Communications in
Analysis and Geometry
Volume 25, Number 3, 545–556, 2017

Density of a minimal submanifold and total curvature of its boundary

Jaigyoung Choe and Robert Gulliver

Given a piecewise smooth submanifold $\Gamma^{n-1} \subset \mathbb{R}^m$ and $p \in \mathbb{R}^m$, we define the *vision angle* $\Pi_p(\Gamma)$ to be the $(n-1)$-dimensional volume of the radial projection of Γ to the unit sphere centered at p. If p is a point on a stationary n-rectifiable set $\Sigma \subset \mathbb{R}^m$ with boundary Γ, then we show the density of Σ at p is $\leq$ the density at its vertex p of the cone over Γ. It follows that if $\Pi_p(\Gamma)$ is less than twice the volume of S^{n-1}, for all $p \in \Gamma$, then Σ is an embedded submanifold. As a consequence, we prove that given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$ and two compact convex hypersurfaces Γ_i of $R_i^n, i = 1, 2$, a nonflat minimal submanifold spanned by $\Gamma := \Gamma_1 \cup \Gamma_2$ is embedded.

1. Introduction

Fenchel [F1] showed that the total curvature of a closed space curve $\gamma \subset \mathbb{R}^m$ is at least 2π, and it equals 2π if and only if γ is a plane convex curve. Fáry [Fa] and Milnor [M] independently proved that a simple knotted regular curve has total curvature larger than 4π. These two results indicate that a Jordan curve which is curved at most *double* the minimum is isotopically simple. But in fact minimal surfaces spanning such Jordan curves must be simple as well. Indeed, Nitsche [N] showed that an analytic Jordan curve in $\mathbb{R}^3$ with total curvature at most 4π bounds exactly one minimal disk. Moreover, Ekholm, White and Wienholtz [EWW] proved that a minimal surface spanning such a Jordan curve in $\mathbb{R}^m$ is embedded.

Given an n-dimensional submanifold M of $\mathbb{R}^m$, there are two well-studied ways of defining the total curvature of M: the higher-dimensional Gauss-Bonnet integral $\int_M \Omega$ as defined in [AW] and [C1]; and the total absolute curvature of M, $\int_M K^* dV_M$ as defined by Chern and Lashof in [CL] (see section 2 below). Chern and Lashof proved that $\int_M K^*\, dV_M \geq 2$, with equality if and only if M is a convex hypersurface in an $(n+1)$-dimensional plane.

The first-named author was supported in part by KRF-2007-313-C00057.

Eells and Kuiper have shown that if $\int_M K^* dV_M < 3$ then M is homeomorphic to $\mathbb{S}^n$ and that if $\int_M K^* dV_M < 4$ then M is homeomorphic to $\mathbb{S}^n$, $\mathbb{R}P^n$, $\mathbb{C}P^{n/2}$, $\mathbb{H}P^{n/4}$ or to $CayP^2$ (for $n = 16$). [EK].

In the light of Ekholm-White-Wienholtz's theorem, it is quite natural to conjecture that *an n-dimensional minimal submanifold $\Sigma \subset \mathbb{R}^m$ spanning a compact connected submanifold Γ^{n-1} with total absolute curvature < 4 is embedded.* In this paper we prove a theorem in the spirit of this conjecture: given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$ and two compact convex hypersurfaces Γ_i^{n-1} of $R_i^n, i = 1, 2$, a nonflat minimal submanifold spanned by $\Gamma := \Gamma_1 \cup \Gamma_2$ is embedded.

In [Fa] Fáry showed that the total curvature of a space curve γ in $\mathbb{R}^m$ is equal to the average over all 2-planes $R^2 \subset \mathbb{R}^m$ of the total curvature of the orthogonal projection of γ onto the R^2. We shall use an extension of Fáry's theorem, due to Langevin and Shifrin [LS], which shows that given an $(n-1)$-dimensional submanifold Γ of $\mathbb{R}^m$, the total absolute curvature of Γ equals the average over all n-planes $R^n \subset \mathbb{R}^m$ of the total absolute curvature of the orthogonal projection of Γ into the n-plane R^n.

2. Total absolute curvature

Consider a submanifold M^n of Euclidean space $\mathbb{R}^m$. As discussed above, in high dimension and codimension we discuss two types of total curvature: one intrinsic (Allendörfer-Weil-Chern-Gauss-Bonnet), and one extrinsic (Chern-Lashof). In this section we shall review Chern-Lashof's total absolute curvature. This total curvature may be understood in terms of Gauss-Kronecker curvature of hypersurfaces.

Let M^n be an oriented hypersurface immersed in $\mathbb{R}^{n+1}$. A unit normal vector ν to M at $p \in M$ defines the Gauss map $G_1 : M \to \mathbb{S}^n$. The determinant of the differential G_1*, or of the second fundamental form of M, is called the *Gauss-Kronecker curvature* of M, which we shall denote GK_M. It follows that for M compact,

$$\int_M GK_M \, dV_M = c_n \deg(G_1), \quad c_n := \mathrm{Vol}(\mathbb{S}^n).$$

Furthermore, if n is even, H. Hopf [H] showed

$$\int_M GK_M \, dV_M = \frac{1}{2} c_n \, \chi(M). \tag{1}$$

Now let M be an n-dimensional submanifold of $\mathbb{R}^m$. The volume form of the unit normal bundle N_1M of M is $dV_M \wedge d\sigma_{m-n-1}$ where the restriction of $d\sigma_{m-n-1}$ to a fiber of N_1M at p is the volume form of the sphere of unit normal vectors at $p \in M$. Define the Gauss map $G_1 : N_1M \to \mathbb{S}^{m-1}$ by $G_1(p,\nu) = \nu$ and let $d\sigma_{m-1}$ be the volume form of $\mathbb{S}^{m-1}$. Then the *Lipschitz-Killing curvature* $G(p,\nu)$ of M at (p,ν) is defined to be the scalar $G(p,\nu)$ such that

$$G_1^*(d\sigma_{m-1}) = G(p,\nu)\, dV_M \wedge d\sigma_{m-n-1}.$$

Then $G(p,\nu)$ is exactly the volume expansion ratio of G_1, that is,

$$G(p,\nu) = \lim_{D\to\{p\}} \frac{\mathrm{Vol}(G_1(D))}{\mathrm{Vol}(D)},$$

where $\mathrm{Vol}(G_1(D))$ denotes the signed volume of $G_1(D)$. In fact, $G(p,\nu)$ has the following geometric interpretation [CL]: $G(p,\nu)$ is equal to the Gauss-Kronecker curvature at p of the orthogonal projection of M onto the $(n+1)$-dimensional plane $L(\nu)$ spanned by T_pM and ν.

Let π be the canonical projection of N_1M into M. The integrals

$$K(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} G(p,\nu)\, d\sigma_{m-n-1} \quad \text{and}$$
$$K^*(p) := \frac{1}{c_{m-1}} \int_{\pi^{-1}(p)} |G(p,\nu)|\, d\sigma_{m-n-1}$$

are called the *total curvature* and the *total absolute curvature* of M at p, respectively. The integrals

$$\tau(M) := \int_M K\, dV_M, \quad \text{and} \quad \tau^*(M) := \int_M K^*\, dV_M$$

are called the *total curvature* and the *total absolute curvature of* M, respectively. Lipschitz and Killing have shown that $K(p)$ is an intrinsic quantity of M at p for n even (see [SS] for a more general result). However, $K(p) = 0$ for n odd. Both $\tau(M)$ and $\tau^*(M)$ remain unchanged even if the ambient space $\mathbb{R}^m$ is embedded into $\mathbb{R}^k$, $k > m$.

For $M^n \subset \mathbb{R}^m$, Fenchel [F2] generalized Hopf's theorem (1):

$$\int_M K\, dV_M = \chi(M). \tag{2}$$

In contrast, Chern and Lashof [CL] proved that

$$\int_M K^*\, dV_M \geq 2, \tag{3}$$

with equality if and only if M is a convex hypersurface in an $(n+1)$-dimensional plane, and that if $\int_M K^* dV_M < 3$ then M is homeomorphic to $\mathbb{S}^n$. Moreover, Morse theory tells us that

$$\int_M K^*\, dV_M \geq \sum_i \beta_i,$$

where β_i is the i-th Betti number of M ([W], Theorem 28).

3. Vision angle versus average density

A minimal submanifold Σ^n in $\mathbb{R}^m$ has the remarkable property that the density of Σ at $p \in \Sigma$ is bounded above by that of the cone $C = p \divideontimes \partial\Sigma$ at its vertex p. (We assume that Σ with its boundary is compact.) Recall that the *density* of Σ is defined as

$$\Theta_\Sigma(p) = \lim_{r\to 0} \frac{\operatorname{Vol}\Big(\Sigma \cap B_r^m(p)\Big)}{\operatorname{Vol}\Big(B_r^n(p)\Big)}.$$

Further, the density of a cone C has the interesting property that it equals the average of the densities of the orthogonal projections of C onto n-planes in $\mathbb{R}^m$. These properties will be verified in this section.

In what follows, we shall write $\overline{\nabla}$ for the Euclidean connection on $\mathbb{R}^m$, and $\nabla = \nabla_M$ for the induced connection on a submanifold M.

Lemma 1. *Let Σ be an n-dimensional minimal submanifold of $\mathbb{R}^m$, p a point of $\mathbb{R}^m$, and C an n-dimensional piecewise smooth cone with vertex p. Define the Euclidean distance function $r(x) = \operatorname{dist}(p,x), x \in \mathbb{R}^m$. Let $Y_1 = r\overline{\nabla} r$ and $Y_2 = r^{1-n}\overline{\nabla} r$, and define $\operatorname{div}_\Sigma Y_i = \operatorname{tr}_\Sigma \overline{\nabla} Y_i = \sum_j \langle \overline{\nabla}_{e_j} Y_i, e_j\rangle$, $\{e_1, \ldots, e_n\}$ being an orthonormal frame of Σ. Then*

(a) *On Σ, $\operatorname{div}_\Sigma Y_1 = n$ and $\operatorname{div}_\Sigma Y_2 \geq 0$;*

(b) *On C, $\operatorname{div}_C Y_1 = n$ and $\operatorname{div}_C Y_2 = 0$.*

We require that C be piecewise smooth, that is, a topological manifold which has a triangulation into simplices that are C^2 up to their boundaries.

Proof. Given an n-dimensional submanifold $M \subset \mathbb{R}^m$, it is well known that

$$\triangle_M x := (\triangle_M x_1, \ldots, \triangle_M x_m) = \vec{H},$$

where $\vec{H}$ is the mean curvature vector of M, the trace of its second fundamental form. Hence the orthogonal coordinate functions $x_1, \ldots, x_m$ of $\mathbb{R}^m$ are harmonic on a minimal submanifold Σ^n of $\mathbb{R}^m$. If we take p as the origin, then since $\vec{H} = 0$ on Σ,

$$\begin{aligned} \mathrm{div}_\Sigma(Y_1) &= \mathrm{div}_\Sigma(r\overline{\nabla} r) = \frac{1}{2}\triangle_\Sigma r^2 + \langle r\overline{\nabla} r, \vec{H}\rangle \\ &= \frac{1}{2}\sum \triangle_\Sigma x_i^2 = \sum x_i \triangle_\Sigma x_i + \sum |\nabla x_i|^2 = n. \end{aligned}$$

On the cone C, since $\vec{H}$ is perpendicular to $r\overline{\nabla} r = x \in C$, we have

$$\begin{aligned} \mathrm{div}_C(Y_1) &= \mathrm{div}_C(r\overline{\nabla} r) = \frac{1}{2}\triangle_C r^2 + \langle r\overline{\nabla} r, \vec{H}\rangle \\ &= \frac{1}{2}\sum \triangle_C x_i^2 = \langle x, \vec{H}\rangle + \sum |\nabla x_i|^2 = n. \end{aligned}$$

On the other hand, for $M = \Sigma$ or C,

$$\begin{aligned} \mathrm{div}_M Y_2 = \mathrm{div}_M(r^{-n} Y_1) &= -nr^{-n-1}\langle \nabla r, Y_1\rangle + r^{-n}\mathrm{div}_M(Y_1) \\ &= nr^{-n}\Big(- |\nabla r|^2 + 1\Big). \end{aligned}$$

Note that $|\nabla r| \leq 1$ on $M = \Sigma$ and $|\nabla r| \equiv 1$ on $M = C$. This completes the proof. □

Theorem 1. *Let Σ be a stationary n-rectifiable set with boundary Γ in $\mathbb{R}^m$, an open dense subset of Σ being a smooth minimal submanifold. Let C be the cone $p ⋇ \Gamma$, $p \in \mathbb{R}^m$. Then*

$$\Theta_\Sigma(p) \leq \Theta_C(p),$$

with equality if and only if $\Sigma = C$ and C is star-shaped with respect to p.

Proof. Compute the first variation of volume with respect to the (Lipschitz continuous) variation vector field

$$Y := r^{1-n}\overline{\nabla} r \quad \text{for } \ r \geq \varepsilon$$

and

$$Y := \varepsilon^{-n} r\overline{\nabla} r \quad \text{for } \ r \leq \varepsilon.$$

Then the first variation of Σ with respect to the flow with velocity field Y [Si, p. 80] is

$$\int_\Sigma \mathrm{div}_\Sigma Y \, dV_\Sigma,$$

which must equal

$$\int_\Gamma \langle Y, \nu_\Sigma \rangle \, dV_\Gamma,$$

where ν_Σ is the outward unit normal vector to Γ tangent to Σ.

Computing the divergence on smooth subsets of the stationary set Σ, we find by Lemma 1 (a)

$$\mathrm{div}_\Sigma Y \geq 0 \quad \text{for } \ r \geq \varepsilon, \tag{4}$$

with equality at points where $\overline{\nabla} r$ lies in the tangent space, and

$$\mathrm{div}_\Sigma Y = n\varepsilon^{-n} \quad \text{for } \ r \leq \varepsilon.$$

It follows that for each small ε,

$$\frac{\mathrm{Vol}(\Sigma \cap B_\varepsilon(p))}{|B_1^n|\varepsilon^n} \leq \frac{1}{n|B_1^n|} \int_\Gamma r^{1-n} \langle \overline{\nabla} r, \nu_\Sigma \rangle \, dV_\Gamma, \quad |B_1^n| := \mathrm{Vol}(B_1^n(0)). \tag{5}$$

Now apply Stokes' theorem to the integral of $\mathrm{div}_C Y$ on C:

$$\int_C \mathrm{div}_C Y \, dV_C = \int_{\partial C} \langle Y, \nu_C \rangle = \int_\Gamma \langle Y, \nu_C \rangle,$$

where ν_C is the outward unit conormal to Γ on C. Therefore, by Lemma 1(b)

$$\frac{\mathrm{Vol}(C \cap B_\varepsilon(p))}{|B_1^n|\varepsilon^n} = \frac{1}{n|B_1^n|} \int_\Gamma r^{1-n} \langle \overline{\nabla} r, \nu_C \rangle \, dV_\Gamma. \tag{6}$$

Note here that

$$0 \leq \langle \overline{\nabla} r, \nu_C \rangle$$

and

$$(7) \qquad \langle \overline{\nabla} r, \nu_\Sigma \rangle \leq \langle \overline{\nabla} r, \nu_C \rangle.$$

Thus, letting $\varepsilon \to 0$ in inequality (5) and equation (6), we get the desired density estimate. If equality holds, then we must have equality in inequalities (4) and (7), which implies $\Sigma = C$ and $\partial r / \partial \nu \geq 0$. □

Definition 1. Let π_p be the radial projection of $\mathbb{R}^m \setminus \{p\}$ onto $\partial B_1(p)$, the unit sphere centered at $p \in \mathbb{R}^m$. Define the *vision angle at* p of an $(n-1)$-rectifiable set $\Gamma \subset \mathbb{R}^m$ by

$$\Pi_p(\Gamma) = \mathrm{Vol}(\pi_p(\Gamma)),$$

and the *vision angle* of Γ by

$$\Pi(\Gamma) = \sup{}_{p \in \mathbb{R}^m} \Pi_p(\Gamma)).$$

Here the volume $\mathrm{Vol}(\pi_p(\Gamma))$ counts multiplicity.

Clearly we have for any $p \in \mathbb{R}^m$ and $C := p ⋇ \Gamma$

$$c_{n-1} \Theta_C(p) = \Pi_p(\Gamma^{n-1}) \leq \Pi(\Gamma), \quad c_{n-1} := \mathrm{Vol}(\mathbb{S}^{n-1}),$$

and hence we get the following corollaries to Theorem 1.

Corollary 1. *If* $\Gamma \subset \mathbb{R}^m$ *is an* $(n-1)$*-dimensional compact manifold, then any stationary rectifiable set* Σ *spanning* Γ *satisfies*

$$c_{n-1} \Theta_\Sigma(p) \leq \Pi_p(\Gamma)$$

for all $p \in \Sigma$.

Corollary 2. *If* $\Gamma \subset \mathbb{R}^m$ *is an* $(n-1)$*-dimensional compact manifold with* $\Pi(\Gamma) < 2c_{n-1}$*, then any immersed minimal submanifold* Σ *spanning* Γ *is embedded.*

Proof. An immersed submanifold Σ with density $\Theta_\Sigma(q) < 2$ at each point $q \in \mathbb{R}^m$ has no self-intersection. □

Remark. It may appear inappropriate to view $\Pi(\Gamma)$ as a total curvature. But it has its own merit, as the following example demonstrates. Define an immersed closed C^1 curve $\gamma \subset \mathbb{R}^2$ (the unit square plus four small loops at the corners) by

$$\gamma = \partial([-1,1]^2) \cup \{(x,y) : |x| > 1, |y| > 1, \\ [(|x|-1)^2 + (|y|-1)^2]^{3/2} = \varepsilon(|x|-1)(|y|-1)\}$$

and define a Jordan curve $\Gamma \subset \mathbb{R}^n$ to be an embedded C^2 curve C^1-close to γ. Then for small ε,

$$\int_\Gamma |\vec{k}| ds > 6\pi, \quad \text{however,} \quad \Pi(\Gamma) \approx 3\pi.$$

Hence by Corollary 2 any immersed minimal surface Σ spanning Γ is embedded since $2c_1 = 4\pi$, although the Ekholm-White-Wienholtz theorem [EWW] cannot give the same conclusion.

Let $G_n(\mathbb{R}^m)$ denote the *Grassmann manifold* of n-planes through the origin in $\mathbb{R}^m$, equipped with the unique $\mathbb{O}(m)$-invariant probability measure, and let $\text{Ave}_{P \in G_n(\mathbb{R}^m)}$ be the average over all $P \in G_n(\mathbb{R}^m)$. Denote by ψ_P the orthogonal projection of $\mathbb{R}^m$ onto $P \in G_n(\mathbb{R}^m)$.

Lemma 2. *Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n \subset \mathbb{R}^m$ centered at the origin O of $\mathbb{R}^m$ and let D be a domain in $\mathbb{S}^{n-1}$. Then*

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = \Theta_{O \divideontimes D}(O).$$

Proof. Assume that $a(D) > 0$ is a positive real number such that

$$\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = a(D) \cdot \Theta_{O \divideontimes D}(O). \tag{8}$$

Letting D shrink to a point $x \in \mathbb{S}^{n-1}$, one can define a function $a : \mathbb{S}^{n-1} \to \mathbb{R}$ given by

$$a(x) := \lim_{D \to \{x\}} \frac{\text{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\}}{\Theta_{O \divideontimes D}(O)}.$$

Then, by means of a partition of unity by functions of small support, one can see that

$$a(D) = \frac{\int_D a(x) dV_{\mathbb{S}^{n-1}}}{\text{Vol}(D)}.$$

Note here that $\mathbb{O}(n)$ is transitive on $\mathbb{S}^{n-1}$ and that the elements of $\mathbb{O}(n)$ preserve the volume form $dV_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$. Therefore one concludes that for all $x \in \mathbb{S}^{n-1}$,

$$a(x) \equiv c \quad \text{for a positive constant } c$$

and hence for any domain $D \subset \mathbb{S}^{n-1}$,

$$a(D) \equiv c.$$

Therefore it follows from equation (8) that

$$\mathrm{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(O \divideontimes D)}(O)\} = c \cdot \Theta_{O \divideontimes D}(O) \tag{9}$$

for any domain $D \subset \mathbb{S}^{n-1}$. However, for almost all $P \in G_n(\mathbb{R}^m)$,

$$\Theta_{\psi_P(O \divideontimes \mathbb{S}^{n-1})}(O) = \Theta_{O \divideontimes \mathbb{S}^{n-1}}(O) = 1.$$

Thus $c = 1$ in equation (9), which completes the proof. □

Theorem 2. *Let $\Gamma^{n-1} \subset \mathbb{R}^m$ be a compact submanifold. Then*

$$\mathit{\Pi}_q(\mathit{\Gamma}^{n-1}) = \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)}\{\mathit{\Pi}_{\psi_P(q)}(\psi_P(\mathit{\Gamma}))\}.$$

Proof. The cone $q \divideontimes \Gamma$ can be thought of as a union of infinitesimal cones $q \divideontimes \Delta\Gamma_i$ and then one can apply Lemma 2 to each $q \divideontimes \Delta\Gamma_i$. Hence

$$\begin{aligned}\Pi_q(\Gamma) &= c_{n-1}\,\Theta_{q \divideontimes \Gamma}(q) \\ &= c_{n-1}\,\mathrm{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Theta_{\psi_P(q \divideontimes \Gamma)}(\psi_P(q))\} \\ &= \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)}\{\Pi_{\psi_P(q)}(\psi_P(\Gamma))\}.\end{aligned}$$

□

We shall also require the following generalization of Fáry's theorem to any dimension n and to any codimension $m - n$, which was proved by Langevin and Shifrin ([LS], Proposition 2.15):

Theorem LS. *Let Γ^{n-1} be a smooth submanifold of $\mathbb{R}^m, m \geq n$. Then*

$$\frac{c_{n-1}}{2}\int_\Gamma K^* dV_\Gamma = \mathrm{Ave}_{P \in G_n(\mathbb{R}^m)}\int_{\psi_P(\Gamma)} |GK_{\psi_P(\Gamma)}|\, dV_{\psi_P(\Gamma)}.$$

4. Embeddedness of minimal submanifolds

It is tempting to propose a higher-dimensional extension of Ekholm-White-Wienholtz's theorem as follows:

Conjecture. If $q \in \Sigma$, a minimal submanifold of $\mathbb{R}^m$ spanning an $(n-1)$-dimensional compact manifold Γ, then

$$\Theta_\Sigma(q) \leq \frac{1}{2}\int_\Gamma K^* dV_\Gamma.$$

If this were known, one could prove the following as well:

If an $(n-1)$-dimensional compact connected manifold Γ satisfies $\int_\Gamma K^* dV_\Gamma < 4$, then any immersed minimal submanifold Σ spanning Γ is embedded.

Conjecture seems to be hard to prove as yet.

However, if we let Γ_i be a compact convex hypersurface of an affine n-plane $R_i^n \subset \mathbb{R}^m$, $i = 1, 2$, and define $\Gamma = \Gamma_1 \cup \Gamma_2$, then we may prove Conjecture for this case. Our proof uses the vision angle of Γ from a point of Σ, and averages over projections onto all n-dimensional subspaces P of $\mathbb{R}^m$. Namely, for $i = 1, 2$,

$$\Pi_{\psi_P(q)}(\psi_P(\Gamma_i)) \leq c_{n-1} = \int_{\psi_P(\Gamma_i)} |GK_{\psi_P(\Gamma_i)}| dV_{\psi_P(\Gamma_i)}, \tag{10}$$

since $\psi_P(\Gamma_i)$ is a convex hypersurface in $\psi_P(R_i^n)$. Here equality holds for all P if and only if q is in R_i^n and inside Γ_i. Thus we have the following:

Theorem 3. *Given two n-planes R_1^n, R_2^n in $\mathbb{R}^m$, let Γ_i be a compact convex hypersurface in R_i^n, $i = 1, 2$. If $\Gamma = \Gamma_1 \cup \Gamma_2$, then any n-dimensional minimal submanifold Σ spanning Γ is either a union of two flat domains of R_i^n or is nonflat and has no self intersection.*

Proof. We may compute that

$$\int_\Gamma K^* dV_\Gamma = \sum_{i=1,2} \int_{\Gamma_i} K^* \, dV_{\Gamma_i} = 4.$$

Thus by inequality (10) and Corollary 1 we have $\Theta_\Sigma \leq 2$. If $\Theta_\Sigma = 2$, inequality (10) and Corollary 1 imply Σ is flat. If $\Theta_\Sigma < 2$, Σ is nonflat and has no self intersection. □

Remark. It should be mentioned that R. Schoen [Sc] proved a theorem which implies a special case of Theorem 3:

If $\Gamma = \Gamma_1 \cup \Gamma_2$ *where* Γ_1, Γ_2 *are* $(n-1)$*-spheres in parallel* n*-planes with the line* ℓ *joining their centers being orthogonal to these hyperplanes, then any immersed minimal submanifold* Σ^n *spanning* Γ *is a hypersurface of revolution with axis* ℓ*. In particular,* Σ *is a catenoid or a pair of plane disks.*

References

[AW] C. B. Allendörfer and A. Weil, *The Gauss-Bonnet theorem for Riemannian polyhedra*, Trans. Amer. Math. Soc. **53** (1943), 101–129.

[C1] S.-S. Chern, *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Annals of Math. **45** (1944), 747–752.

[C2] S.-S. Chern, *On the curvatura integra in a Riemannian manifold*, Annals of Math. **46** (1945), 674–684.

[CL] S.-S. Chern and R. K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.

[EK] J. Eells and N. Kuiper, *Manifolds which are like projective planes*, Institut des Hautes Etudes Sci. Pub. Maths. **14** (1962), 5–46.

[EWW] T. Ekholm, B. White, and D. Wienholtz, *Embeddedness of minimal surfaces with total boundary curvature at most* 4π, Ann. Math. **155** (2002), 109–234.

[Fa] I. Fáry, *Sur la courbure totale d'une courbe gauche faisant un noeud*, Bull. Soc. Math. France **77** (1949), 128–138.

[F1] W. Fenchel, *Über Krümmung und Windung geschlossener Raumkurven*, Math. Ann. **101** (1929), 238–252.

[F2] W. Fenchel, *On total curvature of Riemannian manifolds: I*, J. London Math. Soc. **15** (1940), 15–22.

[H] H. Hopf, *Über die Curvatura integra geschlossener Hyperflächen*, Math. Ann. **95** (1925), 340–367.

[LS] R. Langevin and T. Shifrin, *Polar varieties and integral geometry*, Amer. J. Math. **104** (1982), 553–605.

[M] J. Milnor, *On the total curvature of knots*, Annals of Math. **52** (1950), 248–257.

[N] J. C. C. Nitsche, *A new uniqueness theorem for minimal surfaces*, Arch. Rat. Mech. Anal. **52** (1973), 319–329.

[Sc] Richard M. Schoen, *Uniqueness, symmetry, and embeddedness of minimal surfaces*, J. Differential Geom. **18** (1983), 791–809.

[SS] J. A. Schouten and D. J. Struik, *On curvature and invariants of deformation of a V_m in V_n*, Proc. Kon. Akad. v. Wetensch. Amsterdam **24** (1922), 146–161.

[Si] L. Simon, *Lectures on geometric measure theory*, Proc. Centre for Mathematical Analysis, Australian National University, Vol. 3, 1983.

[W] T. J. Willmore, *Total curvature in Riemannian geometry*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], New York, 1982.

Korea Institute for Advanced Study
Hoegiro 85, Dongdaemun-gu
Seoul 02455, South Korea
E-mail address: choe@kias.re.kr

School of Mathematics, University of Minnesota
127 Vincent Hall, 206 Church St. SE
Minneapolis MN 55455, USA
E-mail address: gulliver@math.umn.edu

Received January 12, 2012