Complete ancient solutions to the Ricci flow with pinched curvature

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We show that any complete ancient solution to the Ricci flow equation with possibly unbounded curvature has constant curvature at each time if its curvature is pinched all the time. This is a slight extension of a result of Brendle, Huisken and Sinestrari for ancient solutions on compact manifolds. In our proof, we adapt their argument relying on the maximum principle with the help of Chen's technique.

1. Introduction

In this paper, we study complete ancient solutions to the Ricci flow equation with possibly unbounded curvature on noncompact manifolds. A *Ricci flow* is a family $g(t)$ of Riemannian metrics on a manifold M evolving along the equation

$$\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t)),$$

where $\mathrm{Ric}(g(t))$ is the Ricci tensor of the metric $g(t)$. We say a Ricci flow $g(t)$ is *complete* when the metric $g(t)$ is complete for each t. An *ancient solution* is a Ricci flow $g(t)$ which exists on the infinite time interval $(-\infty, 0)$. Ancient solutions appear naturally in the study of singularities of Ricci flow. Moreover, they are of geometric interest also in their own right, e.g. [Yo].

Ricci flow has been the object of extensive study in Riemannian geometry since its introduction by Hamilton [Ha]. Some of the relatively recent prominent results about Ricci flow on closed manifolds are the convergence theorems proved by Böhm–Wilking [BW2], Brendle–Schoen [BS], Brendle [Br], etc. We recommend Brendle's monograph [Br4] for this topic.

On the other hand, Ricci flow on noncompact manifolds has been studied and applied as well, e.g. Theorem 18 below. However our knowledge, especially of Ricci flow with unbounded curvature, e.g. Topping [To], is quite limited compared to the compact case and we would like to make a contribution to a better understanding.

We now state the main result of this paper, which slightly extends the result of Brendle–Huisken–Sinestrari [BHS]. Hereafter, except for the next section, we use R or $R_{ijkl} := R(e_i, e_j, e_k, e_l)$ for an orthonormal frame $\{e_i\}$ to denote the curvature tensor or operator, while the scalar curvature is denoted as scal.

Theorem 1 (cf. [BHS]). *Let $g(t), t \in (-\infty, 0)$ be a complete ancient solution to the Ricci flow equation on a manifold M of dimension 3. Suppose that it satisfies with a uniform constant $\rho > 0$ that*

$$\mathrm{Ric}(g(t)) \geq \rho \, \mathrm{scal}_{g(t)} \, g(t) \geq 0$$

on $M \times (-\infty, 0)$. Then $(M, g(t))$ has nonnegative constant curvature for each $t \in (-\infty, 0)$.

Theorem 2 (cf. [BHS]). *Let $g(t), t \in (-\infty, 0)$ be a complete ancient solution to the Ricci flow equation on a manifold M of dimension $n \geq 4$. Suppose that it satisfies with a uniform constant $\rho > 0$ that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq \rho \, \mathrm{scal}_{g(t)} \geq 0$$

for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$ on $M \times (-\infty, 0)$. Then $(M, g(t))$ has nonnegative constant curvature for each $t \in (-\infty, 0)$.

The curvature pinching conditions in the above theorems are related to the assumptions of the convergence theorems of Hamilton [Ha] and Brendle [Br] respectively. Brendle–Huisken–Sinestrari [BHS] proved the above theorems on closed manifolds. We prove our Theorems 1 and 2 by adapting their argument in [BHS] and adopting Chen's argument in [Ch]. While separate preferable proofs were given to the counterparts of Theorems 1 and 2 respectively in [BHS], we shall give a unified proof for them.

The maximum principle is the key of the argument in [BHS]. However it is known that the maximum principle may fail on noncompact manifolds. Nevertheless Chen [Ch] showed that a maximum principle argument is useful for complete Ricci flow especially in dimension ≤ 3, cf. Simon [Si]. Our result is a spin-off of an attempt concerning the maximum principle for Ricci flow on noncompact manifolds in higher dimension.

In Remark 17 below we make some comment on the relevance of our theorems to the compactness theorems for Riemannian manifolds with pinched curvature in the literature.

We then apply our result to Ricci solitons. We call a triple (M, g, V) a *Ricci soliton* if it satisfies

$$\text{Ric}(g) + \frac{1}{2}\mathcal{L}_V g = \lambda g$$

for some real number $\lambda \in \mathbb{R}$, where $\mathcal{L}_V$ denotes the Lie derivative with respect to the vector field $V \in \Gamma(TM)$ on M. It is said to be *shrinking* or *steady* or *expanding* when $\lambda > 0$ or $\lambda = 0$ or $\lambda < 0$ respectively. A Ricci soliton is said to be *complete* if the metric g and the vector field V are complete in their senses. A *gradient Ricci soliton* is a Ricci soliton (M, g, V) for which V is a gradient vector field of a function $f \in C^\infty(M)$.

Complete Ricci solitons generate self-similar solutions to the Ricci flow equation which exist on infinite intervals. Especially complete shrinking and steady Ricci solitons form an important class of ancient solutions. Geometry of gradient Ricci solitons has also been the subject of extensive study by many authors. The following corollary holds for Ricci solitons which are expanding and not necessarily of gradient type as well.

Corollary 3. *Let (M, g, V) be a complete Ricci soliton of dimension $n \geq 3$. Suppose that g satisfies with a uniform constant $\rho > 0$ that*

$$\text{Ric}(g) \geq \rho \,\text{scal}_g\, g \geq 0$$

on M if $n = 3$ or

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq \rho \,\text{scal}_g \geq 0$$

for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and all $\lambda \in [-1, 1]$ on M if $n \geq 4$. Then (M, g) has nonnegative constant curvature.

Brendle [Br2, Br3] obtained stronger results for Einstein manifolds. It would be interesting to ask to what extent we can further our Corollary 3 for Ricci solitons; see also Ni–Wallach [NWa, Remark 5.3]. Some more remarks on Corollary 3 are given in Remark 19 below.

In the next section, we discuss Hamilton's maximum principle for complete Ricci flow on noncompact manifolds. In Section 3, we present the proofs of Theorems 1 and 2 and Corollary 3. The final section is devoted to an observation about complete Ricci flow on noncompact surfaces with nonnegative curvature.

Throughout this paper, $B_t(\cdot, r) \subset M$ denotes an open metric ball of radius $r > 0$. The letter t means that it is defined with the metric $g(t)$ at time t.

2. Maximum principle for Ricci flow on noncompact manifolds

In this section, we make a detour and discuss the maximum principle for Ricci flow on noncompact manifolds. The main theorem of this section is Theorem 6 stated below. Since we do not use it in our proof of Theorems 1 and 2, a reader in a hurry can safely skip to the next section.

In stating our result, we stick to the setting of Section 1 of the paper [BW] by Böhm–Wilking except for allowing our manifold M to be noncompact. A reader can consult [BW, Section 1] for possible ambiguous terminologies.

We suppose that $g(t), t \in [0, \delta]$ is a complete Ricci flow on a possibly noncompact manifold M with possibly unbounded curvature and $\pi : V \to M$ is a vector bundle over M with a time-independent metric k, which induces the norm $|\cdot|_k$ and the distance $d_k(\cdot, \cdot)$ on a fiber $V_p := \pi^{-1}(p)$ for each $p \in M$. Then the Laplacian $\Delta_t R \in \Gamma(V)$ is defined for a section $R \in \Gamma(V)$ and $t \in [0, \delta]$. Let $f : V \to V$ is a locally Lipschitz map which is assumed to preserve fibers V_p for each $p \in M$ and to be invariant under parallel transport in the sense explained in [BW].

We consider a family of sections $R(\cdot, t) \in \Gamma(V), t \in [0, \delta]$ satisfying the evolution equation

$$(4) \qquad \frac{\partial}{\partial t} R(\cdot, t) = \Delta_t R(\cdot, t) + f(R(\cdot, t))$$

and the corresponding ODE

$$(5) \qquad \frac{d}{dt} R_p = f(R_p)$$

on a fiber V_p for each $p \in M$.

We say that a continuous family $\{C(t)\}_{t \in [0, \delta]}$ of subsets of V is preserved by the ODE (5) if any solution $R_p(\cdot)$ to (5) on a fiber V_p satisfies that $R_p(t) \in C_p(t) := C(t) \cap V_p$ for all $t \in [s, \delta]$ provided that $R_p(s) \in C_p(s)$ for some $s \in [0, \delta)$.

Theorem 6 (Dynamical maximum principle for complete Ricci flow). *Let $g(t), t \in [0, \delta]$ be a complete Ricci flow on a manifold M of dimension $n \geq 2$. and $\{C(t)\}_{t \in [0, \delta]}$ be a continuous family of subsets of V which are fiberwise closed convex and invariant under parallel transport is preserved by the ODE (5).*

We assume that $R(\cdot, t) \in \Gamma(V), t \in [0, \delta]$ is a solution to the PDE (4) with homogeneous f in the sense that there exists some $\alpha > 0$ with $f(cR) = c^{\alpha+1} f(R)$ for any $R \in V$ and $c > 0$ and in addition that

$$|R(p, t)|_k \leq \ell(d_t(p, o))$$

with some $o \in M$ and $\ell : \mathbb{R} \to \mathbb{R}$ with $\ell(r)^\alpha / \log r \to 0$ as $r \to \infty$ for any $(p, t) \in M \times [0, \delta]$.

Then $R(p, t) \in C(t)$ for all $(p, t) \in M \times [0, \delta]$ provided that $R(p, 0) \in C(0)$ for all $p \in M$.

Theorem 6 is a form of the dynamical maximum principle for Ricci flow that we are familiar with except the minor unfamiliarity that we do not impose the bounded curvature condition on the Ricci flow $g(t)$ and impose only a mild growth condition on $|R(\cdot, t)|_k$. We suspect that our growth assumption is optimal.

A simple proof of the maximum principle on compact manifolds is given in e.g. Böhm–Wilking [BW, Theorem 1.1], while a thorough treatment of the maximum principle on noncompact manifolds with time-dependent metrics is found in e.g. [Vol2-II, Chapter 12].

We will not use Theorem 6 anywhere in this paper. Wishing to prompt further study of noncompact Ricci flow, we include its proof here.

As in Chen's proof of Theorem 9 stated below as well as our proofs of Theorems 1 and 2, we utilize the following lemma of Perelman [Pe, Lemma 8.3].

Lemma 7 (Perelman [Pe]). *Let $g(t)$ be a complete Ricci flow on a manifold M of dimension $n \geq 2$. Assume that $\mathrm{Ric}(\cdot, t_0) \leq (n - 1)K$ on a ball $B_{t_0}(o, r_0) \subset M$ for some $K \in \mathbb{R}$. Then we have*

$$\left(\frac{\partial}{\partial t} - \Delta_{t_0} \right) d_{t_0}(\cdot, o) \geq -(n - 1) \left(\frac{2}{3} K r_0 + r_0^{-1} \right)$$

outside $B_{t_0}(o, r_0)$ at time t_0. The inequality is understood in the barrier sense.

In our proof of Theorem 6 we combine the arguments of Böhm–Wilking [BW, Theorem 1.1], Brendle [Br4, Theorem 5.10] and Chen [Ch].

Proof of Theorem 6. We take $r_0 > 0$ such that $\mathrm{Ric}(\cdot, t) \leq (n - 1)r_0^{-2}$ on $B_t(o, r_0)$ for each $t \in [0, \delta]$. Choose $\beta \in (0, 1)$ with $\alpha + \beta \geq 1$ and let $\varphi :$

$\mathbb{R} \to [0,1]$ be a nonincreasing C^2-function such that $\varphi \equiv 1$ on $(-\infty, 1/2]$, $\varphi \equiv 0$ on $[1, \infty)$ and $2(\varphi')^2/\varphi - \varphi'' \leq C\varphi^\beta$ for some constant $C < \infty$.

For $S \in V_p$ with $p \in M$ and $t \in [0, \delta]$, let $r_t(S) := d_k(S, C_p(t))$. We fix a large number $A \gg 1$ and consider

$$u(p, t) := \varphi \left(\frac{d_t(p, o) + \frac{5}{3}(n-1)r_0^{-1}t}{A} \right) r_t(R(p, t))$$

and $u(t) := \max_{p \in M} u(p, t)$ for $p \in M$ and $t \in [0, \delta]$.

We claim that $u(\cdot) \leq (C/A^2)^{1/2\alpha}$ on $[0, \delta]$. To verify this claim, we suppose that $u(t_0) > (C/A^2)^{1/2\alpha}$ for some $t_0 \in (0, \delta]$ and consider

$$u'(t) := \limsup_{h \searrow 0} \frac{u(t) - u(t - h)}{h}$$

for $t \in (0, \delta]$.

For a number $B \in \mathbb{R}$ with $\exp(-B\delta) = (C/A^2)^{1/2\alpha}$ we define the time

$$t_B := \inf \left\{ t \in [0, \delta] : u(t) \geq \exp(B(t - 2\delta)) \right\}.$$

Since $u(t) \searrow 0$ as $t \searrow 0$, we have $t_B \in (0, t_0)$,

$$u(t_B) = \exp(B(t_B - 2\delta)) > \exp(-2B\delta) = (C/A^2)^{1/\alpha}$$

and

$$u(t) < \exp(B(t - 2\delta)) = u(t_B)\exp(B(t - t_B))$$

for any $t \in [0, t_B)$. This implies $u'(t_B) \geq B \cdot u(t_B)$.

We take $p_B \in B_{t_B}(o, A)$ with $u(t_B) = u(p_B, t_B)$. We then have

$$u'(t_B) \leq \limsup_{h \searrow 0} \frac{u(t_B) - u(p_B, t_B - h)}{h}$$

$$= \limsup_{h \searrow 0} \frac{\varphi'}{A} \left(\frac{\partial}{\partial t} d_{t_B}(p_B, o) + \frac{5}{3}(n-1)r_0^{-1} \right) r_{t_B}(R(p_B, t_B))$$

$$+ \varphi \limsup_{h \searrow 0} \frac{r_{t_B}(R(p_B, t_B)) - r_{t_B - h}(R(p_B, t_B - h))}{h}.$$

Letting $C_r := \sup_{(p,t) \in M \times [0, \delta]} d_k(0_p, C_p(t)) < \infty$ with $0_p \in V_p$ being the origin of the fiber V_p for $p \in M$, we note $r_{t_B}(R(p_B, t_B)) \leq \ell(A) + C_r$.

At $(p_B, t_B) \in M \times (0, \delta]$ we have

$$0 = \nabla u = r_{t_B} \nabla \varphi + \varphi \nabla r_{t_B}$$

and

$$
\begin{aligned}
0 \geq \Delta u \\
= \left(\frac{\varphi''}{A^2} + \varphi' \frac{\Delta d_{t_B}(\cdot, o)}{A} \right) r_{t_B} + 2 \langle \nabla \varphi, \nabla r_{t_B} \rangle + \varphi k (\nabla r_{t_B}, \Delta_{t_B} R) \\
\geq \left(\frac{\varphi''}{A^2} - 2 \frac{|\nabla \varphi|^2}{\varphi} + \varphi' \frac{\Delta d_{t_B}(\cdot, o)}{A} \right) r_{t_B} + \varphi k (\nabla r_{t_B}, \Delta_{t_B} R) \\
\geq -\frac{C}{A^2} \varphi^\beta r_{t_B} + \varphi' \frac{\Delta d_{t_B}(\cdot, o)}{A} r_{t_B} + \varphi k (\nabla r_{t_B}, \Delta_{t_B} R) \\
> -\frac{C(\ell(A) + C_r)^\alpha}{A^2} \frac{A^2}{C} \varphi r_{t_B} + \varphi' \frac{\Delta d_{t_B}(\cdot, o)}{A} r_{t_B} + \varphi k (\nabla r_{t_B}, \Delta_{t_B} R),
\end{aligned}
$$

cf. the corresponding part in the proof of Lemma 12 below.

From the proof of [BW, Theorem 1.1] we borrow an estimate:

$$
\begin{aligned}
\limsup_{h \searrow 0} & \frac{r_{t_B}(R(p_B, t_B)) - r_{t_B - h}(R(p_B, t_B - h))}{h} \\
\leq \limsup_{h \searrow 0} & \frac{r_{t_B}(R(p_B, t_B)) - r_{t_B - h}(R(p_B, t_B) - h f(R(p_B, t_B)))}{h} \\
& + k(\nabla r_{t_B}, \Delta_{t_B} R(p_B, t_B)) \\
\leq \limsup_{h \searrow 0} & |f(S_h) - f(R(p_B, t_B))|_k + k(\nabla r_{t_B}, \Delta_{t_B} R(p_B, t_B)),
\end{aligned}
$$

where $S_h \in C_{p_B}(t_B - h) \subset V_{p_B}$ is the unique point with

$$|R(p_B, t_B) - h f(R(p_B, t_B)) - S_h|_k = r_{t_B - h}(R(p_B, t_B) - h f(R(p_B, t_B))).$$

We note $|S_h|_k \leq 2\ell(A) + C_r$. Since f is homogeneous, locally Lipschitz and hence Lipschitz on compact subsets, there exists a number $L < \infty$ such that

$$(8) \qquad |f(S_h) - f(R(p_B, t_B))|_k \leq L (2\ell(A) + C_r)^\alpha |S_h - R(p_B, t_B)|_k$$

for small $h > 0$. As $h \searrow 0$ the right hand side of (8) tends to $L(2\ell(A) + C_r)^\alpha r_{t_B}(R(p_B, t_B))$.

If $d_{t_B}(p_B, o) > r_0$, we utilise Lemma 7 to estimate $u'(t_B)$ from above. If $d_{t_0}(p_0, o) \leq r_0$, we have $\varphi \equiv 1$ on a neighborhood of (p_B, t_B) and the terms

with the differential of φ disappear. We then gather that

$$B \cdot u(t_B) \leq u'(t_B) \leq ((\ell(A) + C_r)^\alpha + L(2\ell(A) + C_r)^\alpha) \, u(t_B).$$

If $A \gg 1$ is large enough, this leads to a contradiction. Therefore we infer that $u(\cdot) \leq (C/A^2)^{1/2\alpha}$ on $[0, \delta]$ for any large $A \gg 1$. We let $A \to \infty$ to conclude that $r_t(R(p, t)) = 0$ for any $(p, t) \in M \times [0, \delta]$. This completes the proof of Theorem 6. $\qquad\square$

3. Ancient solutions on noncompact manifolds

In this section, we present proofs of Theorems 1 and 2 stated in the introduction. As in [BHS], Lemma 12 below which is based on the maximum principle argument is the key lemma. Differing from [BHS], we give a unified proof for them.

For $n \geq 2$, we let $\mathcal{C}_B(\mathbb{R}^n)$ be the space of algebraic curvature tensors or operators on $\mathbb{R}^n$, i.e., self-adjoint linear maps of $\mathfrak{so}(n)$ enjoying the Bianchi identity. It carries a natural inner product $\langle \cdot, \cdot \rangle$ which induces a norm $|\cdot|$ and a distance $d(\cdot, \cdot)$ on it.

Before starting the proof, we make some comments. As already remarked several times, our proof of the theorems relies on a maximum principle argument as in [BHS]. It is known that a maximum principle does not always hold for solutions to the heat equation on noncompact manifolds. Nevertheless Chen [Ch, Proposition 2.1] managed to prove the following theorem for complete Ricci flow.

Theorem 9 (Chen [Ch]). *Let $g(t), t \in [0, T)$ be a complete Ricci flow on a possibly noncompact manifold M of dimension $n \geq 2$. If its scalar curvature satisfies that $\mathrm{scal}_{g(0)} \geq -1/K$ on M with $K \in [0, \infty)$, then*

$$\mathrm{scal}_{g(t)} \geq -\frac{1}{(2t/n) + K}.$$

on M for each $t \in [0, T)$. In particular, if $\mathrm{scal}_{g(0)} \geq 0$ on M, then $\mathrm{scal}_{g(t)} \geq 0$ on M for each $t \in [0, T)$. Moreover, any complete ancient solution has nonnegative scalar curvature.

Chen [Ch], cf. Chen–Xu–Zhang [CXZ], also proved that any complete Ricci flow in dimension ≤ 3 preserves nonnegativity of sectional curvature and that any complete ancient solution in dimension ≤ 3 has nonnegative sectional curvature.

It is natural to expect that Chen's argument can be used to see that complete Ricci flow preserves some nonnegative curvature conditions, such as Wilking's condition [Wi], in higher dimension as well. However in higher dimension it would not be true that all ancient solutions, especially Ricci flat metrics, satisfy a fruitful nonnegative curvature condition. This might explain the difficulty in generalizing Chen's result to higher dimension.

In his proof of Theorem 9, Chen [Ch] exploited the quadratic term in the evolution equation of the scalar curvature:

$$\frac{\partial}{\partial t}\mathrm{scal} = \Delta\mathrm{scal} + 2|\mathrm{Ric}|^2.$$

On the other hand, we instead make use of the quadratic term in the evolution equation of the curvature tensor:

$$\frac{\partial}{\partial t}R = \Delta R + Q(R),$$

where $Q(R)$ is defined by

$$Q(R)_{ijkl} = R_{ijpq}R_{klpq} + 2(R_{ipkq}R_{jplq} - R_{iplq}R_{jpkq}).$$

In order to state the main lemma, we consider a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$ satisfying the following property $(*)$ as in [BHS]:

- C is closed convex and $O(n)$-invariant.
- $Q(R)$ lies in the interior of the tangent cone $T_R C$ for any $R \in C \setminus \{0\}$;
- Every curvature tensor $R \in C \setminus \{0\}$ has positive scalar curvature;
- The curvature tensor $I_{ijkl} := \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ of curvature 1 is in the interior of C.

For a closed convex subset $C \subset \mathcal{C}_B(\mathbb{R}^n)$, we let $r(\cdot) := d(\cdot, C)$ and $\pi : \mathcal{C}_B(\mathbb{R}^n) \to C$ be the map which assigns to a point $S \in \mathcal{C}_B(\mathbb{R}^n)$ the closest point in C to S, i.e., $d(S, \pi(S)) = r(S)$. Let $\xi(S) := (S - \pi(S))/|S - \pi(S)| \in \mathcal{C}_B(\mathbb{R}^n)$ for any $S \in \mathcal{C}_B(\mathbb{R}^n) \setminus C$. Note that $\nabla r(S) = \xi(S)$ for any $S \in \mathcal{C}_B(\mathbb{R}^n) \setminus C$.

In the rest of this section, $C \subset \mathcal{C}_B(\mathbb{R}^n)$ denotes a fixed closed cone satisfying property $(*)$.

We start as in [BHS] by noting that for a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$ satisfying property $(*)$ there exist $\Lambda < \infty$ and $\mu > 0$ such that

$$|\mathrm{Ric}(R)| \leq \Lambda\,\mathrm{scal}(R) \quad \text{for any } R \in C$$

and

$$\langle Q(\pi(S)), \xi(S) \rangle \le -3\mu|\pi(S)|^2 \text{ for any } S \in \mathcal{C}_B(\mathbb{R}^n) \setminus C.$$

Lemma 10 (cf. [BHS, Lemma 5]). *There exists a number $\delta > 0$ such that any algebraic curvature tensor $R \in C$ with*

$$S := R + (1 - t\operatorname{scal}(R))I \in \mathcal{C}_B(\mathbb{R}^n) \setminus C$$

for some $t \in (0, \delta]$ satisfies

(11) $$\langle Q(R) - \operatorname{scal}(R)I - 2t|\operatorname{Ric}(R)|^2 I, \xi(S) \rangle \le -\mu|S|^2.$$

Proof. Suppose that $R \in C$ and $S \in \mathcal{C}_B(\mathbb{R}^n) \setminus C$. Since $Q : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)$ is locally Lipschitz and hence Lipschitz on the unit ball in $\mathcal{C}_B(\mathbb{R}^n)$, there exists a number $L < \infty$ such that

$$\begin{aligned}
\langle Q(R), \xi(S) \rangle &= \langle Q(R) - Q(\pi(S)), \xi(S) \rangle + \langle Q(\pi(S)), \xi(S) \rangle \\
&\le L \max\{|R|, |\pi(S)|\}|R - \pi(S)| - 3\mu|\pi(S)|^2
\end{aligned}$$

Since $S \notin C$, we have $t\operatorname{scal}(R) > 1$ and

$$|R - \pi(S)| \le |R - S| = (t\operatorname{scal}(R) - 1)|I| < t\operatorname{scal}(R)|I|.$$

We also have

$$-(\operatorname{scal}(R) + 2t|\operatorname{Ric}(R)|^2)\langle I, \xi(S) \rangle < (1 + 2\Lambda^2)t\operatorname{scal}(R)^2|I|.$$

On the other hand, since $1 - t\operatorname{scal}(R) < 0$ and $\langle S - \pi(S), \pi(S) \rangle = 0$, we have

$$|R| \ge |S| \ge |\pi(S)|$$

and

$$|\pi(S)| \ge |R| - |R - \pi(S)| > \frac{|R|}{2} \ge \frac{1}{2}\sqrt{\frac{n(n-1)}{2}}\operatorname{scal}(R)$$

provided that t is small enough.

Therefore we gather that there exists a small $\delta > 0$ satisfying the desired property. $\qquad\square$

As in [BHS], the following Lemma 12 is the main lemma in our proof of Theorems 1 and 2. As in Chen's proof of Theorem 9, we utilise Perelman's Lemma 7 recalled in the previous section.

Lemma 12 (cf. [BHS, Corollary 7]). *Let $g(t), t \in [0, \delta]$ be a complete Ricci flow on a possibly noncompact manifold M of dimension $n \geq 2$. Suppose that the curvature operator $R_{g(t)}$ of $g(t)$ lies in the cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$ satisfying property $(*)$ on $M \times [0, \delta]$. Then*

$$S_{g(t)} := R_{g(t)} + (1 - t \operatorname{scal}_{g(t)})I \in C$$

on $M \times [0, \delta]$.

Proof. We put $\tilde{Q}(R) := Q(R) - \operatorname{scal}(R)I - 2t|\operatorname{Ric}(R)|^2 I$. Then we have

$$\frac{\partial}{\partial t} S = \Delta S + \tilde{Q}(R)$$

and Lemma 10 states that for any $R \in \mathcal{C}_B(\mathbb{R}^n)$ with $S \notin C$

$$(13) \qquad \langle \nabla r(S), \tilde{Q}(R) \rangle \leq -\mu |S|^2 \leq -\mu \cdot r^2(S).$$

We fix a point $o \in M$ and take $r_0 > 0$ such that $\operatorname{Ric}(\cdot, t) \leq (n-1)r_0^{-2}$ on $B_t(o, r_0)$ for each $t \in [0, \delta]$. Let $\varphi : \mathbb{R} \to [0, 1]$ be a nonincreasing C^2-function such that $\varphi \equiv 1$ on $(-\infty, 1/2]$, $\varphi \equiv 0$ on $[1, \infty)$ and $2(\varphi')^2/\varphi - \varphi'' \leq C_\varphi \sqrt{\varphi}$ for some constant $C_\varphi < \infty$.

We fix a large number $A \gg 1$ and consider

$$u(p, t) := \varphi \left(\frac{d_t(p, o) + \frac{5}{3}(n-1)r_0^{-1}t}{A} \right) r(S(p, t))$$

and $u(t) := \max_{p \in M} u(p, t)$ for $p \in M$ and $t \in [0, \delta]$.

We now claim that $u(\cdot) \leq C/A^2$ with $C := (2C_\varphi/\mu)^2$ on $[0, \delta]$. To verify this claim, we consider

$$u'(t) := \limsup_{h \searrow 0} \frac{u(t) - u(t - h)}{h}$$

for $t \in (0, \delta]$.

We suppose that $u(t_0) > C/A^2$ for some $t_0 \in (0, \delta]$ and take $p_0 \in M$ with $u(t_0) = u(p_0, t_0)$. We then have

$$
\begin{aligned}
u'(t_0) &\leq \limsup_{h \searrow 0} \frac{u(t_0) - u(p_0, t_0 - h)}{h} \\
&= \frac{\varphi'}{A} \left(\frac{\partial}{\partial t} d_{t_0}(p_0, o) + \frac{5}{3}(n-1)r_0^{-1} \right) r(S(p_0, t_0)) \\
&\quad + \varphi \left\langle \nabla r(S(p_0, t_0)), \Delta S(p_0, t_0) + \tilde{Q}(R(p_0, t_0)) \right\rangle.
\end{aligned}
$$

In order to estimate this from above further, we replace $d_{t_0}(\cdot, o)$ by $d_{t_0}(\cdot, o') + d_{t_0}(o', o)$ with $o' \in M$ being a point on a minimal geodesic joining p_0 and o with respect to the metric $g(t_0)$ if $d_{t_0}(\cdot, o)$ is not smooth around $p_0 \in M$.

We set $\underline{u}(p, t) := \varphi(p, t)\underline{r}(S(p, t))$ for $(p, t) \in M \times [0, \delta]$ with $\underline{r}(T) := \langle T, \xi(S(p_0, t_0)) \rangle$ for $T \in \mathcal{C}_B(\mathbb{R}^n)$ being the distance to the hyperplane in $\mathcal{C}_B(\mathbb{R}^n)$ through $\pi(S(p_0, t_0))$ and perpendicular to $\xi(S(p_0, t_0))$. We have $\underline{u}(\cdot, t_0) \leq u(\cdot, t_0)$ on a neighborhood of p_0 and $\underline{u}(p_0, t_0) = u(p_0, t_0)$.

At $(p_0, t_0) \in M \times (0, \delta]$ we have

$$
0 = \nabla \underline{u}(p_0, t_0) = \underline{r}\nabla \varphi + \varphi \nabla \underline{r}
$$

and

$$
\begin{aligned}
0 \geq \Delta \underline{u}(p_0, t_0) &\geq \left(\frac{\varphi''}{A^2} + \varphi' \frac{\Delta d_{t_0}(\cdot, o)}{A} \right) \underline{r} + 2\langle \nabla \varphi, \nabla \underline{r} \rangle + \varphi \langle \Delta S, \xi(S) \rangle \\
&\geq \left(-\frac{C_\varphi}{A^2} \sqrt{\varphi} \underline{r} + \varphi' \frac{\Delta d_{t_0}(\cdot, o)}{A} \right) \underline{r} + \varphi \langle \Delta S, \xi(S) \rangle.
\end{aligned}
$$

If $d_{t_0}(p_0, o) > r_0$, by Lemma 7, Inequality (13) and $u(p_0, t_0) > C/A^2$, we obtain

$$
\begin{aligned}
u'(t_0) &\leq \frac{C_\varphi}{A^2} \sqrt{\varphi} r + \varphi \left\langle \nabla r, \tilde{Q}(R) \right\rangle \\
&\leq \frac{1}{2} \mu \varphi r^2 + \frac{2C_\varphi^2}{A^4 \mu} - \mu \varphi r^2 \\
&< -\frac{1}{2} \mu (\varphi r)^2 + \frac{2C_\varphi^2}{A^4 \mu} < 0.
\end{aligned}
$$

If $d_{t_0}(p_0, o) \leq r_0$, we have $\varphi \equiv 1$ on a neighborhood of $(p_0, t_0) \in M \times (0, \delta]$ and $u'(t_0) < 0$ as well. This implies that $u(t) > C/A^2$ for all $t \in (0, t_0]$, which contradicts to $u(0) = 0$.

Therefore we infer that $u(\cdot) \leq C/A^2$ on $[0, \delta]$ for any large $A \gg 1$. We let $A \to \infty$ to conclude $r(S(p, t)) = 0$ for any $(p, t) \in M \times [0, \delta]$. This completes the proof of Lemma 12. $\square$

Corollary 14 (cf. [BHS, Corollary 8]). *Let $g(t), t \in (-\infty, 0)$ be a complete ancient solution to the Ricci flow on a possibly noncompact manifold M of dimension $n \geq 2$. Suppose that the curvature tensor $R_{g(t)}$ of $g(t)$ lies in a cone $C \subset \mathcal{C}_B(\mathbb{R}^n)$ satisfying property $(*)$ on $M \times (-\infty, 0)$. Then*

$$R_{g(t)} - \delta \operatorname{scal}_{g(t)} I \in C$$

on $M \times (-\infty, 0)$.

Proof. The proof is identical to that of [BHS, Corollary 8] except for a few words to adjust to the noncompact manifolds. We fix a point $(p, \tau) \in M \times (-\infty, 0)$ and a number $\sigma > 0$ to consider a Ricci flow

$$\tilde{g}(t) := \sigma g\left(\frac{t - \delta}{\sigma} + \tau\right), \quad t \in [0, \delta].$$

It follows from Lemma 12 that

$$R_{\tilde{g}} + (1 - t \operatorname{scal}_{\tilde{g}})I \in C$$

on $M \times [0, \delta]$ and hence

$$R_g(p, \tau) + (\sigma - \delta \operatorname{scal}_g(p, \tau))I \in C.$$

Since (p, τ) and σ are taken arbitrarily, we finish the proof by letting $\sigma \to 0$. $\square$

We consider a continuous family $\{C(s)\}_{s \in \mathcal{I}}$ of cones in $\mathcal{C}_B(\mathbb{R}^n)$ for some interval $\mathcal{I} \subset \mathbb{R}$ such that $C(s) \subset \mathcal{C}_B(\mathbb{R}^n)$ is a cone satisfying property $(*)$ for each $s \in \mathcal{I} \setminus \{\inf \mathcal{I}\}$ and $\{C(s)\}_{s \in \mathcal{I}}$ converges to the cone $\mathbb{R}_+ I \subset \mathcal{C}_B(\mathbb{R}^n)$ of curvature operators of constant nonnegative curvature as $s \to \sup \mathcal{I}$ in the pointed Hausdorff topology. We call such a family a *pinching family*, although this is a little more restrictive than in [BW2].

Lemma 15 (cf. [BHS, Theorem 9]). *Let $g(t), t \in (-\infty, 0)$ be a complete ancient solution to the Ricci flow equation on a possibly noncompact manifold M of dimension $n \geq 2$. Suppose that $\{C(s)\}_{s \in \mathcal{I}}$ is a pinching family in $\mathcal{C}_B(\mathbb{R}^n)$ and that $R_{g(t)} \in C(s)$ on $M \times (-\infty, 0)$ for some $s \in \mathcal{I} \setminus \{\inf \mathcal{I}\}$. Then $(M, g(t))$ has nonnegative constant curvature for each $t \in (-\infty, 0)$.*

Proof. This is an immediate consequence of Corollary 14 as in [BHS]. □

We are now in a position to complete our proof of Theorems 1 and 2.

Proof of Theorem 1. In dimension 3, we know that the family $\{C(s)\}_{s\in[0,1)}$ given by

$$C(s) := \left\{ R \in \mathcal{C}_B(\mathbb{R}^3) : \operatorname{Ric}(R) \geq s\frac{\operatorname{scal}(R)}{3}\cdot\operatorname{id}\right\}$$

is a pinching family, e.g. [BW2, Example]. By assumption, $R_{g(t)} \in C(3\rho)$ on $M \times (-\infty, 0)$. Now Theorem 1 follows immediately from Lemma 15. □

Proof of Theorem 2. The rest of our proof is completely the same as that of [BHS, Theorem 2]. We give it here for reader's convenience.

For $s \in (0, \infty)$ we consider the set $\tilde{C}(s) \subset \mathcal{C}_B(\mathbb{R}^n)$ of curvature tensors $R \in \mathcal{C}_B(\mathbb{R}^n)$ such that

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424}$$
$$- 2\lambda\mu R_{1234} + \frac{1}{s}(1 - \lambda^2)(1 - \mu^2)\operatorname{scal}(R) \geq 0$$

for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$.

Note that $\{\tilde{C}(s)\}_{s\in(0,\infty)}$ converges as $s \to \infty$ in the pointed Hausdorff topology to the cone $\hat{C} \subset \mathcal{C}_B(\mathbb{R}^n)$ of curvature tensors of nonnegative complex sectional curvature, i.e., $R \in \hat{C}$ if and only if

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0$$

for all orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ and all $\lambda, \mu \in [-1, 1]$.

Let $l_{a,b} : \mathcal{C}_B(\mathbb{R}^n) \to \mathcal{C}_B(\mathbb{R}^n)$ be the linear map introduced by Böhm–Wilking [BW2]. It is defined for all $a, b \in \mathbb{R}$ and is the identity map when $a = b = 0$.

By assumption, $R_{g(t)} \in l_{a,b}(\tilde{C}(s))$ on $M \times (-\infty, 0)$ for all small $a, b, s > 0$. It follows from the combination of [BW2, Proposition 3.2] and [BHS, Proposition 10] that $l_{a,b}(\tilde{C}(s)) \subset \mathcal{C}_B(\mathbb{R}^n)$ with

$$2a = 2b + (n - 2)b^2 \quad\text{and}\quad b \in \left(0, \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}\right]$$

and arbitrary $s \in (0, \infty)$ is a cone satisfying property $(*)$. Corollary 14 yields that $R_{g(t)} - \delta \operatorname{scal}_{g(t)} I \in l_{a,b}(\tilde{C}(s))$ for all small $a, b > 0$ with the above condition and $s \in (0, \infty)$.

Let $\{\hat{C}(s)\}_{s\in[0,\infty)}$ be the pinching family with $\hat{C}(0) = \hat{C}$ considered by Brendle–Schoen [BS]. We then have $R_{g(t)} \in \hat{C}(s)$ on $M \times (-\infty, 0)$ for all small $s > 0$ and Lemma 15 yields that $(M, g(t))$ has constant curvature for each $t \in (0, \infty)$. Now the proof of Theorem 2 is complete. $\qquad\square$

We state a corollary of our Theorems 1 and 2.

Let $\lambda \in (0,1)$. We say an algebraic curvature tensor $R \in \mathcal{C}_B(\mathbb{R}^n)$ has *λ-pinched flag curvature* if it satisfies, with $R_e(X, X) := R(e, X, e, X)$ for $e, X \in \mathbb{R}^n$, that

$$|R_e(X, X)| \geq \lambda|R_e(Y, Y)|$$

for any nonzero vectors $e \in \mathbb{R}^n \setminus \{0\}$ and $X, Y \in \mathbb{R}^n \setminus \{0\}$ which are perpendicular to e. We say a Riemannian metric g on a manifold M has *λ-pinched flag curvature* if the curvature tensor $R_g \in \mathcal{C}_B(T_pM)$ has λ-pinched flag curvature at any point $p \in M$.

Ni–Wilking [NWi] proved that an algebraic curvature tensor of dimension ≥ 4 with nonnegative $1/4$-pinched flag curvature has nonnegative complex sectional curvature.

It might be worth stating the following, cf. [BS2, Corollary 7.5].

Corollary 16. *Let $g(t), t \in (-\infty, 0)$ be a complete ancient solution to the Ricci flow equation on a manifold M of dimension ≥ 3. Suppose that $g(t)$ has nonnegative $(1+\varepsilon)/4$-pinched flag curvature for some $\varepsilon > 0$ on $M \times (-\infty, 0)$. Then $(M, g(t))$ has constant nonnegative curvature for each $t \in (-\infty, 0)$.*

Proof. With the result of Ni–Wilking [NWi] it is easy to check that any algebraic curvature tensor with nonnegative $(1+\varepsilon)/4$-pinched flag curvature satisfies the curvature pinching condition in Theorem 1 or 2. $\qquad\square$

Remark 17. We here discuss a relevance of our Theorem 1 to the following theorem.

Theorem 18 (Chen–Zhu [CZ]). *Any 3-dimensional complete Riemannian manifold (M^3, g) with bounded nonnegative sectional curvature satisfying*

$$\mathrm{Ric}(g) \geq \rho \, \mathrm{scal}_g \, g \geq 0$$

for a constant $\rho > 0$ is either compact or flat.

The most crucial reason for the boundedness of the curvature in the statement of Theorem 18 is that we need to run the Ricci flow and use the maximum principle in the proof. Recently Cabezas-Rivas–Wilking [CRW] showed that the Ricci flow runs if the initial complete metric has possibly unbounded nonnegative complex sectional curvature. In viewing these results, it might be plausible that ancient solutions in our Theorem 1 are a priori compact and hence our Theorem 1 is reduced to [BHS, Theorem 1]. However at present we do not know how to prove this.

In higher dimension, Brendle–Schoen [BS2, Theorem 7.4], cf. Ni–Wu [NWu], Chen–Zhu [CZ], proved a similar compactness theorem for complete manifolds with bounded curvature satisfying a pinched curvature condition stronger than that of Theorem 2.

It remains to give a proof of Corollary 3 stated in the introduction.

Proof of Corollary 3. For shrinking and steady Ricci solitons, this corollary is an immediate consequence of Theorems 1 and 2, since they are isometric to time-slice of ancient solutions to the Ricci flow equation as in these theorems.

For a complete expanding Ricci soliton (M, g, V), we consider the immortal solution $g(t) := -\lambda t(\varphi_t)^* g, t \in (0, \infty)$ on M, where $\{\varphi_t\}_{t\in(0,\infty)}$ is a family of diffeomorphisms of M constructed from $V \in \Gamma(TM)$. Since it is a self-similar solution, we can show that Corollary 14 holds for this immortal solution $g(t), t \in (0, \infty)$ and apply the same argument as in the proof of Theorems 1 and 2 to finish the proof. $\square$

Remark 19.

1) If the manifold M is compact in the statement of Corollary 3, this corollary follows from the sphere theorems of Hamilton [Ha] and Brendle [Br]. It is also known that the only compact steady and expanding Ricci solitons are Einstein manifolds. Therefore our contribution of Corollary 3 is for noncompact Ricci solitons.

2) The curvature condition in Corollary 3 implies that the Ricci curvature is nonnegative. Ni [Ni, Proposition 1.1] proved that any complete non-flat gradient shrinking Ricci soliton with nonnegative Ricci curvature has uniformly positive scalar curvature, i.e., $\inf_M \mathrm{scal}_g > 0$. Thus any gradient shrinking Ricci soliton satisfying the assumption of Corollary 3 turns out to be compact due to the theorem of Myers.

4. Ricci flow with nonnegative curvature on surfaces

In this final section, we prove the following proposition concerning ancient solutions on surfaces, which are somewhat excluded from the previous sections. This proposition popped out during the discussion with E. Cabezas-Rivas before this work started and a similar result can be found in Cabezas-Rivas–Wilking [CRW].

Proposition 20. *Let $g(t), t \in (0, T]$ be a complete Ricci flow on a surface M. Suppose that $g(0)$ has nonnegative curvature. Then there exists a constant $C < \infty$ such that*

$$(21) \qquad\qquad 0 \leq \mathrm{scal}(p, t) \leq \frac{CT}{t}$$

for all $(p, t) \in M \times (0, T)$. Moreover, any complete ancient solution $g(t), t \in (-\infty, 0)$ on a surface has bounded curvature on any subinterval $(-\infty, T] \subset (-\infty, 0)$.

Ancient solutions to the Ricci flow equation on surfaces with bounded curvature have been classified; see the introduction of the paper [DHS] by Daskalopoulos–Hamilton–Sesum. Therefore by Proposition 20 we now have the classification of complete ancient solutions to the Ricci flow equation on surfaces.

It might be interesting to compare this proposition to the constructions of immortal solutions with unbounded curvature by Giesen–Topping [GT] and Cabezas-Rivas–Wilking [CRW].

In our proof of Proposition 20, we use the following proposition proved in the author's thesis [Yo].

Proposition 22 ([Yo, Proposition 8.3]). *Let $g(t), t \in (0, T]$ be a complete Ricci flow on a surface M. Suppose that $g(t)$ has nonnegative curvature and satisfies the trace Harnack inequality:*

$$(23) \qquad \frac{\partial}{\partial t}\mathrm{scal} + \frac{\mathrm{scal}}{t} + 2\langle \nabla\mathrm{scal}, V \rangle + 2\,\mathrm{Ric}(V, V) \geq 0$$

for any vector $V \in TM$ on $M \times (0, \delta]$. Then Inequality (21) holds for all $(p, t) \in M \times (0, T]$ with $C := \sup_M \mathrm{scal}(\cdot, T) < \infty$.

Hamilton [Ha2] proved that any complete Ricci flow with bounded non-negative curvature operator satisfies the Harnack inequality.

In Proposition 22 it is actually shown that the Ricci flow has finite asymptotic scalar curvature ratio, cf. [CLN, Lemma 9.7]. The *asymptotic scalar curvature ratio* (ASCR) of a noncompact Riemannian manifold (M, g) is defined as

$$\limsup_{x \to \infty} \operatorname{scal}_g(x) d(x, o)^2$$

with $o \in M$ being a fixed point. The value of ASCR is independent of the choice of the base point $o \in M$.

Proposition 22 was proved in [Yo] by a combination of Hamilton's point picking lemma and finite bump theorem [Ha3], e.g. Chow–Lu–Ni [CLN].

Proof of Proposition 20. Since the first inequality in (21) follows from Theorem 9 of Chen [Ch] stated above, only the second inequality remains to prove.

A theorem of Croke–Karcher [CK] states that

$$\inf_{p \in M} \operatorname{vol}_0(B_0(p, 1)) > 0$$

and Cohn-Vossen's inequality implies that

$$\int_{B_t(p,1)} \operatorname{scal}(\cdot, t) d\mathrm{vol}_t \le \int_M \operatorname{scal}(\cdot, t) d\mathrm{vol}_t \le 8\pi$$

for any $p \in M$ and $t \in [0, T]$.

Since $\frac{\partial}{\partial t} g(t) = -\operatorname{scal}_{g(t)} g(t) \le 0$, we have $B_0(p, 1) \subset B_t(p, 1)$ and

$$\operatorname{vol}_t(B_t(p, 1)) \ge \operatorname{vol}_t(B_0(p, 1))$$
$$= \operatorname{vol}_0(B_0(p, 1)) - \int_0^t \int_{B_0(p,1)} \operatorname{scal}(\cdot, s) d\mathrm{vol}_s ds$$
$$\ge \operatorname{vol}_0(B_0(p, 1)) - \int_0^t \int_{B_s(p,1)} \operatorname{scal}(\cdot, s) d\mathrm{vol}_s ds$$

for any $p \in M$ and $t \in [0, T]$ and hence

$$\inf_{(p,t) \in M \times [0,\varepsilon]} \operatorname{vol}_t(B_t(p, 1)) > 0$$

with some small $\varepsilon > 0$, cf. [CRW].

If $t \operatorname{scal}(\cdot, t)$ is not bounded on $M \times [0, \varepsilon]$, Perelman's point-picking and rescaling argument in [Pe, Section 10] and Hamilton's compactness theorem

yield a complete ancient solution $g_\infty(t), t \in (-\infty, 0)$ with bounded curvature on a surface M_∞ with

$$\lim_{r \to \infty} \frac{\mathrm{vol}_t(B_t(p_\infty, r))}{r^2} > 0$$

for some/any $(p_\infty, t) \in M_\infty \times (-\infty, 0]$. This contradicts to Perelman's result [Pe, Corollary 11.3].

We put

$$T' := \sup \left\{ \hat{T} \in (0, T] : t \, \mathrm{scal} \text{ is bounded on } M \times [0, \hat{T}] \right\}.$$

Then $g(t)$ satisfies the Harnack inequality (23) on $M \times (0, T']$ and it tells us that $t \, \mathrm{scal}(p, t)$ is nondecreasing in t for each $p \in M$. Proposition 22 implies that Inequality (21) holds on $M \times (0, T']$ with $C := \sup_M \mathrm{scal}(\cdot, T') < \infty$.

In case $T' < T$, we repeat the above argument to see that

$$\inf_{(p,t) \in M \times [T', T' + \varepsilon']} \mathrm{vol}_t(B_t(p, 1)) > 0$$

and the curvature is bounded on $M \times [T', T' + \varepsilon']$ for some small $\varepsilon' > 0$. Therefore we conclude that $T' = T$ and Inequality (21) was proved.

For a complete ancient solution $g(t), t \in (-\infty, 0)$, the trace Harnack inequality (23) says that $\mathrm{scal}(\cdot, t)$ is point-wise nondecreasing in t. This completes the proof of the proposition. $\qquad\square$

Acknowledgments

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