

A strong Harnack inequality for graphs

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We introduce the curvature (and the flow curvature) for graphs, which allow us to prove several Harnack inequalities for general graphs (which are not necessarily regular). We first show that for a graph with curvature κ , the combinatorial eigenfunction f associated with eigenvalue λ satisfies, for any edge $\{x, y\}$,

$$|f(x) - f(y)|^2 \leq (8\lambda + 4\kappa)d_{max}$$

where $\max_x |f(x)| = \max_x f(x) = 1$ and d_{max} denotes the maximum degree. The above inequality is used to prove a strengthened inequality which implies that the ‘stretches’ of edges determined by eigenfunctions are particularly ‘small’ (in terms of the associated eigenvalue and curvature) near the maximum point. Namely, for a graph with curvature κ , the combinatorial eigenfunction f associated with eigenvalue λ satisfies, for any edge $\{x, y\}$,

$$|f(x) - f(y)|^2 \leq \frac{c_1\lambda + c_2\kappa}{\beta - 1} (\beta - f(x))^2$$

where $c_5 \geq \beta \geq 1 + c_3\lambda + c_4\kappa$ and c 's are constants depending only on the maximum degree of the graph.

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1. Introduction

In spectral graph theory, the main subjects of study are eigenvalues and eigenfunctions. Compared with eigenvalues, the treatments of eigenfunctions are relatively scarce. One of the main tools for dealing with eigenfunctions is the Harnack inequality. In [5], it was shown that for a d -regular homogeneous graph, the eigenfunction f associated with eigenvalue λ satisfies, for a vertex x and a neighbor y of x ,

$$(1) \quad |f(x) - f(y)|^2 \leq 8\lambda d \max_z |f(z)|^2$$

provided that the graph is so called “*Ricci flat*”.

In this paper, we consider general graphs which are not necessarily regular. We define “frames” on graphs which are used to define the notion of the Ricci curvature for general graphs. Different from several existing notions of Ricci curvatures for graphs, the definitions of Ricci curvatures for graphs here are crucial for proving the strong version of the Harnack inequalities. It is desirable to show that the stretches $|f(x) - f(y)|$ of any given edge $\{x, y\}$, where f is a combinatorial eigenfunction, can be upper bounded by a linear combination of the associated eigenvalue and the curvature. In particular, for edges incident to vertices close to the vertex which achieving the maximum value of f in absolute values, the bound can be shown to be even tighter. An example of a cycle C_n on n vertices is useful for illustrating such effects. Namely, for the eigenfunction $f(j) = \cos(2\pi j/n)$ associated with eigenvalue $\lambda = 1 - \cos(2\pi/n)$, the stretch of any edge is bounded above by $c\sqrt{\lambda}$ which is of order $1/n$ while at the maximum point $f(0) = 1$, the stretch of any incident edge is of order $1/n^2$. Our goal is to establish general Harnack inequalities of a similar flavor for general graphs.

We will first establish the following Harnack inequality. For a graph with curvature κ , the combinatorial eigenfunction f associated with eigenvalue λ

satisfies, for any edge $\{x, y\}$,

$$(2) \quad |f(x) - f(y)|^2 \leq (8\lambda + 4\kappa)d_{max}$$

where $\max_x |f(x)| = \max_x f(x) = 1$ and d_{max} denotes the maximum degree.

The above Harnack inequality will then be used to derive a sharper inequality for edges close to the maximum point. We will show that for every vertex x and a neighbor y of x ,

$$(3) \quad |f(x) - f(y)|^2 \leq \frac{c_1\lambda + c_2\kappa}{\beta - 1} (\beta - f(x))^2$$

where $c_5 \geq \beta \geq 1 + c_3\lambda + c_4\kappa$ and c 's are constants depending only on the maximum degree.

We note that (3) provides a much sharper upper bound than (2) when $f(x)$ is close to 1. If we choose $\beta = 2$, then (3) essentially implies (2) albeit with different constant factors. We remark that (3) is essentially a discrete version of the gradient estimates given by Li and Yau [8] in the corresponding problem for compact Riemannian manifolds.

The paper is organized as follows: In Section 2, we examine *Ricci flat* graphs which provide the motivation for defining the Ricci curvature for general graphs. In Section 3, we define *frames* which will then be used to define two notions of Ricci curvature for graphs. We briefly survey several previously known notions of Ricci curvature for graphs and comparisons are given among different notions of curvature. In particular, the *curvature* and *flow curvature* for graphs that we define here are motivated and required in the proofs of the main theorems for establishing the Harnack inequalities. In Section 4 we deal with the tedious but essential ways for bounding lower order terms which usually can easily be eliminated in continuous settings by taking limits but in the discrete cases, various estimates depend on controlling the lower order terms which need to be taken account carefully (and delicately). In Section 5, we give the main proof for the strong Harnack inequality for Ricci flat homogeneous graphs upon which the subsequent proof for general graphs is based. In Section 6, we give the Harnack inequality for general graphs and the proof for the strong Harnack inequality is given in Section 7. In Section 8, we illustrate some consequences of Harnack and strong Harnack inequalities by deducing several eigenvalue-diameter inequalities.

2. Laplace operator and Ricci flat homogenous graphs

Let G denote a graph with vertex set V and edge set E . For a vertex x in V , the degree d_x is the number of neighbors y such that $\{x, y\}$ is in E . We write $x \sim y$ if $\{x, y\}$ is an edge in E .

The Laplace operator Δ acting on $\{f : V \rightarrow \mathbb{R}\}$ is defined by

$$\Delta f(x) = \frac{1}{d_x} \sum_{\substack{y \\ y \sim x}} (f(x) - f(y)).$$

An eigenfunction f with associated eigenvalue λ satisfies

$$(4) \quad \Delta f(x) = \lambda f(x).$$

A homogeneous graph Γ refers to a graph whose vertex set consisting of elements or cosets of a group \mathcal{G} . The edges in Γ is determined by a subset S of \mathcal{G} such that $x \sim y$ if and only if $y = gx$ where g is in S . We say that the edge generating set S is symmetric if $a \in S$ if and only if $a^{-1} \in S$. When S is symmetric, then Γ is an undirected graph. It is well known that the eigenvalues of a homogeneous graph can be explicitly expressed (see [3]).

Lemma 1. *In a homogeneous graph Γ determined by a symmetric edge generating set S of a group \mathcal{G} , the eigenvalues λ_i of the Laplace operator can be written as*

$$\lambda_i = 1 - \frac{1}{|S|} \sum_{a \in S} \rho_i(a)$$

where ρ_i is a character of \mathcal{G} .

A homogeneous graph Γ with edge generating set S is said to be *Ricci flat* if for every $x \in V$ and for $a \in S$, we have

$$(5) \quad \{abx : b \in S\} = \{bax : b \in S\}$$

For $f, g : V \rightarrow \mathbb{R}$ and $a \in S$, we define

$$(6) \quad \nabla_a f(x) = f(x) - f(ax)$$

$$(7) \quad \int_a g(x) = \frac{1}{|S|} \sum_{a \in S} g(ax).$$

Therefore we can write $\Delta f = \int_a \nabla_a f(x)$.

Lemma 2. *In a Ricci flat homogeneous graph Γ with a symmetric edge generating set S , for any function $f : V \rightarrow \mathbb{R}$, we have, for a fixed $a \in S$ and any vertex x ,*

$$\int_b \nabla_a \nabla_b f = \int_b \nabla_b \nabla_a f.$$

Proof. For a fixed $a \in S$, we consider

$$\begin{aligned} \int_b \nabla_b \nabla_a f(x) &= \int_b \left(f(x) - f(ax) - (f(bx) - f(abx)) \right) \\ &= \int_b \left(f(x) - f(bx) - (f(ax) - f(abx)) \right) \\ &= \int_b \left(f(x) - f(bx) - (f(ax) - f(bax)) \right) \quad \text{using (5)} \\ &= \int_b \nabla_a \nabla_b f(x). \end{aligned}$$

□

Lemma 3. *In a homogeneous graph Γ with a symmetric edge generating set S , for an eigenfunction $f : V \rightarrow \mathbb{R}$ with associated eigenvalue λ , we define*

$$\begin{aligned} F(x) &= \int_a |\nabla_a f(x)|^2 \\ (8) \quad &= \frac{1}{|S|} \sum_{a \in S} (f(x) - f(ax))^2. \end{aligned}$$

Then, for a fixed $a \in S$ and a vertex x , we have

$$(9) \quad \Delta F(x) = 2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2Q(x)$$

where

$$(10) \quad Q(x) = \int_b \int_a \nabla_b f(x) (\nabla_a \nabla_b f(x) - \nabla_b \nabla_a f(x)).$$

Proof. We consider

$$\begin{aligned} |\nabla_b f(x)|^2 - |\nabla_b f(ax)|^2 &= (f(x) - f(bx))^2 - (f(ax) - f(bax))^2 \\ &= 2(f(x) - f(bx))(f(x) - f(bx) - f(ax) + f(bax)) \\ &\quad - (f(x) - f(bx) - f(ax) + f(bax))^2 \\ &= 2\nabla_b f(x) \cdot \nabla_a \nabla_b f(x) - (\nabla_a \nabla_b f(x))^2 \end{aligned}$$

Since $F(x) = \int_b |\nabla_b f(x)|^2$, we have

$$\begin{aligned} \nabla_a F(x) &= \int_b |\nabla_b f(x)|^2 - |\nabla_b f(ax)|^2 \\ &= \int_b (2\nabla_b f(x) \cdot \nabla_a \nabla_b f(x)) - \int_b (\nabla_a \nabla_b f(x))^2 \end{aligned}$$

Therefore we have

$$\begin{aligned} \Delta F(x) &= \int_a \nabla_a F(x) \\ &= \int_a \int_b (2\nabla_b f(x) \cdot \nabla_a \nabla_b f(x) - (\nabla_a \nabla_b f(x))^2) \\ &= \int_b (2\nabla_b f(x) \left(\int_a \nabla_a \nabla_b f(x) \right)) - \int_a \int_b (\nabla_a \nabla_b f(x))^2. \end{aligned}$$

From the definition of Q in (10), we have

$$\begin{aligned} \Delta F(x) &= \int_b 2\nabla_b f(x) \nabla_b \left(\int_a \nabla_a f(x) \right) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2Q(x) \\ &= \int_b 2\nabla_b f(x) \nabla_b (\lambda f(x)) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2Q(x) \\ &= 2\lambda \int_b |\nabla_b f(x)|^2 - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2Q(x) \\ &= 2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2Q(x) \end{aligned}$$

as desired. □

Another useful identity is the following:

Lemma 4.

$$(11) \quad \Delta f^2(x) = 2\lambda f^2(x) - F(x).$$

Proof. We note that

$$\begin{aligned} \Delta f^2(x) &= \int_a \nabla_a f^2(x) \\ &= \int_a 2f(x) \nabla_a f(x) - \int_a |\nabla_a f(x)|^2 \\ &= 2\lambda f^2(x) - F(x). \end{aligned}$$

□

3. The Ricci curvature for a general graph

Our strategy for defining the curvature for a general graph is based on Lemma 2 and Lemma 3. We will define *frames* on a general graph so that the methods for homogeneous graphs can be applied. In addition we will use the function $Q(x)$ as defined in (10) to define the curvature.

We consider a general weighted graph Γ and the weight of an edge $\{u, v\}$ in $E(\Gamma)$ is denoted by $w_{u,v}$. For a vertex v , the set of neighbors of v is denoted by $N(v)$.

$$N(v) = \{u \in V(\Gamma) : \{u, v\} \in E(\Gamma)\}.$$

The degree of v is denoted by d_v , which is the sum of edge weights $w_{u,v}$ over all u .

3.1. Defining frames

Before we proceed to define the Ricci curvature for a general graph Γ , we will first define *frames* on a graph G . Let \mathcal{A} be a set and μ be a probability distribution on \mathcal{A} . We say a graph Γ has an (\mathcal{A}, μ) -*gravitation* via $\varphi = \{\varphi_v : v \in V(\Gamma)\}$ if $\varphi_v : \mathcal{A} \rightarrow N(v)$ is an onto mapping satisfying the property for each vertex u in $N(v)$, the pre-image of u in φ_v , denoted by $\varphi_v^{-1}(u)$, satisfies

$$(12) \quad \mu(\varphi_v^{-1}(u)) = \frac{w_{u,v}}{d_v}.$$

We call \mathcal{A} a *frame* of the gravitation and φ a set of *frame maps*. For example, for the grid graph as the cartesian product of two cycles, we can take the frame to be $\mathcal{A} = \{\text{east, south, west, north}\}$ and each of the four edges incident to a vertex v is assigned a direction accordingly. For a homogeneous graph with edge generating set S , we can take $\mathcal{A} = S$ and, for a vertex x , we define $\varphi_x(a) = ax$.

In general, \mathcal{A} is not necessarily finite. For example, we can choose \mathcal{A} to be the unit interval $[0, 1]$ for the graph which consists of two cycles intersecting at one vertex, as illustrated in Figure 1. We can define φ so that, for $i \neq 1, 20, 21, 22, 41$, $\varphi_{v_i}^{-1}(v_{i+1}) = [0, 1/2)$ and $\varphi_{v_i}^{-1}(v_{i-1}) = [1/2, 1]$. In addition, we set $\varphi_{v_1}^{-1}(v_2) = [0, 1/2) = \varphi_{v_{20}}^{-1}(v_{21}) = \varphi_{v_{22}}^{-1}(v_{23}) = \varphi_{v_{41}}^{-1}(v_{21})$, $\varphi_{v_1}^{-1}(v_{21}) = [1/2, 1] = \varphi_{v_{20}}^{-1}(v_{19}) = \varphi_{v_{22}}^{-1}(v_{21}) = \varphi_{v_{41}}^{-1}(v_{40})$, and $\varphi_{v_{21}}^{-1}(v_1) = [0, 1/4)$, $\varphi_{v_{21}}^{-1}(v_{20}) = [1/4, 1/2)$, $\varphi_{v_{21}}^{-1}(v_{22}) = [1/2, 3/4]$, and $\varphi_{v_{21}}^{-1}(v_{41}) = [3/4, 1]$.

For a graph Γ with an (\mathcal{A}, μ) -gravitation via φ , we can define the gradient $\nabla_a f$ for $f : V(\Gamma) \rightarrow \mathbb{R}$ as follows: For $v \in V(\Gamma)$,

$$\nabla_a f(v) = f(v) - f(\varphi_v(a)).$$

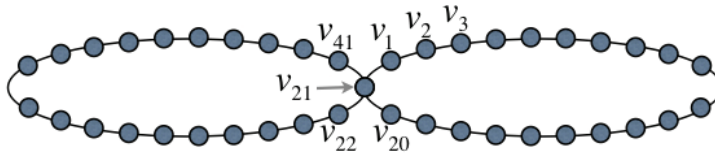


Figure 1: A graph which consists of two cycles intersecting at one vertex.

For $g : \mathcal{A} \rightarrow \mathbb{R}$, we write

$$\int_a g(a) = \int_{a \in \mathcal{A}} g(a) \mu(a).$$

The Laplace operator Δ for a general graph can be rewritten as follows:

$$\begin{aligned} \Delta f(x) &= \int_a \nabla_a f(x) \\ &= \sum_{y \sim x} (f(x) - f(y)) \int_{\varphi_x(a)=y} \mu(a) \\ &= \frac{1}{d_x} \sum_{y \sim x} (f(x) - f(y)) w_{x,y} \end{aligned}$$

We will consider

$$\begin{aligned} (13) \quad F(x) &= \int_a |\nabla_a f(x)|^2 \\ &= \sum_{y \sim x} (f(x) - f(y))^2 \int_{\varphi_x(a)=y} \mu(a) \\ &= \frac{1}{d_x} \sum_{y \sim x} (f(x) - f(y))^2 w_{x,y}. \end{aligned}$$

Therefore for two adjacent vertices x and y , we have

$$(14) \quad |f(x) - f(y)|^2 \leq \frac{d_x}{w_{x,y}} F(x) \leq d_{max} F(x)$$

where

$$(15) \quad d_{max} = \max_{x,y} \frac{d_x}{w_{x,y}}.$$

For a simple graph, d_{max} is just the maximum degree. We will call d_{max} the maximum degree for general weighted graphs as well.

A graph with an (\mathcal{A}, μ) -gravitation via φ , is said to be Ricci flat if for every vertex v , we have

$$(16) \quad \int_b \nabla_a \nabla_b f(v) = \int_b \nabla_b \nabla_a f(v).$$

From the graph in Figure 1 and the set of framing maps as defined above, the graph is almost Ricci flat in the sense that for a and for v_i where $i \neq 1, 20, 21, 22, 41$, equation (16) holds. However, for $a \in [1/2, 3/4]$ and $i = 1$, we have

$$(17) \quad \begin{aligned} & \int_b \nabla_a \nabla_b f(v_1) - \int_b \nabla_b \nabla_a f(v_1) \\ &= \frac{1}{4}f(v_1) + \frac{1}{4}f(v_{20}) - \frac{1}{4}f(v_{22}) + \frac{1}{4}f(v_{41}). \end{aligned}$$

3.2. Defining two notions of curvature

We say that κ is the *curvature* of a graph if κ is the least value such that

$$(18) \quad Q(x) = \int_a \nabla_a f(x) \int_b \left(\nabla_b \nabla_a f(x) - \nabla_a \nabla_b f(x) \right) \leq \kappa \int_a |\nabla_a f(x)|^2$$

holds for all x and eigenfunctions $f : V \rightarrow \mathbb{R}$. We remark that this definition could be defined for all functions f , although in later proofs only eigenfunctions are required.

We note that κ can be positive or negative. Sometimes we write $\kappa = -K_1$ where K_1 is consistent with the notion of Ricci curvature in spectral geometry. We remark that we mostly are dealing with curvature K_1 negative (i.e., the discrete version of Ricci curvature of the negative type).

A direct consequence of the definition in (18) is the following fact which will be useful later:

Lemma 5. *Let Γ be a graph with curvature κ and frames \mathcal{A} and let $f : V \rightarrow \mathbb{R}$ denote an eigenfunction with associated eigenvalue λ . Then for a fixed $a \in S$ and a vertex x , the function F , as defined in (13), satisfies*

$$(19) \quad \Delta F(x) \leq 2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa F(x).$$

Proof. The proof is quite similar to that of Lemma 3 and will be omitted.

For two measures f, g on V , the transportation distance $\text{tdist}(f, g)$ of f and g is defined by

$$(20) \quad \text{tdist}(f, g) = \inf_{\phi} \int_z \text{dist}(f(z), g(\phi(z))) dz$$

where the infimum ranges over all measure preserving maps $\phi : [0, 1] \rightarrow [0, 1]$.

We say a graph Γ has *flow-curvature* κ' if there exists an (\mathcal{A}, μ) -gravitation with frame maps φ such that for every vertex $v \in V(\Gamma)$ and $a \in \mathcal{A}$, we have

$$(21) \quad \frac{1}{4} \int_b \text{dist}(\varphi_a \varphi_b(v), \varphi_b \varphi_a(v)) \leq \kappa'.$$

where dist denotes the graph distance.

Remark 1. $0 \leq \kappa' < 1$ since

$$\begin{aligned} \text{dist}(\varphi_a \varphi_b(v), \varphi_b \varphi_a(v)) &\leq \text{dist}(\varphi_a \varphi_b(v), v) + \text{dist}(\varphi_b \varphi_a(v), v) \\ &= 2 + 2 = 4. \end{aligned}$$

Remark 2. If a graph is Ricci-flat, then it follows that $\kappa = \kappa' = 0$.

Remark 3. For the graph in Figure 1, let us define

$$t_i = \text{tdist}(\varphi_a \varphi_b(v_i), \varphi_b \varphi_a(v_i))$$

for $i = 1, 2, \dots, 41$. It is easy to check that for any $a \in [0, 1/4]$, we have $t_i = 0$ for $i \neq 1, 20, 21, 22, 41$, $t_1 = t_{20} = t_{21} = t_{22} = t_{41} = 2$. Therefore the curvature of the graph is $\kappa' = 1/2$.

Remark 4. For a special function such as an eigenfunction $g(v_j) = \sin(2\pi j/21)$ for $j = 1, \dots, 41$, it is easy to check that g satisfies (21) for $\kappa' = 1/4$.

A function $f : V(\Gamma) \rightarrow \mathbb{R}$ is said to be ϵ -Lipschitz, if for any edge $\{x, y\}$ in $E(\Gamma)$, we have

$$(22) \quad |f(x) - f(y)| \leq \epsilon$$

From the definition of the curvature, we can have the following:

Lemma 6. *Suppose a graph Γ has a flow-curvature κ' . Then an ϵ -Lipschitz function $f : V \rightarrow \mathbb{R}$ satisfies*

$$(23) \quad \left| \int_b (\nabla_a \nabla_b f(v) - \nabla_b \nabla_a f(v)) \right| \leq \kappa' \epsilon$$

$$(24) \quad \text{and } Q \leq \kappa' \epsilon^2.$$

Proof. From the definition of the flow-curvature and ϵ -Lipschitz, we have

$$\begin{aligned} & \left| \int_b (\nabla_a \nabla_b f(v) - \nabla_b \nabla_a f(v)) \right| \\ = & \left| \int_b (f(v) - f(\varphi_b(v)) - (f(\varphi_a(v)) - f(\varphi_b \varphi_a(v))) \right. \\ & \left. - \int_b (f(v) - f(\varphi_a(v)) - (f(\varphi_b(v)) - f(\varphi_a \varphi_b(v))) \right| \\ \leq & \int_b |f(\varphi_b \varphi_a(v)) - f(\varphi_a \varphi_b(v))| \\ \leq & \int_b \text{dist}(\varphi_b \varphi_a(v), \varphi_a \varphi_b(v)) \epsilon \\ \leq & \kappa' \epsilon \end{aligned}$$

as desired. □

Remark 5. *From Lemma 6, we have*

$$\kappa \leq \kappa' \epsilon^2$$

if the eigenfunctions are ϵ -Lipschitz.

3.3. Comparing several notions of curvature for graphs

In the recent literature, there are several definitions of the curvature for a graph. One notion of the graph curvature is a discrete version of the Ricci curvature defined by Bakry and Emery (see [6] and also in [7]). The original definition of the Bakry-Emery curvature is somewhat complicated. A simplified version of the discrete Bakry-Emery curvature in [6] can be stated as follows: For all functions $f : V(\Gamma) \rightarrow \mathbb{R}$ and for all $v \in V(\Gamma)$, the graph is said to have curvature $-K$ and the dimension m if for all vertices,

we have

$$(25) \quad \begin{aligned} & \frac{1}{4} \left(\Delta F(x) + 2\Delta(f(x)\Delta f(x)) - 2f(x)\Delta\Delta f(x) - 2(\Delta f(x))^2 \right) \\ & \leq -\frac{(\Delta f(x))^2}{m} - \frac{K}{2}F(x) \end{aligned}$$

where $F(x)$ is as defined in (13).

Remark 6. *The above definition is quite strong. For example, in the graph in Figure 1, for the function f with $f(v_1) = f(v_{20}) = f(v_{21}) = f(v_{22})$ and $f(v_{41}) = -f(v_1) = \alpha$ for any $\alpha > 0$, then for $v = v_1$ we have $\nabla_a f(v) = 0$ so that the inequalities (25) implies that $K \leq 0$.*

Remark 7. *By using Lemmas 3 and 4, we can rewrite (25) as follows:*

$$\begin{aligned} & \frac{1}{4} \left(\Delta F(x) + 2\Delta(f(x)\Delta f(x)) - 2f(x)\Delta\Delta f(x) - 2(\Delta f(x))^2 \right) \\ & = \frac{1}{4} \left(-2 \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 4Q(x) \right). \end{aligned}$$

Thus, the Bakry-Emery curvature is just

$$Q(x) - \frac{1}{2} \int_a \int_b (\nabla_a \nabla_b f(x))^2 \leq -\frac{(\Delta f(x))^2}{m} - \frac{K}{2}F(x).$$

Hence, we see that the Bakry-Emery curvature $-K$ is quite close to 2κ as defined in (18).

Another notion of curvature of a graph was defined by Ollivier [10]. For two vertices x and y in a graph, the probability distributions on the neighborhoods $N(x)$ and $N(y)$ are denoted by $p_x = P(x, \cdot)$ and $p_y = P(y, \cdot)$, respectively, where $P = D^{-1}A$ is the transition probability matrix. The coarse Ricci curvature is defined in [10] by

$$(26) \quad \kappa_c(x, y) = 1 - \frac{\text{tdist}(p_x, p_y)}{\text{dist}(x, y)}$$

where tdist denotes the transportation distance between two measures as defined in (20), (also see [10]).

Remark 8. The definition of curvature in (21) can be interpreted as having the frame map ϕ_a serving a similar role as the transportation distance.

Suppose we assume that the frame map ϕ_a achieves the transportation distance for every vertex v between the measure p_v (on the neighborhood of v) and p_u where $u = \phi_a(v)$ is a neighbor of u . Under this assumption, we have

$$\begin{aligned} \int_b \text{dist}(\varphi_a\varphi_b(v), \varphi_b\varphi_a(v)) &= \text{tdist}(\mu_v, \mu_u) - 1 \\ &= -\kappa_c(u, v). \end{aligned}$$

4. Useful facts for graphs with frames

In a graph G , suppose the combinatorial eigenfunction f is associated with eigenvalue λ , satisfying $\max_z |f(z)| = \max_z f(z) = 1$. For some positive $\beta > 1$, we define, for each vertex x ,

$$(27) \quad \mathcal{F}(x) = \frac{F(x)}{(\beta - f(x))^2} = \frac{\int_a |\nabla_a f(x)|^2}{(\beta - f(x))^2}.$$

In this section we assume a graph G has an (\mathcal{A}, μ) -gravitation and a function f , defined on the vertex set of G , is ϵ -Lipschitz. Different from the usual methods in differential geometry, the ‘lower-ordered’ terms do not ‘vanish and have to be dealt with. Here we use several useful identities.

Lemma 7. *For*

$$X = \nabla_a \left(\frac{1}{(\beta - f(x))^2} \right) = \frac{1}{(\beta - f(x))^2} - \frac{1}{(\beta - f(ax))^2},$$

we have

$$(28) \quad X \left(1 + \frac{\nabla_a f(x)}{\beta - f(x)} \right)^2 = \frac{2\nabla_a f(x)}{(\beta - f(x))^3} + \frac{|\nabla_a f(x)|^2}{(\beta - f(x))^4}.$$

Proof.

$$\begin{aligned} X &= \frac{1}{(\beta - f(x))^2} - \frac{1}{(\beta - f(ax))^2} \\ &= \frac{(f(x) - f(ax))(2\beta - f(x) - f(ax))}{(\beta - f(x))^2(\beta - f(ax))^2} \\ &= \frac{(\nabla_a f(x))(2(\beta - f(x)) + f(x) - f(ax))}{(\beta - f(x))^2(\beta - f(ax))^2} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{2\nabla_a f(x)}{\beta - f(x)} + \frac{|\nabla_a f(x)|^2}{(\beta - f(x))^2} \right) \frac{1}{(\beta - f(ax))^2} \\
&= \left(\frac{2\nabla_a f(x)}{\beta - f(x)} + \frac{|\nabla_a f(x)|^2}{(\beta - f(x))^2} \right) \left(\frac{1}{(\beta - f(x))^2} - X \right).
\end{aligned}$$

Therefore (28) is proved. □

We set

$$(29) \quad Y = \frac{2\nabla_a f(x)}{\beta - f(x)} + \frac{|\nabla_a f(x)|^2}{(\beta - f(x))^2}.$$

Then (28) can be rewritten as:

$$(30) \quad X(1 + Y) = \frac{Y}{(\beta - f(x))^2}.$$

Lemma 8.

$$\begin{aligned}
(31) \quad &\nabla_a \mathcal{F}(x) \left(1 + \frac{\nabla_a f(x)}{\beta - f(x)} \right)^2 \\
&= \frac{\nabla_a F(x)}{(\beta - f(x))^2} + \frac{2F(x)\nabla_a f(x)}{(\beta - f(x))^3} + \frac{F(x)(\nabla_a f(x))^2}{(\beta - f(x))^4} \\
&= \frac{\nabla_a F(x) + F(x)Y}{(\beta - f(x))^2}.
\end{aligned}$$

Proof. We follow the notation in Lemma 7 and consider

$$\begin{aligned}
(32) \quad \nabla_a \mathcal{F}(x) &= \left(\frac{F(x)}{(\beta - f(x))^2} - \frac{F(ax)}{(\beta - f(ax))^2} \right) \\
&= \frac{F(x) - F(ax)}{(\beta - f(x))^2} + F(ax) \left(\frac{1}{(\beta - f(x))^2} - \frac{1}{(\beta - f(ax))^2} \right) \\
&= \frac{\nabla_a F(x)}{(\beta - f(x))^2} + (F(x) - \nabla_a F(x))X \\
&= \frac{\nabla_a F(x)}{(\beta - f(x))^2} + (F(x) - \nabla_a F(x)) \left(\frac{Y}{(\beta - f(x))^2} - XY \right)
\end{aligned}$$

by using Lemma 7. From (35), we have

$$\begin{aligned} & \nabla_a \mathcal{F}(x)(1+Y) \\ &= \frac{\nabla_a F(x)(1+Y)}{(\beta-f(x))^2} + (F(x) - \nabla_a F(x)) \left(\frac{Y(1+Y)}{(\beta-f(x))^2} - XY(1+Y) \right) \\ &= \frac{\nabla_a F(x)(1+Y)}{(\beta-f(x))^2} + (F(x) - \nabla_a F(x)) \left(\frac{Y(1+Y) - Y^2}{(\beta-f(x))^2} \right) \\ &= \frac{\nabla_a F(x) + F(x)Y}{(\beta-f(x))^2} \end{aligned}$$

and (31) follows. □

Lemma 9.

$$\begin{aligned} & \nabla_a \mathcal{F}(x) \left(1 + \frac{\nabla_a f(x)}{\beta-f(x)} \right)^2 \left(1 - \frac{\nabla_a f(x)}{\beta-f(x)} \right) \\ &= \frac{\nabla_a F(x)}{(\beta-f(x))^2} + \frac{2F(x)\nabla_a f(x)}{(\beta-f(x))^3} - \frac{\nabla_a F(x)\nabla_a f(x)}{(\beta-f(x))^3} \\ & \quad - \frac{F(x)|\nabla_a f(x)|^2}{(\beta-f(x))^4} - \frac{F(x)(\nabla_a f(x))^3}{(\beta-f(x))^5} \end{aligned}$$

Proof. From the definition of Y in (29), we note that

$$1 + Y = \left(1 + \frac{\nabla_a f(z)}{\beta-f(z)} \right)^2 \geq 0.$$

From (31) and (35), we have

$$\begin{aligned} & \nabla_a \mathcal{F}(x) \left(1 + \frac{\nabla_a f(x)}{\beta-f(x)} \right)^2 \left(1 - \frac{\nabla_a f(x)}{\beta-f(x)} \right) \\ &= \nabla_a \mathcal{F}(x)(1+Y) \left(1 - \frac{\nabla_a f(x)}{\beta-f(x)} \right) \\ &= \left(\frac{\nabla_a F(x)}{(\beta-f(x))^2} + \frac{F(x)Y}{(\beta-f(x))^2} \right) \left(1 - \frac{\nabla_a f(x)}{\beta-f(x)} \right) \\ &= \frac{\nabla_a F(x)}{(\beta-f(x))^2} + \frac{2F(x)\nabla_a f(x)}{(\beta-f(x))^3} - \frac{F(x)|\nabla_a f(x)|^2}{(\beta-f(x))^4} - \frac{\nabla_a F(x)\nabla_a f(x)}{(\beta-f(x))^3} \\ & \quad - \frac{F(x)(\nabla_a f(x))^3}{(\beta-f(x))^5}. \end{aligned}$$

□

5. A strong Harnack inequality for Ricci flat homogeneous graphs

We will prove the following theorem for homogeneous graphs which are Ricci flat. This is a special case of Theorem 5. Although homogeneous graphs are special, the proof of Theorem 1 captures the essence of the proof for the general case without the extra complication. In the subsequent sections, we will deal with general graphs with non-zero Ricci curvature later.

Theorem 1. *Let Γ be a Ricci flat homogeneous graph with a symmetric edge generating set S with $|S| = d$, and let eigenfunction f be associated with eigenvalue λ satisfying $\max_z |f(z)| = \max_z f(z) = 1$. Then, for a vertex x and a neighbor y of x , we have*

$$(33) \quad |f(x) - f(y)|^2 \leq c\lambda(\beta - f(x))^2$$

$$(34) \quad \text{and} \quad \mathcal{F}(x) = \frac{1}{d} \int_a |\nabla f(x)|^2 \leq \frac{c}{d} \lambda(\beta - f(x))^2$$

where $c = 24\beta d^2 / (\beta - 1)$ and $\beta \geq 1 + 32\lambda d^2$.

Before proving Theorem 1, we note that Theorem 1 is best possible in the sense that for the cycles C_{2n} , we consider the eigenfunction $f(v_j) = \cos(\frac{2j\pi}{n})$ for $j = 0, 1, \dots, 2n - 1$. In this case, the eigenvalue $\lambda = 1 - \cos(\frac{2\pi}{n}) \sim \frac{2\pi^2}{n^2}$. We consider the edge $\{v_0, v_1\}$ and note that $|f(v_0) - f(v_1)| \sim \frac{2\pi}{n^2}$ which is of the same order as λ . This illustrates that (33) is tight up to a constant factor.

Proof of Theorem 1. For $\beta > 1 + 32\lambda d^2$, we consider \mathcal{F} :

$$\mathcal{F}(x) = \frac{F(x)}{(\beta - f(x))^2} = \frac{\int_a |\nabla_a f|^2(x)}{(\beta - f(x))^2}.$$

Let x denote a vertex which achieves the maximum of \mathcal{F} . Without loss of generality, we assume $f(x) > 0$ (since otherwise we consider $-f$ instead). We also have $\nabla_a \mathcal{F}(x) \geq 0$ for all $a \in \mathcal{A}$. Now we use Lemma 9 and we have

$$\begin{aligned}
 0 &\leq \nabla_a \mathcal{F}(x) \left(1 + \frac{\nabla_a f(x)}{\beta - f(x)}\right)^2 \left(1 - \frac{\nabla_a f(x)}{\beta - f(x)}\right) \\
 &= \frac{\nabla_a F(x)}{(\beta - f(x))^2} + \frac{2F(x)\nabla_a f(x)}{(\beta - f(x))^3} - \frac{F(x)|\nabla_a f(x)|^2}{(\beta - f(x))^4} \\
 &\quad - \frac{\nabla_a F(x)\nabla_a f(x)}{(\beta - f(x))^3} - \frac{F(x)(\nabla_a f(x))^3}{(\beta - f(x))^5}.
 \end{aligned}$$

Now we average over $a \in \mathcal{A}$.

$$\begin{aligned}
 (35) \quad 0 &\leq \frac{\int_a \nabla_a F(x)}{(\beta - f(x))^2} - \frac{\int_a \nabla_a F(x)\nabla_a f(x)}{(\beta - f(x))^3} - \frac{F(x) \int_a |\nabla_a f(x)|^2}{(\beta - f(x))^4} \\
 &\quad + \frac{F(x)}{(\beta - f(x))^2} \int_a \left(\frac{2\nabla_a f(x)}{\beta - f(x)} - \frac{(\nabla_a f(x))^3}{(\beta - f(x))^3} \right).
 \end{aligned}$$

We assume

$$(36) \quad \delta = \sup_z \frac{|\nabla_a f(z)|}{\beta - f(z)}.$$

Using the result in [5] as in (1) and the assumption for β , we have $\delta \leq 1/(4d) < 1$.

Using (9), we continue from (35):

$$\begin{aligned}
 (37) \quad 0 &\leq \frac{\int_a \nabla_a F(x)}{(\beta - f(x))^2} - \frac{\int_a \nabla_a F(x)\nabla_a f(x)}{(\beta - f(x))^3} - \frac{F(x) \int_a |\nabla_a f(x)|^2}{(\beta - f(x))^4} \\
 &\quad + \frac{2F(x)\lambda f(x)}{(\beta - f(x))^3} + \frac{\delta F(x) \int_a |\nabla_a f(x)|^2}{(\beta - f(x))^4}.
 \end{aligned}$$

We now use Lemma 2 and (9) for (37).

(38)

$$\begin{aligned}
 0 &\leq \frac{\int_a \nabla_a F(x)}{(\beta - f(x))^2} - \frac{\int_a \nabla_a F(x) \nabla_a f(x)}{(\beta - f(x))^3} - \frac{(1 - \delta)F(x) \int_a |\nabla_a f(x)|^2}{(\beta - f(x))^4} \\
 &\quad + \frac{2\lambda f(x)F(x)}{(\beta - f(x))^3} \\
 &= \frac{2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2}{(\beta - f(x))^2} - \frac{(1 - \delta)F(x) \int_a |\nabla_a f(x)|^2}{(\beta - f(x))^4} + \frac{2\lambda f(x)F(x)}{(\beta - f(x))^3} \\
 &\quad - \int_a \int_b \frac{(2\nabla_b f(x) \nabla_a \nabla_b f(x) - (\nabla_a \nabla_b f(x))^2) \nabla_a f(x)}{(\beta - f(z))^3} \\
 &\leq \frac{4\beta}{\beta - 1} \lambda \mathcal{F}(x) - (1 - \delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta - f(x)} \right)^2 \\
 &\quad - (1 - \delta) \int_a \int_b \frac{(\nabla_a f(x))^2 (\nabla_b f(x))^2}{(\beta - f(x))^4} \\
 &\quad - \int_a \int_b \frac{2\nabla_b f(x) \nabla_a \nabla_b f(x) \nabla_a f(x)}{(\beta - f(x))^3}.
 \end{aligned}$$

Now, continuing from (38), we have:

$$\begin{aligned}
 (39) \quad 0 &\leq \frac{4\beta}{\beta - 1} \lambda \mathcal{F}(x) - \int_a \int_b (1 - \delta) \left(\frac{\nabla_a \nabla_b f(x)}{\beta - f(x)} + \frac{\nabla_a f(x) \nabla_b f(x)}{(\beta - f(x))^2 (1 - \delta)} \right)^2 \\
 &\quad + \int_a \int_b \frac{(\nabla_a f(x) \nabla_b f(x))^2}{(\beta - f(x))^4} \left(\frac{1}{1 - \delta} - 1 + \delta \right) \\
 &\leq \frac{4\beta}{\beta - 1} \lambda \mathcal{F}(x) - (1 - \delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta - f(x)} + \frac{\nabla_a f(x) \nabla_b f(x)}{(\beta - f(x))^2 (1 - \delta)} \right)^2 \\
 &\quad + \frac{2\delta}{1 - \delta} \int_a \int_b \frac{(\nabla_a f(x) \nabla_b f(x))^2}{(\beta - f(x))^4}.
 \end{aligned}$$

We note that

$$\begin{aligned}
 &\int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta - f(x)} + \frac{\nabla_a f(x) \nabla_b f(x)}{(\beta - f(x))^2 (1 - \delta)} \right)^2 \\
 &\geq \int_a \int_{b=a} \left(\frac{\nabla_a \nabla_b f(x)}{\beta - f(x)} + \frac{\nabla_a f(x) \nabla_b f(x)}{(\beta - f(x))^2 (1 - \delta)} \right)^2 \\
 &\geq \frac{1}{d} \int_a \left(\frac{\nabla_a \nabla_a f(x)}{\beta - f(x)} + \frac{(\nabla_a f(x))^2}{(\beta - f(x))^2 (1 - \delta)} \right)^2.
 \end{aligned}$$

We continue from (39) and we have

$$\begin{aligned}
 (40) \quad 0 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - \frac{1-\delta}{d} \int_a \left(\frac{\nabla_a \nabla_a f(x)}{\beta-f(x)} + \frac{|\nabla_a f(x)|^2}{(\beta-f(x))^2(1-\delta)} \right)^2 \\
 &\quad + \frac{2\delta}{1-\delta} \mathcal{F}^2(x) \\
 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - \frac{1-\delta}{d} \left(\frac{\int_a \nabla_a \nabla_a f(x)}{\beta-f(x)} + \frac{\mathcal{F}(x)}{1-\delta} \right)^2 + \frac{2\delta}{1-\delta} \mathcal{F}^2(x).
 \end{aligned}$$

From Lemma 1, we can write $\lambda = \int_a (1 - \rho(a))$ for some character ρ of \mathcal{G} and we have

$$\Delta f(x) = \int_a \nabla_a f(x) = \int_a (f(x) - f(ax)) = \int_a (1 - \rho(a)) f(x) = \lambda f(x).$$

Since the edge generating set A is symmetric, we can rewrite

$$\lambda = \int_a (1 - \rho(a)) = \int_a (1 - \operatorname{Re} \rho(a))$$

where $\operatorname{Re} z$ denotes the real part of z .

Hence,

$$\begin{aligned}
 (41) \quad \int_a \nabla_a \nabla_a f(x) &= \int_a (f(x) - 2f(ax) + f(a^2x)) \\
 &= \int_a (1 - \rho(a))^2 f(x) \\
 &= \int_a (1 - 2 \operatorname{Re} \rho(a) + (\operatorname{Re} \rho(a))^2 - (\operatorname{Im} \rho(a))^2) f(x) \\
 &\geq \int_a (-2 \operatorname{Re} \rho(a) + 2(\operatorname{Re} \rho(a))^2) f(x) \\
 &\geq -2 \int_a \operatorname{Re} \rho(a) f(x) + 2 \left(\int_a \operatorname{Re} \rho(a) \right)^2 f(x) \\
 &= -2 \left(\int_a \operatorname{Re} \rho(a) \right) \left(1 - \int_a \operatorname{Re} \rho(a) \right) f(x) \\
 &= -2(1 - \lambda) \lambda f(x).
 \end{aligned}$$

We now consider two possibilities:

Case 1: Suppose $\int_a \nabla_a \nabla_a f(x) \geq 0$. From (40), we have

$$(42) \quad 0 \leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x).$$

Case 2: Suppose $\int_a \nabla_a \nabla_a f(x) \leq 0$. Expanding (40) and using (41), we have

$$(43) \quad \begin{aligned} 0 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - \frac{2 \int_a \nabla_a \nabla_a f(x) \mathcal{F}(x)}{d(\beta-f(x))} - \frac{\mathcal{F}^2(x)}{1-\delta} + \frac{2\delta}{1-\delta} \mathcal{F}^2(x) \\ &\leq \frac{4\beta + 4(1-\lambda)/d}{\beta-1} \lambda \mathcal{F}(x) - \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x) \\ &\leq \frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) - \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x) \end{aligned}$$

since $d > 1$.

Now, we see that (43) implies

$$\left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}(x) \leq \frac{6\beta}{\beta-1} \lambda.$$

By the assumption on β , we have $\delta < 1/(4d)$ which implies

$$\left(\frac{1-\delta}{d} - \frac{\delta}{1-\delta} \right) \geq \frac{1}{4d}.$$

Hence, we have, for any x ,

$$\mathcal{F}(x) \leq \frac{24\beta d}{\beta-1} \lambda.$$

From the definition of δ , we have

$$\frac{\delta^2}{d} \leq \mathcal{F}(x) \leq \delta^2.$$

Therefore

$$\delta^2 \leq \mathcal{F}(x)d \leq \frac{24\beta d^2}{\beta-1} \lambda$$

This proves

$$\delta^2 \leq \frac{24\beta d^2}{\beta-1} \lambda.$$

□

6. A Harnack inequality for general graphs

We first prove the Harnack inequalities for general graphs with bounded curvature. This will later be needed in our proofs for strong Harnack inequalities in Section 7.

Theorem 2. *Let Γ be a graph with curvature κ and frames \mathcal{A} , and let $f : V \rightarrow \mathbb{R}$ be an eigenfunction satisfying $\sup_z |f(z)| = \sup_z f(z) = 1$ and with associated eigenvalue λ . Then*

$$\int_a |\nabla_a f(x)|^2 \leq 8\lambda + 4\kappa$$

and for any edge $\{x, y\}$, f satisfies

$$|f(x) - f(y)|^2 \leq 8\lambda d_{max} + 4\kappa d_{max}$$

where d_{max} denotes the maximum degree.

Proof. From Lemma 5, we have

$$(44) \quad \Delta F(x) \leq 2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa F(x).$$

We consider

$$\begin{aligned} & \Delta(F(x) + (4\lambda + 2\kappa_1)f^2(x)) \\ & \leq 2\lambda F(x) + 2\kappa F(x) + (4\lambda + 2\kappa)(2\lambda f^2(x) - F(x)) \\ & \leq (8\lambda + 4\kappa)\lambda - 2\lambda F(x). \end{aligned}$$

Let z denote the vertex which achieves the maximum of $F(x) + (4\lambda + 2\kappa)f^2(x)$ over all $x \in V(\Gamma)$. Then we have

$$\begin{aligned} 0 & \leq \Delta(F(z) + (4\lambda + 2\kappa)\lambda f^2(z)) \\ & \leq (8\lambda + 4\kappa)\lambda - 2\lambda F(z). \end{aligned}$$

Therefore we have

$$(45) \quad F(z) \leq 4\lambda + 2\kappa.$$

For all x , we have

$$\begin{aligned} F(x) &\leq F(x) + (4\lambda + 2\kappa)f^2(x) \\ &\leq F(z) + 4\lambda + 2\kappa \\ &\leq 8\lambda + 4\kappa \end{aligned}$$

The bound for $|f(x) - f(y)|^2$ follows from (14). This completes the proof of Theorem 2. \square

In the remainder of this section, we deal with graphs with bounded flow-curvature,

Lemma 10. *Let Γ be a graph with a flow-curvature κ' and frames \mathcal{A} , and let an eigenfunction $f : V \rightarrow \mathbb{R}$ have associated eigenvalue λ . Then, the function*

$$(46) \quad F(x) = \int_a |\nabla_a f(x)|^2$$

satisfies, for a fixed $a \in \mathcal{A}$ and a vertex x ,

$$(47) \quad \begin{aligned} \nabla_a F(x) &\leq 2 \int_b \left(\nabla_b f(x) \cdot \nabla_b \nabla_a f(x) \right) \\ &\quad - \int_b \left(\nabla_a \nabla_b f(x) \right)^2 + 2\kappa' \epsilon |\nabla_a f(x)| \end{aligned}$$

and

$$(48) \quad \Delta F(x) \leq 2\lambda F(x) - \int_a \int_b \left(\nabla_a \nabla_b f(x) \right)^2 + 2\kappa' \epsilon \sqrt{F(x)},$$

provided that f is ϵ -Lipschitz.

Proof. The proof is quite similar to that of Lemma 3. By using Lemma 6, we have

$$\begin{aligned} \Delta F(x) = \int_a \nabla_a F(x) &\leq 2 \int_b \nabla_b f(x) \int_a \nabla_b \nabla_a f(x) - \int_a \int_b \left(\nabla_a \nabla_b f(x) \right)^2 \\ &\quad + 2 \int_b |\nabla_b f(x)| \kappa' \epsilon. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\int_b |\nabla_b f(x)| \leq \left(\int_b |\nabla_b f(x)|^2 \right)^{1/2} = \sqrt{F(x)}.$$

Thus, we have

$$\begin{aligned} \Delta F(x) &\leq 2 \int_b \nabla_b f(x) \nabla_b \left(\int_a \nabla_a f(x) \right) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa' \epsilon \sqrt{F(x)} \\ &= 2 \int_b \nabla_b f(x) \nabla_b (\lambda f(x)) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa' \epsilon \sqrt{F(x)} \\ &= 2\lambda \int_b |\nabla_b f(x)|^2 - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa' \epsilon \sqrt{F(x)} \\ &= 2\lambda F(x) - \int_a \int_b (\nabla_a \nabla_b f(x))^2 + 2\kappa' \epsilon \sqrt{F(x)} \end{aligned}$$

as desired. □

Theorem 3. *Let Γ be a graph with a flow-curvature κ' and frames \mathcal{A} and let an eigenfunction $f : V \rightarrow \mathbb{R}$ satisfying $\sup |f(x)| = \sup f(x) = 1$ have an associated eigenvalue λ . Then $F = \int_a |\nabla f(x)|^2$ satisfies*

$$(49) \quad F(x) \leq 8\lambda + 8\kappa' \sqrt{d_{max}} + \frac{2\kappa'^2 d_{max}}{\lambda}$$

where d_{max} denotes the maximum degree.

Proof. We use Lemma 10 and for $d' = \sqrt{d_{max}}$, we consider

$$\begin{aligned} &\Delta(F(x) + (4\lambda + 2\kappa'd')f^2(x)) \\ &\leq 2\lambda F(x) + 2\kappa' \epsilon \sqrt{F(x)} + (4\lambda + 2\kappa'd')(2\lambda f^2(x) - F(x)) \\ &\leq (8\lambda + 4\kappa'd')\lambda - 2\lambda F(x) + 2\kappa' \epsilon \sqrt{F(x)} - 2\kappa'd'F(x). \end{aligned}$$

Let z denote the vertex which achieves the maximum of $F(x) + (4\lambda + 2\kappa'd')f^2(x)$ over all $x \in V(\Gamma)$. Then we have

$$\begin{aligned} 0 &\leq \Delta(F(z) + (4\lambda + 2\kappa'd')\lambda f^2(z)) \\ &\leq (8\lambda + 4\kappa'd')\lambda - 2\lambda F(z) + 2\kappa' \epsilon \sqrt{F(z)} - 2\kappa'd'F(z). \end{aligned}$$

Therefore we have

$$(50) \quad (\lambda + \kappa'd')F(z) - \kappa' \epsilon \sqrt{F(z)} \leq \lambda(4\lambda + 2\kappa'd').$$

Let y denote a vertex which achieves the maximum value of $F(x)$. Then, we have

$$F(z) \leq F(y) \leq F(y) + (4\lambda + 2\kappa'd')f^2(y) \leq F(z) + 4\lambda + 2\kappa'd'.$$

Substituting into (50), we have

$$(\lambda + \kappa d')(F(y) - 4\lambda - 2\kappa' d') - \kappa' \epsilon \sqrt{F(y)} \leq \lambda(4\lambda + 2\kappa' d').$$

Since $\epsilon \leq \sqrt{d_{max} F(y)} = d' \sqrt{F(y)}$, we have

$$\lambda(F(y) - 4\lambda - 2\kappa' d') - \kappa' d'(4\lambda + 2\kappa' d') \leq \lambda(4\lambda + 2\kappa' d').$$

This implies

$$F(y) \leq 8\lambda + 8\kappa' \sqrt{d_{max}} + \frac{2\kappa'^2 d_{max}}{\lambda}.$$

□

As an immediate consequence of Theorem 3, we have the following:

Theorem 4. *In a graph G with a flow-curvature κ' with frames \mathcal{A} , for an eigenfunction $f : V \rightarrow \mathbb{R}$ satisfying $\sup |f(x)| = 1$ and with associated eigenvalue λ , then for any edge $\{x, y\}$, f satisfies*

$$(51) \quad |f(x) - f(y)|^2 \leq 8\lambda d_{max} + 8\kappa'(d_{max})^{3/2} + \frac{2\kappa'^2 d_{max}^2}{\lambda}$$

where d_{max} denotes the maximum degree in G .

7. A strong Harnack inequality for general graphs

Theorem 5. *Let Γ be a graph with curvature κ and frames \mathcal{A} . Let $f : V \rightarrow \mathbb{R}$ denote an eigenfunction with the associated eigenvalue λ , satisfying $\max_z |f(z)| = \max_z f(z) = 1$. Assume $\int_a \nabla_a \nabla_a f(x) \geq -2\lambda(1 - \lambda)f(x)$. Then for a vertex x and a neighbor y of x , we have*

$$(52) \quad |f(x) - f(y)|^2 \leq \left(\frac{24\beta d_{max}^2}{\beta - 1} \lambda + 4\kappa d_{max}^2 \right) (\beta - f(x))^2$$

and $F(x) = \int_a |\nabla f(x)|^2 \leq \left(\frac{24\beta d_{max}}{\beta - 1} \lambda + 4\kappa d_{max} \right) (\beta - f(x))^2.$

where $\beta \geq 1 + 32\lambda d_{max}^2$ and d_{max} denotes the maximum degree.

Proof. The proof follows a similar line as in the proof of Theorem 1 for homogeneous Ricci flat graphs. Because of the frames \mathcal{A} , the treatment for general graphs is not so different from the case for homogeneous graphs.

Instead of being Ricci flat, we will deal with the additional contributions from our definition of curvature. We sketch the proof here.

All notations are the same as in the proof of Theorem 1 and in particular, we also have (37). However, (38) is to be replaced by the following:

$$\begin{aligned}
 (53) \quad 0 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - (1-\delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta-f(x)} \right)^2 \\
 &\quad - (1-\delta) \int_a \int_b \frac{(\nabla_a f(x))^2 (\nabla_b f(x))^2}{(\beta-f(x))^4} \\
 &\quad - \int_a \int_b \frac{2\nabla_b f(x) \nabla_a \nabla_b f(x) \nabla_a f(x)}{(\beta-f(x))^3} + \kappa \frac{F(x)}{(\beta-f(x))^2} \\
 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - (1-\delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta-f(x)} \right)^2 \\
 &\quad - (1-\delta) \int_a \int_b \frac{(\nabla_a f(x))^2 (\nabla_b f(x))^2}{(\beta-f(x))^4} \\
 &\quad - \int_a \int_b \frac{2\nabla_b f(x) \nabla_a \nabla_b f(x) \nabla_a f(x)}{(\beta-f(x))^3} + 2\kappa \mathcal{F}(x).
 \end{aligned}$$

The additional terms involving κ also appear in (39), (40) and (41). Instead of (43), we have the following:

$$(54) \quad 0 \leq \frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) - \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x) + 2\kappa \mathcal{F}(x).$$

This implies

$$\left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}(x) \leq \frac{6\beta}{\beta-1} \lambda + 2\kappa.$$

From the definition of δ and (14), we have

$$\frac{\delta^2}{d_{max}} \leq \mathcal{F}(x) \leq \delta^2.$$

By the assumption on β , we have $\delta < 1/(4d_{max})$ which implies

$$(55) \quad \left(\frac{1-\delta}{d_{max}} - \frac{2\delta}{1-\delta} \right) \geq \frac{1}{4d_{max}}.$$

Therefore we have

$$\mathcal{F}(x) \leq \frac{24\beta d_{max}}{\beta - 1} \lambda + 8\kappa d_{max}$$

and

$$\delta^2 \leq \mathcal{F}(x) d_{max} \leq \frac{24\beta d_{max}^2}{\beta - 1} \lambda + 8\kappa d_{max}^2.$$

This proves

$$\delta^2 \leq \frac{24\beta d_{max}^2}{\beta - 1} \lambda + 8\kappa d_{max}^2.$$

□

We note that the assumption $\int_a \nabla_a \nabla_a f(x) \geq -\lambda(1 - \lambda)f(x)$ can be weakened. As long as the absolute value of $\frac{\int_a \nabla_a \nabla_a f(x)}{\beta - f(x)}$ is less than a fraction of $\mathcal{F}(x)$, some modified version of Theorem 5 can still be derived.

We can also establish the following strong Harnack inequality for graphs with bounded flow-curvature.

Theorem 6. *Let Γ be a graph with flow-curvature $-K = \kappa'$ and frames \mathcal{A} . Let $f : V \rightarrow \mathbb{R}$ denote an eigenfunction with the associated eigenvalue λ satisfying $\max_z |f(z)| = \max_z f(z) = 1$. Suppose $\int_a \nabla_a \nabla_a f(x) > -2\lambda(1 - \lambda)$. Then for a vertex x , we have*

$$(56) \quad \mathcal{F}(x) \leq \frac{48\beta d_{max}}{\beta - 1} \lambda + \frac{\kappa'^2 d_{max}}{\beta^2 \lambda} + \frac{\kappa'^3 d_{max}^{3/2}}{\beta^2 \lambda^2} + \frac{\kappa'^4 d_{max}^2}{2\beta^2 \lambda^3}$$

and for y adjacent to x , we have

$$(57) \quad \frac{(f(x) - f(y))^2}{(\beta - f(x))^2} \leq \frac{48\beta d_{max}^2}{\beta - 1} \lambda + \frac{\kappa'^2 d_{max}^2}{\beta^2 \lambda} + \frac{\kappa'^3 d_{max}^{5/2}}{\beta^2 \lambda^2} + \frac{\kappa'^4 d_{max}^3}{2\beta^2 \lambda^3}$$

where d denotes the maximum degree.

Proof. The proof here is a modified version of the proof of Theorem 5. We follow the notations used previously. (53) is to be replaced by the following:

$$\begin{aligned}
 (58) \quad 0 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - (1-\delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta-f(x)} \right)^2 \\
 &\quad - (1-\delta) \int_a \int_b \frac{(\nabla_a f(x))^2 (\nabla_b f(x))^2}{(\beta-f(x))^4} \\
 &\quad - \int_a \int_b \frac{2\nabla_b f(x) \nabla_a \nabla_b f(x) \nabla_a f(x)}{(\beta-f(x))^3} + \frac{2\kappa' \epsilon \int_a |\nabla_a f(x)|}{(\beta-f(x))^2} \\
 &\leq \frac{4\beta}{\beta-1} \lambda \mathcal{F}(x) - (1-\delta) \int_a \int_b \left(\frac{\nabla_a \nabla_b f(x)}{\beta-f(x)} \right)^2 \\
 &\quad - (1-\delta) \int_a \int_b \frac{(\nabla_a f(x))^2 (\nabla_b f(x))^2}{(\beta-f(x))^4} \\
 &\quad - \int_a \int_b \frac{2\nabla_b f(x) \nabla_a \nabla_b f(x) \nabla_a f(x)}{(\beta-f(x))^3} + \frac{2\kappa' \epsilon \sqrt{\mathcal{F}(x)}}{\beta-f(x)}.
 \end{aligned}$$

Instead of (54), we have the following:

$$(59) \quad 0 \leq \frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) - \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x) + \frac{2\kappa' \epsilon \sqrt{\mathcal{F}(x)}}{\beta-1}.$$

From (55), we have

$$(60) \quad \frac{\mathcal{F}^2(x)}{4d_{max}} \leq \left(\frac{1-\delta}{d} - \frac{2\delta}{1-\delta} \right) \mathcal{F}^2(x) \leq \frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) + \frac{2\kappa' \epsilon \sqrt{\mathcal{F}(x)}}{\beta-1}.$$

We now consider two cases:

Case 1: Suppose

$$\frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) \geq \frac{2\kappa' \epsilon \sqrt{\mathcal{F}(x)}}{\beta-1}.$$

We then have

$$\frac{\mathcal{F}^2(x)}{4d_{max}} \leq \frac{12\beta}{\beta-1} \lambda \mathcal{F}(x) \quad \text{and} \quad \mathcal{F}(x) \leq \frac{48\beta d_{max}}{\beta-1} \lambda.$$

Case 2: Suppose

$$\frac{6\beta}{\beta-1} \lambda \mathcal{F}(x) < \frac{2\kappa' \epsilon \sqrt{\mathcal{F}(x)}}{\beta-1}.$$

In this case, we have

$$6\beta \lambda \sqrt{\mathcal{F}(x)} < 2\kappa' \epsilon$$

which implies

$$\begin{aligned} \mathcal{F}(x) &\leq \frac{\kappa'^2 \epsilon^2}{9\beta^2 \lambda^2} \leq \frac{\kappa'^2}{9\beta^2 \lambda^2} \left(8\lambda d_{max} + 8\kappa' d_{max}^{3/2} + \frac{2\kappa'^2 d_{max}^2}{\lambda} \right) \\ &\leq \frac{\kappa'^2 d_{max}}{\beta^2 \lambda} + \frac{\kappa'^3 d_{max}^{3/2}}{\beta^2 \lambda^2} + \frac{\kappa'^4 d_{max}^2}{2\beta^2 \lambda^3} \end{aligned}$$

by using Theorem 4.

Together, we have

$$\mathcal{F}(x) \leq \frac{48\beta}{\beta - 1} \lambda + \frac{\kappa'^2 d_{max}}{\beta^2 \lambda} + \frac{\kappa'^3 d_{max}^{3/2}}{\beta^2 \lambda^2} + \frac{\kappa'^4 d_{max}^2}{2\beta^2 \lambda^3}$$

and for y adjacent to x , we have

$$\frac{(f(x) - f(y))^2}{(\beta - f(x))^2} \leq \frac{48\beta d_{max}}{\beta - 1} \lambda + \frac{\kappa'^2 d_{max}^2}{\beta^2 \lambda} + \frac{\kappa'^3 d_{max}^{5/2}}{\beta^2 \lambda^2} + \frac{\kappa'^4 d_{max}^3}{2\beta^2 \lambda^3}$$

□

8. Eigenvalue inequalities using Harnack and strong Harnack inequalities

For a graph G with diameter D , we can use the theorems in the preceding sections to establish lower bounds for eigenvalues as follows: Suppose that the combinatorial eigenfunction f associated with λ satisfies $\max_z |f(z)| = \max_z f(z) = 1$ and $\sum_z f(z) d_z = 0$. We consider two vertices u and v with $f(u) = 1$ and $f(v) < 0$. We choose a path P joining u and v of length at most D , say, $P = (x_0, x_1, \dots, x_t)$ where $u = x_0$ and $v = x_t$ where $t \leq D$. We will use such a path together with the Harnack and strong Harnack inequalities to establish lower bounds for eigenvalue λ

8.1. Eigenvalue inequalities using the Harnack inequalities

Theorem 7. *For a graph with curvature κ , maximum degree d_{max} and diameter D , we have*

$$\lambda \geq \frac{1}{8D^2 d_{max}} - \frac{\kappa}{2}.$$

Proof. We consider the path P as above. We have

$$1 \leq f(u) - f(v) \leq \sum_{i=0}^{t-1} (f(x_i) - f(x_{i+1})) \leq D\sqrt{8\lambda d_{max} + 4\kappa d_{max}}$$

by using the Harnack inequality in Theorem 2. This implies

$$\lambda \geq \frac{1}{8D^2 d_{max}} - \frac{\kappa}{2}.$$

□

Theorem 8. *For a graph with flow-curvature κ' , maximum degree d_{max} and diameter D , we have*

$$\lambda \geq \min \left\{ \frac{1 + \sqrt{1 - 128\kappa'^2 D^4 d_{max}^3}}{16}, \kappa' d_{max}^{1/2} \right\}$$

Proof. We following the notation in the proof of Theorem 7. We have

$$\begin{aligned} 1 &\leq f(u) - f(v) \\ &\leq \sum_{i=0}^{t-1} (f(x_i) - f(x_{i+1})) \\ &\leq D\sqrt{8\lambda d_{max} + 8\kappa' d_{max}^{3/2} + \frac{2\kappa'^2 d_{max}^2}{\lambda}} \end{aligned}$$

by using the Harnack inequality in Theorem 4. We consider two possibilities:

Case 1: Suppose $8\kappa' d_{max}^{3/2} \geq \frac{2\kappa'^2 d_{max}^2}{\lambda}$. Then we have $\lambda \geq \kappa' d_{max}^{1/2}$.

Case 2: Suppose $8\kappa' d_{max}^{3/2} < \frac{2\kappa'^2 d_{max}^2}{\lambda}$. Hence

$$\begin{aligned} 0 &\leq 8D^2 d_{max} \lambda^2 - \lambda + 4\kappa'^2 D^2 d_{max} \\ \text{and } \lambda &\geq \frac{1 + \sqrt{1 - 128\kappa'^2 D^4 d_{max}^3}}{16}. \end{aligned}$$

Together we have

$$\lambda \geq \min \left\{ \frac{1 + \sqrt{1 - 128\kappa'^2 D^4 d_{max}^3}}{16}, \kappa' d_{max}^{1/2} \right\}.$$

□

8.2. Eigenvalue inequalities using the strong Harnack inequalities

We can slightly improve the eigenvalue inequalities in Section 8.1 by using the strong Harnack inequalities. The method follows from the results of Li and Yau in [8].

Theorem 9. *For a graph with curvature κ , maximum degree d_{max} and diameter D , we have*

$$\lambda \geq \frac{1}{24D^2d_{max}} e^{-\sqrt{1-4\kappa D^2d_{max}}}.$$

Proof. We follow the previous definition given at the beginning of Section 8. In addition, we consider the function

$$H(x) = \log(\beta - f(x)).$$

Along the path P , we have

$$\begin{aligned} \log\left(\frac{\beta}{\beta-1}\right) &\leq \sum_{i=0}^{t-1} (\log(\beta - f(x_i)) - \log(\beta - f(x_{i+1}))) \\ &\leq \sum_{i=0}^{t-1} \frac{f(x_i) - f(x_{i+1})}{\beta - f(x_i)} \\ &\leq D\sqrt{\frac{24\beta d_{max}\lambda}{\beta-1} + 4\kappa d_{max}} \end{aligned}$$

by using Theorem 5. This implies

$$\lambda \geq \frac{\beta-1}{24\beta d_{max}D^2} \left(\left(\log \frac{\beta}{\beta-1}\right)^2 - 4\kappa d_{max}D^2 \right).$$

We choose β so that

$$\log\left(\frac{\beta}{\beta-1}\right) = \sqrt{1-4\kappa D^2d_{max}}.$$

Therefore we have

$$\lambda \geq \frac{1}{24D^2d_{max}} e^{-\sqrt{1-4\kappa D^2d_{max}}}.$$

□

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