

Almost graphical hypersurfaces become graphical under mean curvature flow

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Consider a mean curvature flow of hypersurfaces in Euclidean space, that is initially graphical inside a cylinder. There exists a period of time during which the flow is graphical inside the cylinder of half the radius. Here we prove a lower bound on this period depending on the Lipschitz-constant of the initial graphical representation. This is used to deal with a mean curvature flow that lies in a slab and is initially graphical inside a cylinder except for a small set. We show that such a flow will become graphical inside the cylinder of half the radius. The proofs are mainly based on White’s regularity theorem.

1	Introduction	589
2	Arbitrary co-dimension	597
3	Hypersurfaces	607
	Appendix A	618
	References	622

1. Introduction

1.1. Overview

Setting. Let $t_1 \in \mathbb{R}$ and $t_2 \in (t_1, \infty]$. Consider a (properly embedded) mean curvature flow $(M_t)_{t \in [t_1, t_2]}$ in some open set $U \subset \mathbb{R}^{n+k}$. By this we mean the following: There exist a submanifold M of \mathbb{R}^{n+k} and a smooth function $\Psi : [t_1, t_2] \times M \rightarrow \mathbb{R}^{n+k}$ such that $\Psi_t := \Psi(t, \cdot)$ is an embedding with $M_t =$

$\Psi_t[M]$ for all $t \in [t_1, t_2]$. Also Ψ satisfies

$$(1.1) \quad \partial_t \Psi(t, p) = \mathbf{H}(M_t, \Psi(t, p))$$

for all $(t, p) \in (t_1, t_2) \times M$, where \mathbf{H} denotes the mean curvature vector. Furthermore we assume there exists an open set $V \subset \mathbb{R}^{n+k}$ with $\bar{U} \subset V$, $\mathcal{H}^n(V \cap M_t) < \infty$ and $\partial M_t \cap V = \emptyset$ for all $t \in [t_1, t_2]$.

Consider open sets $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^k$. The manifold M_t is called graphical inside the cylinder $\Omega_1 \times \Omega_2$, if there exists an $f_t : \Omega_1 \rightarrow \mathbb{R}^k$ such that

$$(1.2) \quad M_t \cap (\Omega_1 \times \Omega_2) = \text{graph}(f_t).$$

Let $t_0 \in [t_1, t_2]$, $x_0 \in M_{t_0}$ and $\epsilon \in (0, 1)$. One can choose coordinates such that x_0 is the origin and the tangent space is $\mathbb{R}^n \times \{0\}^k$. Parametrizing over the tangent space yields an $r > 0$ such that M_t is graphical inside the open cylinder $\mathbf{B}^n(0, 2r) \times \mathbf{B}^k(0, 2r)$ for all $t \in [t_0 - r, t_0 + r] \cap [t_1, t_2]$ and some parametrization with $\sup |f_{t_0}| \leq \epsilon r$, $\sup |Df_{t_0}| \leq \epsilon$. In particular (1.2) always holds locally for small enough Ω_1, Ω_2 . Note that lower bounds on r can be related to curvature estimates on (M_t) .

Problem 1. Consider a mean curvature flow $(M_t)_{t \in [0, \tau]}$. Assume M_0 is graphical inside the open cylinder $\mathbf{B}^n(0, 2) \times \mathbf{B}^k(0, 2)$ for some graphical representation f_0 . By continuity in t , M_t will remain graphical inside $\mathbf{B}^n(0, 1) \times \mathbf{B}^k(0, 1)$ for $t \in [0, \kappa)$ for some $\kappa > 0$. We are interested in lower bounds on κ that only depend on \mathbf{n}, \mathbf{k} and $\text{lip}(f_0)$.

Problem 2. Again consider a mean curvature flow $(M_t)_{t \in [0, \tau]}$. Assume M_0 is almost graphical inside the open cylinder $\mathbf{B}^n(0, 2) \times \mathbf{B}^k(0, 2)$ and also lies in a narrow slab $M_0 \subset \mathbb{R}^n \times \mathbf{B}^k(0, \gamma)$. By almost graphical we mean that there exist an $E \subset \mathbf{B}^n(0, 2)$ and an $f_0 : \mathbf{B}^n(0, 2) \setminus E \rightarrow \mathbb{R}^k$ such that

$$M_0 \cap \left((\mathbf{B}^n(0, 2) \setminus E) \times \mathbf{B}^k(0, 2) \right) = \text{graph}(f_0)$$

and E is small in some sense. We are looking for conditions on M_0 and E such that M_t is graphical inside $\mathbf{B}^n(0, 1) \times \mathbf{B}^k(0, 1)$ for $t \in [\epsilon, \kappa)$ for some $0 < \epsilon < \kappa \leq \tau$.

Note that if $\text{lip}(f_0)$ is small, there already exist satisfying statements for both problems, see below. Hence we are especially interested in the case $\text{lip}(f_0) \in [1, \infty)$.

History and known results. The mean curvature flow was introduced by Brakke [Bra78] in the setting of geometric measure theory. Starting with Huisken’s work [Hui84] the smooth mean curvature flow came more into focus. Graphical mean curvature flow of hypersurfaces, was studied by Ecker and Huisken in [EH89], [EH91]. Among other things, they proved a local gradient bound [EH91, 2.1] and a local curvature estimate [EH91, 3.1, 3.2], under the assumption that the flow is graphical over a period of time. These estimates will be of importance for our results. Wang [Wan04] generalised the estimates from Ecker and Huisken [EH91, 2.1, 3.1] to higher co-dimension in [Wan04, 3.2, 4.1], where one additionally has to assume that the Lipschitz-constant of the graphical representation is small. Wang observed, that such an extra assumption should be necessary in view of the minimal cone constructed by Lawson and Ossermann [LO77, 7.1].

The local regularity theorem by White [Whi05] yields curvature bounds in case the Gaussian density ratios are close to one. In particular it can be used for initially not graphical mean curvature flows. Ilmanen, Neves and Schulze used White’s regularity theorem to solve Problem 1 if $\text{lip}(f_0) \leq l_0(\mathbf{n}, \mathbf{k})$, see [INS14, 1.5]. Changing the proof of [INS14, 1.5] a bit yields a solution to Problem 2, if one assumes slab-height γ , $\text{lip}(f_0)$ and $\mathcal{H}^n(M_0 \cap (E \times \mathbf{B}^k(0, 2)))$ to be bounded by some $\gamma_0(\epsilon, \mathbf{n}, \mathbf{k})$.

Curvature bounds by Chen and Yin yield a solution to Problem 1, if $\sup |D^2 f_0| \leq K$ and $\sup |Df_0| \leq l_1(K, \mathbf{n}, \mathbf{k})$, see [CY07, 7.5]. Moreover curvature bounds by Brendle and Huisken yield a solution to Problem 1, if $\|f_0\|_{C^4} \leq \beta_0$ for some constant β_0 , see [BH12, 2.2].

Results of the present article. In case the co-dimension is one, we will answer Problems 1 and 2 without assuming small gradient of f_0 . Referring to Problem 1 we prove the following:

Theorem 1.1. *For every $L \in (0, \infty)$ exists an $\kappa_L = \kappa_L(L, \mathbf{n})$ such that the following holds: Let $(M_t)_{t \in [0, \tau)}$ be a mean curvature flow in the cylinder $\mathbf{B}^n(0, 2) \times \mathbf{B}^1(0, 2)$. Suppose there exists an $f_0 : \mathbf{B}^n(0, 2) \rightarrow \mathbb{R}$ with $\text{lip}(f_0) \leq L$ and $\sup |f_0| \leq \frac{1}{2}$*

$$M_0 \cap \mathbf{B}^n(0, 2) \times \mathbf{B}^1(0, 2) = \text{graph}(f_0)$$

Set $I := (0, \kappa_L) \cap (0, \tau)$. Then there exists a $g : I \times \mathbf{B}^n(0, 1) \rightarrow \mathbb{R}$ with

$$M_t \cap \mathbf{B}^n(0, 1) \times \mathbf{B}^1(0, 1) = \text{graph}(g(t, \cdot))$$

for all $t \in I$. See also Theorem 3.3 for a more general version.

- Remark.** 1) The local gradient and curvature estimates by Ecker and Huisken [EH91, 2.1, 3.2(ii)] yield estimates on $|Dg|$ and $|D^2g|$.
- 2) See Example (3.7) for a mean curvature flow that is initially graphical inside $\mathbf{B}^n(0, 2) \times \mathbf{B}^1(0, 2)$ with gradient 0, but becomes non-graphical inside $\mathbf{B}^n(0, 2) \times \mathbf{B}^1(0, 2)$ arbitrarily fast. Also the flow becomes non-graphical inside $\mathbf{B}^n(0, 1) \times \mathbf{B}^1(0, 1)$ after finite time.
- 3) It is not clear if the dependency of κ_L on L is necessary.
- 4) Also it is not clear, whether this result can be adopted to higher co-dimension. In particular the cone constructed by Lawson and Ossermann [LO77, 7.1], has to be taken into account.

Referring to Problem 2 we prove the following:

Theorem 1.2. *There exists a $\kappa_2 = \kappa_2(\mathbf{n})$ and for all $L \in (0, \infty)$, $\epsilon \in (0, \kappa_2)$ exists a $\gamma_2 = \gamma_2(L, \epsilon, \mathbf{n})$ such that the following holds: Let $(M_t)_{t \in [0, \tau]}$ be a mean curvature flow in $\mathbf{B}^{n+1}(0, 3)$. Assume*

$$M_0 \cap \mathbf{B}^{n+1}(0, 3) \subset \mathbb{R}^n \times \mathbf{B}^1(0, \gamma_2).$$

Suppose there exist a closed subset $E \subset \overline{\mathbf{B}^{n-1}(0, 2)} \times \mathbf{B}^1(0, \gamma_2)$ and a function $f_0 : \mathbf{B}^n(0, 2) \setminus E \rightarrow \mathbb{R}$ with $\text{lip}(f_0) \leq L$ such that

$$\begin{aligned} M_0 \cap ((\mathbf{B}^n(0, 2) \setminus E) \times \mathbf{B}^1(0, 2)) &= \text{graph}(f_0), \\ \mathcal{H}^n(M_0 \cap (E \times \mathbf{B}^1(0, 2))) &\leq \gamma_2. \end{aligned}$$

Set $I := [\epsilon, \kappa_2) \cap (0, \tau)$. Then there exists a $g : I \times \mathbf{B}^n(0, 1) \rightarrow \mathbb{R}$ with

$$M_t \cap (\mathbf{B}^n(0, 1) \times \mathbf{B}^1(0, 1)) = \text{graph}(g(t, \cdot))$$

for all $t \in I$. See also Theorem 3.8 for a more general version, which includes estimates for $|g|$, $|Dg|$ and $|D^2g|$.

Remark. As for Theorem 1.1 it is not clear, if the dependency of γ_2 on L is necessary and whether the result can be adopted to higher co-dimension.

Outline of the proof of Theorem 1.1. First note that for arbitrary $\delta_0 > 0$ we remain graphical inside $\mathbf{B}^n(0, 2 - \delta_0) \times \mathbf{B}^1(0, 2 - \delta_0)$ for some period of time $[0, s_0]$. Let c_L denote values that only depend on L and \mathbf{n} . Suppose we have graphical representation inside $\mathbf{B}^n(0, r) \times \mathbf{B}^1(0, r)$ for some period of time $[0, s]$. Using the curvature bound by Ecker and Huisken [EH91,

3.2(ii)] we obtain that inside $\mathbf{B}^{\mathbf{n}}(0, r - \sqrt{s}) \times \mathbf{B}^1(0, r - \sqrt{s})$ the curvature of M_s is bounded by $(c_L \sqrt{s})^{-1}$. This lets us cover M_s with balls of radius $c_L l_0 \sqrt{s}$, such that we can parametrize over the tangent space inside each of these balls with a parametrization with gradient smaller than l_0 . Here l_0 denotes a constant we obtain from the regularity result by Ilmanen, Neves and Schulze [INS14, 1.5]. Hence we can use [INS14, 1.5] in each of the balls, to maintain the graphical representation over the tangent spaces for a period of time $[s, (1 + c_L)s]$. Combining these graphical representations we see that M_t is graphical inside $\mathbf{B}^{\mathbf{n}}(0, r - \sqrt{s}) \times \mathbf{B}^1(0, r - \sqrt{s})$ for all $t \in [s, (1 + c_L)s]$. Thus we can maintain graphical representation inside a shrinking cylinder, where the shrinking is controlled by L and \mathbf{n} . This yields the result. Note that in the process we use that the gradient is uniformly bounded by $4L$, which is provided by estimate [EH91, 2.1] from Ecker and Huisken.

Outline of the proof of Theorem 1.2. Using Theorem 1.1 we see that for some small $s_1 \in (0, \epsilon)$ the manifold M_{s_1} is still almost graphical inside $\mathbf{B}^{\mathbf{n}}(0, \frac{3}{2}) \times \mathbf{B}^1(0, \frac{3}{2})$, where the possibly non-graphical part has increased to some $\tilde{E} \times \mathbf{B}^1(0, \frac{3}{2})$. By our initial assumptions we can bound $\mathcal{H}^{\mathbf{n}}(M_{s_1} \cap (\tilde{E} \times \mathbf{B}^1(0, \frac{3}{2})))$ in terms of γ_2 . Also, in view of the slab condition, using the curvature bound by Ecker and Huisken [EH91, 3.2(ii)] yields, that the gradient on the graphical part of M_{s_1} is bounded in terms of γ_2 as well. Thus choosing γ_2 small enough, we are almost in the situation to use the regularity result by Ilmanen, Neves and Schulze [INS14, 1.5]. Modifying the proof of [INS14, 1.5] a bit, yields the result.

Organisation of the paper. In Section 1.2 the notation is introduced. This is followed by recalling some basic properties and an application of White’s regularity theorem in Section 2.1. Then we show, in Section 2.2, that the work of Ilmanen, Neves and Schulze [INS14] already solves Problems 1 and 2 in the case of small gradient. This also yields a solution to Problem 1 for bounded curvature, presented in Section 2.3. Afterwards we restrict ourselves to the case of one co-dimension. In Section 3.1 we prove Theorem 1.1, which solves Problem 1 for bounded gradient. Finally in Section 3.2 we prove Theorem 1.2, which solves Problem 2 for bounded gradient.

1.2. Notation and definitions

We set $\mathbb{R}^+ := \{x \in \mathbb{R}, x \geq 0\}$, $\mathbb{N} := \{1, 2, 3, \dots\}$ and $(a)_+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. We fix $\mathbf{n}, \mathbf{k} \in \mathbb{N}$. Quantities that only depend on \mathbf{n} and/or \mathbf{k} are considered constant. Such a constant may be denoted by C or c , in particular

the value of C and c may change in each line. We denote the canonical basis of \mathbb{R}^{n+k} and \mathbb{R}^n by $(\mathbf{e}_i)_{1 \leq i \leq n+k}$ and $(\hat{\mathbf{e}}_i)_{1 \leq i \leq n}$ respectively.

Let $n, k \in \mathbb{N}$. Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be linear. We denote by A^* the adjoint of A and write $A(v) = Av$. We consider the following norms

$$|A|^2 := \text{trace}(A^*A), \quad \|A\| := \sup_{v \in \mathbb{R}^n \setminus \{0\}} |A(v)| |v|^{-1}.$$

Let T be an n -dimensional subspace of \mathbb{R}^{n+k} . Set $T^\perp = \{x \in \mathbb{R}^{n+k} : x \cdot v = 0 \forall v \in T\}$. By $T_{\natural} : \mathbb{R}^{n+k} \rightarrow T$ we denote the projection onto T , i.e. if $T = \text{span}(\tau_i)_{1 \leq i \leq n}$, $T^\perp = \text{span}(\nu_j)_{1 \leq j \leq k}$ then T_{\natural} is given by $T_{\natural}\tau_i = \tau_i$ and $T_{\natural}\nu_j = 0$ for all $1 \leq i \leq n, 1 \leq j \leq k$.

For $R \in (0, \infty)$ and $b \in \mathbb{R}^n$ we set

$$\mathbf{B}^n(b, R) := \{x \in \mathbb{R}^n : |x - b| < R\}, \quad \mathbf{B}^n(b, 0) := \emptyset, \quad \mathbf{B}^n(b, \infty) := \mathbb{R}^n,$$

For $r, h \in [0, \infty]$ and $a = (\hat{a}, \tilde{a}) \in \mathbb{R}^n \times \mathbb{R}^k$ set

$$\mathbf{C}(a, r, h) := \mathbf{B}^n(\hat{a}, r) \times \mathbf{B}^k(\tilde{a}, h), \quad \mathbf{B}(a, r) := \mathbf{B}^{n+k}(a, r).$$

Consider an open subset $\Omega \subset \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}, t_1 < t_2$. For a function $\phi : (t_1, t_2) \times \Omega \rightarrow \mathbb{R}$ we denote by $\partial_t \phi$ the partial derivative in time (in (t_1, t_2)), by $D_i \phi$ the partial derivatives in space $1 \leq i \leq n$ and set $D\phi := (D_1 \phi, \dots, D_n \phi)$.

Submanifolds of \mathbb{R}^{n+k} . Let M denote an (n -dimensional, \mathcal{C}^2 -regular) submanifold of \mathbb{R}^{n+k} . By this we mean that for each $y \in M$ there are open sets $U, V \in \mathbb{R}^{n+k}$ with $y \in U, 0 \in V$ and a diffeomorphism $\psi \in \mathcal{C}^2(U, V)$ such that $\psi(0) = y$ and

$$\psi(\Omega) = M \cap U, \quad \Omega := V \cap \mathbb{R}^n \times \{0\}^k.$$

For $\hat{p} \in \Omega$ and $x = \psi(\hat{p})$ we define

$$\tau_i(\hat{p}) := D_i \psi(\hat{p})$$

for $1 \leq i \leq n$. The τ_i are linearly independent vectors spanning the tangent space $\mathbf{T}(M, x) = \mathbf{T}_x M := \text{span}(\tau_i(\hat{p}))_{1 \leq i \leq n}$. The first fundamental form at

p called $g(\hat{p}) \in \mathbb{R}^{n \times n}$ is defined by

$$g_{ij}(\hat{p}) := \tau_i(\hat{p}) \cdot \tau_j(\hat{p}).$$

The inverse $(g(\hat{p}))^{-1}$ always exists and its components will be denoted by $g^{ij}(\hat{p})$.

For $\phi \in \mathcal{C}^1(U, \mathbb{R})$ and $X \in \mathcal{C}^1(U, \mathbb{R}^{n+k})$ we define

$$\begin{aligned} \nabla_i^M \phi(x) &:= \sum_{j=1}^n g^{ij} D_i(\phi \circ \psi) \Big|_{\hat{p}} \cdot \tau_j(\hat{p}), & \nabla^M \phi(x) &:= \sum_{i=1}^n \nabla_i^M \phi(x) \tau_i(\hat{p}) \\ \operatorname{div}_M X(x) &:= \sum_{i=1}^{n+k} (\nabla^M(\mathbf{e}_i \cdot X(x))) \cdot \mathbf{e}_i = \sum_{i=1}^{n+k} (DX(x) \tau_i(\hat{p})) \cdot \tau_i(\hat{p}) \end{aligned}$$

We define the second fundamental form $A(\hat{p}) \in \mathbb{R}^{n \times n \times (n+k)}$ by

$$A_{ij}(\hat{p}) := \left(\mathbf{T}(M, x)^\perp \right)_{\hat{p}} D_i D_j \psi(\hat{p}) \in \mathbb{R}^{n+k}.$$

Moreover we define the mean curvature vector

$$\mathbf{H}(M, x) := \sum_{i,j=1}^n g^{ij}(\hat{p}) A_{ij}(\hat{p})$$

and the norm

$$|\mathbf{A}(M, x)|^2 := \sum_{i,j,i',j'=1}^n g^{ii'}(\hat{p}) g^{jj'}(\hat{p}) A_{ij}(\hat{p}) \cdot A_{i'j'}(\hat{p}).$$

Note that these two quantities are both independent of the choice of ψ .

M induces a Borel measure on \mathbb{R}^{n+k} via $\mu_M(A) := \mathcal{H}^n(M \cap A)$ for all $A \subset \mathbb{R}^{n+k}$. For $I \subset J \subset \mathbb{R}$, $A \subset \mathbb{R}^{n+k}$, a family $(M_t)_{t \in I}$ of submanifolds and a function $\phi : J \times A \rightarrow \mathbb{R}^N$ we set $\int_{M_t \cap A} \phi := \int_A \phi(t, x) d\mu_{M_t}(x)$, supposed this expression exists. We will often just write μ or μ_t instead of μ_M or μ_{M_t} .

Graphical submanifolds. Let M be a submanifold of \mathbb{R}^{n+k} . Consider the special case

$$M \cap \mathbf{C}(y, R, \Gamma) = \operatorname{graph}(f)$$

for $y = (\hat{y}, \tilde{y}) \in M$, $R, \Gamma \in (0, \infty)$ and some $f \in \mathcal{C}^2(\mathbf{B}^n(\hat{y}, R), \mathbb{R}^k)$. The first derivative of f is bounded by the tilt of the tangent space $\operatorname{tilt}(M, y) :=$

$\|(\mathbb{R}^n \times \{0\}^k)_\natural - \mathbf{T}(M, y)_\natural\|^2$ in the following way (see [All72, 8.9(5)])

$$(1.3) \quad \text{tilt}(M, y) \leq \|Df(\hat{y})\|^2 \leq \frac{\text{tilt}(M, y)}{1 - \text{tilt}(M, y)}.$$

The second derivative of f is related to the curvature \mathbf{A} in the following way

$$(1.4) \quad c \frac{|D^2 f(\hat{y})|^2}{(1 + |Df(\hat{y})|^2)^3} \leq |\mathbf{A}(M, y)|^2 \leq C|D^2 f(\hat{y})|^2.$$

Consider $F \in \mathcal{C}^2(\mathbf{B}^n(\hat{y}, R), \mathbb{R}^{n+k})$ given by $F(\hat{x}) := (\hat{x}, f(\hat{x}))$. We denote by $JF(\hat{x}) := \sqrt{DF(\hat{x})^T DF(\hat{x})}$ the Jacobian of F at \hat{x} . For a function $\phi : \mathbf{C}(y, R, \Gamma) \rightarrow \mathbb{R}^N$ we have

$$\int_{M \cap \mathbf{C}(y, R, \Gamma)} \phi = \int_{F^{-1}[M \cap \mathbf{C}(y, R, \Gamma)]} \phi(F(\hat{x})) JF(\hat{x}) d\mathcal{L}^n(\hat{x}),$$

supposed this expression exists.

In the special case $k = 1$ the following holds: The normal space $\mathbf{T}(M, y)^\perp$ is spanned by the normal vector

$$(1.5) \quad \nu(y) = \frac{(-Df(\hat{y}), 1)}{\sqrt{1 + |Df(\hat{y})|^2}}.$$

The tilt is given by

$$(1.6) \quad \|(\mathbb{R}^n \times \{0\})_\natural - \mathbf{T}(M, y)_\natural\|^2 = 1 - (\nu(y) \cdot \mathbf{e}_{n+1})^2 = \frac{|Df(\hat{y})|^2}{1 + |Df(\hat{y})|^2}.$$

Comments on the definition of mean curvature flow. By the smoothness of $\Psi : [t_1, t_2) \times M \rightarrow \mathbb{R}^{n+k}$, we mean that there exist a $t_0 \in (-\infty, t_1)$, an $\Omega \subset \mathbb{R}^{n+k}$ open and a $\bar{\Psi} \in \mathcal{C}^\infty((t_0, t_2) \times \Omega, \mathbb{R}^{n+k})$ such that $M \subset \Omega$ and $\bar{\Psi}|_{[t_1, t_2) \times M} \equiv \Psi$. In particular all M_t are smooth submanifolds of \mathbb{R}^{n+k} and all graphical representations are smooth as well.

Actually we only use Ψ is smooth on (t_1, t_2) , Ψ_{t_1} is locally Lipschitz (Sections 2.1 and 2.2) or \mathcal{C}^2 (Sections 2.3, 3.1 and 3.2) and $\int_{M_{t_1}} \phi = \lim_{t \searrow t_1} \int_{M_t} \phi$ for all $\phi \in \mathcal{C}_c^2(\mathbb{R}^{n+k})$.

2. Arbitrary co-dimension

2.1. Local regularity

The following integrated version of the mean curvature flow equation (1.1) is from Brakke [Bra78, 3.5]. For smooth mean curvature flow this follows from the evolution equations by Huisken [Hui84, 3.6] (see also Ecker [Eck04, 4.6]).

Proposition 2.1 ([Bra78, 3.5]). *Consider an open, bounded subset $U \subset \mathbb{R}^{n+k}$ and let $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty]$. Let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in U and $\phi \in \mathcal{C}^2((t_1, t_2) \times \mathbb{R}^{n+k}, \mathbb{R})$ with $\cup_{t \in (t_1, t_2)} M_t \cap \{\phi_t \neq 0\} \subset U$. Then for all $s \in (t_1, t_2)$*

$$(2.1) \quad \frac{d}{dt} \Big|_{t=s} \int_{M_t} \phi_t = \int_{M_t} \left(\partial_t \Big|_{t=s} \phi_t + \operatorname{div}_{M_s}(D\phi_s) - |\mathbf{H}_s|^2 \phi_s \right),$$

where $\phi_t(x) = \phi(t, x)$.

The above Proposition shows that if $(\partial_t - \operatorname{div}_{M_t} D)\phi \leq 0$, the M_t integral of ϕ is monotonically non-increasing. If a test function η satisfies the inequality $(\partial_t - \operatorname{div}_{M_t} D)\eta \leq 0$, but does not have bounded support, we can use the following Proposition to localize η . See also [Bra78, 3.6], [Eck04, 3.17] and [EH89].

Proposition 2.2. *Let $\varrho \in (0, \infty)$, $L \in [0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty]$, $y_0 \in \mathbb{R}^{n+k}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{B}(y_0, \varrho)$. Consider a function $\eta \in \mathcal{C}^{0,1}((t_1, t_2) \times \mathbf{B}(y_0, \varrho), [0, 1]) \cap \mathcal{C}^2(\{\eta > 0\})$ which satisfies*

$$(2.2) \quad (\partial_t - \operatorname{div}_{M_t} D)\eta(t, x) \leq 0, \quad |\nabla^{M_t} \eta(t, x)| \leq L$$

for all $(t, x) \in (t_1, t_2) \times \mathbf{B}(y_0, \varrho) \cap \{\eta > 0\}$ with $x \in M_t$. Define a function $\Upsilon \in \mathcal{C}^{0,1}((t_1, t_2) \times \mathbf{B}(y_0, \varrho), [0, 1])$ by

$$\Upsilon(t, x) := ((\varrho^2 - |x - y_0|^2) \eta(t, x) - (2n + 4L\varrho)(t - t_1))_+.$$

Then for all $t \in (t_1, t_2)$

$$(2.3) \quad \frac{d}{dt} \int_{\mathbb{R}^{n+k}} \Upsilon^3(t, x) d\mu_t(x) \leq 0.$$

Proof. We may assume $y_0 = 0$. Set $\mathcal{M} := \{(t, x) \in (t_1, t_2) \times \mathbb{R}^{n+k} : x \in M_t\}$. Note that we have $\Upsilon^3 \in \mathcal{C}^2((t_1, t_2) \times \mathbf{B}(y_0, \varrho), [0, 1])$ and $\{\Upsilon > 0\} \subset \{\eta > 0\} \cap \mathbf{B}(0, \varrho)$. Using $\operatorname{div}_{M_t}(x) = \mathbf{n}$, $\eta \leq 1$ and (2.2) we can estimate

$$(\partial_t - \operatorname{div}_{M_t} D) \Upsilon^3(t, x) \leq 3\Upsilon^2(\varrho^2 - |x|^2) (\partial_t - \operatorname{div}_{M_t} D) \eta(t, x) \leq 0$$

on $\mathcal{M} \cap \{\Upsilon > 0\}$. Continuity of the derivatives of Υ^3 implies that this estimate holds on all of \mathcal{M} . Also we know $\cup_{t \in (t_1, t_2)} \{\Upsilon(t, \cdot) > 0\} \subset \mathbf{B}(0, \varrho)$. Using Proposition 2.1 then establishes the result. \square

In the very important special case $\eta \equiv 1$ and $L = 0$ the Υ from above proposition will be called φ . This test function already appears in [Bra78, ch.3].

Definition 2.3. Let $\rho \in (0, \infty)$, $x_0 \in \mathbb{R}^{n+k}$, $t_0 \in \mathbb{R}$ be fixed. Define $\varphi_\rho \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbb{R}^{n+k})$ by

$$\varphi_\rho(t, x) := (1 - \rho^{-2}(|x|^2 + 2\mathbf{n}t))_+, \quad \varphi_{(t_0, x_0), \rho}(t, x) := \varphi_\rho(t - t_0, x - x_0).$$

Applying Proposition 2.2 directly yields a monotonicity formula for φ_ρ , which implies an a-priori measure bound for balls.

Corollary 2.4 ([Bra78, 3.7],[Eck04, 4.9]). *Let $\varrho \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_2 \in (s_1, \infty)$, $y_0 \in \mathbb{R}^{n+k}$ and let $(M_t)_{t \in [s_1, s_2]}$ be a mean curvature flow in $\mathbf{B}(y_0, \varrho)$. Then*

$$\frac{d}{dt} \int_{\mathbb{R}^{n+k}} \varphi_{(s_1, y_0), \varrho}^3(t, x) d\mu_t(x) \leq 0$$

for all $t \in (s_1, s_2)$. Also we have

$$\mathcal{H}^n(M_t \cap \mathbf{B}(y_0, 2^{-1}\varrho)) \leq 8\mathcal{H}^n(M_{s_1} \cap \mathbf{B}(y_0, \varrho))$$

for all $t \in [s_1, s_1 + (8\mathbf{n})^{-1}\varrho^2] \cap [s_1, s_2)$.

Proposition 2.2 also implies that a local height bound is maintained. Though the bound is increasing linear in time.

Corollary 2.5. *There exists a $C \in (1, \infty)$ such that the following holds: Let $R, r_0 \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$, $x_0 \in \mathbb{R}^{n+k}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a*

mean curvature flow in $\mathbf{B}(x_0, 2R)$. Suppose

$$M_{t_1} \cap \mathbf{B}(x_0, 2R) \subset \mathbf{C}(x_0, 2R, r_0).$$

Then for all $t \in [t_1, t_2)$ and $r(t) := r_0 + C(t - t_1)R^{-1}$ we have

$$M_t \cap \mathbf{C}(x_0, R, R) \subset \mathbf{C}(x_0, R, r(t)).$$

Proof. We may assume $t_1 = 0$, $x_0 = 0$ and $R = 1$. Use Proposition 2.2 with $\varrho = 2$, $L = 2^{-1}$ and

$$\eta(t, (\hat{x}, \tilde{x})) := 2^{-1} (|\tilde{x}| - r_0)_+, \quad \zeta(x) := (4 - |x|^2)_+.$$

Then by (2.3) we have

$$\int_{\mathbb{R}^{n+k}} (\zeta(x)\eta(t, (\hat{x}, \tilde{x})) - Ct)_+ d\mu_t \leq \int_{\mathbb{R}^{n+k}} (\zeta(x)\eta(0, (\hat{x}, \tilde{x})))_+ d\mu_0$$

By assumption the right hand side is zero. For $x \in \mathbf{C}(0, 1, 1)$ we have $\zeta \geq 2$, hence by definition of η we obtain the result. \square

The main ingredient to obtain graphical representation, will be White’s regularity theorem [Whi05]. The version stated here follows from the proof by Ecker [Eck04, 5.6] and Huisken’s monotonicity formula [Hui90, 3.1].

Definition 2.6. Let $x_0 \in \mathbb{R}^{n+k}$, $t_0 \in \mathbb{R}$ be fixed. Define the heat kernel $\Phi_{(t_0, x_0)} \in C^\infty((-\infty, t_0) \times \mathbb{R}^{n+k})$ by

$$\Phi_{(t_0, x_0)}(t, x) := (4\pi(t_0 - t))^{-\frac{n}{2}} \exp\left(\frac{|x - x_0|^2}{4(t - t_0)}\right).$$

Remark 2.7. For every n -dimensional subspace T of \mathbb{R}^{n+k} we have

$$\int_T \Phi_{(t_0, x_0)}(t, x) d\mathcal{H}^n(x) = 1$$

for all $t \in (-\infty, t_0)$.

Theorem 2.8 ([Whi05]). *There exist $C \in (1, \infty)$ and $d_0 \in (0, 1)$ such that the following holds: Let $r_0 \in (0, \infty)$, $t_0 \in \mathbb{R}$, $t_1 \in (-\infty, t_0 - 8r_0^2]$, $t_2 \in$*

(t_0, ∞) , $x_0 \in \mathbb{R}^{n+k}$, $R_0 = C\sqrt{t_0 - t_1}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{B}(x_0, R_0)$. Suppose $x_0 \in M_{t_0}$ and

$$(2.4) \quad \int_{\mathbf{B}(x_0, R_0)} \Phi_{(s,x)}(t_1, y) d\mu_{t_1}(y) \leq 1 + d_0$$

for all $(s, x) \in [t_0 - 4r_0^2, t_0] \times \mathbf{B}(x_0, 2r_0)$. Then

$$(2.5) \quad |\mathbf{A}(M_t, x)| \leq Cr_0^{-1}$$

for all $t \in [t_0 - r_0^2, t_0]$ and $x \in M_t \cap \mathbf{B}(x_0, r_0)$.

Proof. We may assume $x_0 = 0$ and $t_0 = 0$. Consider $\rho := \frac{R_0}{4}$. Note that $\rho = \frac{C}{4}\sqrt{-t_1} \geq 2r_0$, for C large enough. Consider arbitrary $(s, x) \in [-4r_0^2, 0] \times \mathbf{B}(0, 2r_0)$ and $t \in [s - 4r_0^2, s)$. We can estimate $2n(s - t_1) \leq \frac{8n\rho^2}{C^2}$. This yields $\text{spt}\varphi_{(s,x),\rho}(t_1, \cdot) \subset \mathbf{B}(0, R_0)$ and $\sup|\varphi_{(s,x),\rho}(t_1, \cdot)| \leq 1 + d_0$, where we chose C large enough depending on d_0 . Hence using (2.4) we can estimate

$$\int_{M_{t_1}} \Phi_{(s,x)}\varphi_{(s,x),\rho} \leq (1 + d_0) \int_{M_{t_1} \cap \mathbf{B}(0, R_0)} \Phi_{(s,x)} \leq (1 + d_0)^2 \leq 1 + 4d_0.$$

Let $t \in [s - 4r_0^2, s)$, in particular $t_1 \leq t$. Hence Huisken’s monotonicity formula [Hui90, 3.1] (see also Ecker’s localised version [Eck04, 4.8]) yields

$$\int_{M_t} \Phi_{(s,x)}\varphi_{(s,x),\rho} \leq \int_{M_{t_1}} \Phi_{(s,x)}\varphi_{(s,x),\rho} \leq 1 + 4d_0.$$

Now we can proceed as in [Eck04, 5.6]. □

2.2. Stay graphical for small gradient

Here we give the proof of Theorem 2.9, which is the local version of Theorem 1.5 from Ilmanen, Neves and Schulze [INS14, 1.5]. Basically this section is a slight variation of Section 9 of [INS14]. In particular we point out Proposition 2.10, which can be easily obtained from the proof of [INS14, 1.5]. For the convenience of the reader we include all the details.

Theorem 2.9 ([INS14, 1.5]). *There exist constants $C \in (1, \infty)$ and $l_0 \in (0, 1)$ such that the following holds: Let $\rho, \tau \in (0, \infty)$, $\delta \in (0, 1]$, $l \in [0, l_0]$, $(M_t)_{t \in [0, \tau]}$ be a mean curvature flow in $\mathbf{C}(a, 2\rho, 2\rho)$ and let $a = (\hat{a}, \tilde{a}) \in M_0$.*

Suppose there exists an $f : \mathbf{B}^n(\hat{a}, 2\rho) \rightarrow \mathbb{R}^k$ with $\sup |Df| \leq l$ and

$$(2.6) \quad M_0 \cap \mathbf{C}(a, 2\rho, 2\rho) = \text{graph}(f).$$

Set $I := (0, l_0\rho^2) \cap (0, \tau)$. Then there exists a $g : I \times \mathbf{B}^n(\hat{a}, \rho) \rightarrow \mathbb{R}^k$ with

$$(2.7) \quad M_t \cap \mathbf{C}(a, \rho, \rho) = \text{graph}(g(t, \cdot))$$

and

$$(2.8) \quad \begin{aligned} \sup |g(t, \cdot) - \tilde{a}| &\leq 2l\rho + C\rho^{-1}t, \\ \sup |Dg(t, \cdot)| &\leq C\sqrt[4]{l + \rho^{-2}t}, \\ \sup |D^2g(t, \cdot)| &\leq Ct^{-\frac{1}{2}} \end{aligned}$$

for all $t \in I$.

Proposition 2.10 ([INS14, 1.5]). *There exist $C \in (1, \infty)$ and $\beta_0 \in (0, 1)$ such that the following holds: Let $\beta \in (0, \beta_0)$, $\varrho \in (0, \infty)$, $s_1 \in \mathbb{R}$, $s_0 := s_1 + \beta^2\varrho^2$, $s_2 \in (s_0, \infty)$, $z_0 = (\hat{z}_0, \tilde{z}_0) \in \mathbb{R}^{n+k}$ and let $(M_t)_{t \in [s_1, s_2]}$ be a mean curvature flow in $\mathbf{C}(z_0, 4\varrho, 4\varrho)$. Set $J := [s_0, s_0 + \beta_0^2\varrho^2] \cap [s_0, s_2]$. Assume*

$$(2.9) \quad M_{s_1} \cap \mathbf{C}(z_0, 4\varrho, 4\varrho) \subset \mathbf{C}(z_0, 4\varrho, \beta^2\varrho),$$

$$(2.10) \quad M_{s_0} \cap \mathbf{C}(z_0, \varrho, \varrho) \neq \emptyset$$

Suppose there exist an open set $D_0 \subset \mathbf{B}^n(\hat{z}_0, 4\varrho)$ and an $f_0 : D_0 \rightarrow \mathbb{R}^k$ such that

$$(2.11) \quad M_{s_1} \cap D_0 \times \mathbf{B}^k(\tilde{z}_0, \varrho) = \text{graph}(f_0),$$

$$(2.12) \quad \sup |Df_0| \leq \beta_0,$$

$$(2.13) \quad \mathcal{H}^n(M_{s_1} \cap S_0) \leq \beta_0\beta^n\varrho^n,$$

where $S_0 := (\mathbf{B}^n(\hat{z}_0, 4\varrho) \setminus D_0) \times \mathbf{B}^k(\tilde{z}_0, \varrho)$. Then there exists a function $g : J \times \mathbf{B}^n(\hat{z}_0, \varrho) \rightarrow \mathbb{R}^k$ with

$$(2.14) \quad M_t \cap \mathbf{C}(z_0, \varrho, \varrho) = \text{graph}(g(t, \cdot))$$

and

$$(2.15) \quad \begin{aligned} \sup |g(t, \cdot) - \tilde{z}_0| &\leq C\varrho^{-1}(t - s_1), \\ \sup |Dg(t, \cdot)| &\leq C\sqrt[4]{\varrho^{-2}(t - s_1)}, \\ \sup |D^2g(t, \cdot)| &\leq C(t - s_1)^{-\frac{1}{2}} \end{aligned}$$

for all $t \in J$.

Remark 2.11. 1) A similar result for Brakke flows can be found in the author’s thesis [Lah14, 11.7].

2) Assumption (2.10) can be replaced by

$$(2.16) \quad \mathbf{B}^n(\hat{a}_0, \sqrt{\beta}\varrho) \subset D_0 \quad \text{for some } \hat{a}_0 \in \mathbf{B}^n(\hat{z}_0, \varrho).$$

3) If M_0 has bounded curvature and $n \geq k + 1$, assumption (2.13) follows from $\mathbf{B}^n(\hat{z}_0, 4\varrho) \setminus D_0 \subset \mathbf{B}^n(\hat{y}_0, \beta\varrho)$, see Lemma A.5.

Proof of Proposition 2.10. We may assume $s_1 = 0$, $z_0 = 0$ and $\varrho = 1$. The idea is, that the almost graphical representation implies (2.4). Hence Theorem 2.8 yields curvature bounds. Combining these bounds with the slab condition implies, that the flow has to be a union of graphs. Finally by continuity in time and assumption (2.10) we see, that we actually have exactly one graph.

Let $t_0 \in J$ be fixed but arbitrary. In view of (2.9), Corollary 2.5 with $R = 2$, $r_0 = \beta^2$ yields

$$(2.17) \quad M_t \cap \mathbf{C}(0, 2, 2) \subset \mathbf{C}(0, 2, C_1 t_0)$$

for all $t \in [0, t_0]$ for some constant $C_1 \in (1, \infty)$.

Consider $F_0(\hat{v}) := (\hat{v}, f_0(\hat{v}))$. Assumptions (2.11)-(2.13) imply

$$(2.18) \quad \int_{M_0 \cap \mathbf{B}(0,4)} \phi \leq (1 + C\beta_0) \int_{D_0} \phi(F_0(\hat{v})) d\mathcal{L}^n(\hat{v}) + \sup |\phi| \beta_0 \beta^n$$

for all bounded $\phi \in C^\infty(\mathbb{R}^{n+k})$. Here we used that by (2.12) we have $JF_0 \leq 1 + C\beta_0$. Also we used that by (2.9) the set $M_0 \cap \mathbf{B}(0, 4)$ is contained in $(D_0 \times \mathbf{B}^k(0, 1)) \cup S_0$.

Temporarily fix an arbitrary $y \in M_{t_0} \cap \mathbf{C}(0, 2, 2)$. We want to use Theorem 2.8 with $x_0 = y$, $r_0 = \frac{\sqrt{t_0}}{8}$, $R_0 = 1$ and $t_1 = 0$. Thus for arbitrary $(s, x) \in [t_0 - \frac{t_0}{2}, t_0] \times \mathbf{B}(y, \frac{\sqrt{t_0}}{2})$, we have to show

$$(2.19) \quad \int_{M_0 \cap \mathbf{B}(0,4)} \Phi_{(s,x)} \leq 1 + d_0,$$

where d_0 is from Theorem 2.8. Here we used $C\sqrt{t_0} \leq C\beta_0 \leq 1$ for β_0 small enough. Also we used $\mathbf{B}(y, 1) \subset \mathbf{B}(0, 4)$. By Definition 2.6 we have

$$\int_{D_0} \Phi_{(s,x)}(0, F_0(\hat{v})) d\mathcal{L}^n(\hat{v}) \leq \int_{\mathbf{B}^n(0,4\rho_0)} \Phi_{(s,(\hat{x},0))}(0, (\hat{v}, 0)) d\mathcal{L}^n(\hat{v}) \leq 1.$$

As well as $\sup |\Phi_{(s,x)}(0, \cdot)| \leq (4\pi s)^{-\frac{n}{2}} \leq (2\pi t_0)^{-\frac{n}{2}} \leq \beta^{-n}$. Then for β_0 small enough, inequality (2.18) with $\phi = \Phi_{(s,x)}(0, \cdot)$ establishes (2.19). Thus Theorem 2.8 yields

$$(2.20) \quad |\mathbf{A}(M_{t_0}, y)| \leq C_2 t_0^{-\frac{1}{2}}$$

for all $y \in M_{t_0} \cap \mathbf{C}(0, 2, 2)$ for some constant $C_2 \in (1, \infty)$.

In view of (2.17) we can now apply Lemma A.4 with $r = 1$, $K^2 = C_2 t_0^{-\frac{1}{2}}$ and $\xi^2 = C_1 t_0$ to obtain an $m_0 \in \mathbb{N} \cup \{0\}$ such that

$$(2.21) \quad M_{t_0} \cap \mathbf{C}(0, 1, 1) = \bigcup_{i=1}^{m_0} \text{graph}(g_i)$$

for functions $g_i : \mathbf{B}^n(0, 1) \rightarrow \mathbb{R}^k$ that satisfy (2.15) for $t = t_0$. Here we used $t_0 \leq \beta_0^2$ and chose β_0 small enough.

We want to show $m_0 \leq 1$. Consider φ from Definition 2.3 and set $K := \int \varphi^3(0, (\hat{x}, 0)) d\mathcal{L}^n(\hat{x})$. We have

$$\begin{aligned} m_0 K &\leq \int_{\mathbf{B}(0,1)} \varphi^3(0, (\hat{x}, 0)) d\mu_{t_0}(x) \leq \int_{\mathbf{B}(0,1)} \varphi^3(t_0, x) d\mu_{t_0}(x) + C\beta_0 \\ &\leq \int_{\mathbf{B}(0,1)} \varphi^3(0, x) d\mu_0(x) + C\beta_0 \leq (K + C\beta_0), \end{aligned}$$

which implies $m_0 < 2$ for β_0 small enough. Here we used (2.21) for the first inequality, (2.17) and $t_0 \leq \beta_0^2$ for the second, Corollary 2.4 for the third and (2.18) for the fourth. Now (2.21) yields a $g_{t_0} : \mathbf{B}^n(0, 1) \rightarrow \mathbb{R}^k$ which satisfies (2.15) and such that

$$(2.22) \quad M_{t_0} \cap \mathbf{C}(0, 1, 1) = \text{graph}(g_{t_0}) \text{ or } M_{t_0} \cap \mathbf{C}(0, 1, 1) = \emptyset.$$

This statement holds for all $t_0 \in J$. Thus by continuity of $\mu_t(\mathbf{C}(0, 1, 1))$ and (2.10) we see, that always the first alternative has to be true, then (2.22) implies the result. \square

Proof of Theorem 2.9. We may assume $a = 0$ and $\rho = 1$. Then in particular $f(0) = 0$ and $\sup |f| \leq 2l$. Fix $t \in I$. Let $\hat{z} \in \mathbf{B}^n(0, 1)$ be arbitrary and set

$$\rho_0 := 2^{-2} \sqrt{(l+t)^{-1}t}, \quad z := (\hat{z}, f(\hat{z})).$$

Note that $4\rho_0 + |\hat{z}| < 2$ and $4\rho_0 + |\hat{z}| < 2$. In view of (2.6) and by $\text{lip}(f) \leq l$ we see

$$(2.23) \quad M_0 \cap \mathbf{C}(z, 4\rho_0, 2) = M_0 \cap \mathbf{C}(z, 4\rho_0, 4l\rho_0).$$

We want to use Proposition 2.10 with $z_0 = z$, $s_1 = 0$, $s_2 = \tau$, $\beta = 2\sqrt{l}$ and $\varrho = \rho_0$. Note that $C\sqrt{l_0} \leq \beta_0$ for l_0 small enough depending on β_0 . Then there exists a function $g_{\hat{z}} : \mathbf{B}^n(\hat{z}, \rho_0) \rightarrow \mathbb{R}^k$ with

$$(2.24) \quad M_t \cap \mathbf{C}(z, \rho_0, \rho_0) = \text{graph}(g_{\hat{z}}) \text{ or } M_t \cap \mathbf{C}(z, \rho_0, \rho_0) = \emptyset.$$

Moreover we have

$$(2.25) \quad \begin{aligned} \sup |g_{\hat{z}}| &\leq \sup |f| + Ct \leq 2l + Ct \\ \sup |Dg_{\hat{z}}| &\leq C\sqrt[4]{\rho_0^{-2}t} \leq C\sqrt[4]{l+t}, \quad \sup |D^2g_{\hat{z}}| \leq Ct^{-\frac{1}{2}} \end{aligned}$$

where we used $\rho_0 = 2^{-2} \sqrt{(l+t)^{-1}t}$. Here the height bound follows from Corollary 2.5 combined with (2.6), $\text{lip}(f) \leq l$ and $0 \in M_0$.

As $t \leq Cl_0\rho_0^2$ and l_0 can be chosen small, we can use Corollary 2.4 and (2.23), to see that (2.24) actually holds in the larger cylinder

$$M_t \cap \mathbf{C}((\hat{z}, 0), \rho_0, 1) = \text{graph}(g_{\hat{z}}) \text{ or } M_t \cap \mathbf{C}((\hat{z}, 0), \rho_0, 1) = \emptyset.$$

Choosing different $\hat{z} \in \mathbf{B}^n(0, 1)$, we obtain a graphical representation $g_t : \mathbf{B}^n(0, 1) \rightarrow \mathbb{R}^k$ which satisfies (2.8) and such that

$$(2.26) \quad M_t \cap \mathbf{C}(0, 1, 1) = \text{graph}(g_t) \text{ or } M_t \cap \mathbf{C}(0, 1, 1) = \emptyset.$$

Statement (2.26) holds for all $t \in I$. Thus by continuity of $\mu_t(\mathbf{C}(0, 1, 1))$ and (2.6) we see, that always the first alternative has to be true, then (2.26) implies the result. □

Proof of Remark 2.11.2. We may assume $s_1 = 0$ and $\varrho = 1$. Consider the setting of Proposition 2.10 with (2.10) replaced by (2.16). Then Theorem 2.9 with $\tau = s_2$, $a = (\hat{a}_0, f_0(\hat{a}_0))$, $l = l_0$ and $\rho = \sqrt{\beta}$ implies, that (2.10) holds nevertheless. Here we used $\beta \leq \beta_0$ and chose β_0 small. □

2.3. Stay graphical for bounded curvature

Consider an initial manifold that is graphical with possibly large gradient, but bounded curvature. Then we can use Theorem 2.9 locally to obtain the statement below.

Proposition 2.12. *There exists a $C \in (1, \infty)$ and for all $\kappa \in (0, 1)$, $K \in [1, \infty)$ there exists a $\sigma_1 \in (0, 1)$ such that the following holds: Let $\varrho, \tau \in (0, \infty)$, $\Gamma \in (\frac{\varrho}{2}, \infty)$, $t_0 \in \mathbb{R}$, $z_0 = (\hat{z}_0, \tilde{z}_0) \in \mathbb{R}^{n+k}$ and let $(M_t)_{t \in [t_0, t_0 + \tau]}$ be a mean curvature flow in $\mathbf{C}(0, 2\varrho, \Gamma + \varrho)$. Suppose there exists $u : \mathbf{B}^n(\hat{z}_0, \varrho) \rightarrow \mathbb{R}^k$ with $\sup |u - \tilde{z}_0| < \Gamma - \frac{\varrho}{2}$ and*

$$(2.27) \quad M_{t_0} \cap \mathbf{C}(z_0, 2\varrho, \Gamma + \varrho) = \text{graph}(u).$$

Moreover suppose

$$(2.28) \quad \|\mathbf{T}(M_{t_0}, x)_{\natural} - (\mathbb{R}^n \times \{0\}^k)_{\natural}\| \leq 1 - 2\kappa,$$

$$(2.29) \quad |\mathbf{A}(M_{t_0}, x)| \leq K\varrho^{-1}$$

for all $x \in M_{t_0} \cap \mathbf{C}(z_0, 2\varrho, \Gamma + \varrho)$. Set $I := (t_0, t_0 + \sigma_1\varrho^2) \cap (t_0, t_0 + \tau)$. Then there exists a $v : I \times \mathbf{B}^n(\hat{z}_0, \varrho) \rightarrow \mathbb{R}^k$ with

$$(2.30) \quad M_t \cap \mathbf{C}(z_0, \varrho, \Gamma) = \text{graph}(v(t, \cdot))$$

for all $t \in I$. Moreover we have

$$(2.31) \quad \|\mathbf{T}(M_t, x)_{\natural} - (\mathbb{R}^n \times \{0\}^k)_{\natural}\| \leq 1 - \kappa,$$

$$(2.32) \quad |\mathbf{A}(M_t, x)| \leq C(t - t_0)^{-\frac{1}{2}}$$

for all $t \in I$ and for all $x \in M_t \cap \mathbf{C}(z_0, \varrho, \Gamma)$.

Remark 2.13. See Chen and Yin [CY07, 7.5] for a better curvature estimate than (2.32).

Proof. We may assume $z_0 = 0$, $t_0 = 0$ and $\varrho = 1$. Fix $s \in I$. Let $x = (\hat{x}, \tilde{x}) \in M_s \cap \mathbf{C}(0, 1, \Gamma)$ be arbitrary. Consider r_0 and z such that

$$r_0 := (c_1\kappa)^5 K^{-1}, \quad z = (\hat{z}, \tilde{z}) \in M_0 \cap \mathbf{B}(x, r_0)$$

for some constant $c_1 \in (0, 1)$, which will be chosen below. Note that such a z always exists, by Corollary 2.4 and $s \leq \sigma_1 \leq cr_0^2$. By (2.27) and for c_1 small

enough we can estimate

$$(2.33) \quad |\tilde{x}| \leq |\tilde{z}| + r_0 \leq \sup |u| + \frac{1}{2} < \Gamma.$$

Let $S \in \mathbf{SO}(\mathbf{n} + \mathbf{k})$ be such that $S[\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}] = \mathbf{T}(M_0, z)$. In view of (2.27) and (2.29) we can apply Corollary A.2 with $R = K^{-1}(c_1\kappa)^4$, $a = z$, $L = C\kappa^{-1}$ and $\alpha = (c_1\kappa)^4$ to obtain a $g_0 : \mathbf{B}^{\mathbf{n}}(0, K^{-1}(c_1\kappa)^4) \rightarrow \mathbb{R}^{\mathbf{k}}$ with

$$(2.34) \quad M_0 \cap \mathbf{B}(\hat{z}, 8r_0) \subset S(\text{graph}(g_0)) + z \subset M_0.$$

Here we estimated $(c_1\kappa)^4 \leq \alpha_0$ and $8r_0 \leq (C\kappa^{-1})^{-1} K^{-1}(c_1\kappa)^4$, for c_1 small enough. Also we used that by (1.3) and (2.28) we have $\sup |Du| \leq C\kappa^{-1}$. Corollary A.2 then implies the following bounds

$$(2.35) \quad \sup |g_0| \leq CK^{-1}(c_1\kappa)^8 \leq r_0. \quad \sup |Dg_0| \leq (c_1\kappa)^4 \leq l_0,$$

where l_0 is the constant from Theorem 2.9. Here we used $r_0 = (c_1\kappa)^5 K^{-1}$, $\kappa \leq 1$ and we chose c_1 small enough.

For $t \in [0, \tau]$ set $N_t := S^{-1}[M_t - z]$. In view of (2.34) and (2.35) we can apply Theorem 2.9 with (M_t) replaced by (N_t) , $a = 0$, $\rho = 4r_0$, $\Gamma = 4r_0$ and $l = (c_1\kappa)^4$. Thus we obtain a $g_s : \mathbf{B}^{\mathbf{n}}(0, 2r_0) \rightarrow \mathbb{R}^{\mathbf{k}}$ with $N_s \cap \mathbf{C}(0, 2r_0, 2r_0) = \text{graph}(g_s)$. In particular as $z \in M_0 \cap \mathbf{B}(x, r_0)$, there exists a $v := (\hat{v}, g_s(\hat{v})) \in \text{graph}(g_s)$ such that $x = Sv + z$. Also g_s satisfies

$$(2.36) \quad \sup |Dg_s| \leq C\sqrt{(c_1\kappa)^4 + s} \leq Cc_1\kappa, \quad \sup |D^2g_s| \leq Cs^{-\frac{1}{2}}.$$

Here we used $s \leq \sigma_1 \leq cr_0^2$. By $S[\mathbb{R}^{\mathbf{n}}] = \mathbf{T}(M_0, z)$, $S[\mathbf{T}(N_s, v)] = \mathbf{T}(M_s, x)$, (2.28), (2.36) and (1.3) we obtain

$$\begin{aligned} \|\mathbf{T}(M_s, x)_{\natural} - (\mathbb{R}^{\mathbf{n}})_{\natural}\| &\leq \|\mathbf{T}(N_s, v)_{\natural} - (\mathbb{R}^{\mathbf{n}})_{\natural}\| + \|(\mathbb{R}^{\mathbf{n}})_{\natural} - \mathbf{T}(M_0, z)_{\natural}\| \\ &\leq Cc_1\kappa + 1 - 2\kappa \leq 1 - \kappa, \end{aligned}$$

where we identified $\mathbb{R}^{\mathbf{n}}$ with $\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}$ and we chose c_1 small enough. Similarly using (2.36) and (1.4) yields

$$|\mathbf{A}(M_s, x)| = |\mathbf{A}(N_s, v)| \leq Cs^{-\frac{1}{2}}.$$

As s and x were arbitrary this already establishes (2.31) and (2.32). In view of (2.33) and (2.31) we can use Lemma A.3 with $[t_1, t_2] = [0, \sigma_1]$, $r = 1$ and $\Gamma_0 = \Gamma$ to obtain the existence of the desired v . Here we used that by (2.27) the m_0 from Lemma A.3 has to be 1. □

3. Hypersurfaces

Here we consider mean curvature flows of hypersurfaces. In particular all results from the first part carry over with $\mathbf{k} = 1$. Having co-dimension one allows the usage of the local estimates by Ecker and Huisken from [EH91]. We state the two theorems we need below.

Theorem 3.1 ([EH91, 2.1]). *Let $\varrho \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$, $x_0 \in \mathbb{R}^{\mathbf{n}+1}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{B}(x_0, \varrho)$. Set $\varrho(t) := \sqrt{(\varrho^2 - 2\mathbf{n}(t - t_1))_+}$ and suppose*

$$\nu_t(x) \cdot \mathbf{e}_{\mathbf{n}+1} > 0$$

for all $x \in M_t \cap \mathbf{B}(x_0, \varrho(t))$ for all $t \in [t_1, t_2]$. Set $v(t, x) := (\nu_t(x) \cdot \mathbf{e}_{\mathbf{n}+1})^{-1}$. Then

$$v(t, x) (1 - \varrho^{-2} (|x - x_0|^2 + 2\mathbf{n}(t - t_1))) \leq \sup_{\hat{x} \in \mathbf{B}^{\mathbf{n}}(\hat{x}_0, \varrho)} v(t_1, x)$$

holds for all $x \in \mathbf{B}(x_0, \varrho(t))$ for all $t \in [t_1, t_2]$.

Theorem 3.2 ([EH91, 3.2(ii)]). *There exists a $C \in (1, \infty)$ such that the following holds: Let $\varrho, \Gamma \in (0, \infty)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty]$, $x_0 \in \mathbb{R}^{\mathbf{n}+1}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{C}(x_0, 2\varrho, \Gamma)$. Suppose there exists an $f : (t_1, t_2) \times \mathbf{B}^{\mathbf{n}}(\hat{x}_0, 2\varrho) \rightarrow \mathbb{R}$ such that*

$$M_t \cap \mathbf{C}(x_0, 2\varrho, \Gamma) = \text{graph}(f(t, \cdot))$$

for all $t \in (t_1, t_2)$. Let $s_1 \in (t_1, t_2)$. Then

$$|\mathbf{A}(M_s, (\hat{x}, f(t, \hat{x})))|^2 \leq C ((s - s_1)^{-1} + \varrho^{-2}) \sup_{t \in [s_1, s]} \sup_{\mathbf{B}^{\mathbf{n}}(\hat{x}_0, 2\varrho)} (1 + |Df(t, \cdot)|^2)^2$$

holds for all $x \in \mathbf{B}^{\mathbf{n}}(x_0, \varrho)$ for all $s \in (s_1, t_2)$.

3.1. Stay graphical for bounded gradient

In this section we show Theorem 3.3, which implies Theorem 1.1. The main ingredient of the proof is Lemma 3.5, which originally appears in the author’s thesis [Lah14, 12.11]. This Lemma combines the curvature bound from Ecker and Huisken [EH91, 3.2(ii)] with Proposition 2.12, to maintain the graphical representation of a mean curvature flow that has been graphical over a period of time.

Theorem 3.3. *For every $L \in [1, \infty)$ there exists a $\Lambda \in (1, \infty)$ such that the following holds: Let $\rho, \tau \in (0, \infty)$, $\delta \in (0, 1]$, $\Gamma \in (2\delta\rho, \infty)$, $t_0 \in \mathbb{R}$, $\hat{a} \in \mathbb{R}^n$, $a := (\hat{a}, 0)$ and let $(M_t)_{t \in [t_0, t_0 + \tau]}$ be a mean curvature flow in $\mathbf{C}(a, \rho, \Gamma)$. Suppose there exists an $f : \mathbf{B}^n(\hat{a}, \rho) \rightarrow \mathbb{R}$ with $\sup |f| \leq \Gamma - 2\delta\rho$, $\sup |Df| \leq L$ and*

$$(3.1) \quad M_{t_0} \cap \mathbf{C}(a, \rho, \Gamma) = \text{graph}(f).$$

Set $\sigma(t) := \Lambda\sqrt{t - t_0}$. Then for all $t \in (t_0, t_0 + \tau)$ such that $\sigma(t) < \delta\rho$ there exists an $g_t : \mathbf{B}^n(\hat{a}, \rho - \sigma(t)) \rightarrow \mathbb{R}$ with $\sup |g_t| \leq \sup |f| + \sigma(t)$, $\sup |Dg_t| \leq 4L$ and

$$(3.2) \quad M_t \cap \mathbf{C}(a, \rho - \sigma(t), \Gamma - \sigma(t)) = \text{graph}(g_t).$$

- Remark 3.4.**
- 1) The curvature estimate by Ecker and Huisken [EH91, 3.2(ii)](see Theorem 3.2) can be used to obtain bounds on the curvature and $|D^2g_t|$ (via (1.4)).
 - 2) The bounds on $|g_t|$, $|Dg_t|$ may be improved if you are further inside the graphical cylinder. Use Corollary 2.5 or the gradient estimate by Ecker and Huisken [EH91, 2.1](see Theorem 3.1) respectively.
 - 3) For $a = 0$, $t_0 = 0$, $\rho = \Gamma = 2$ and $\delta = \frac{1}{4}$ this implies Theorem 1.1. Here we use that for $t \in (0, \kappa_L]$ we have $\sigma(t) < \frac{1}{2}$, for $\kappa_L < (2\Lambda)^{-2}$.
 - 4) See also Example 3.7, which shows, that it is reasonable that the set where graphical representation is maintained shrinks in time.

The proof of Theorem 3.3 is based on the following Lemma which is taken from [Lah14, 12.11]. Here we give a much shorter proof.

Lemma 3.5 ([Lah14, 12.11]). *For every $L_1 \in [1, \infty)$ there exists a $\lambda_1 \in (0, 1)$ such that the following holds: Let $\rho_1, R_1 \in (0, \infty)$, $\Gamma_1 \in (R_1, \infty)$, $s_0 \in \mathbb{R}$, and let $(M_t)_{t \in [s_0 - R_1^2, s_0 + \lambda_1^2 R_1^2]}$ be a mean curvature flow in $\mathbf{C}(0, \rho_1 + R_1, \Gamma_1 + R_1)$. Set $J := (s_0 - R_1^2, s_0)$. Suppose there exists $u : J \times \mathbf{B}^n(0, \rho_1 + R_1) \rightarrow \mathbb{R}$ with $\sup |u| < \Gamma_1 - R_1$, $\sup |Du| \leq L_1$ and*

$$(3.3) \quad M_t \cap \mathbf{C}(0, \rho_1 + R_1, \Gamma_1 + R_1) = \text{graph}(u(t, \cdot))$$

for all $t \in J$. Set $I := (s_0, s_0 + \lambda_1^2 R_1^2)$. Then there exists a function $v : I \times \mathbf{B}^n(0, R_1) \rightarrow \mathbb{R}$ with $\sup |v| \leq \sup |u| + R_1$, $\sup |Dv| \leq 4L_1$ and

$$(3.4) \quad M_t \cap \mathbf{C}(0, \rho_1, \Gamma_1) = \text{graph}(v(t, \cdot))$$

for all $t \in I$.

Proof. We may assume $s_0 = 0$ and $R_1 = 4$. Let $z \in \mathbf{B}^n(0, \rho_1) \times \{0\}$ be arbitrary. Note that $1 + \sup |Du|^2 \leq 2L_1^2$, so (1.6) yields

$$(3.5) \quad \left\| \mathbf{T}(M_{t_0}, x)_{\natural} - \left(\mathbb{R}^n \times \{0\}^{\mathbf{k}} \right)_{\natural} \right\| = 1 - \frac{1}{1 + |Du(t_0, x)|^2} \leq 1 - 2(2L_1)^{-2}$$

for all $x \in M_t \cap \mathbf{C}(z, 4, \Gamma_1 + 4)$ for all $t \in [-16, 0]$.

Also by Theorem 3.2 with $t_2 = 0$, $t_1 = -16$, $R = 2$ and $\Gamma = \Gamma_1 + 4$ we obtain

$$(3.6) \quad |\mathbf{A}(M_t, (\hat{x}, f(t, \hat{x})))| \leq CL_1^2 =: K$$

for all $\hat{x} \in \mathbf{B}^n(\hat{z}, 2)$ and all $t \in (-1, 0)$.

In view of (3.5), (3.6) and (3.3) we can apply Proposition 2.12 with $t_0 = -2^{-1}\sigma_1$, $\varrho = 1$, $\Gamma = \Gamma_1$ and $\kappa = (2L_1)^{-2}$. As $z \in \mathbf{B}^n(0, \rho_1) \times \{0\}$ was arbitrary, this establishes the result. Here we chose $\lambda_1^2 \leq 2^{-1}\sigma_1$. \square

Proof of Theorem 3.3. We may assume $t_0 = 0$ and $a = 0$. Let $\epsilon \in (0, 2^{-2}\delta]$. For $t \in [0, \tau)$ set

$$\begin{aligned} \rho_\epsilon(t) &:= \rho - \epsilon\rho - 2^{-2}\Lambda\sqrt{t}, \\ \Gamma_\epsilon(t) &:= \Gamma - \epsilon\rho - 2^{-2}\Lambda\sqrt{t}, \\ \gamma_\epsilon(t) &:= \sup |f| + \epsilon\rho + 2^{-2}\Lambda\sqrt{t}. \end{aligned}$$

A time $t \in [0, \tau)$ is called proper, if $\sigma(t) \geq \delta\rho$, or there exists a function $f_{\epsilon,t} : \mathbf{B}^n(0, \rho_\epsilon(t)) \rightarrow \mathbb{R}$ such that

$$(3.7) \quad M_t \cap \mathbf{C}(0, \rho_\epsilon(t), \Gamma_\epsilon(t)) = \text{graph}(f_{\epsilon,t}),$$

$$(3.8) \quad \sup |f_{\epsilon,t}| \leq \gamma_\epsilon(t), \quad \sup |Df_{\epsilon,t}| \leq 4L.$$

Note that if $\sigma(t) < \delta\rho$, we have $\rho_\epsilon(t) > 0$. We consider the set

$$I_\epsilon := \{s \in (0, \tau) : t \text{ is proper for all } t \in [0, s)\}.$$

For $s \in I_\epsilon$ with $\sigma(s) < \delta\rho$ we have $\sqrt{s} < \Lambda^{-1}\delta\rho$. Thus as $\epsilon \leq \frac{\delta}{4}$ and $\sup |f| \leq \Gamma - 2\delta\rho$ we can estimate

$$(3.9) \quad \gamma_\epsilon(s) < \sup |f| + \frac{\delta\rho}{2} \leq \Gamma - \frac{3\delta\rho}{2} \leq \Gamma_\epsilon(s) - \Lambda\sqrt{s}$$

for all $s \in I_\epsilon$ with $\sigma(s) < \delta\rho$.

By continuity in time and $\mathbf{C}(0, \rho - \epsilon, \Gamma - \epsilon) \subset\subset \mathbf{C}(0, \rho, \Gamma)$ we have $I_\epsilon \neq \emptyset$. Consider $s \in I_\epsilon$ with $s < \tau$. We want to show, that there exists an $s_2 \in (s, \tau]$ such that $(0, s_2] \subset I_\epsilon$. If $\sigma(s) \geq \delta\rho$ we directly see $(0, \tau] = I_\epsilon$. Thus assume $\sigma(s) < \delta\rho$. Let $\lambda_1 \in (0, 1)$ be from Lemma 3.5, chosen for $L_1 = 4L$. Set

$$s_2 := \min \{ (1 + 2^{-3}\lambda_1^2)s, \tau \}, \quad s_0 := s_2 - 2^{-2}\lambda_1^2s,$$

in particular $2^{-1}s < s_0 < s < s_2 < 2s_0$ and $\sqrt{s} - \sqrt{s_0} \geq c\lambda_1^2\sqrt{s}$. By definition of $\rho_\epsilon, \Gamma_\epsilon$ and γ_ϵ this yields

$$(3.10) \quad \rho_\epsilon(s_0) - \sqrt{s} \geq \rho_\epsilon(s) + 4\sqrt{\mathbf{n}s} =: \rho_1,$$

$$(3.11) \quad \Gamma_\epsilon(s_0) \geq \Gamma_\epsilon(s) + \sqrt{s},$$

$$(3.12) \quad \gamma_\epsilon(s_0) \leq \gamma_\epsilon(s) - \sqrt{s},$$

where we chose Λ large depending on λ_1 .

By (3.8) we have $\sup |Df_{\epsilon,t}| \leq 4L$ for all $t \in [0, s_0]$. Also, by (3.8), (3.9) and (3.12) we obtain

$$(3.13) \quad \sup |f_{\epsilon,t}(\hat{x})| \leq \sup_{t \in [0, s_0]} \gamma_\epsilon(t) = \gamma_\epsilon(s_0) \leq \gamma_\epsilon(s) - \sqrt{s} < \Gamma_\epsilon(s) - \sqrt{s}.$$

Combining (3.7) with (3.10), (3.11) and (3.13) we have

$$M_t \cap \mathbf{C}(0, \rho_1 + \sqrt{s}, \Gamma_\epsilon(s) + \sqrt{s}) = \text{graph}(f_{\epsilon,t})$$

for all $t \in (0, s_0]$, where we consider the restrictions of $f_{\epsilon,t}$ to $\mathbf{B}^{\mathbf{n}}(\hat{y}, \rho_1 + \sqrt{s})$. Then by Lemma 3.5 with $L_1 = 4L, R_1 = 2^{-1}\sqrt{s}$ and $\Gamma_1 = \Gamma_\epsilon(s)$ there exists a $g_\epsilon : (0, s_2) \times \mathbf{B}^{\mathbf{n}}(0, \rho_1) \rightarrow \mathbb{R}$ with

$$(3.14) \quad M_t \cap \mathbf{C}(0, \rho_1, \Gamma_\epsilon(s)) = \text{graph}(g_\epsilon(t, \cdot))$$

for all $t \in (0, s_2)$. Moreover $\sup |g_\epsilon| \leq \gamma_\epsilon(s)$.

It remains to show that $\sup |Dg_\epsilon| \leq 4L$. Consider arbitrary $t_2 \in (0, s_2)$ and $\hat{x}_0 \in \mathbf{B}^{\mathbf{n}}(0, \rho_\epsilon(s))$. Set $x_0 = (\hat{x}_0, (g_\epsilon(t_2, \hat{x}_0))$ and $\varrho := \rho_1 - \rho_\epsilon(s) = 4\sqrt{\mathbf{n}s}$. Then by $\sup |g_\epsilon| \leq \gamma_\epsilon(s)$, (3.9) and for Λ large enough we have

$$\mathbf{B}(x_0, \varrho) \subset \mathbf{C}(0, \rho_1, \Gamma_\epsilon(s)).$$

In view of (3.14), (3.1), (1.5) and $\sqrt{1 + \sup |Df|^2} \leq 2L$ we can now use Theorem 3.1 to obtain $|Dg_\epsilon(t_2, \hat{x}_0)| \leq 4L$. Thus we have $s_2 \in I_\epsilon$. Note that

s_2 is either τ or $(1 + 2^{-3}\lambda_1^2)s$, so the amount by which we enlarge the time interval is actually increasing with s . As $s \in I_\epsilon$ was arbitrary and $I_\epsilon \neq \emptyset$ this yields $\tau \in I_\epsilon$. Note that

$$\rho_\epsilon(t) \geq \rho - \sigma(t) - \epsilon\rho, \quad \Gamma_\epsilon(t) \geq \Gamma - \sigma(t) - \epsilon\rho, \quad \gamma_\epsilon(t) \leq \sup |f| + \sigma(t) + \epsilon\rho.$$

As ϵ can be chosen arbitrary small we established the result. □

Additionally assuming a height bound in Theorem 3.3, yields that the graphical representation will have small gradient after some time.

Lemma 3.6. *For every $L_0 \in [1, \infty)$ there exist $\Lambda_1 \in (1, \infty)$ and $\gamma_1 \in (0, 1)$ such that the following holds: Let $\gamma \in (0, \gamma_1]$, $\rho_0 \in (0, \infty)$, $t_1 \in \mathbb{R}$, $\hat{z}_0 \in \mathbb{R}^n$, $z_0 := (\hat{z}_0, 0)$, $s_1 := t_1 + \gamma^4 \rho_0^2$, $t_2 \in (s_1, \infty)$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{C}(z_0, 4\rho_0, 4\rho_0)$. Suppose there exists an $u : \mathbf{B}^n(\hat{z}_0, 4\rho_0) \rightarrow \mathbb{R}$ with $\sup |Du| \leq L_0$, $\sup |u| \leq \gamma^4 \rho_0$ and*

$$(3.15) \quad M_{t_1} \cap \mathbf{C}(z_0, 4\rho_0, 4\rho_0) = \text{graph}(u).$$

Then there exists a $v : \mathbf{B}^n(\hat{z}_0, \rho_0) \rightarrow \mathbb{R}$ with

$$(3.16) \quad M_{s_1} \cap \mathbf{C}(z_0, \rho_0, \rho_0) = \text{graph}(v)$$

and

$$(3.17) \quad \sup |v| \leq \Lambda_1 \gamma^4 \rho_0, \quad \sup |Dv| \leq \Lambda_1 \gamma, \quad \sup |D^2v| \leq \Lambda_1 \gamma^{-2} \rho_0^{-1}.$$

Proof. We may assume $t_1 = 0$, $z_0 = 0$ and $\rho_0 = 1$. Let Λ be from Theorem 3.3, chosen for $L = L_0$. Then we can estimate $\Lambda\sqrt{2s_1} \leq \frac{1}{4}$ and $\sup |u| \leq \frac{1}{2}$, where we used $\gamma \leq \gamma_1$ and chose γ_1 small depending on Λ . Hence we can use Theorem 3.3 with $\rho = \Gamma = 4$, $a = 0$, $t_0 = 0$, $\tau = t_2$, $L = L_0$ and $\delta = \frac{1}{4}$. This yields a function $g : [0, 2s_1] \times \mathbf{B}^n(0, 3) \rightarrow \mathbb{R}$ with $\sup |g(\hat{x})| < \rho_0$,

$$(3.18) \quad M_t \cap \mathbf{C}(0, 3, 3) = \text{graph}(g(t, \cdot))$$

for all $t \in [0, 2s_1]$. Now we can apply Theorem 3.2 with $\varrho = 1$, $\Gamma = 3$ and arbitrary $x_0 \in \mathbf{B}^n(0, 2) \times \{0\}$, to obtain

$$(3.19) \quad |\mathbf{A}(M_{s_1}, x)| \leq CL_0^4 s_1^{-\frac{1}{2}} = CL_0^4 \gamma^{-2}$$

for all $x \in M_{s_1} \cap \mathbf{C}(0, 2, 3)$.

As $\sup |u| \leq \gamma^4$, Corollary 2.5 with $r_0 = \gamma^4$, $R = 2$ and $x_0 = 0$ implies

$$(3.20) \quad M_t \cap \mathbf{C}(0, 2, 2) \subset \mathbf{C}(0, 2, C\gamma^4)$$

for all $t \in [t_1, s_1]$.

Let v be the restriction of $g(s_1, \cdot)$ to $\mathbf{B}^n(0, 1)$. In particular (3.18) implies (3.16). In view of (3.19) and (3.20) we can apply Lemma A.4 with $r = 1$, $\xi = C\gamma^2$ and $K = CL_0^2\gamma^{-1}$ to see that (3.17) holds. Here we estimated $K\xi \leq CL_0^2\gamma \leq \alpha_1$. Note that by (3.18) we have that $m_0 = 1$, so the function from (A.17) has to coincide with v . \square

In the following example we construct a mean curvature flow (actually a curve shortening flow) that initially intersects the cylinder $(-2, 2) \times (-2, 2)$ in the straight line $(-2, 2) \times \{0\}$, but becomes non-graphical inside that cylinder immediately and also becomes non-graphical inside the cylinder of half the radius after finite time.

Example 3.7. Consider $\epsilon \in (0, 4^{-1})$, $\mathbf{S}^1 := \partial\mathbf{B}^2(0, 1)$ and $A \subset \mathbb{R}^2$ with

$$[-2, 2] \times [0, 2] \subset A \subset [-(2 + \epsilon), 2 + \epsilon] \times [0, 2 + \epsilon]$$

and such that $\partial A = \Psi_0[\mathbf{S}^1]$ for a smooth embedding Ψ_0 . In particular $M_0 := \partial A$ is graphical inside $(-2, 2) \times (-2, 2)$ with gradient zero. By the work of Grayson [Gra87] there exists a unique mean curvature flow $(M_t)_{t \in [0, T]}$ starting from M_0 and shrinking to a round point for $t \nearrow T$. By choice of A and Brakke’s sphere comparison results [Bra78, 3.7, 3.9] (see also [Eck04, 3.3]) we have

$$(3.21) \quad \mathbf{B}^2((0, 1), r(t)) \cap M_t = \emptyset, \quad M_t \subset \mathbf{B}^2((0, 1), R(t))$$

for all $t \in [0, T)$, where $R(t) := \sqrt{3 + 3\epsilon - 2t}$ and $r(t) := \sqrt{1 - 2t}$.

In particular $\frac{1}{2} \leq T \leq \frac{3+3\epsilon}{2} \leq 2$. Statement (3.21) also yields that $M_{2\epsilon} \subset \mathbf{B}^2((0, 1), \sqrt{3 - \epsilon})$. Then $M_{2\epsilon}$ cannot be graphical inside $(-2, 2) \times (-2, 2)$. Moreover as (M_t) shrinks to a round point, there exists an $s \in (0, T)$ such that $M_s \subset \mathbf{B}^2(a, \epsilon)$ for some $a \in \mathbb{R}^2$. Then $s < 2$ and for any $h \in (0, 2]$ the manifold M_s cannot be graphical inside $(-2\epsilon, 2\epsilon) \times (-h, h)$.

3.2. Become graphical

In this section we show Theorem 3.8, which implies Theorem 1.2. The idea is that for an initially almost graphical mean curvature flow that lies in a slab,

the gradient on the shrinking graphical part is decreasing (by Lemma 3.6) and the measure of the growing non-graphical part can be controlled. It turns out that there is a time when both the gradient and this measure are small enough to apply Proposition 2.10, which yields a graphical representation.

Theorem 3.8. *There exist constants $C \in (1, \infty)$ and $\sigma_0 \in (0, 1)$ and for every $L \in [1, \infty)$ there exist $\Lambda_0 \in (1, \infty)$ and $\lambda_0 \in (0, 1)$ such that the following holds: Let $\lambda \in (0, \lambda_0]$, $\rho \in (0, \infty)$, $h, t_1 \in \mathbb{R}$, $t_0 := t_1 + \Lambda_0 \lambda^2 \rho^2$, $t_2 \in (t_0, \infty)$, $a = (\hat{a}, \tilde{a}) \in \mathbb{R}^{n+1}$ and let $(M_t)_{t \in [t_1, t_2]}$ be a mean curvature flow in $\mathbf{C}(a, 2\rho, 2\rho)$. Assume*

$$(3.22) \quad M_{t_1} \cap \mathbf{C}(a, 2\rho, 2\rho) \subset \mathbf{C}(a, 2\rho, \lambda^{2n}\rho).$$

Suppose there exist an open subset $D^n \subset \mathbf{B}^n(\hat{a}, 2\rho)$ and an $f : D^n \rightarrow \mathbb{R}$ with $\sup |Df| \leq L$ such that

$$(3.23) \quad M_{t_1} \cap (D^n \times \mathbf{B}^1(\tilde{a}, \rho)) = \text{graph}(f),$$

and for $E^n := \mathbf{B}^n(\hat{a}, 2\rho) \setminus D^n$

$$(3.24) \quad E^n \subset (\mathbb{R}^{n-1} \times \mathbf{B}^1(h, \lambda^n \rho)),$$

$$(3.25) \quad \mathcal{H}^n(M_{t_1} \cap (E^n \times \mathbf{B}^1(\tilde{a}, \rho))) \leq \lambda^n \rho^n.$$

Set $I := [t_0, t_0 + \sigma_0 \rho^2) \cap [t_0, t_2)$. Then there exists a $g : I \times \mathbf{B}^n(\hat{a}, \rho) \rightarrow \mathbb{R}$ with

$$(3.26) \quad M_t \cap \mathbf{C}(a, \rho, \rho) = \text{graph}(g(t, \cdot))$$

and

$$(3.27) \quad \begin{aligned} \sup |g(t, \cdot) - \tilde{a}| &\leq C \rho_0^{-1} (t - t_1), \\ \sup |Dg(t, \cdot)| &\leq C \sqrt[4]{\rho_0^{-2} (t - t_1)}, \\ \sup |D^2g(t, \cdot)| &\leq C (t - t_1)^{-\frac{1}{2}} \end{aligned}$$

for all $t \in I$.

Remark 3.9. Using this Theorem with $a = 0$, $\rho = 1$, $h = 0$, $t_1 = 0$, $t_2 = \tau$ and $\lambda^2 = \Lambda_0^{-1} \epsilon$ implies Theorem 1.2 with $\gamma_2 = (\Lambda_0^{-1} \epsilon)^n$ and $\kappa_2 := c \sigma_0$.

Proof. We may assume $a = 0$, $t_1 = 0$ and $\rho = 4$. Set

$$\begin{aligned} \rho_1 &:= 2^{-4}\lambda^n, \quad B^1 := \mathbf{B}^1(0, 1) = (-1, 1), \\ D_1^n &:= \mathbf{B}^n(0, 5) \setminus (\mathbb{R}^{n-1} \times \overline{\mathbf{B}^1(h, 8\lambda^n)}) \\ E_1^n &:= \mathbf{B}^n(0, 5) \cap (\mathbb{R}^{n-1} \times \overline{\mathbf{B}^1(h, 8\lambda^n)}). \end{aligned}$$

Consider $z = (\hat{z}, 0) \in D_1^n \times \{0\}$. Then by (3.24) and $\lambda \leq \lambda_0 \leq 1$ we have

$$\mathbf{C}(z, 4\rho_1, 4\rho_1) \subset D^n \times B^1.$$

In view of (3.22) and (3.23) we can use Lemma 3.6 with $L_0 = L$, $z_0 = z$, $\rho_0 = \rho_1$, $\gamma = 4\lambda^{\frac{n}{4}}$. Here we chose λ_0 small enough such that $4\lambda^{\frac{n}{4}} \leq \gamma_0$, which depends on L . Set

$$s_1 := \gamma^4 \rho_1^2 = \lambda^{3n}.$$

Lemma 3.6 yields a $v_z : \mathbf{B}^n(\hat{z}, \rho_1) \rightarrow \mathbb{R}$ with $\sup |Dv_z| \leq 4\Lambda_1 \lambda^{\frac{n}{4}}$ and

$$M_{s_1} \cap \mathbf{C}(z, \rho_1, \rho_1) = \text{graph}(v_z).$$

Thus, as $\hat{z} \in D_1^n$ was arbitrary, we can combine the v_z to obtain a $v_2 : D_1^n \rightarrow \mathbb{R}$ with

$$(3.28) \quad M_{s_1} \cap (D_1^n \times \mathbf{B}^1(0, \rho_1)) = \text{graph}(v_2)$$

and

$$(3.29) \quad \sup |Dv_2| \leq 4\Lambda_1 \lambda^{\frac{n}{4}}.$$

By (3.22) we can use Corollary 2.5 with $R = 1$ and $x_0 \in \mathbf{B}^n(0, 6) \times \{0\}$ as well as $R = 4$ and $x_0 = 0$, to obtain

$$(3.30) \quad M_t \cap \mathbf{C}(0, 6, 1) \subset \mathbf{C}(0, 6, r(t))$$

$$(3.31) \quad M_t \cap \mathbf{C}(0, 4, 4) \subset \mathbf{C}(0, 4, r(t))$$

for $r(t) := \lambda^{2n} + Ct$ for all $t \in [0, t_2)$. Note that $r(s_1) = C\lambda^{2n} \leq 2^{-4}\lambda^n = \rho_1$, where we used $\lambda \leq \lambda_0$ and λ_0 small. Hence with (3.28) we can conclude

$$(3.32) \quad M_{s_1} \cap (D_1^n \times B^1) = \text{graph}(v_2).$$

We want to use Proposition 2.10. In order to do so, the M_{s_1} -measure in $E_1^n \times B^1$ has to be small. The idea is, that by Proposition 2.2 the M_{s_1} -measure in $E_1^n \times B^1$ is bounded by the M_0 -measure in some larger set $Y \times B^1$. This set can then be estimated with (3.23) and (3.25).

By definition of E_1^n and the slab statement (3.30) we have

$$(3.33) \quad M_{s_1} \cap (E_1^n \times B^1) \subset \mathbf{B}(0, 6) \cap (\mathbb{R}^{n-1} \times \mathbf{B}^2((h, 0), C_1 \lambda^n))$$

for some constant $C_1 \in (1, \infty)$. Consider

$$Y^n := \mathbf{B}^n(0, 8) \cap (\mathbb{R}^{n-1} \times \mathbf{B}^1(h, 2C_1 \lambda^n)).$$

We want to show

$$(3.34) \quad \mathcal{H}^n(M_{s_1} \cap (E_1^n \times B^1)) \leq C \mathcal{H}^n(M_0 \cap (Y^n \times B^1)).$$

For $(u, v) \in \mathbb{R}^{n-1} \times \mathbb{R}^2$ set $Q(u, v) := v$. Consider $\eta, \zeta \in \mathcal{C}^{0,1}(\mathbb{R} \times \mathbf{B}(0, 8))$ given by

$$\begin{aligned} \eta(t, x) &:= (1 - (2C_1 \lambda^n)^{-2} (|Q(x)|^2 + 2\mathbf{n}t))_+ \\ \zeta(t, x) &= \zeta(x) := (8^2 - |x|^2)_+. \end{aligned}$$

Note that $\eta \in \mathcal{C}^2(\{\eta > 0\})$ and $\{\zeta(\cdot)\eta(0, \cdot) > 0\} \subset Y \times B^1$. We also have $(\partial_t - \operatorname{div}_{M_t} D)\eta \leq 0$ on $\{\eta > 0\}$, as $\operatorname{div}_{M_t} DQ \leq \mathbf{n}$. Then Proposition 2.2 with $y_0 = 0$, $L = \lambda^{-2\mathbf{n}}$ and $\varrho = 8$ yields

$$(3.35) \quad \int_{\mathbb{R}^{n+1}} (\zeta(x)\eta(s_1, x) - 2\mathbf{n}\lambda^{-2\mathbf{n}}s_1)^3 d\mu_{s_1} \leq C \mathcal{H}^n(M_0 \cap (Y^n \times B^1)).$$

For $x \in M_{s_1} \cap (E_1^n \times B^1)$ estimate $\zeta(x)\eta(s_1, x) - 2\mathbf{n}\lambda^{-2\mathbf{n}}s_1 \geq 1$, where we used (3.33), $s_1 = \lambda^{3\mathbf{n}}$, $\lambda \leq \lambda_0$ and λ_0 small. Hence (3.35) implies (3.34).

Let $C_L \in (1, \infty)$ be a constant depending on L , which may increase in each step. By $\sup |Df| \leq L$ we have $JF \leq C_L$. Then in view of assumption (3.23) we obtain

$$(3.36) \quad \mathcal{H}^n(M_0 \cap ((Y^n \cap D^n) \times B^1)) \leq C_L \mathcal{L}^n(Y^n) \leq C_L \lambda^n.$$

On the other hand as $Y^n \subset \mathbf{B}^n(0, 8)$ we see $Y^n \setminus D^n \subset E^n$. Thus by assumption (3.25) we can estimate

$$(3.37) \quad \mathcal{H}^n(M_0 \cap ((Y^n \setminus D^n) \times B^1)) \leq C \lambda^n.$$

Combining (3.34) with (3.36) and (3.37) we conclude

$$(3.38) \quad \mathcal{H}^n (M_{s_1} \cap (E_1^n \times B^1)) \leq C_L \lambda^n.$$

Set $\beta := \sqrt{t_0 - s_1} = \sqrt{\Lambda_0 \lambda^2 - \lambda^{3n}}$ and $\rho_0 := 2^{-3}$. In particular

$$(3.39) \quad c\Lambda_0 \lambda^2 \leq \beta^2 \leq \Lambda_0 \lambda^2 \leq \beta_0^2,$$

and $I = [t_0, t_0 + 16\sigma_0] \subset (s_1 + \beta^2 \rho_0^2, s_1 + \beta_0^2 \rho_0^2)$, where β_0 is the constant from Proposition 2.10. Here we used $\lambda \leq \lambda_0$, λ_0 small enough depending on Λ_0 and σ_0 small depending on β_0 .

Let $z = (\hat{z}, 0) \in \mathbf{B}^n(0, 4) \times \{0\}$ be arbitrary. We want to use Proposition 2.10 with $z_0 = z$, $s_2 = t_2$ and $D_0 = \mathbf{B}^n(\hat{z}, 2^{-2}) \setminus E_1^n$. In view of (3.39) and for Λ_0 large enough, the slab statement (3.30) with $r(s_1) \leq C\lambda^2$ implies (2.9). By choice of D_0 and (3.32) also (2.11) holds. Choosing λ_0 small enough depending on Λ_1 of Lemma 3.6 for $L_0 = L$, we see that (3.29) implies (2.12). Choosing Λ_0 large depending on L we can use (3.39) and (3.38) to obtain (2.13). At last we consider the points $\hat{a}_0^\pm := \hat{z} \pm 2^{-4} \hat{\mathbf{e}}_n$. Then

$$|(\hat{a}_0^+ - \hat{a}_0^-) \cdot \hat{\mathbf{e}}_n| = 2^{-3} > 4^{n+1} \lambda^n + 2^{-3} \sqrt{\beta},$$

where we used (3.39), $\lambda \leq \lambda_0$ and chose λ_0 small enough. Now by definition of E_1^n we see that (2.16) holds for $\hat{a}_0 = \hat{a}_0^+$ or \hat{a}_0^- . Proposition 2.10 then yields a $g_z : I \times \mathbf{B}^n(\hat{z}, 2^{-4}) \rightarrow \mathbb{R}$ with

$$(3.40) \quad M_t \cap \mathbf{C}(z, 2^{-4}, 2^{-4}) = \text{graph}(g_z(t, \cdot)).$$

Note that for $t \in I$ we have $t \geq t - s_1 \geq \frac{t}{2}$ and $r(t) \leq 2^{-4}$, where we estimated $\lambda \leq \lambda_0$ and we chose λ_0 small enough. Hence (2.15) implies (3.27) with g replaced by g_z and by (3.31) we see, that (3.40) actually holds for the larger cylinder $\mathbf{C}(z, 2^{-4}, 4)$. As $z \in \mathbf{B}^n(0, 4) \times \{0\}$ was arbitrary, the g_z can be combined to obtain the desired g . □

Theorem 3.8 naturally implies a global result.

Corollary 3.10. *For every $L \in [1, \infty)$ there exists a $\Lambda_3 \in (1, \infty)$ such that the following holds: Let $\varrho \in (0, \infty)$, $K \in [1, \infty)$, $t_0 = \Lambda_3 K^2 \varrho^2$, $T \in (t_0, \infty]$*

and let $(M_t)_{t \in [0, T]}$ be a mean curvature flow in \mathbb{R}^{n+1} . Assume

$$(3.41) \quad M_0 \subset \mathbb{R}^n \times \mathbf{B}^1(0, \varrho).$$

Suppose there exists a closed subset $E_0 \subset \mathbb{R}^{n-1} \times \mathbf{B}^1(0, \varrho)$ and a function $f : \mathbb{R}^n \setminus E_0 \rightarrow \mathbb{R}$ with $\sup |Df| \leq L$ and

$$(3.42) \quad M_0 \cap ((\mathbb{R}^n \setminus E_0) \times \mathbb{R}) = \text{graph}(f).$$

Also suppose

$$(3.43) \quad \mathcal{H}^n(M_0 \cap ((E_0 \cap \mathbf{B}^n(\hat{a}, 4\varrho)) \times \mathbb{R})) \leq K\varrho^n$$

for all $\hat{a} \in \mathbb{R}^{n-1} \times \{0\}$. Then there exists a $g : [t_0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$(3.44) \quad M_t = \text{graph}(g(t, \cdot)) \quad \text{for all } t \in [t_0, T)$$

Proof. We may assume $\varrho = 1$. First note that by the comparison principle and (3.41) we have $M_t \subset \mathbb{R}^n \times \mathbf{B}^1(0, 1)$ for all $t \in [0, T)$. This also follows from Corollary 2.5 with $R \rightarrow \infty$. Choose $\sigma_0, \Lambda_0 = \Lambda_0(L)$ and $\lambda_0 = \lambda_0(L)$ according to Theorem 3.8. Set

$$\lambda_1^2 := \min\{\lambda_0^2, (2\Lambda_0)^{-1}\sigma_0\}, \quad \rho := C_0\lambda_1^{-2n}K$$

for some constant $C_0 \in (1, \infty)$ which will be chosen below.

Let $\hat{x} \in \mathbb{R}^n$ be arbitrary and set $A := \mathbf{B}^n(\hat{x}, 5\rho) \cap (\mathbb{Z}^{n-1} \times \{0\})$. Then $\#A \leq C\rho^{n-1}$ and $\mathbf{B}^n(\hat{x}, 4\rho) \cap E_0 \subset \bigcup_{\hat{a} \in A} \mathbf{B}^n(\hat{a}, 4)$. Hence by (3.43) we have

$$\mathcal{H}^n(M_0 \cap ((E_0 \cap \mathbf{B}^n(\hat{x}, 4\rho)) \times \mathbb{R})) \leq C\rho^{n-1}K \leq \lambda_1^{2n}\rho^n,$$

where we used $\lambda_1 \leq 1$ and chose C_0 large enough. Using Theorem 3.8 with $t_1 = 0, t_2 = \lambda_0^2\rho^2, x_0 = (\hat{x}, 0), \lambda = \lambda_1, h = 0$ and $D = \mathbf{B}^n(\hat{x}, 4\rho) \setminus E_0$ yields a $g_{\hat{x}} : \mathbf{B}^n(\hat{x}, \rho) \rightarrow \mathbb{R}$ with

$$M_{t_0} \cap \mathbf{C}(\hat{x}, \rho, \rho) = \text{graph}(g_{\hat{x}}).$$

Here we used $t_0 = \Lambda_3 K^2 = \Lambda_0 \lambda_1^2 \rho^2$ for $\Lambda_3 = C_0^2 \Lambda_0 \lambda_1^{2-4n}$. As $\rho \geq 1$ and $\hat{x} \in \mathbb{R}^n$ was arbitrary, we can combine the $g_{\hat{x}}$ to obtain a $g_{t_0} : \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$M_{t_0} = \text{graph}(g_{t_0}).$$

Now by [EH91, 5.1] we know there exists a unique graphical solution for all times, which establishes the result. □

Appendix A.

This appendix contains some details on the relation between height, tilt, curvature and graphical representation of a manifold M . In particular we show, how curvature bounds combined with a slab condition yield graphical representability. The main ingredient is the following result from Colding and Minicozzi.

Proposition A.1 ([CM11, 2.4]). *There exist $C \in (1, \infty)$ and $\alpha_0 \in (0, 1)$ such that the following holds: Let $\alpha \in (0, \alpha_0]$, $R \in (0, \infty)$ and let M be a submanifold of $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$ with $0 \in M$ and $\mathbf{T}(M, 0) = \mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}$. Suppose*

$$(A.1) \quad \partial M \cap \mathbf{B}(0, 2R) = \emptyset,$$

$$(A.2) \quad \sup_{x \in M \cap \mathbf{B}(0, 2R)} |\mathbf{A}(M, x)| \leq \alpha R^{-1}.$$

Then there exists a $g \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(0, 2R), \mathbb{R}^{\mathbf{k}})$ with $\text{graph}(g) \subset M$ and

$$(A.3) \quad g(0) = 0, \quad Dg(0) = 0,$$

$$(A.4) \quad \max\{R^{-1} \sup |g|, \sup |Dg|, R \sup |D^2g|\} \leq C\alpha.$$

Corollary A.2. *There exist $C \in (1, \infty)$ and $\alpha_0 \in (0, 1)$ such that the following holds: Let $\alpha \in (0, \alpha_0]$, $R \in (0, \infty)$, $L \in [1, \infty)$, $\hat{a} \in \mathbb{R}^{\mathbf{n}}$. Consider an $f \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(\hat{a}, 2R), \mathbb{R}^{\mathbf{k}})$ with $\sup |Df| \leq L$. Set $M := \text{graph}(f)$ and assume*

$$(A.5) \quad \sup_{x \in M} |\mathbf{A}(M, x)| \leq \alpha R^{-1}.$$

Set $a := (\hat{a}, f(\hat{a}))$. Consider $S \in \mathbf{SO}(\mathbf{n} + \mathbf{k})$ with $S[\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}] = \mathbf{T}(M, a)$. Then there exists a local parametrisation $g \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(0, R), \mathbb{R}^{\mathbf{k}})$ such that

$$(A.6) \quad M \cap (\mathbf{B}^{\mathbf{n}}(\hat{a}, (2L)^{-1}R) \times \mathbb{R}^{\mathbf{k}}) \subset S(\text{graph}(g)) + a \subset M.$$

Furthermore g satisfies

$$(A.7) \quad g(0) = 0, \quad Dg(0) = 0,$$

$$(A.8) \quad \max\{R^{-1} \sup |g| + \sup |Dg| + R \sup |D^2g|\} \leq C\alpha.$$

Proof. Let α_0 be from Proposition A.1. As $\alpha \leq \alpha_0$ we can apply Proposition A.1 to the manifold $S^{-1}[M - a]$ to obtain a function $g \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(0, R), \mathbb{R}^{\mathbf{k}})$, which satisfies (A.7), (A.8) and $\text{graph}(g) \subset S^{-1}[M - a]$.

Set $N := S[\text{graph}(g)] + a$. We already know $N \subset M$ and want to show $M \cap (\mathbf{B}^n(\hat{a}, (2L)^{-1}R) \times \mathbb{R}^k) \subset N$. It suffices to prove

$$(A.9) \quad M \cap (\mathbf{B}^n(\hat{a}, (2L)^{-1}R) \times \mathbb{R}^k) \cap \partial N = \emptyset.$$

Suppose this is false and there exists a point y_0 in this set. Then we find sequences $(\hat{w}_m)_{m \in \mathbb{N}}$ and $(\hat{y}_m)_{m \in \mathbb{N}}$ with $\lim_{m \rightarrow \infty} y_m = y_0$, $\lim_{m \rightarrow \infty} |\hat{w}_m| = R$ and such that for all $m \in \mathbb{N}$ we have $\hat{w}_m \in \mathbf{B}^n(0, R)$, $\hat{y}_m \in \mathbf{B}^n(\hat{a}, (2L)^{-1}R)$ and

$$S(\hat{w}_m, g(\hat{w}_m)) + a = (\hat{y}_m, f(\hat{y}_m)) =: y_m$$

But for $|\hat{w}_m|$ we can estimate

$$\begin{aligned} |\hat{w}_m|^2 &\leq |S(\hat{w}_m, g(\hat{w}_m))|^2 = |y_m - a|^2 = |\hat{y}_m - \hat{a}|^2 + |f(\hat{y}_m) - f(\hat{a})|^2 \\ &\leq (2L)^{-2}R^2 + L^2|\hat{y}_m - \hat{a}|^2 \leq (1 + L^2)(2L)^{-2}R^2 \leq \frac{R^2}{2} \end{aligned}$$

for all $m \in \mathbb{N}$. Here we used $a = (\hat{a}, f(\hat{a}))$, $\sup |Df| \leq L$ and $L \geq 1$. Thus we obtain a contradiction. This proves (A.9) which establishes the result. \square

We observe that, if for a family of submanifolds (M_t) the tangent space $\mathbf{T}(M_t, x)$ is never perpendicular to \mathbb{R}^n , then (M_t) consists of a constant number of sheets.

Lemma A.3. *Let $r, \Gamma_0 \in (0, \infty)$, $\kappa_0 \in [0, 1)$, $t_1 \in \mathbb{R}$, $t_2 \in (t_1, \infty)$. Let M be a submanifold of \mathbb{R}^{n+k} and consider $\Psi \in \mathcal{C}([t_1, t_2] \times M, \mathbb{R}^{n+k})$ such that $\Psi_t := \Psi(t, \cdot) \in \mathcal{C}^1(M, \mathbb{R}^{n+k})$ are embeddings for all $t \in [t_1, t_2)$. Set $M_t := \Psi_t[M]$. Suppose*

$$(A.10) \quad \|\mathbf{T}(M_t, x)_{\mathfrak{h}} - (\mathbb{R}^n \times \{0\}^k)_{\mathfrak{h}}\| \leq \kappa_0 < 1$$

for all $t \in [t_1, t_2)$ and all $x \in M_t \cap \mathbf{C}(0, r, \Gamma_0)$. Also suppose

$$(A.11) \quad \begin{aligned} \partial M_t \cap \mathbf{B}^n(0, r) \times \overline{\mathbf{B}^k(0, \Gamma_0)} &= \emptyset, \\ M_t \cap \mathbf{B}^n(0, r) \times \partial \mathbf{B}^k(0, \Gamma_0) &= \emptyset \end{aligned}$$

for all $t \in [t_1, t_2)$. Then there exists an $m_0 \in \mathbb{N} \cup \{0\}$ such that

$$(A.12) \quad \# \left\{ x \in M_t \cap \{\hat{y}\} \times \mathbf{B}^k(0, \Gamma_0) \right\} = m_0$$

for all $\hat{y} \in \mathbf{B}^n(0, \rho)$ and all $t \in [t_1, t_2)$.

Proof. Fix $t \in [t_1, t_2)$. For $\hat{x} \in \mathbf{B}^n(0, r)$ set

$$A(\hat{x}) := \left\{ a \in M_t \cap \{\hat{x}\} \times \mathbf{B}^k(0, \Gamma_0) \right\}, \quad N(\hat{x}) := \#A(\hat{x}).$$

Note that by (A.10) we can use the implicit function theorem to see that the points in $A(\hat{x})$ are discrete. Thus, by (A.11) and as $\mathbf{B}^k(0, \Gamma_0)$ is bounded we observe that $N(\hat{x})$ has to be finite.

Consider arbitrary $\hat{y} \in \mathbf{B}^n(0, r)$. We want to show there exists some $\delta = \delta(\hat{y})$ such that

$$(A.13) \quad N(\hat{x}) = N(\hat{y}) =: N$$

for all $\hat{x} \in \mathbf{B}^n(\hat{y}, \delta)$. There exist N different $a_i = (\hat{y}, h_i) \in A(\hat{y})$. In view of (A.10) we can use the implicit function theorem, to obtain a $\delta_1 \in (0, r)$ and $g_i \in \mathcal{C}^2(\mathbf{B}^n(\hat{x}_i, 2\delta_1), \mathbb{R}^k)$ with $g_i(\hat{y}) = h_i$, such that

$$M_t \cap \mathbf{B}(a_i, 2\delta_1) \subset \text{graph}(g_i) \subset M_t \cap \mathbf{C}(0, r, \Gamma_0)$$

for all $i = 1, \dots, N$. Note that δ_1 can be chosen such that the $\text{graph}(g_i)$ are disjoint. In particular this shows " \geq " in (A.13) for $\delta \leq \delta_1$. Let $W := (\{\hat{y}\} \times \mathbf{B}^k(0, \Gamma_0)) \setminus \bigcup_{i=1}^N \mathbf{B}(a_i, \delta_1)$, so $W \cap M_t = \emptyset$. Then by (A.11) and compactness of W there exists a $\delta \in (0, \delta_1]$ such that " \leq " holds in (A.13), which proves (A.13).

Set $m(t) := N(0)$. By (A.13) we see $\{N(\hat{y}) = m(t)\}$ is open and closed in $\mathbf{B}^n(0, r)$. As $\mathbf{B}^n(0, r)$ is connected this shows (A.12) for fixed t . By continuity in time, this establishes the result. \square

Combining this result with Proposition A.1 yields

Lemma A.4. *There exist $C \in (1, \infty)$ and $\alpha_1 \in (0, 1)$ such that the following holds: Let $K, r \in (0, \infty)$, $\xi \in (0, K]$ and let M be a submanifold of \mathbb{R}^{n+k} . Assume $K\xi \leq \alpha_1$*

$$(A.14) \quad M \cap \mathbf{C}(0, 2r, 2r) \subset \mathbf{C}(0, 2r, \xi^2 r),$$

$$(A.15) \quad \partial M \cap \mathbf{C}(0, 2r, 2r) = \emptyset.$$

Also suppose

$$(A.16) \quad \sup_{x \in M \cap \mathbf{C}(0, 2r, 2r)} |\mathbf{A}(M, x)| \leq K^2 r^{-1},$$

Then there exists an $m_0 \in \mathbb{N} \cup \{0\}$ and $g_i \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(0, r), \mathbb{R}^{\mathbf{k}})$ such that

$$(A.17) \quad M \cap \mathbf{C}(0, r, r) = \bigcup_{i=1}^{m_0} \text{graph}(g_i),$$

where the union on the right is disjoint and we set $\bigcup_{i=1}^0 G_i := \emptyset$. Moreover we have

$$(A.18) \quad \sup |g_i| \leq C\xi^2 r, \quad \sup |Dg_i| \leq CK\xi, \quad \sup |D^2g_i| \leq CK^2 r^{-2}$$

for all $i = 1, \dots, m_0$.

Proof. We may assume $r = 1$. Let $a \in M \cap \mathbf{C}(0, 1, 1)$ be arbitrary. We want to show

$$(A.19) \quad \left\| (\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}})_{\natural} - \mathbf{T}(M, a)_{\natural} \right\| \leq CK\xi.$$

Consider $S \in \mathbf{SO}(\mathbf{n} + \mathbf{k})$ with $S(\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}) = \mathbf{T}(M, a)$. Set $\tau_i := S(\mathbf{e}_i)$, $i \in \{1, \dots, \mathbf{n}\}$. For ξ_0 small enough we can apply Proposition A.1 with $R := (2K)^{-1}\xi$ and $\alpha = K\xi$ to the manifold $S^{-1}[M - a]$, to obtain a function $g \in \mathcal{C}^2(\mathbf{B}^{\mathbf{n}}(0, 2R), \mathbb{R}^{\mathbf{k}})$ with $g(0) = 0$, $\sup |g| \leq C\xi^2$ and $S[\text{graph}(g) + a] \subset M$. Here we used $K\xi \leq \alpha_1$ and chose $\alpha_1 \leq \alpha_0$.

For $i \in \{1, \dots, \mathbf{n}\}$ define $v_i \in \text{graph}(g)$ and $w_i \in \mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}$ by

$$v_i := (R\hat{\mathbf{e}}_i, g(R\hat{\mathbf{e}}_i)), \quad w_i := Sv_i - \left(\{0\}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{k}} \right)_{\natural} (Sv_i)$$

Using $S(\mathbf{e}_i) = \tau_i$ and (A.14) we can estimate

$$|R\tau_i - w_i| \leq |S(0, g(R\hat{\mathbf{e}}_i))| + \left| \left(\{0\}^{\mathbf{n}} \times \mathbb{R}^{\mathbf{k}} \right)_{\natural} (Sv_i) \right| \leq C\xi^2.$$

Now set $\tilde{w}_i := R^{-1}w_i$ for $i \in \{1, \dots, \mathbf{n}\}$. As $R = (2K)^{-1}\xi$, this yields $|\tau_i - \tilde{w}_i| \leq CK\xi$. The (\tilde{w}_i) lie in $\mathbb{R}^{\mathbf{n}} \times \{0\}^{\mathbf{k}}$ and the (τ_i) form an orthonormal basis of $\mathbf{T}(M, 0)$, so this establishes (A.19) (see [All72, 8.9(5)]).

In view of (A.19) and $CK\xi \leq C\alpha_1 < 1$, we can use Lemma A.3 with $M_t \equiv M$ and $\Gamma_0 = r$ to obtain the g_i . For the bounds in (A.18) use (A.14), (A.16) and (A.19) combined with (1.3) and (1.4). \square

We end the appendix by giving an alternative condition to (2.13) or (3.25).

Lemma A.5. *There exists a $C \in (1, \infty)$ such that the following holds: Let $p \in (\mathbf{n}, \infty)$, $\varrho, K \in (0, \infty)$, $\gamma, \delta \in (0, 1]$, $y_0 \in \mathbb{R}^{\mathbf{n}+\mathbf{k}}$ and let M be a submanifold of $\mathbb{R}^{\mathbf{n}+\mathbf{k}}$. Assume $\partial M \cap \mathbf{B}(y_0, 2\varrho) = \emptyset$,*

$$(A.20) \quad M \cap \mathbf{B}(y_0, 2\varrho) \subset \mathbf{C}(y_0, 2\varrho, \gamma\varrho)$$

and

$$(A.21) \quad \varrho^{-\mathbf{n}} \mathcal{H}^{\mathbf{n}}(M \cap \mathbf{B}(y_0, 2\varrho)) + \varrho^{p-\mathbf{n}} \int_{\mathbf{B}(y_0, 2\varrho)} |\mathbf{H}(M, x)|^p d\mu_M(x) \leq K.$$

Then $\mathcal{H}^{\mathbf{n}}(M \cap \mathbf{C}(y_0, \delta\varrho, \varrho)) \leq C2^p(p - \mathbf{n})^{-1}K(\gamma + \delta)^{\mathbf{k}}\delta^{\mathbf{n}-\mathbf{k}}\varrho^{\mathbf{n}}$.

Proof. We may assume $y_0 = 0$ and $\varrho = 2$. Set $A := \mathbf{C}(0, 2\delta, 2\gamma) \cap \delta\mathbb{Z}^{\mathbf{n}+\mathbf{k}}$. Let $a \in A$, then $\mathbf{B}(a, 1) \subset \mathbf{B}(0, 4)$. By (A.21) and Simon’s monotonicity formula [Sim83, 4.3.2] with $R = 1$, $r = 2\delta$ and $\Gamma^p = K$ we can estimate

$$(2\delta)^{-\mathbf{n}} \mu_M(\mathbf{B}(a, 2\delta)) \leq \left((\mu_M(\mathbf{B}(a, 1)))^{\frac{1}{p}} + \frac{K^{\frac{1}{p}}(1 - \delta^{\frac{p-\mathbf{n}}{p}})}{p - \mathbf{n}} \right)^p \leq \frac{2^p K}{p - \mathbf{n}}.$$

By assumption (A.20) we have $M \cap \mathbf{C}(0, \delta, 2) \subset \bigcup_{a \in A} \mathbf{B}(a, 2\delta)$ and also $\#A \leq C(\gamma\delta^{-1} + 1)^{\mathbf{k}}$. This establishes the result. □

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