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Stable cohomotopy Seiberg-Witten invariants of connected sums of four-manifolds with positive first Betti number II: Applications

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This is a sequel to our article [16] where a generalization of a non-vanishing theorem for stable cohomotopy Seiberg-Witten invariants is proved. The main purpose of the current article is to give various applications of the non-vanishing theorem to the differential geometry and topology of 4-manifolds, including existence of exotic smooth structures, smooth connected sum decompositions of 4-manifolds and computations of Perelman's $\bar{\lambda}$ invariant.

1. Introduction

Let X be a closed smooth Riemannian 4-manifold X with $b^+(X) > 1$, where $b^+(X)$ (resp. $b^-(X)$) denotes the dimension of the maximal positive (resp. negative) definite linear subspace in the second cohomology of X. In what follows, $\chi(X)$ and $\tau(X)$ denote respectively the Euler characteristic and the signature of X. We set $c_1^2(X) := 2\chi(X) + 3\tau(X)$. We also denote respectively Seiberg-Witten invariant [41] and stable cohomotopy Seiberg-Witten invariant [3, 4] of X by SW_X and BF_X.

To state the main results of the current article, we recall

Definition 1 ([16]). A closed oriented smooth 4-manifold X with $b^+(X) > 1$ is called BF-admissible if the following three conditions are satisfied.

1) There exists a spin^c-structure Γ_X with $SW_X(\Gamma_X) \equiv 1 \pmod 2$ and $c_1^2(\mathcal{L}_{\Gamma_X}) = 2\chi(X) + 3\tau(X)$, where $c_1(\mathcal{L}_{\Gamma_X})$ is the first Chern class of $\mathcal{L}_{\Gamma_X}$, where $\mathcal{L}_{\Gamma_X}$ is the complex line bundle associated with Γ_X.
2) $b^+(X) - b_1(X) \equiv 3 \pmod 4$.
3) $\mathfrak{S}^{ij}(\Gamma_X) := \frac{1}{2}\langle c_1(L_{\Gamma_X}) \cup \mathfrak{e}_i \cup \mathfrak{e}_j, [X]\rangle \equiv 0 \pmod 2$ for all i, j,

where $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_s$ be a set of generators of $H^1(X, \mathbb{Z})$, $s = b_1(X)$, and $[X]$ is the fundamental class of X_i and $\langle \cdot, \cdot \rangle$ is the pairing between cohomology and homology.

There exist BF-admissible 4-manifolds with $b_1 > 0$ in profusion [5, 16]. Let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$, where $m = 1, 2, 3$. We proved in [16] that BF_X of connected sums $X := \#_{m=1}^n X_m$ are not trivial, where $n = 2, 3$.

The main purpose of the current article is to give various applications of the non-vanishing theorem [16] to the geometry and topology of 4-manifolds. The first application concerns exotic smooth structures on 4-manifolds. By Donaldson-Freedman [10, 12] classification of homeomorphism types of simply connected 4-manifolds and Taubes's theorem [39] on the non-triviality of Seiberg-Witten invariants of symplectic 4-manifolds, one can see that every simply connected, non-spin symplectic 4-manifolds X with $b^+(X) > 1$ admits at least one exotic smooth structure. It is natural to ask if the exotic smooth structure can survive after the connected sum of other 4-manifolds. In this direction, we will prove

Theorem A. *Let X be a closed, simply connected, non-spin, symplectic 4-manifold with $b^+(X) \equiv 3 \pmod 4$, homeomorphic but not diffeomorphic to $Y = p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$. Let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$ for $m = 1, 2$. Then, after connect summing both X and Y with either X_1 or $X_1 \# X_2$, we still get homeomorphic but not diffeomorphic 4-manifolds.*

The second application concerns smooth connected sum decompositions of 4-manifolds. Taubes's non-vanishing theorem [39] for Seiberg-Witten invariants of symplectic 4-manifolds implies that any closed symplectic 4-manifold X has no decompositions of the form $Y_1 \# Y_2$ with $b^+(Y_1) > 0$, $b^+(Y_2) > 0$. Then one can ask whether connected sums $\#_{m=1}^n X_m$ of symplectic 4-manifolds X_m have no decompositions of the form $\#_{m=1}^N Y_m$ with $b^+(Y_m) > 0$, $N > n$. However, one can construct counter examples to this question. Indeed, it is known [31] that connected sums $X \# \mathbb{CP}^2$ of any simply connected, elliptic surfaces X and $\mathbb{CP}^2$ is diffeomorphic to $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ for some integers $p, q \geq 0$. However, we will prove

Theorem B. *For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$ and set $X := \#_{m=1}^n X_m$, where $n = 2, 3$. Then X is never diffeomorphic to $\#_{m=1}^N Y_m$ with $b^+(Y_m) > 0$ and $N > n$.*

The third application concerns computations of the values of differential geometric invariants. In his celebrated work [34–36] on the Ricci flow, Perelman introduced a functional which is called $\mathcal{F}$-functional and also introduced an invariant of any closed manifold with arbitrary dimension, which is called the $\bar{\lambda}$ invariant. The $\bar{\lambda}$ invariant is arisen naturally from $\mathcal{F}$-functional. For 3-manifolds which do not admit positive scalar curvature metrics, Perelman computed its value. See Section 8 in [35]. Li [29] introduced a family of functionals of the type of $\mathcal{F}$-functional with monotonicity property under the Ricci flow. See also [33]. Inspired by these works of Perelman and Li, we will introduce, for any real number $k \in \mathbb{R}$, an invariant $\bar{\lambda}_k$ of any closed manifold. We shall call it $\bar{\lambda}_k$ invariant. In particular, $\bar{\lambda}_1 = \bar{\lambda}$ holds. In dimension 4, we will prove

Theorem C. *For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$ and N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. Then, for $n = 2, 3$ and any real number $k \geq \frac{2}{3}$, $\bar{\lambda}_k$ invariants of $M := (\#_{m=1}^n X_m)\#N$ satisfy $\bar{\lambda}_k(M) \leq 0$ and*

$$\left| \frac{\bar{\lambda}_k(M)}{k} \right|^2 \geq 32\pi^2 \sum_{m=1}^{n} c_1^2(X_m).$$

If moreover X_m are minimal Kähler surfaces for $m = 1, 2, 3$ and N admits a Riemannian metric of non-negative scalar curvature, then the equality holds.

We will also prove that $\bar{\lambda}_k$ invariant is closely related to other differential geometric invariants including the Yamabe invariant.

As the fourth application, we will obtain an obstruction to the existence of Einstein metrics on 4-manifolds:

Theorem D. *Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$. For $n = 2, 3$, $M := (\#_{m=1}^n X_m)\#N$ cannot admit any Einstein metric if*

$$4n - c_1^2(N) \geq \frac{1}{3} \sum_{m=1}^{n} c_1^2(X_m).$$

Theorem D was used to construct the first examples of 4-manifolds with non-zero simplicial volume and satisfying the strict Gromov-Hitchin-Thorpe inequality, but admitting infinitely many distinct smooth structures for which no compatible Einstein metric exists. See [8] for more details.

All results in the current article firstly appeared as a part of our arXiv preprint [15]. We decided to divide the preprint [15] into two articles. The current article is the second part of it.

2. Proofs of Theorems A and B

Let X be a closed oriented smooth 4-manifold with a Riemannian metric g and a spin^c-structure Γ_X. The Seiberg-Witten equations [41] perturbed by a self dual 2-form η are the following equations for $(\phi, A) \in \Gamma(S^+_{\Gamma_X}) \times \mathcal{A}(\mathcal{L}_{\Gamma_X})$:

$$D_A\phi = 0, \quad F_A^+ = q(\phi) + \eta,$$

where $\Gamma(S^+_\Gamma)$ is the space of sections of the determinant line bundle $S^+_{\Gamma_X}$, $\mathcal{A}(\mathcal{L}_{\Gamma_X})$ is the space of $U(1)$-connections on $\mathcal{L}_{\Gamma_X}$, $D_A : \Gamma(S^+_{\Gamma_X}) \to \Gamma(S^-_{\Gamma_X})$ is the twisted Dirac operator, F_A^+ is the self-dual part of the curvature F_A and $q(\phi)$ is the trace-free part of the endomorphism $\phi \otimes \phi^*$ of $S^+_{\Gamma_X}$. Let $\mathcal{G}$ be the group of smooth maps from X to S^1, then $\mathcal{G}$ acts on $\Gamma(S^+_{\Gamma_X}) \times \mathcal{A}(\Gamma_X)$ by $\gamma(\phi, A) := (\gamma\phi, A - 2\gamma^{-1}d\gamma)$, where $\gamma \in \mathcal{G}$ and $(\phi, A) \in \Gamma(S^+_{\Gamma_X}) \times \mathcal{A}(\mathcal{L}_{\Gamma_X})$. This action preserves the solution space $\mathcal{S}_{\Gamma_X}(g, \eta)$ of the perturbed Seiberg-Witten equations. We call the quotient space

$$\mathcal{M}^{SW}_{\Gamma_X}(g, \eta) := \mathcal{S}_{\Gamma_X}(g, \eta)/\mathcal{G}$$

the Seiberg-Witten moduli space. In the following, we denote any element of $\mathcal{M}^{SW}_{\Gamma_X}(g, \eta)$ by $[\phi, A]$.

On the other hand, let Z_l be a 4-manifold and Γ_l be a spin^c-structure on Z_l for $l = 1, 2$. We define a spin^c-structure $\Gamma_1 \# \Gamma_2$ on $Z_1 \# Z_2$ as follows. Take a point $z_l \in Z_l$ and a small open disk D_l in Z_l with center z_l. Denote the closure of D_l by $\overline{D}_l$. We can take an isomorphism φ between $\Gamma_1|_{\overline{D}_1}$ and $\Gamma_2|_{\overline{D}_2}$. The isomorphism φ is unique up to homotopy. Then Γ_1, Γ_2 and φ naturally induce a spin^c-structure $\Gamma_1 \# \Gamma_2$ on $Z_1 \# Z_2 = (Z_1 \setminus D_1) \cup_{S^3} (Z_2 \setminus D_2)$. The isomorphism class of $\Gamma_1 \# \Gamma_2$ is independent of the choice of φ.

A key ingredient in the proofs of Theorems A and B is the following technical lemma:

Lemma 2. *Let Z_ℓ be a closed oriented smooth 4-manifold with $b^+(Z_\ell) > 0$ where ℓ is an arbitrary positive integer. Let Γ_ℓ be a spin^c-structure on Z_ℓ. Put $Z := \#_{\ell=1}^N Z_\ell$, $\Gamma_Z := \#_{\ell=1}^N \Gamma_\ell$ for some $N \geq 1$. Assume that the Seiberg-Witten moduli space $\mathcal{M}^{SW}_{\Gamma_Z}(g, \eta)$ is not empty for all Riemannian metrics g and self-dual 2-forms η on Z. Then the virtual dimension of $\mathcal{M}^{SW}_{\Gamma_Z}(g, \eta)$*

satisfies

$$\dim \mathcal{M}^{SW}_{\Gamma_Z}(g,\eta) \geq N-1.$$

Proof. To simplify notations we consider the case $N=3$. The proof is a standard neck stretching argument. See Section 9.3.2 of [11] for a similar discussion in the instanton case.

Take points $z_1 \in Z_1$, $z_2, z_2' \in Z_2$, $z_3 \in Z_3$ and small open disks D_1, D_2, D_2', D_3 centered at these points. We put

$$\begin{aligned}
\hat{Z}_1 &= (Z_1 \backslash D_1) \cup S^3 \times \mathbb{R}_{\geq 0},\\
\hat{Z}_2 &= S^3 \times \mathbb{R}_{\leq 0} \cup (Z_2 \backslash D_2 \cup D_2') \cup S^3 \times \mathbb{R}_{\geq 0},\\
\hat{Z}_3 &= S^3 \times \mathbb{R}_{\leq 0} \cup (Z_3 \backslash D_4),
\end{aligned}$$

and for each $T>0$ we define

$$\begin{aligned}
\hat{Z}_1(T) &= \hat{Z}_1 \backslash S^3 \times [2T,\infty),\\
\hat{Z}_2(T) &= \hat{Z}_2 \backslash \big(S^3 \times (-\infty,-2T) \cup S^3 \times [2T,\infty)\big),\\
\hat{Z}_3(T) &= \hat{Z}_3 \backslash S^3 \times (-\infty,-2T].
\end{aligned}$$

There is an identification

$$\begin{array}{cccc}
\varphi_T: & S^3 \times (T,2T) & \cong & S^3 \times (-2T,-T)\\
 & (y,t) & \longmapsto & (y,t-3T).
\end{array}$$

Gluing $\hat{Z}_1(T), \hat{Z}_2(T), \hat{Z}_3(T)$ by using φ_T, we have a manifold $Z(T)$ which is diffeomorphic to the connected sum $Z = \#_{\ell=1}^3 Z_\ell$. We take Riemannian metrics $\hat{g}_\ell$ on $\hat{Z}_\ell$ which coincide with $g_{S^3}+dt^2$ on the ends. Here g_{S^3} is the standard metric on S^3. These metrics naturally induce a Riemannian metric $g(T)$ on $Z(T)$.

Let $\mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_\ell,\hat{\eta}_\ell)$ be the moduli spaces of monopoles on $\hat{Z}_\ell$ which converge to the trivial monopole on S^3 for all ℓ. Here $\hat{\Gamma}_\ell$ are spin^c-structures on $\hat{Z}_\ell$ induced by Γ_ℓ. Since $b^+(\hat{Z}_\ell)>0$, we can choose self-dual 2-forms $\hat{\eta}_\ell$ such that $\mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_\ell,\hat{\eta}_\ell)$ contain no reducible monopoles and are smooth of the expected dimension or empty. Moreover we may suppose that the supports of $\hat{\eta}_\ell$ do not intersect the ends of $\hat{Z}_\ell$. See Proposition 4.4.1 in [32]. Extending $\hat{\eta}_\ell$ trivially, we consider $\hat{\eta}_\ell$ as self-dual 2-forms on $Z(T)$, and we get a self-dual 2-form $\eta(T) := \hat{\eta}_1+\hat{\eta}_2+\hat{\eta}_3$ on $Z(T)$. Then we have

$$\dim \mathcal{M}^{SW}_{\Gamma_Z}(g(T),\eta(T)) = \sum_{l=1}^{3} \dim \mathcal{M}^{SW}_{\hat{\Gamma}_l}(\hat{g}_l,\hat{\eta}_l) + 2. \tag{1}$$

Here $\dim \mathcal{M}^{SW}_{\Gamma_Z}(g(T),\eta(T))$, $\dim \mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_\ell,\hat{\eta}_\ell)$ are the virtual dimensions of the moduli spaces. This is derived from the excision principle of index of elliptic differential operators. See Example 4.5.13 and Section 4.6.1 of [32] for the formula (1). See also (7.2.47) and Section 9.3.2 of [11] for the instanton case. We can also see this from the theory of gluing of monopoles. For large T, coordinates of $\mathcal{M}^{SW}_{\Gamma_Z}(g(T),\eta(T))$ are given by coordinates of $\mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_\ell,\hat{\eta}_\ell)$ and gluing parameters. Since $Z(T)$ has two necks, the space of gluing parameters is $U(1)\times U(1)$ and it is 2-dimensional. Hence we have the formula (1).

Let $\{T^\alpha\}_{\alpha=1}^\infty$ be a strictly increasing sequence of positive numbers which goes to infinity. By the assumption we have made in the statement of the lemma, $\mathcal{M}^{SW}_{\Gamma_Z}(g(T^\alpha),\eta(T^\alpha))$ are non-empty for any α. Take any element $[\phi^\alpha,A^\alpha]\in \mathcal{M}^{SW}_{\Gamma_Z}(g(T^\alpha),\eta(T^\alpha))$ for each α. Then there is a subsequence $\{[\phi^{\alpha'},A^{\alpha'}]\}_{\alpha'}$ which converges to some $([\phi_1^\infty,A_1^\infty],[\phi_2^\infty,A_2^\infty],[\phi_3^\infty,A_3^\infty])\in \mathcal{M}^{SW}_{\hat{\Gamma}_1}(\hat{g}_1,\hat{\eta}_1)\times \mathcal{M}^{SW}_{\hat{\Gamma}_2}(\hat{g}_2,\hat{\eta}_2)\times \mathcal{M}^{SW}_{\hat{\Gamma}_3}(\hat{g}_3,\hat{\eta}_3)$. In particular, $\mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_l,\hat{\eta}_\ell)$ are non-empty. Since $\mathcal{M}^{SW}_{\hat{\Gamma}_\ell}(\hat{g}_\ell,\hat{\eta}_\ell)$ are non-empty and smooth of the expected dimension, their virtual dimensions are at least zero. From (1), we have

$$\dim \mathcal{M}^{SW}_{\Gamma_Z}(g(T),\eta(T))\geq 2.$$

The virtual dimension of $\mathcal{M}^{SW}_{\Gamma_Z}(g,\eta)$ is independent of g,η, so we have obtained

$$\dim \mathcal{M}^{SW}_{\Gamma_Z}(g,\eta)\geq 2$$

for any (g,η). The proof of the general case for arbitrary $N\geq 1$ is obtained similarly. □

By using Lemma 2, we prove Theorem B:

Proof. For $m=1,2,3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m)-b_1(X_m)>1$. Take a spin^c-structure Γ_m on each X_m with $c_1^2(\mathcal{L}_{\Gamma_m})=2\chi(X_m)+3\tau(X_m)$, with $SW_X(\Gamma_m)\equiv 1 \pmod 2$ and with $\mathfrak{S}^{ij}(\Gamma_m)\equiv 0 \pmod 2$. Here $\mathcal{L}_{\Gamma_m}$ is the determinant line bundle of Γ_m and see Definition 1 for the definition of $\mathfrak{S}^{ij}(\Gamma_m)$. Let $n=2$ or 3 and put $X=\#_{m=1}^n X_m$, $\Gamma_X=\#_{m=1}^n\Gamma_m$. Notice that it follows from Theorem 23 of [16] that the spin cobordism Seiberg-Witten invariant $SW_X^{spin}(\Gamma_X,L)$ is non-trivial since X_m are BF-admissible 4-manifolds with $b^+(X_m)-b_1(X_m)>1$. This implies that $\mathcal{M}^{SW}_{\Gamma_X}(g,\eta)$ are non-empty for all g,η. Suppose that X has a decomposition $X=\#_{m=1}^N Y_m$ with $b^+(Y_m)>0$ and $N>n$. Then Lemma 2 implies $\dim \mathcal{M}^{SW}_{\Gamma_X}(g,\eta)\geq N-1$. On the other hand, the dimension of $\mathcal{M}^{SW}_{\Gamma_X}(g,\eta)$

is equal to $n-1$. We can see this from the dimension formula:

$$
\begin{aligned}
(2) \qquad \dim \mathcal{M}^{SW}_{\Gamma_X}(g,\eta) &= \frac{1}{4}\Big(c_1^2(\mathcal{L}_{\Gamma_X}) - 2\chi(X) - 3\tau(X)\Big) \\
&= \frac{1}{4}\sum_{m=1}^{n}\Big(c_1^2(\mathcal{L}_{\Gamma_m}) - 2\chi(X_m) - 3\tau(X_m)\Big) + n - 1 \\
&= n-1,
\end{aligned}
$$

where we have used the following formulas:

$$
\begin{aligned}
c_1^2(\mathcal{L}_{\Gamma_X}) &= \sum_{m=1}^{n} c_1^2(\mathcal{L}_{\Gamma_m}), \\
c_1^2(\mathcal{L}_{\Gamma_m}) &= 2\chi(X_m) + 3\tau(X_m), \\
\chi(X) &= \sum_{m=1}^{n} \chi(X_m) - 2(n-1), \\
\tau(X) &= \sum_{m=1}^{n} \tau(X_m).
\end{aligned}
$$

Therefore we have $N \le n$. Since we assumed that $N > n$, this is a contradiction. Hence we have proved Theorem B. □

Lemma 2 also enables us to prove Theorem A:

Proof. Let X' be X_1 or $X_1 \# X_2$. Since X is homeomorphic to Y, $X \# X'$ is homeomorphic to $Y \# X'$. We prove that $X \# X'$ is not diffeomorphic to $Y \# X'$. Let Γ be the spin^c-structure on $X \# X'$ induced by almost complex structures on X and X_m. Then the dimension of Seiberg-Witten moduli space $\mathcal{M}^{SW}_{\Gamma}(g,\eta)$ associated with Γ is 1 or 2 by the formula (2) and it follows from Theorem 23 in [16] that $\mathcal{M}^{SW}_{\Gamma}(g,\eta)$ is non-empty for any g,η. On the other hand, by the assumption that $b^+(X) \equiv 3 \bmod 4$, Y is the connected sum $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ with $p \ge 3$. Hence we can write $Y \# X'$ as $\#_{\ell=1}^4 Z_\ell$ with $b^+(Z_\ell) > 0$. Suppose that $X \# X'$ is diffeomorphic to $Y \# X'$. Then it follows from Lemma 2 that the dimension of the moduli space $\mathcal{M}^{SW}_{\Gamma}(g,\eta)$ is at least 3. Hence we have a contradiction since the dimension of the moduli space is 1 or 2. We have proved Theorem A as desired. □

Remark 3. We say that a closed oriented smooth 4-manifold X is almost completely decomposable if $X \# \mathbb{CP}^2$ is diffeomorphic to $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$ for some integers $p,q \ge 0$. It is known [30, 31] that smooth hypersurfaces in $\mathbb{CP}^3$ and simply connected elliptic surfaces are almost completely decomposable.

Let X be a closed, simply-connected, non-spin, almost completely decomposable and BF-admissible symplectic 4-manifold. Put $p := b^+(X)$, $q := b^-(X)$ and $Y := p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$. Then X is homemorphic to Y and $X\#\mathbb{CP}^2$ is diffeomorphic to $Y\#\mathbb{CP}^2$. On the other hand, if X_1 is a BF-admissible 4-manifold with $b^+(X_1) - b_1(X_1) > 1$, then $X\#X_1$ is homeomorphic to $Y\#X_1$ but not diffeomorphic to $Y\#X_1$ by Theorem A.

3. Proofs of Theorems C and D

3.1. LeBrun's curvature bounds and Theorem D

Let X be a closed oriented smooth 4-manifold with $b^+(X) \geq 2$. An element $\mathfrak{a} \in H^2(X,\mathbb{Z})/\text{torsion} \subset H^2(X,\mathbb{R})$ is called monopole class [14, 21, 26, 27] of X if there exists a spin^c-structure Γ_X with $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X}) = \mathfrak{a}$ which has the property that the corresponding Seiberg-Witten equations have a solution for every Riemannian metric on X. Here $c_1^{\mathbb{R}}(\mathcal{L}_{\Gamma_X})$ is the image of the first Chern class $c_1(\mathcal{L}_{\Gamma_X})$ of the complex line bundle $\mathcal{L}_{\Gamma_X}$ in $H^2(X,\mathbb{R})$. The nontriviality of stable cohomotopy Seiberg-Witten invariants implies the existence of monopole classes (see Proposition 6 in [14]). On the other hand, LeBrun [24, 26, 27] proved that the existence of monopole classes implies curvature bounds which have many differential geometric applications.
A key ingredient in the proofs of Theorems C and D is the following result:

Theorem 4. *For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$ and suppose that N is a closed oriented smooth 4-manifold with $b^+(N) = 0$. For $n = 2, 3$, any Riemannian metric g on $M := \left(\#_{m=1}^n X_m\right)\#N$ satisfies*

$$\int_M s_g^2 d\mu_g \geq 32\pi^2 \sum_{m=1}^n c_1^2(X_m), \tag{3}$$

$$\frac{1}{4\pi^2}\int_M \left(2|W_g^+|^2 + \frac{s_g^2}{24}\right) d\mu_g \geq \frac{2}{3}\sum_{n=1}^m c_1^2(X_m), \tag{4}$$

$$\int_M |r_g|^2 d\mu_g \geq 8\pi^2 \left[4n - c_1^2(N) + \sum_{m=1}^n c_1^2(X_m)\right]. \tag{5}$$

where s_g, W_g^+ and r_g denote respectively the scalar curvature, the self-dual part of the Weyl curvature and Ricci curvature of g.

Proof. Use the non-vanishing theorem in [16] and ideas of proofs of results proved in [14]. In particular, see Proposition 2, Proposition 4, Proposition 6, Corollary 8, Proposition 10, Corollary 11 and Proposition 15 in [14]. □

We have the following Gauss-Bonnet-type formula (see [6]) for any closed oriented Riemannian 4-manifold (X, g):

$$c_1^2(X) = \frac{1}{4\pi^2}\int_X \left(2|W_g^+|^2 + \frac{s_g^2}{24} - \frac{|\overset{\circ}{r}_g|^2}{2}\right) d\mu_g, \tag{6}$$

where $\overset{\circ}{r}_g$ is the trace-free part of the Ricci curvature r_g of g. Using (4), (6) and the argument of the proof of Theorem D in [14], we can prove Theorem D in Introduction. We leave the details to the reader. As a corollary of Theorem D, we obtain

Corollary 5. *For $m = 1, 2$, let X_m be simply connected symplectic 4-manifolds with $b^+(X_m) \equiv 3 \pmod 4$. Consider connected sums $M := (\#_{m=1}^n X_m)\#k(\Sigma_h \times \Sigma_g)\#\ell_1(S^1 \times S^3)\#\ell_2\overline{\mathbb{CP}}^2$, where $n, k \geq 1$ satisfying $n + k \leq 3$, $\ell_1, \ell_2 \geq 0$ and g, h are odd integers > 1. Then M cannot admit any Einstein metric if*

$$4(n + \ell_1 + k) + \ell_2 \geq \frac{1}{3}\left(\sum_{m=1}^{n} c_1^2(X_m) + 4k(1-h)(1-g)\right).$$

See also [8] where an interesting application of this result was given.

3.2. Computation of several differential geometric invariants

In this section, we shall compute the values of several differential geometric invariants. The main results in this subsection are Theorems 7 and 8 below.

Let X be a closed oriented Riemannian manifold X of dimension $n \geq 3$ and $\gamma := [g] = \{ug \mid u : X \to \mathbb{R}^+\}$ a conformal class of an arbitrary metric g. Trudinger, Aubin, and Schoen [2, 28, 37, 40, 42] proved every conformal class on X contains a Riemannian metric of constant scalar curvature. Such a metric $\hat{g}$ can be constructed by minimizing the Einstein-Hilbert functional:

$$\hat{g} \mapsto \frac{\int_X s_{\hat{g}}\, d\mu_{\hat{g}}}{\left(\int_X d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}},$$

among all metrics conformal to g, where $s_{\hat{g}}$ is the scalar curvature of the metric $\hat{g}$ and $d\mu_{\hat{g}}$ is the volume form with respect to $\hat{g}$. Notice that, by setting $\hat{g} = u^{4/(n-2)}g$, the following identity holds:

$$\frac{\int_X s_{\hat{g}}\, d\mu_{\hat{g}}}{\left(\int_X d\mu_{\hat{g}}\right)^{\frac{n-2}{n}}} = \frac{\int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2\right] d\mu_g}{\left(\int_X u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}}.$$

Associated to each conformal class $\gamma := [g]$, we define the Yamabe constant of the conformal class γ in the following way:

$$Y_\gamma = \inf_{u \in C^\infty_+(X)} \frac{\int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2\right] d\mu_g}{\left(\int_X u^{2n/(n-2)} d\mu_g\right)^{(n-2)/n}}, \tag{7}$$

where $C^\infty_+(X)$ is the set of all positive functions $u : X \to \mathbb{R}^+$. Trudinger-Aubin-Schoen theorem teaches us that this number is actually realized as the constant scalar curvature of some unit-volume metric in each conformal class γ. A constant-scalar-curvature metric of this type is called a Yamabe minimizer. For each conformal class γ, we consider an associated number Y_γ, which is called the Yamabe constant of the conformal class γ and defined as follows:

$$Y_\gamma = \inf_{h \in \gamma} \frac{\int_X s_h\, d\mu_h}{\left(\int_X d\mu_h\right)^{\frac{n-2}{n}}}.$$

Kobayashi [19] and Schoen [38] independently introduced the following invariant of X:

$$\mathcal{Y}(X) = \sup_{\gamma \in \mathcal{C}} Y_\gamma,$$

where $\mathcal{C}$ is the set of all conformal classes on X. This is now commonly known as the Yamabe invariant of X. It is known [19] that $\mathcal{Y}(X) \le 0$ if and only if X does not admit a metric of positive scalar curvature. The Yamabe invariant is closely related to the diffeomorphism invariant [7, 26] defined by

$$\mathcal{I}_s(X) := \inf_{g \in \mathcal{R}_X} \int_X |s_g|^{n/2} d\mu_g, \tag{8}$$

where the space of all Riemannian metrics on X is denoted by $\mathcal{R}_X$. For every closed n-manifold with $n \ge 3$ admitting non-negative scalar curvature, i.e., $\mathcal{Y}(X) \ge 0$, we have $\mathcal{I}_s(X) = 0$. The following equality holds whenever

$\mathcal{Y}(X) \leq 0$ (see [14, 25]):

$$\mathcal{I}_s(X) = |\mathcal{Y}(X)|^{n/2}. \tag{9}$$

Hence, $\mathcal{I}_s(X)$ of a closed 4-manifold X with $\mathcal{Y}(X) \leq 0$ satisfies

$$\mathcal{I}_s(X) = |\mathcal{Y}(X)|^2 = \inf_{g \in \mathcal{R}_X} \int_X s_g^2 d\mu_g. \tag{10}$$

On the other hand, consider the following quantity:

$$\mathcal{K}(X) := \sup_{g \in \mathcal{R}_X} \left(\left(\min_{x \in X} s_g \right) (vol_g)^{n/2} \right),$$

where $vol_g = \int_X d\mu_g$ is the total volume with respect to g. Kobayashi (Corollary 1.7 in [19]) pointed out that the following equality holds whenever $\mathcal{Y}(X) \leq 0$:

$$\mathcal{K}(X) = \mathcal{Y}(X). \tag{11}$$

Proposition 6. *Let X_m, N and M be as in Theorem 4. Then, for $n = 2, 3$, $\mathcal{K}(M) = \mathcal{Y}(M) \leq 0$ and*

$$\mathcal{I}_s(M) = |\mathcal{Y}(M)|^2 = |\mathcal{K}(M)|^2 \geq 32\pi^2 \sum_{m=1}^{n} c_1^2(X_m).$$

Proof. Notice that BF_M of the 4-manifold M is not trivial by the non-vanishing theorem proved in [16]. This implies that M cannot admit any Riemannian metric of positive scalar curvature. This tells us that $\mathcal{Y}(M) \leq 0$ holds. Therefore, the bound (3), (10) and (11) imply the desired result. □

The following result can be seen as a generalization of both Theorems A and B in [14] to the case where $b_1 > 0$:

Theorem 7. *Let N be a closed oriented smooth 4-manifold with $b^+(N) = 0$ and with a Riemannian metric of non-negative scalar curvature. For $m = 1, 2, 3$, let X_m be BF-admissible, minimal Kähler surfaces with $b^+(X_m) - b_1(X_m) > 1$. Then the following holds for $M := (\#_{m=1}^n X_m)\#N$, where $n = 2, 3$:*

$$\mathcal{I}_s(M) = |\mathcal{Y}(M)|^2 = |\mathcal{K}(M)|^2 = 32\pi^2 \sum_{m=1}^{n} c_1^2(X_m). \tag{12}$$

Proof. Use Proposition 6 and the idea of the proof of Theorem B in [14]. See also Proposition 12 and Proposition 13 in [14]. □

It is natural to consider the following Ricci curvature version of the invariant (8):

$$\mathcal{I}_r(X) := \inf_{g \in \mathcal{R}_X} \int_X |r_g|^{n/2} d\mu_g. \tag{13}$$

There is the following relation between (8) and (13):

$$\mathcal{I}_r(X) \geq n^{-n/4} \mathcal{I}_s(X), \tag{14}$$

and that equality holds if the Yamabe invariant is both non-positive and realized by an Einstein metric (see [26]). The failure of the equality gives a quantitative obstruction to Yamabe's program for finding Einstein metrics. Therefore, it is quite interesting to investigate when the above inequality (14) becomes strict.

Theorem 8. *Let N be a closed oriented smooth 4-manifold equipped with an anti-self-dual metric of positive scalar curvature. For $m = 1, 2, 3$, let X_m be BF-admissible, minimal Kähler surfaces with $b^+(X_m) - b_1(X_m) > 1$. Then the following holds for $M := (\#_{m=1}^n X_m) \# N$, where $n = 2, 3$:*

$$\mathcal{I}_r(M) = 8\pi^2 \left[4n - c_1^2(N) + \sum_{m=1}^{n} c_1^2(X_m) \right]. \tag{15}$$

Proof. Use (5) and the idea of the proof of Theorem C in [14]. □

The above hypotheses regarding N and Proposition 1 in [22] force that $b^+(N) = 0$. Hence we have $c_1^2(N) = 2\chi(N) + 3\tau(N) = 4 - 4b_1(N) + 5b^+(N) - b^-(N) = 4 - 4b_1(N) - b^-(N) \leq 4$. By this, (12) and (15), the strict inequality holds whenever $n = 2, 3$:

$$\mathcal{I}_r(M) > \frac{1}{4} \mathcal{I}_s(M).$$

Hence, the Yamabe sup-inf on this connected sums never realized by an Einstein metric. On the other hand, since $k\overline{\mathbb{CP}}^2 \# \ell(S^1 \times S^3)$ admits anti-self-dual metrics of positive scalar curvature [17, 23], Theorem 8 implies

Corollary 9. *Let X_m be as in Theorem 8. Then, for $n = 2, 3$, and any integers $k, \ell \geq 0$,*

$$\mathcal{I}_r\Big((\#_{m=1}^n X_m)\#k\overline{\mathbb{CP}}^2\#\ell(S^1 \times S^3)\Big) = 8\pi^2 \left[k + 4(n+\ell-1) + \sum_{m=1}^n c_1^2(X_m)\right].$$

3.3. A variant of Perelman's $\bar{\lambda}$ invariant

The main results of this subsection are Theorems 13 and 15 below. In particular, Theorem C follows from Theorem 15.

Let X be a closed oriented Riemannian manifold of dimension $n \geq 3$ and g be any Riemannian metric on X. We shall denote the space of all Riemannian metrics on X by $\mathcal{R}_X$ and the space of all C^∞ functions on X by $C^\infty(X)$. Then, Perelman's $\mathcal{F}$-functional [34] is the following functional $\mathcal{F} : \mathcal{R}_X \times C^\infty(X) \to \mathbb{R}$ defined by

$$\mathcal{F}(g, f) := \int_X (s_g + |\nabla f|^2) e^{-f} d\mu_g.$$

One of the fundamental discoveries of Perelman is that the Ricci flow can be viewed as the gradient flow of $\mathcal{F}$-functional. Moreover, $\mathcal{F}$-functional is nondecreasing under the Ricci flow coupled with conjugate heat equation. It is then known that, for a given metric g, there exists a unique minimizer of $\mathcal{F}$-functional under the constraint $\int_X e^{-f} d\mu_g = 1$. Hence it is so natural to consider the following functional $\lambda : \mathcal{R}_X \to \mathbb{R}$ which is called Perelman's λ-functional:

$$\lambda(g) := \inf_f \left\{ \mathcal{F}(g, f) \,\middle|\, \int_X e^{-f} d\mu_g = 1 \right\}.$$

It turns out that $\lambda(g)$ is the least eigenvalue of the elliptic operator $4\Delta_g + s_g$, where $\Delta = d^* d = -\nabla \cdot \nabla$ is the positive-spectrum Laplace-Beltrami operator associated with g. Following Perelman [18, 34–36], we consider the scale-invariant quantity $\lambda(g)(vol_g)^{2/n}$. Then, Perelman's $\bar{\lambda}$ invariant of X is defined by

$$\bar{\lambda}(X) = \sup_{g \in \mathcal{R}_X} \lambda(g)(vol_g)^{2/n}.$$

Inspired by recent interesting works of Cao [9] and Li [29], we will introduce one parameter family $\bar{\lambda}_k$ of smooth invariants, where $k \in \mathbb{R}$. We shall call it $\bar{\lambda}_k$ invariant. In particular, $\bar{\lambda}_1 = \bar{\lambda}$ holds. We introduce the following definition which is essentially due to Li [29]. The definition in the case where

$k \geq 1$ is nothing but Definition 41 in [29]. We also notice that the following definition was also appeared in [33]:

Definition 10 ([29, 33]). Let X be a closed oriented Riemannian manifold with dimension ≥ 3. Then, we define the following functional $\mathcal{F}_k : \mathcal{R}_X \times C^\infty(X) \to \mathbb{R}$:

$$\mathcal{F}_k(g, f) := \int_X \Big(k s_g + |\nabla f|^2 \Big) e^{-f} d\mu_g, \tag{16}$$

where k is a real number $k \in \mathbb{R}$. We shall call this $\mathcal{F}_k$-functional.

$\mathcal{F}_1$-functional is nothing but Perelman's $\mathcal{F}$-functional. Li [29] showed that all functionals $\mathcal{F}_k$ with $k \geq 1$ have the monotonicity properties under the Ricci flow coupled with conjugate heat equation. As was already mentioned in [18, 29] essentially, for a given metric g and $k \in \mathbb{R}$, there exists a unique minimizer of $\mathcal{F}_k$-functional under the constraint $\int_X e^{-f} d\mu_g = 1$. In fact, by using a direct method of the elliptic regularity theory [13], one can see that the following infimum is always attained:

$$\lambda(g)_k := \inf_f \left\{ \mathcal{F}_k(g, f) \,\middle|\, \int_X e^{-f} d\mu_g = 1 \right\}.$$

Notice that $\lambda(g)_k$ is the least eigenvalue of the elliptic operator $4\Delta_g + k s_g$. It is natural to introduce the following quantity:

Definition 11. For any real number $k \in \mathbb{R}$, $\bar{\lambda}_k$ invariant of X is defined by

$$\bar{\lambda}_k(X) = \sup_{g \in \mathcal{R}_X} \lambda(g)_k (vol_g)^{2/n}.$$

We have the following result which is a generalization of Theorem A in [1]:

Proposition 12. *Suppose that X is a smooth closed n-manifold, $n \geq 3$. Then the following holds:*

$$\bar{\lambda}_k(X) = \begin{cases} k\mathcal{Y}(X) & \text{if } \mathcal{Y}(X) \leq 0 \text{ and } k \geq \frac{n-2}{n-1}, \\ +\infty & \text{if } \mathcal{Y}(X) > 0 \text{ and } k > 0. \end{cases}$$

Proof. Suppose that γ is a conformal class on a closed oriented Riemannian manifold X of dimension $n \geq 3$ with $Y_\gamma \leq 0$. Let $g \in \gamma$, and let $\hat{g} = u^{4/(n-2)} g$

be the Yamabe minimizer in γ. Assume that $k \geq \frac{n-2}{n-1}$ holds. By (7) and the hypothesis that $Y_\gamma \leq 0$, we have

$$0 \geq \int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2 \right] d\mu_g = Y_\gamma \left(\int_X u^{2n/(n-2)} d\mu_g \right)^{(n-2)/n}. \tag{17}$$

On the other hand, the eigenvalue $\lambda(g)_k$ can be expressed in terms of Raleigh quotient as follows:

$$\lambda(g)_k = \inf_{u \in C^\infty_+(X)} \frac{\int_X \left[k s_g u^2 + 4|\nabla u|^2 \right] d\mu_g}{\int_X u^2 d\mu_g}. \tag{18}$$

Therefore, we get

$$\begin{aligned} \lambda(g)_k \int_X u^2 d\mu_g &\leq \int_X \left[k s_g u^2 + 4|\nabla u|^2 \right] d\mu_g \\ &\leq k \int_X \left[s_g u^2 + 4\frac{n-1}{n-2}|\nabla u|^2 \right] d\mu_g, \end{aligned}$$

where we used the hypothesis that $k \geq \frac{n-2}{n-1}$, i.e., $\frac{1}{k} \leq \frac{n-1}{n-2}$. This bound and (17) tells us that

$$\begin{aligned} \lambda(g)_k \int_X u^2 d\mu_g &\leq k Y_\gamma \left(\int_X u^{2n/(n-2)} d\mu_g \right)^{(n-2)/n} \\ &\leq k Y_\gamma (vol_g)^{-2/n} \int_X u^2 d\mu_g \end{aligned}$$

where notice that, since $Y_\gamma \leq 0$, the last step follows from the Hölder inequality:

$$\int f_1 f_2 \, d\mu \leq \left(\int |f_1|^p d\mu \right)^{1/p} \left(\int |f_2|^q d\mu \right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

with $f_1 = 1$, $f_2 = u^2$, $p = n/2$, and $q = n/(n-2)$. Moreover, equality holds precisely when u is constant, namely, precisely when g has constant scalar curvature. Since we shows that

$$\frac{1}{k}\lambda(g)_k (vol_g)^{2/n} \leq Y_\gamma$$

for every $g \in \gamma$, and since equality occurs if g is the Yamabe minimizer, it follows that

$$Y_\gamma = \frac{1}{k}\left(\sup_{g\in\gamma} \lambda(g)_k (vol_g)^{2/n}\right).$$

The above proof tells us that, under $\mathcal{Y}(X) \le 0$ and for any real number $k \ge \frac{n-2}{n-1}$, each constant scalar curvature metric maximizes $\frac{1}{k}\lambda_k(vol)^{2/n}$ in its conformal class. Given any maximizing sequence $\hat{g}_i$ for $\frac{1}{k}\lambda_k(vol)^{2/n}$, we may construct a new maximizing sequence g_i consisting of unit volume constant scalar curvature metrics by conformal rescaling. However, for any such sequence, the constant number s_{g_i} is viewed either as $\{Y_{[g_i]}\}$ or as $\{\frac{1}{k}\lambda(g_i)_k(vol)^{2/n}_{g_i}\}$. Therefore, the suprema over the space of all Riemannian metrics of $Y_{[g]}$ and $\frac{1}{k}\lambda(g_k)(vol)^{2/n}_g$ must coincide, i.e., $k\mathcal{Y}(X) = \bar{\lambda}_k(X)$.
On the other hand, suppose that $\mathcal{Y}(X) > 0$ and $k > 0$. For any smooth non-constant function $f : X \to \mathbb{R}$, Kobayashi [19] has shown that there exists a unit-volume metric g on M with $s_g = f$. For any sufficiently large positive constant L, take a smooth non-constant function $f : X \to \mathbb{R}$ such that $\min_{x\in X} f \ge L$. Then the above result of Kobayashi tells us that there is a metric g on M with $s_g = f$ and $vol_g = 1$. Notice that $\min_{x\in X} s_g = \min_{x\in X} f \ge L$ holds. On the other hand, we have

$$\begin{aligned}\frac{\int_X \left[ks_g u^2 + 4|\nabla u|^2\right] d\mu_g}{\int_X u^2 d\mu_g} &\ge \frac{\int_X k s_g u^2 d\mu_g}{\int_X u^2 d\mu_g} \ge \frac{\int_X k(\min_{x\in X} s_g)u^2 d\mu_g}{\int_X u^2 d\mu_g}\\ &= k\left(\min_{x\in X} s_g\right) = k\left(\min_{x\in X} f\right) \ge kL.\end{aligned}$$

This bound and (18) imply that $\lambda(g)_k \ge kL$. Since $vol_g = 1$, this bound tells us the following holds:

$$\bar{\lambda}_k(X) := \sup_g \lambda(g)_k (vol_g)^{n/2} \ge \sup_{g, vol_g = 1} \lambda(g)_k (vol_g)^{n/2} \ge kL.$$

Therefore, we get $\bar{\lambda}_k(X) = +\infty$ by taking $L \to +\infty$. □

Proposition 12, (9) and (11) imply

Theorem 13. *Let X be a smooth compact n-manifold with $n \geq 3$ and assume that X does not admit any Riemannian metric of positive scalar curvature. Then the following holds for any real number $k \geq \frac{n-2}{n-1}$:*

$$\mathcal{I}_s(M) = |\mathcal{Y}(M)|^{\frac{n}{2}} = |\mathcal{K}(M)|^{\frac{n}{2}} = \left|\frac{\bar{\lambda}_k(M)}{k}\right|^{\frac{n}{2}}. \tag{19}$$

It is well known that the value of the Yamabe invariant is sensitive to the choice of smooth structures of a 4-manifold. We have the following:

Corollary 14. *The number of distinct values that invariants $\mathcal{Y}(X)$, $\mathcal{I}_s(X)$, $\mathcal{K}(X)$ and $\bar{\lambda}_k(X)$, where $k \geq \frac{2}{3}$, can take on the smooth structures in a fixed homeomorphism type of simply connected 4-manifolds X is unbounded.*

Proof. Use Theorem 5 in [20] and Theorem 13 above. □

Finally, Proposition 6, Theorem 7 and Theorem 13 imply the following result which includes Theorem C as a special case:

Theorem 15. *For $m = 1, 2, 3$, let X_m be BF-admissible 4-manifolds with $b^+(X_m) - b_1(X_m) > 1$ and N be a closed oriented smooth 4-manifold with $b^+(N) = 0$. Consider connected sums $M := (\#_{m=1}^n X_m)\#N$, where $n = 2, 3$. Then, for any real number $k \geq \frac{2}{3}$, $\mathcal{Y}(M) = \mathcal{K}(M) \leq 0$, $\bar{\lambda}_k(M) \leq 0$ and the following holds:*

$$\mathcal{I}_s(M) = |\mathcal{Y}(M)|^2 = |\mathcal{K}(M)|^2 = \left|\frac{\bar{\lambda}_k(M)}{k}\right|^2 \geq 32\pi^2 \sum_{m=1}^{n} c_1^2(X_m).$$

If moreover X_m are minimal Kähler surfaces for $m = 1, 2, 3$ and N admits a Riemannian metric of non-negative scalar curvature, then the equality holds at the last inequality.

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