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Scalar curvatures of Hermitian metrics on the moduli space of Riemann surfaces

Yunhui Wu

In this article we show that any finite cover of the moduli space of closed Riemann surfaces of g genus with $g \geqslant 2$ does not admit any complete finite-volume Hermitian metric of non-negative scalar curvature. Moreover, we also show that the total mass of the scalar curvature of any almost Hermitian metric, which is equivalent to the Teichmüller metric, on any finite cover of the moduli space is negative provided that the scalar curvature is bounded from below.

1. Introduction

Let S_g be a closed surface of genus g with $g \geqslant 2$, $\mathrm{Mod}(S_g)$ be the mapping class group and T_g be the Teichmüller space of S_g. The space T_g is a contractible complex manifold of complex dimension $3g-3$, which carries various $\mathrm{Mod}(S_g)$-invariant metrics which descend into metrics on the moduli space $\mathbb{M}_g$ of S_g with respective properties. For examples, the Teichmüller metric, Kobayashi metric and Caratheódory metric are complete and Finsler. The Weil-Petersson metric is Kähler [1], incomplete [6, 21] and has negative sectional curvatures [22]. The Asymptotic Poincaré metric, Induced Bergman metric, Kähler-Einstein metric, McMullen metric, Ricci metric, and perturbed Ricci metric are complete and Kähler. In [13, 14, 17, 25], the authors showed that the metrics listed above except the Weil-Petersson metric are equivalent.

It is shown that the perturbed Ricci metric [13, 14] has pinched negative Ricci curvature. So does the scalar curvature of the perturbed Ricci metric. The McMullen metric [17] has negative scalar curvature at certain points since the metric, restricted on certain thick part of the moduli space, is the Weil-Petersson metric. The Kähler-Einstein metric on T_g, constructed by Cheng-Yau in [4], has constant negative scalar curvature. However, in [8] Farb and Weinberger show that any finite cover M of the moduli space $\mathbb{M}_g$

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($g \geqslant 2$) admits a complete finite-volume Riemannian metric of (uniformly bounded) positive scalar curvature. They also show that this metric is not quasi-isometric to the Teichmüller metric. And they conjecture (see Conjecture 4.6 in [7]) that

Conjecture 1.1 (Farb-Weinberger). *Let S_g be a surface of genus g with $g \geqslant 2$. Then any finite cover M of the moduli space $\mathbb{M}_g$ of S_g does not admit a finite volume Riemannian metric of (uniformly bounded) positive scalar curvature in the quasi-isometry class of the Teichmüller metric.*

Let M be any finite cover of $\mathbb{M}_g$. The natural complex structure on the Teichmüller space descends into a complex structure on M. In this paper we will focus complete Hermitian metrics on M with this complex structure. Our first result is

Theorem 1.2. *Let S_g be a surface of genus g with $g \geqslant 2$ and M be a finite cover of the moduli space $\mathbb{M}_g$ of S_g. Then for any complete finite-volume Hermitian metric $\|\cdot\|$ on M, the scalar curvature* Sca *of $(M, \|\cdot\|)$ satisfies*

$$\inf_{p \in (M, \|\cdot\|)} \mathrm{Sca}(p) < 0.$$

The infimum of the scalar curvature can be arbitrary close to zero if we take a rescaling of any metric in the theorem above provided that the scalar curvature has a lower bound. As introduced before there is a list of canonical metrics which are equivalent (or quasi-isometric) to the Teichmüller metric $\|\cdot\|_T$ (see [13, 14, 17, 25]). Our second aim is the following uniform negative upper bound on the infimum of the scalar curvature of a Hermitian metric on a given class.

Theorem 1.3. *Let S_g be a surface of genus g with $g \geqslant 2$ and M be a finite cover of the moduli space $\mathbb{M}_g$ of S_g. Given two constants $k_1, k_2 > 0$, then for any Hermitian metric $\|\cdot\|$ on M with $k_1\|\cdot\| \leqslant \|\cdot\|_T \leqslant k_2\|\cdot\|$, there exists a constant $K(k_1, g) > 0$ only depending on k_1 and g such that the scalar curvature satisfies*

$$\inf_{p \in (M, \|\cdot\|)} \mathrm{Sca}(p) \leqslant -K(k_1, g) < 0.$$

Both Theorem 1.2 and Theorem 1.3 tell that the scalar curvature of any Hermitian metric, which is equivalent to the Teichmüller metric, on the moduli space is negative at certain points. It is natural to ask *whether the* *total mass of the scalar curvature could be positive*. Our last result tells that this is impossible if we assume the metric is almost Hermitian and its scalar curvature has a lower bound.

Theorem 1.4. *Let S_g be a surface of genus g with $g \geqslant 2$ and M be a finite cover of the moduli space $\mathbb{M}_g$ of S_g. Assume that $\|\cdot\|$ is an almost Hermitian metric on M satisfying $\|\cdot\| \asymp \|\cdot\|_T$ and the scalar curvature of $(M, \|\cdot\|)$ is bounded from below, then the total curvature satisfies*

$$\int_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p)\,\mathrm{dVol}(p) < 0.$$

Theorem 1.4 applies to the Asymptotic Poincaré metric, Induced Bergman metric, McMullen metric and Ricci metric, which is new. Actually it also applies to any metric in the the convex hull of the Kähler-Einstein metric, perturbed Ricci metric and the four metrics above.

It is *interesting* to know *whether there exists a Hermitian metric $\|\cdot\|$ on a finite cover M of $\mathbb{M}_g$ such that $\|\cdot\|$ is equivalent to the Teichmüller metric and $(M, \|\cdot\|)$ has non-negative scalar curvature outside some compact subset of M*. We hope the method in this paper is helpful for this question.

Remark 1.5. Recently in a joint work [12] with K. Liu, we confirm Conjecture 1.1. Moreover, we use some recent accomplishments in [3, 13, 14, 17] on the geometry of Teichmüller space as bridges to prove the following result, which is analogous to Gromov-Lawson's theorem in [10] for nonpositive curved Riemannian manifolds.

Theorem 1.6 (Liu-W). *Let S_g be a closed Riemann surface of genus g with $g \geqslant 2$ and M be a finite cover of the moduli space $\mathbb{M}_g$ of S_g. Then for any Riemannian metric $\|\cdot\|$ on M with $\|\cdot\| \succ \|\cdot\|_T$ we have*

$$\inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p) < 0.$$

Where $\|\cdot\| \succ \|\cdot\|_T$ means that $\|\cdot\| \geqslant k\cdot\|\cdot\|_T$ for some constant $k > 0$.

1.1. Plan of the paper

In Section 2 we review some recent developments on the canonical metrics on the moduli space of surfaces and recall one formula of S. S. Chern which is crucial for this article. In Section 3 we establish Theorem 1.2. Theorem 1.3 is proved in Section 4. And we will finish the proof of Theorem 1.4 in Section 5.

2. Notations and Preliminaries

2.1. Surfaces

Let S_g be a closed surface of g genus with $g \geqslant 2$, and M_{-1} denote the space of Riemannian metrics with constant curvature -1, and $X = (S_g, \sigma|dz|^2)$ be an element in M_{-1}. The group Diff_0 of diffeomorphisms of S_g isotopic to the identity, acts by pull-back on M_{-1}. The Teichmüller space T_g of S_g is defined by the quotient space

$$T_g = M_{-1}/\,\mathrm{Diff}_0\,.$$

Let Diff_+ be the group of orientation-preserving diffeomorphisms of S_g. The mapping class group $\mathrm{Mod}(S_g)$ is defined as

$$\mathrm{Mod}(S_g) = \mathrm{Diff}_+ \,/\, \mathrm{Diff}_0\,.$$

The moduli space $\mathbb{M}_g$ of S_g is defined by the quotient space

$$\mathbb{M}_g = T_g/\mathrm{Mod}(S_g).$$

The Teichmüller space has a natural complex structure, and its holomorphic cotangent space $T^*_X T_g$ is identified with the *quadratic differentials*

$$QD(X) = \{\phi(z)dz^2\}$$

while its holomorphic tangent space is identified with the *harmonic Beltrami differentials*

$$HBD(X) = \left\{ \frac{\overline{\phi(z)}}{\sigma(z)} \frac{d\overline{z}}{dz} \right\}.$$

Recall that the *Teichmüller metric* $\|\cdot\|_T$ on T_g is defined as

$$\left\| \frac{\overline{\phi(z)}}{\sigma(z)} \right\|_T := \sup_{\psi dz^2 \in QD(X),\ \int_X |\psi|=1} \mathrm{Re} \int_X \frac{\overline{\phi(z)}}{\sigma(z)} \cdot \psi(z) \frac{dz \wedge d\overline{z}}{-2\mathbf{i}}.$$

The *Weil-Petersson metric* $\|\cdot\|_{WP}$ is the Hermitian metric on T_g arising from the the *Petersson scalar product*

$$< \varphi, \psi >= \int_X \frac{\varphi(z) \cdot \overline{\psi(z)}}{\sigma(z)} \frac{dz \wedge d\overline{z}}{-2\mathbf{i}}$$

via duality.

Both the Teichmüller metric and the Weil-Petersson metric are $\mathrm{Mod}(S_g)$-invariant.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be any two metrics on T_g. We call $\|\cdot\|_1$ is *controlled above* by $\|\cdot\|_2$ if there exists a positive constant k such that

$$\|\cdot\|_1 \leqslant k\|\cdot\|_2$$

which is denoted by $\|\cdot\|_1 \prec \|\cdot\|_2$.

The Cauchy-Schwarz inequality and the Gauss-Bonnet formula tell us that

$$\|\cdot\|_{WP} \prec \|\cdot\|_T.$$

We call the two metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ are *equivalent* (or *quasi-isometric*) if

$$\|\cdot\|_1 \prec \|\cdot\|_2 \quad \textit{and} \quad \|\cdot\|_2 \prec \|\cdot\|_1.$$

We denote it by $\|\cdot\|_1 \asymp \|\cdot\|_2$.

It is not hard to see that $\|\cdot\|_{WP}$ is not equivalent to $\|\cdot\|_T$ because the Weil-Petersson metric is incomplete and the Teichmüller metric is complete.

2.2. Kähler metrics on $\mathbb{M}_g$

In this subsection we briefly review some properties of the following three Kähler metrics $\mathbb{M}_g$: the McMullen metric, the Ricci metric, and the perturbed Ricci metric. They will be applied to prove Theorem 1.3 in Section 4.

2.2.1. McMullen metric. In [17] McMullen constructed a new metric $\|\cdot\|_M$ on $\mathbb{M}_g$, called the *McMullen metric*. More precisely, let $Log : \mathbb{R}_+ \to [0,\infty)$ be a smooth function such that

(1) $Log(x) = \log(x)$ if $x \geqslant 2$;

(2) $Log(x) = 0$ if $x \leqslant 1$.

For suitable choices of small constants $\epsilon, \delta > 0$, the Kähler form of the McMullen metric is

$$\omega_M = \omega_{WP} - \mathbf{i}\delta \sum_{\ell_\gamma(X)<\epsilon} \partial\overline{\partial} Log \frac{\epsilon}{\ell_\gamma}$$

where the sum is taken over primitive short geodesics γ on X. Restricted on certain thick part of $\mathbb{M}_g$ the McMullen metric is exactly the Weil-Petersson metric. McMullen in [17] proved that this metric is Kähler hyperbolic and has bounded geometry. He also showed that

Theorem 2.1 (McMullen). *On the moduli space* $\mathbb{M}_g$, $\|\cdot\|_M \asymp \|\cdot\|_T$.

2.2.2. Ricci metric and perturbed Ricci metric. In [19, 22] it is shown that the Weil-Petersson metric has negative sectional curvature. The negative Ricci curvature tensor defines a new metric $\|\cdot\|_\tau$ which is called the *Ricci metric* on $\mathbb{M}_g$. Trapani in [20] proved $\|\cdot\|_\tau$ is a complete Kähler metric.

In [13] Liu-Sun-Yau perturbed the Ricci metric along the Weil-Petersson direction to give new metrics on $\mathbb{M}_g$ which are called the *perturbed Ricci metrics* denoted by $\|\cdot\|_{LSY}$. More precisely, let ω_τ be the Kähler form of the Ricci metric, for any constant $C > 0$, the Kähler form of the perturbed Ricci metric is

$$\omega_{LSY} = \omega_\tau + C \cdot \omega_{WP}.$$

In [13] the authors showed that both $\|\cdot\|_\tau$ and $\|\cdot\|_{LSY}$ have bounded geometry. By using Yau's generalized Schwarz Lemma [24] they also showed that

Theorem 2.2 (Liu-Sun-Yau). *On the moduli space* $\mathbb{M}_g$,

$$\|\cdot\|_{LSY} \asymp \|\cdot\|_\tau \asymp \|\cdot\|_M.$$

Furthermore, in one of their subsequent papers [14] they showed that

Theorem 2.3 (Liu-Sun-Yau). *With a suitable choice of the constant* C, *there exists two positive numbers* C_1, C_2 *such that the Ricci curvature of the perturbed Ricci metric* $\|\cdot\|_{LSY}$ *satisfies*

$$-C_1 \leqslant Ric_{\|\cdot\|_{LSY}} \leqslant -C_2 < 0.$$

Moreover, they also showed in [14] that the perturbed Ricci metric $\|\cdot\|_{LSY}$ has negatively pinched holomorphic sectional curvatures, which is known to be the first complete metric on the moduli space with this property. And they use this property and the Schwarz-Yau lemma to prove a list of canonical metrics on the moduli space are equivalent.

2.3. The ratio of intermediate volume elements

In this subsection we briefly review a formula of Chern in [5] which is crucial for this article. For the notations and computations one can also refer to [16] for more details.

Let M and N be two $2n$-dimensional Hermitian manifolds and $f : M \to N$ be a holomorphic mapping. Let $\{\theta_i\}_{1\leqslant i\leqslant n}, \{\omega_\alpha\}_{1\leqslant \alpha\leqslant n}$ be the unitary coframe fields of M and N respectively. There exists complex numbers $a_{\alpha i}$ such that

$$f^*\omega_\alpha = \sum_{i=1}^{n} a_{\alpha i}\theta_i. \tag{2.1}$$

A direct computation gives that

$$f^*(\omega_\alpha \wedge \overline{\omega}_\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{\alpha i}\overline{a}_{\alpha j}\theta_i \wedge \overline{\theta}_j. \tag{2.2}$$

By raising equation (2.2) to the n^{th} power, the *ratio of intermediate volume elements* v is defined as

$$v := \frac{f^*(\sum_{\alpha=1}^{n} \omega_\alpha \wedge \overline{\omega}_\alpha)^n}{(\sum_{i=1}^{n} \theta_i \wedge \overline{\theta}_i)^n}. \tag{2.3}$$

Linear algebra gives that

$$v = \frac{f^*(\mathrm{dVol}_N)}{\mathrm{dVol}_M} = D \cdot \overline{D} \tag{2.4}$$

where

$$D = \det(a_{\alpha\beta}). \tag{2.5}$$

Now we are ready to state Chern's formula which is crucial for this article.

Theorem 2.4 (Chern). *Let Δ be the Laplace operator of M. Then we have*

$$\frac{\Delta v}{4} = \sum_{k=1}^{n} D_k \cdot \overline{D_k} + \frac{v}{2}\left(\mathrm{Sca} - \sum_{1\leqslant \alpha,\beta,k\leqslant n} a_{\alpha k}\overline{a}_{\beta k}\widetilde{Ric}_{a\overline{\beta}}\right)$$

where $\sum_{k=1}^{n} D_k \cdot \overline{D_k}$ is a nonnegative function on M, Sca is the scalar curvature of M and $\widetilde{Ric}_{a\overline{\beta}}$ is the Ricci tensor of N. For the the proof of Theorem 2.4 one can refer to [5] (or Corollary 4.4 in [16]) for details.

We remark here that the Ricci tensor $\widetilde{Ric}_{a\overline{\beta}}$ in Chern's formula is the first Chern-Ricci curvature which may be different from the standard Riemannian Ricci curvature of N if N is not Kähler. One can see a recent paper of K. Liu and X. Yang [15] for more details in this direction. However, in the remaining part of this paper we will choose N to be the a finite manifold cover of the moduli space $\mathbb{M}_g$ endowed with the Liu-Sun-Yau metric $\|\cdot\|_{LSY}$ which is Kähler. So the second Chern-Ricci curvature of $\|\cdot\|_{LSY}$ agrees with the Riemannian Ricci curvature of $\|\cdot\|_{LSY}$, which is negatively pinched by Theorem 2.3.

3. Proof of Theorem 1.2

In this section we will prove Theorem 1.2. Let M be any finite cover of the moduli space $\mathbb{M}_g$. If necessary we take a finite cover of M again, still denoted by M, such that M is a manifold. We lift the perturbed Ricci metric $\|\cdot\|_{LSY}$ in Theorem 2.3 onto M. Let $\|\cdot\|$ be a complete finite-volume Hermitian metric on M. We consider the identity map

$$i : (M, \|\cdot\|) \to (M, \|\cdot\|_{LSY}).$$

It is clear that i is holomorphic. We let v be the ratio of intermediate volume elements for i as in equation (2.3).

Lemma 3.1. *For any $p \in (M, \|\cdot\|)$,*

$$v(p) > 0.$$

Proof. For any $p \in (M, \|\cdot\|)$, $v(p) > 0$ follows from the fact that $i : (M, \|\cdot\|) \to (M, \|\cdot\|_{LSY})$ is biholomorphic. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Assume that $\inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p) \geqslant 0$. We will argue it by getting a contradiction. Let Δ be the Laplace operator on $(M, \|\cdot\|)$.

Claim 1: $\Delta v(q) > 0$ for any $q \in (M, \|\cdot\|)$.

Proof of Claim 1. First by Theorem 2.3 we know that there exists a constant $C_2 > 0$ such that

$$Ric_{\|\cdot\|_{LSY}} \leqslant -C_2 < 0. \tag{3.1}$$

From Lemma 3.1 we know that $v > 0$. Since $\inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p) \geqslant 0$, by Theorem 2.4 we have

$$\Delta v \geqslant -2v \left(\sum_{1\leqslant\alpha,\beta,k\leqslant n} a_{\alpha k}\overline{a}_{\beta k}\widetilde{Ric}_{\alpha\overline{\beta}} \right). \tag{3.2}$$

Since $v > 0$, inequalities (3.1) and (3.2) lead to

$$\begin{aligned} \Delta v &\geqslant 2C_2 v \cdot \left(\sum_{1\leqslant\alpha,k\leqslant n} |a_{\alpha k}|^2 \right) \\ &\geqslant 2C_2(3g-3)v^{\frac{3g-2}{3g-3}} \quad \text{(by the AM-GM inequality)} \\ &> 0. \end{aligned} \tag{3.3}$$

□

The remaining argument is inspired by the proof of Theorem 9.1 in [2] and Section 8.12 in [18]. We remark that it is not known that v is bounded from above, so the following Claim (2) can not directly follow from the results in [23] or method (2) in the proof of Theorem 1.3 in Section 4.

Claim 2: v is a constant on $(M, \|\cdot\|)$.

Proof of Claim 2. Let g_t denote the flow generated by the vector field ∇v. Since $(M, \|\cdot\|)$ is complete, g_t is defined for all $t \geqslant 0$.

Assume that v is not a constant and let $p_0 \in M$ such that $\nabla v(p_0) \neq 0$. Along the flow line of g_t starting at p_0, v is increasing since for all $s_2 > s_1 \geqslant 0$,

$$v(g_{s_2}(p_0)) - v(g_{s_1}(p_0)) = \int_{s_1}^{s_2} \|\nabla v(g_t(p_0))\| dt \geqslant 0. \tag{3.4}$$

That is

$$v(g_{s_2}(p_0)) \geqslant v(g_{s_1}(p_0)) \quad \forall s_2 > s_1 \geqslant 0. \tag{3.5}$$

Since we assume that $\nabla v(p_0) \neq 0$, let $s_2 = 1$ and $s_1 = 0$ we have

$$v(g_1(p_0)) > v(p_0) > 0. \tag{3.6}$$

Therefore there exists a small enough constant $r_0 > 0$, depending on p_0, such that

$$\inf_{q \in B(p_0, r_0)} v(g_1(q)) > \sup_{q \in B(p_0, r_0)} v(q) \tag{3.7}$$

where $B(p_0, r_0)$ is the geodesic ball centered at p_0 of radius r_0.

In particular we have

$$B(p_0, r_0) \cap g_1(B(p_0, r_0)) = \emptyset. \tag{3.8}$$

Inequality (3.4) and equation (3.8) give that

$$B(p_0, r_0) \cap g_n(B(p_0, r_0)) = \emptyset \quad \forall n \in \mathbb{Z}^+. \tag{3.9}$$

Which also implies

$$g_n(B(p_0, r_0)) \cap g_m(B(p_0, r_0)) = \emptyset \quad \forall n \neq m \in \mathbb{Z}^+. \tag{3.10}$$

Otherwise there exist two positive integers $n_0 > m_0 \geqslant 1$ and $q_1, q_2 \in B(p_0, r_0)$ such that $g_{n_0}(q_1) = g_{m_0}(q_2)$. Since g_t is a flow, $g_{n_0 - m_0}(q_1) = q_2$ which contradicts equation (3.9).

On the other hand, for any $t_0 > 0$ (we use Proposition 18.18 in [11]), we have

$$\begin{aligned} \frac{d\,\mathrm{Vol}(g_t(B(p_0, r_0)))}{dt}|_{t=t_0} &= \int_{B(p_0, r_0)} \frac{d}{dt}|_{t=t_0} g_t^*(\mathrm{dVol}) \\ &= \int_{B(p_0, r_0)} g_{t_0}^*(\mathcal{L}_{\nabla v}(\mathrm{dVol})) \\ &= \int_{B(p_0, r_0)} g_{t_0}^*(\mathrm{div}(\nabla(v))\,\mathrm{dVol}) \\ &= \int_{g_{t_0}(B(p_0, r_0))} \Delta v \,\mathrm{dVol}\,. \end{aligned} \tag{3.11}$$

From Claim (1) we have

$$\frac{d\,\mathrm{Vol}(g_t(B(p_0, r_0)))}{dt}\Big|_{t=t_0} > 0, \quad \forall t_0 > 0. \tag{3.12}$$

That is the flow g_t is volume increasing.

Thus, equation (3.10) and inequality (3.12) give that

$$\begin{aligned}\mathrm{Vol}((M, \|\cdot\|)) &\geqslant \mathrm{Vol}(\cup_{k=1}^{\infty} g_k(B(p_0, r_0))) \\ &= \sum_{k=1}^{\infty} \mathrm{Vol}(g_k(B(p_0, r_0))) \\ &\geqslant \sum_{k=1}^{\infty} \mathrm{Vol}(B(p_0, r_0)) \\ &= \infty\end{aligned} \tag{3.13}$$

which contradicts our assumption that $(M, \|\cdot\|)$ has finite volume. □

It is clear that Claim (1) and Claim (2) can not simultaneously hold. Therefore, the proof is completed. □

4. Proof of Theorem 1.3

As the same in Section 3 we let M be any finite manifold cover of the moduli space $\mathbb{M}_g$ and $\|\cdot\|_{LSY}$ be the perturbed Ricci metric in Theorem 2.3. Let k_1, k_2 be two positive constants and $\|\cdot\|$ be a Hermitian metric on M with

$$k_1\|\cdot\| \leqslant \|\cdot\|_T \leqslant k_2\|\cdot\|.$$

From Theorem 2.1 and Theorem 2.2, up to some uniform constants, we may assume that

$$k_1\|\cdot\| \leqslant \|\cdot\|_{LSY} \leqslant k_2\|\cdot\|. \tag{4.1}$$

Use the same notations in Section 3 we let v be the ratio of intermediate volume elements for i as in equation (2.3).

Lemma 4.1. *For any $p \in (M, \|\cdot\|)$, we have*

$$v(p) \geqslant k_1^{6g-6} > 0.$$

In particular, $(M, \|\cdot\|)$ has finite volume.

Proof. Since $k_1\|\cdot\| \leqslant \|\cdot\|_{LSY}$, linear algebra gives that

$$k_1^{6g-6}\, \mathrm{dVol}_{\|\cdot\|} \leqslant \mathrm{dVol}_{\|\cdot\|_{LSY}}. \tag{4.2}$$

The conclusion follows from equation (2.4) and inequality (4.2).

Since $(M, \|\cdot\|_{LSY})$ has finite volume, inequality (4.2) tells that $(M, \|\cdot\|)$ also has finite volume. $\square$

Lemma 4.2. *$(M, \|\cdot\|)$ is complete.*

Proof. It directly follows from our assumption that $\|\cdot\|_T \leqslant k_2\|\cdot\|$ and the fact that the Teichmüller metric is complete. $\square$

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Theorem 2.3 tells that there exists a constant $C_2 > 0$ such that

$$Ric_{\|\cdot\|_{LSY}} \leqslant -C_2 < 0. \tag{4.3}$$

Set $k_3 := \inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p)$. We may assume that $k_3 \neq -\infty$ otherwise we are done. From Lemma 3.1 we know that $v > 0$. Thus, from Theorem 2.4 we have

$$\begin{aligned}\Delta v &\geqslant 2v \cdot k_3 - 2v \left(\sum_{1\leqslant\alpha,\beta,k\leqslant n} a_{\alpha k}\overline{a}_{\beta k}\widetilde{Ric}_{\alpha\overline{\beta}} \right) \\ &\geqslant 2v(k_3 + C_2(3g-3)v^{\frac{1}{3g-3}}) \quad \text{(by the AM-GM inequality).}\end{aligned}$$

From Lemma 4.1 we have

$$\Delta v \geqslant 2v \cdot (k_3 + C_2(3g-3)k_1^2). \tag{4.4}$$

We choose $K(k_1, g) = \frac{C_2(3g-3)k_1^2}{2} > 0$.

Claim : $\inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p) \leqslant -K(k_1, g) < 0$.

Proof of Claim. Assume it is not. That is

$$\inf_{p\in(M,\|\cdot\|)} \mathrm{Sca}(p) > -K(k_1, g). \tag{4.5}$$

From Lemma 4.1, inequalities (4.4) and (4.5) we have

$$\Delta v \geqslant C_2 \cdot (3g-3) \cdot k_1^{6g-4} > 0. \tag{4.6}$$

There are more information coming from the conditions of Theorem 1.3. We provide two different methods to finish the proof of the claim.

Method (1). Since $(M, \|\cdot\|)$ is complete and has finite volume, it follows from inequality (4.6) and the same argument as in the *proof of Claim (2)* in Section 3 that v is a constant which contradicts inequality (4.6). $\square$

Method (2). First from the right side of inequality (4.1) and by using a same argument in the proof of Lemma 4.1 there exists a positive constant C_3 such that

$$\sup_{p\in(M,\|\cdot\|)} v(p) \leqslant C_3. \tag{4.7}$$

Fix $p_0 \in M$ and let $B(p_0, r)$ be the closed geodesic ball of $(M, \|\cdot\|)$ centered at p_0 of radius r. Then for any $t > 0$ there exists a bump function $f(x) : (M, \|\cdot\|) \to [0, \infty)$ which is a Lipschitz continuous function and a constant $C_4 > 0$ such that

(i) $f \equiv 1$ on $B(p_0, t)$ and $f \equiv 0$ on $M - B(p_0, 2t)$.

(ii) $\|\nabla f\| \leqslant \frac{C_4}{t}$ a.e. on M.

(For the existence of such bump functions one can refer to [23]).

First since $v > 0$ and $\Delta v > 0$ we have

$$\Delta v^2 \geqslant 2\|\nabla v\|^2. \tag{4.8}$$

Let $<,>$ be the Riemannian inner product associated to the metric $\|\cdot\|$. The Stokes' theorem and inequality (4.8) give

$$\begin{aligned} 0 &= \int_{B(p_0,2t)} \operatorname{div}(f^2\nabla(v^2)) \\ &\geqslant 4\int_{B(p_0,2t)} f\cdot v\cdot <\nabla f, \nabla v> +2\int_{B(p_0,2t)} f^2\cdot\|\nabla v\|^2. \end{aligned}$$

The Cauchy-Schwarz inequality leads to

$$\begin{aligned} \int_{B(p_0,2t)} f^2\cdot\|\nabla v\|^2 &\leqslant -2\int_{B(p_0,2t)} f\cdot v\cdot <\nabla f, \nabla v> \\ &\leqslant 2\sqrt{\int_{B(p_0,2t)} f^2\cdot\|\nabla v\|^2}\cdot\sqrt{\int_{B(p_0,2t)} v^2\cdot\|\nabla f\|^2}. \end{aligned}$$

That is

$$\int_{B(p_0,2t)} f^2\cdot\|\nabla v\|^2 \leqslant 4\int_{B(p_0,2t)} v^2\cdot\|\nabla f\|^2. \tag{4.9}$$

Since $f \equiv 1$ on $B(p_0, t)$ and $\|\nabla f\| \leqslant \frac{C_4}{t}$ a.e. on M, we replace inequality (4.9) by

$$\int_{B(p_0,t)} \|\nabla v\|^2 \leqslant \frac{4C_4^2}{t^2} \int_{B(p_0,2t)} v^2. \tag{4.10}$$

By inequality (4.7),

$$\int_{B(p_0,t)} \|\nabla v\|^2 \leqslant \frac{4C_3^2 C_4^2}{t^2} \operatorname{Vol}(B(p_0, 2t)). \tag{4.11}$$

Since we assume that $(M, \|\cdot\|)$ has finite volume, there exists a constant $C_5 > 0$ such that

$$\int_{B(p_0,t)} \|\nabla v\|^2 \leqslant \frac{C_5}{t^2}. \tag{4.12}$$

Since $(M, \|\cdot\|)$ is complete and open, we let $t \to \infty$, it follows from inequality (4.12) that $\nabla v \equiv 0$ on M. That is, v is a constant on M which contradicts inequality (4.6). $\square$

Then the claim follows by any one of the arguments above. For the second method above, we use the same argument as in [23]. $\square$

It is clear that the conclusion follows from the claim. $\square$

5. Proof of Theorem 1.4

Recall that a Hermitian manifold $(M, <,>)$ is *almost Hermitian* if there exists an almost complex structure $\mathbb{J}$ on M such that

$$< \mathbb{J} \circ V, \mathbb{J} \circ W > = < V, W >$$

for all tangent vectors V and W. It is clear that a Kähler manifold is almost Hermitian.

If we assume that both M and N in Subsection 2.3 are almost Hermitian, Goldberg and Harél in [9] proved that the term $\sum_{k=1}^{n} D_k \cdot \overline{D_k}$ in Chern's formula (see Theorem 2.4) satisfies

$$\sum_{k=1}^{n} D_k \cdot \overline{D_k} = \frac{\|\nabla v\|^2}{4v}.$$

Thus, a direct computation for Theorem 2.4 gives that

$$\Delta \log(v) = 2 \cdot \left(\mathrm{Sca} - \sum_{1 \leqslant \alpha,\beta,k \leqslant n} a_{\alpha k} \overline{a}_{\beta k} \widetilde{Ric}_{\alpha\overline{\beta}} \right). \tag{5.1}$$

(One can see formula (10) in [9] for details.)

Before we prove Theorem 1.4 let us recall a theorem of S.-T. Yau (see Theorem 1 in [23]) which is crucial for this section.

Let N be a complete Riemannian manifold and Δ be the Laplace operator of N. Assume that u and h are two functions on N satisfying the following equation

$$\Delta \log u = h. \tag{5.2}$$

Then the following theorem says that

Theorem 5.1 (Yau). *Suppose h is bounded from below by a constant and $0 < \int_N h(p)\, \mathrm{dVol}(p) \leqslant \infty$. Then $\int_N u^n(p)\, \mathrm{dVol}(p) = \infty$ for $n > 0$, unless u is a constant function.*

Let M be a finite cover of the moduli space and $\|\cdot\|$ be an almost Hermitian metric with $\|\cdot\| \asymp \|\cdot\|_T$. We use the same notations here as in Section 3. In our setting we let

$$N = (M, \|\cdot\|), \quad u = v$$

and

$$h = 2 \cdot \left(\mathrm{Sca} - \sum_{1 \leqslant \alpha,\beta,k \leqslant (3g-3)} a_{\alpha k} \overline{a}_{\beta k} \widetilde{Ric}_{\alpha\overline{\beta}} \right).$$

Since the perturbed Ricci metric $\|\cdot\|_{LSY}$ is Kähler and $(M, \|\cdot\|)$ is almost Hermitian, formula (5.1) exactly tells us that

$$\Delta \log v = h. \tag{5.3}$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. First since $\|\cdot\| \asymp \|\cdot\|_T$, From Theorem 2.1 and Theorem 2.2, we may assume that

$$k_1\|\cdot\| \leqslant \|\cdot\|_{LSY} \leqslant k_2\|\cdot\| \tag{5.4}$$

where k_1, k_2 are two positive constants.

The proof of Lemma 4.1 gives that

$$k_1^{6g-6} \leqslant v \leqslant k_2^{6g-6}. \tag{5.5}$$

By using a similar argument in the previous proofs, from Theorem 2.3 we have, for any $p \in (M, \|\cdot\|)$,

$$\begin{aligned} h(p) &= 2\cdot\left(\mathrm{Sca}(p) - \sum_{1\leqslant\alpha,\beta,k\leqslant(3g-3)} a_{\alpha k}(p)\overline{a}_{\beta k}(p)\widetilde{Ric}_{a\overline{\beta}}(p)\right) \\ &\geqslant 2\cdot \inf_{p\in(M,\|\cdot\|))} \mathrm{Sca}(p) + 2C_2\left(\sum_{1\leqslant\alpha,\beta\leqslant(3g-3)} |a_{\alpha\beta}(p)|^2\right) \\ &\geqslant 2\cdot \inf_{p\in(M,\|\cdot\|))} \mathrm{Sca}(p) + 2C_2(3g-3)v^{\frac{1}{3g-3}} \quad \text{(by the AM-GM inequality)} \\ &> 2\cdot \inf_{p\in(M,\|\cdot\|))} \mathrm{Sca}(p) \\ &> -\infty \end{aligned}$$

where we apply the assumption that the scalar curvature of $(M, \|\cdot\|)$ is bounded from below for the last step. That is, h is bounded from below on $(M, \|\cdot\|)$.

We finish the proof through the following two cases.

Case (1). v is a constant on $(M, \|\cdot\|)$.

If v is a constant, by equation (5.3) we have $h \equiv 0$. That is,

$$\mathrm{Sca} = \sum_{1\leqslant\alpha,\beta,k\leqslant n} a_{\alpha k}\overline{a}_{\beta k}\widetilde{Ric}_{a\overline{\beta}}. \tag{5.6}$$

It follows from Theorem 2.3, equation (5.6) and the AM-GM inequality that for all $p \in (M, \|\cdot\|)$,

$$\begin{aligned}\text{Sca}(p) &\leqslant -C_2 \left(\sum_{1\leqslant \alpha,\beta \leqslant (3g-3)} |a_{\alpha\beta}(p)|^2 \right) \\ &\leqslant -C_2(3g-3)v(p)^{\frac{1}{3g-3}} \\ &< 0.\end{aligned} \tag{5.7}$$

Then it is clear that the total curvature satisfies

$$\int_{p\in(M,\|\cdot\|)} \text{Sca}(p)\,\text{dVol}(p) < 0.$$

Case (2). v is not a constant on $(M, \|\cdot\|)$.

Since $(M, \|\cdot\|_{LSY})$ has finite volume, from inequality (5.5) we have

$$\int_{p\in(M,\|\cdot\|)} v^n(p)\,\text{dVol}(p) < \infty \quad \forall n > 0. \tag{5.8}$$

We already show that h is bounded from below on $(M, \|\cdot\|)$. Thus, from Theorem 5.1 we know that

$$\int_{p\in(M,\|\cdot\|)} h(p)\,\text{dVol}(p) \leqslant 0. \tag{5.9}$$

That is,

$$\begin{aligned}&\int_{p\in(M,\|\cdot\|)} \text{Sca}(p)\,\text{dVol}(p) \\ &\leqslant \int_{p\in(M,\|\cdot\|)} \sum_{1\leqslant \alpha,\beta,k \leqslant (3g-3)} a_{\alpha k}(p)\overline{a}_{\beta k}(p)\widetilde{Ric}_{\alpha\overline{\beta}}(p)\,\text{dVol}(p).\end{aligned}$$

By Theorem 2.3 and the AM-GM inequality we have

$$\begin{aligned}\int_{p\in(M,\|\cdot\|)} \text{Sca}(p)\,\text{dVol}(p) &\leqslant -\int_{p\in(M,\|\cdot\|)} C_2 \left(\sum_{1\leqslant \alpha,\beta \leqslant (3g-3)} |a_{\alpha\beta}(p)|^2 \right) \text{dVol}(p) \\ &\leqslant -\int_{p\in(M,\|\cdot\|)} C_2(3g-3)v(p)^{\frac{1}{3g-3}}\,\text{dVol}(p) \\ &< 0\end{aligned}$$

where we apply the fact that $v > 0$ for the last step. □

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Department of Mathematics, Rice University
Houston, Texas, 77005-1892, USA
Current addrsss:
Yau Mathematical Sciences Center, Tsinghua University
Haidian District, Beijing 100084, China
E-mail address: yunhui_wu@mail.tsinghua.edu.cn

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