} u_{\nu_F}F(\nu)dA.\end{aligned}$$

Thus

$$\int_\Omega 1 + \varepsilon\Delta_F u dV = |\Omega| + \varepsilon\int_{\partial\Omega} u_{\nu_F}F(\nu)dA. \tag{4.13}$$

It follows from (4.8), (4.12) and (4.13) that

$$\int_{\partial\Omega} u_{\nu_F}^2 H_F F(\nu)dA \le \frac{n-1}{n}\int_{\Omega\setminus N_u}(\Delta_F u)^2 dV = \frac{n-1}{n}|\Omega|.$$

Using the Hölder inequality, we have

$$\begin{aligned}|\Omega|^2 &= \left(\int_{\Omega\setminus N_u}\Delta_F u dV\right)^2 = \left(\int_{\partial\Omega} u_{\nu_F}F(\nu)dA\right)^2\\ &\le \int_{\partial\Omega} u_{\nu_F}^2 H_F F(\nu)dA\int_{\partial\Omega}\frac{F(\nu)}{H_F}dA\\ &\le \frac{n-1}{n}|\Omega|\int_{\partial\Omega}\frac{F(\nu)}{H_F}dA,\end{aligned}$$

which implies the desired inequality

$$\int_{\partial\Omega}\frac{F(\nu)}{H_F}dA \ge \frac{n}{n-1}|\Omega|.$$

We are remained with the equality case. By examining the equality in the anisotropic ABP estimate, we find

$$(D(\nabla_F u))_{ij} = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial \xi_j}\frac{1}{2}F^2(Du)\right) = \frac{1}{n}\delta_{ij} \text{ in } \Omega \setminus N_u.$$

Thus outside N_u, $l(Du(x)) = \frac{1}{n}(x-b)$ and in turn $Du(x) = \frac{1}{n}D(\frac{1}{2}(F^0)^2)(x-b)$ for some $b \in \mathbb{R}^n$. Since Du is continuous in Ω, we have

$$Du(x) = \frac{1}{n}D\left(\frac{1}{2}(F^0)^2\right)(x-b)$$

for all points in Ω. It follows that $u(x) = \frac{1}{2n}(F^0)^2(x-b)+c$. Note that $\partial\Omega = \{u=0\} = \{x \in \mathbb{R}^n | \frac{1}{2n}(F^0)^2(x-b)+c=0\}$. We conclude that $\partial\Omega$ must be a rescaling and translation of $\mathcal{W} = \{x \in \mathbb{R}^n \mid F^0(x) = 1\}$. □

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