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Volume-preserving flow by powers of the m-th mean curvature in the hyperbolic space

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This paper concerns closed hypersurfaces of dimension $n(\geq 2)$ in the hyperbolic space $\mathbb{H}^{n+1}_{\kappa}$ of constant sectional curvature κ evolving in direction of its normal vector, where the speed is given by a power $\beta(\geq 1/m)$ of the mth mean curvature plus a volume preserving term, including the case of powers of the mean curvature and of the Gauß curvature. The main result is that if the initial hypersurface satisfies that the ratio of the biggest and smallest principal curvatures is close enough to 1 everywhere, depending only on n, m, β and κ, then under the flow this is maintained, there exists a unique, smooth solution of the flow for all times, and the evolving hypersurfaces converge exponentially to a geodesic sphere of $\mathbb{H}^{n+1}_{\kappa}$, enclosing the same volume as the initial hypersurface.

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1. Introduction

Let M^n be a smooth, compact oriented manifold of dimension $n(\geq 2)$ without boundary, $(N^{n+1}, \bar{g})$ be an $(n+1)$-dimensional complete Riemannian manifold, and $X_0 : M^n \to N^{n+1}$ a smooth immersion. Consider a one-parameter family of smooth immersions: $X_t : M^n \to N^{n+1}$ evolving according to

$$\begin{cases} \frac{\partial}{\partial t}\mathrm{X}\,(p,t) = \{\bar{F}(t) - F(\lambda(\mathcal{W}\,(p,t)))\}\nu\,(p,t)\,, \quad p \in M^n, \\ X(\cdot,0) = X_0(\cdot), \end{cases} \tag{1.1}$$

where $\nu\,(p,t)$ is the outer unit normal to $M_t = X_t(M^n)$ at the point $\mathrm{X}\,(p,t)$ in the tangent space TN^{n+1}, $\mathscr{W}_{-\nu}\,(p,t) = -\mathscr{W}_{\nu}\,(p,t)$ is the matrix of the Weingarten map on the tangent space TM^n induced by X_t, λ is the map from $T^*M^n \otimes TM^n$ to $\mathbb{R}^n$ which gives the eigenvalues of the map $\mathscr{W}$, F is a smooth symmetric function, and $\bar{F}(t)$ is the average of F on M_t:

$$\bar{F}(t) = \frac{\int_{M_t} F(\lambda(\mathscr{W}))\mathrm{d}\mu_t}{\int_{M_t} \mathrm{d}\mu_t}, \tag{1.2}$$

where $\mathrm{d}\mu_t$ denotes the surface area element of M_t. As is clear from the presence of the global term $\bar{F}(t)$ in equation (1.1), the flow keeps the volume of the domain Ω_t enclosed by M_t constant.

This paper considers the flow (1.1) with the speed $F(\lambda)$ given by a power of an mth mean curvature, namely

$$F(\lambda_1, \ldots, \lambda_n) = H_m^{\beta}, \tag{1.3}$$

where $(\lambda_1, \ldots, \lambda_n)$ are the principal curvatures of the evolving hypersurfaces M_t, and for any $m = 1, \ldots, n$, the mth mean curvature H_m is the average of the mth elementary symmetric functions E_m, namely

$$H_m = \binom{n}{m}^{-1} E_m = \frac{m!(n-m)!}{n!} \sum_{1\leq i_1<\cdots<i_m\leq n} \lambda_{i_1} \cdots \lambda_{i_m}. \tag{1.4}$$

Obviously $H_1 = H/n$ and $H_n = K$, where H and K denote the mean curvature and the Gauß-Kronecker curvature respectively.

For the flow (1.1) without the volume constraint term $\bar{F}(t)$, in the case when N^{n+1} is the Euclidean space $\mathbb{R}^{n+1}$, there are many papers which consider the evolution of convex hypersurfaces, of particular interest here is in the analysis of the flow (1.1) where the speed $F(\lambda)$ is homogeneous of degree one in the principal curvatures, beginning with a classical result of Huisken [31] who proved that any closed convex hypersurface under mean curvature flow shrinks to a round point in finite time (Huisken's theorem may be considered as an extension of the theorem of Gage and Hamilton [26] to dimensions bigger than one), and including the similar results on the nth root of Gauß-Kronecker curvature [20], the square root of scalar curvature [21], and a large family of other speeds [3, 9]. The first such result with degree of homogeneity greater than one was due to Chow [20] who considered flow by powers of the Gauß-Kronecker curvature. He proved that the evolving hypersurfaces by K^β with $\beta \geq 1/n$ become spherical as they shrink to a point provided the initial hypersurface M_0 is sufficiently pinched. We also mention that Tso [54] showed the same result for the Gauß-Kronecker curvature flow and Andrews [6] proved that the limit of the solutions under K^β-flow with $\beta \in (\frac{1}{n+2}, \frac{1}{n}]$ evolves purely by homothetic contraction to a point in finite time. Later such results were proved by Schulze [51] for the flow by powers of the mean curvature, by Alessandroni and Sinestrari [1] for the flow by powers of the scalar curvature, by Andrews and McCoy [11] for the flow of convex hypersurfaces with pinched principal curvatures by high powers of curvature, and for such flows in the special case of surfaces in three-dimensional spaces [5, 10, 48, 51], where the lower dimension allows a more complete understanding of the equation for the evolution of the second fundamental form. However, when N^{n+1} is a more general Riemannian manifold, there are few results on the behavior of these flows: Huisken [32] extended the result of [31] in $\mathbb{R}^{n+1}$ to compact hypersurfaces in general Riemannian manifolds with suitable bounds on curvature. Andrews [4] has considered a flow which takes any compact hypersurface with principal curvatures greater than $\sqrt{c}$ with $c > 0$ in a Riemannian background space with sectional curvatures at least $-c$, and converges to a round point in finite time.

The volume-preserving versions of these flows are the flows (1.1)–(1.2) with an extra term $\bar{F}(t)$ which balances the contraction. In the case of volume-preserving mean curvature flow, Huisken [33] showed that convex hypersurfaces remain convex for all time and converge exponentially fast to round spheres (the corresponding result for curves in the plane is due to

Gage [25]), while Andrews [7] extended this result to the smooth anisotropic mean curvature flow, and McCoy showed similar results for the surface area preserving mean curvature flow [43] and the mixed volume preserving mean curvature flows [44]. The volume-preserving flow has been used to study constant mean curvature surfaces between parallel planes [12, 13] and to find canonical foliations near infinity in asymptotically flat spaces arising in general relativity [34] (Rigger [47] showed analogous results in the asymptotically hyperbolic setting). While for convex curvature flows with general extra terms, the hypersurfaces may contract, converge or expand in terms of the magnitude of the extra terms [37, 38]. If the initial hypersurface is sufficiently close to a fixed Euclidean sphere (possibly non-convex), Escher and Simonett [24] proved that the flow converges exponentially fast to a round sphere, a similar result for average mean convex hypersurfaces with initially small traceless second fundamental form is due to Li [39]. For a general ambient manifold, Alikakos and Freire [2] proved long time existence and convergence to a constant mean curvature surface under the hypotheses that the initial hypersurface is close to a small geodesic sphere and that it satisfies some non-degenerate conditions. Cabezas-Rivas and Miquel exported the Euclidean results of [12, 13] to revolution hypersurfaces in a rotationally symmetric space [18], and showed the same results as Huisken [33] for a hyperbolic background space [17] by assuming the initial hypersurface is horospherically convex (the definition will be given later).

On the other hand, there are few results on speeds different from the mean curvature: McCoy [45] proved the convergence to a sphere for a large class of functions F homogeneous of degree one (including the case $F = H_m^\beta$ with $m\beta = 1$), Makowski showed that the mixed volume preserving curvature flow for a function F homogeneous of degree one, starting with a compact and strictly horosphere-convex hypersurface in the hyperbolic space exponentially converges to a geodesic sphere [41], and the volume preserving curvature flow in Lorentzian manifolds for F as a function with homogeneous of degree one exponentially converges to a hypersurface of constant F-curvature [42] (moreover, stability properties and foliations of such a hypersurface were also examined). In 2010 Cabezas-Rivas and Sinestrari [19] studied the deformation of convex hypersurfaces in $\mathbb{R}^{n+1}$ by a speed of the form (1.4) for some power $\beta \geq 1/m$. In this way F is a homogeneous function of the curvatures with a degree $m\beta \geq 1$. In particular, they proved the following

Theorem 1.1. *For $m \in \{1, \ldots, n\}$, $m\beta \geq 1$ there exists a positive constant $\mathcal{C} = \mathcal{C}(n, m, \beta) < 1/n^n$ such that the following holds: If the initial hypersurface of $\mathbb{R}^{n+1}$ is pinched in the sense that*

$$K(p) > \mathcal{C}H^n(p) > 0 \quad \text{for all} \quad p \in M^n, \tag{1.5}$$

then the flow (1.1)–(1.3) with F given by (1.4), has a unique and smooth solution for all times, inequality (1.5) remains true everywhere on the evolving hypersurfaces M_t for all $t > 0$ and the M_t's converge, exponentially in the C^∞-topology, to a round sphere enclosing the same volume as M_0.

However, the results of [19] do not closely relate to the ambient space, we face the challenges of extending the above results to hypersurface in more general ambient spaces. But not every Riemannian manifold is well suited to deal with the situation analogous to the setting in Euclidean spaces. We want to consider the case that the ambient space is a simply connected Riemannian manifold of constant sectional curvature $\kappa(< 0)$ whose flow behaves quite differently compared to the Euclidean space to a certain extent.

Set $a = \sqrt{|\kappa|}$. The ambient space N_κ^{n+1} is isometric to the hyperbolic space $\mathbb{H}_\kappa^{n+1}$ of radius $1/a$:

$$\mathbb{H}_\kappa^{n+1} := \left\{ p \in L^{n+2} : \langle p, p \rangle = -\frac{1}{a^2} \right\}.$$

Here $(L^{n+2}, \langle \cdot, \cdot \rangle)$ denotes the $(n+2)$-dimensional Lorentz-Minkowski space. To consider the flow (1.1)–(1.3) in N_κ^{n+1} is then equivalent to considering the flow (1.1)–(1.3) in $\mathbb{H}_\kappa^{n+1}$. Now, it is necessary to provide some definitions as in [14, 17] as follows.

Definition 1.2. A horosphere $\mathcal{H}$ of $\mathbb{H}_\kappa^{n+1}$ is the limit of a geodesic sphere of $\mathbb{H}_\kappa^{n+1}$ as its center goes to the infinity along a fixed geodesic ray.

Definition 1.3. An horoball $\mathscr{H}$ is the convex domain whose boundary is a horosphere.

Definition 1.4. A hypersurface M of $\mathbb{H}_\kappa^{n+1}$ is said to be convex by horospheres (h-convex for short) if it bounds a domain Ω satisfying that for every $p \in M = \partial\Omega$, there is a horosphere $\mathcal{H}$ of $\mathbb{H}_\kappa^{n+1}$ through p such that Ω is contained in $\mathscr{H}$ of $\mathbb{H}_\kappa^{n+1}$ bounded by $\mathcal{H}$.

Remark 1.5. In fact, Currier in [22] showed that h-convex immersions of smooth compact hypersurfaces are embedded spheres, and Borisenko and

Miquel in [14] showed that horosphere $\mathcal{H}$ of $\mathbb{H}^{n+1}_\kappa$ is weakly (strictly) h-convex if and only if all its principal curvatures are (strictly) bounded from below by a at each point.

Most of the literature mentioned above requires a pinching condition on the initial hypersurface, so that parabolic maximum principles, an important tool in the investigation of evolution equations, can be used to deduce that they can converge and become spherical in shape as the final time is approached under these flows. It is well-known that in hyperbolic spaces the negative curvature of the background space produces terms such that the maximum principles either fail or become more subtle for our flow (1.1)–(1.3). So for our purposes a challenge in the hyperbolic ambient setting is how to find a suitable pinching condition on the initial hypersurfaces. However in the hyperbolic space there is an intuitive example, as pointed out by Cabezas-Rivas and Miquel in [17]: a geodesic sphere, moving outward in the radial direction with the speed H_m^β, its normal curvature decreases, and it becomes nearer and nearer to that of a horosphere, but it never gets h-convex. This fact leads us to hope for the result by choosing a suitable convex hypersurface which is sufficiently positively curved to overcome the obstructions from the negative curvature imposed by the ambient spaces like that of space of Cabezas-Rivas and Sinestrari [19]. More precisely, denote the shifted second fundamental form by $\tilde{h}_{ij} := h_{ij} - ag_{ij}$, then the shifted mean curvature $\tilde{H} = H - na$ and the shifted Gauß curvature $\tilde{K} = \det\{\tilde{h}_i^j\}$. In this paper the shifted geometric quantities are distinguished by a tilde. Compared with the pinching condition $K(p) > \mathcal{C}H^n(p) > 0$ on the initial hypersurface in Theorem 1.1, which is analogous to the initial pinching condition in Chow [20] and Schulze [51], it is natural to impose a pinching condition: $\tilde{K} > C^*\tilde{H}^n > 0$ on the initial hypersurfaces of $\mathbb{H}^{n+1}_\kappa$, where C^* is a suitable positive constant. It is shown in Section 4 that the condition $\tilde{K} > C^*\tilde{H}^n > 0$ on a closed hypersurface implies in particular the h-convexity of the hypersurface. The aim of this paper is to achieve such extension of the above Theorem 1.1 of Cabezas-Rivas and Sinestrari [19] in the hyperbolic case. Precisely, we prove the following

Theorem 1.6 (main theorem). *For $m \in \{1, \ldots, n\}$, $m\beta \geq 1$ there exists a positive constant $C^* = C^*(a, n, m, \beta) < 1/n^n$ such that the following holds: If the initial hypersurface of $\mathbb{H}^{n+1}_\kappa$ is pinched in the sense that*

$$\tilde{K}(p) > C^*\tilde{H}^n(p) > 0 \quad \textit{for all} \quad p \in M^n, \tag{1.6}$$

then the flow (1.1)–(1.3) *with* F *given by* (1.4)*, has a unique and smooth solution for all times, inequality* (1.6) *remains true everywhere on the evolving hypersurfaces* M_t *for all* $t > 0$ *and the* M_t*'s converge, exponentially in the* C^∞*-topology, to a geodesic sphere of* $\mathbb{H}^{n+1}_\kappa$ *enclosing the same volume as* M_0*.*

Our analysis follows the framework of [19], we make modifications to consider our problem for the background space. The rest of the paper is organized as follows: Section 2 first gives some useful preliminary results employed in the remainder of the paper. Section 3 contains details of the short time existence of the flow (1.1)–(1.3) and the induced evolution equations of some important geometric quantities and the corresponding shifted quantities, this requires only minor modifications of Euclidean case due to the background curvature. In Section 4 applying the maximum principle to the evolution equation of the shifted quantity $\tilde{K}/\tilde{H}^n$ gives that if the initial hypersurface is pinched good enough then this is preserved for $t > 0$ as long as the flow (1.1)–(1.3) exists. This is a fundamental step in our procedure as in most of the literature quoted above. Furthermore, Section 5 proves the uniform bound of the speed F by following a method which was firstly used by Tso [54]. Using more sophisticated results for fully nonlinear elliptic and parabolic partial differential equations, Section 6 obtains uniform bounds on all derivatives of the curvature and proves long time existence of the flow (1.1)–(1.3). Finally Section 7, following the idea in [3], obtains the lower bound for $\tilde{H}$, which we infer from a Harnack inequality due to Krylov ([35]), the estimates obtained so far will then allow us to prove that these evolving hypersurfaces converge to a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$ smoothly and exponentially.

2. Notations and preliminary results

From now on, we use the same notation as in [18, 31, 50] in local coordinates $\{x^i\}, 1 \le i \le n$, near $p \in M^n$ and $\{y^\alpha\}, 0 \le \alpha, \beta \le n$, near $F(p) \in \mathbb{H}^{n+1}_\kappa$. Denote by a bar all quantities on $\mathbb{H}^{n+1}_\kappa$, for example by $\bar{g} = \{\bar{g}_{\alpha\beta}\}$ the metric, by $\bar{g}^{-1} = \{\bar{g}^{\alpha\beta}\}$ the inverse of the metric, by $\bar{\nabla}$ the covariant derivative, by $\bar{\Delta}$ the rough Laplacian, and by $\bar{\mathrm{R}} = \{\bar{\mathrm{R}}_{\alpha\beta\gamma\delta}\}$ the Riemannian curvature tensor. Components are sometimes taken with respect to the tangent vector fields $\partial_\alpha (= \frac{\partial}{\partial y^\alpha})$ associated with a local coordinate $\{y^\alpha\}$ and sometimes with respect to a moving orthonormal frame e_α, where $\bar{g}(e_\alpha, e_\beta) = \delta_{\alpha\beta}$. The corresponding geometric quantities on M^n will be denoted by g (the induced metric), $g^{-1}, \nabla, \Delta, \mathrm{R}, \partial_i$ and e_i. Then further important quantities are the second fundamental form $A(p) = \{h_{ij}\}$ and the Weingarten map $\mathscr{W} = \{g^{ik}h_{kj}\} = \{h^i_j\}$ as a symmetric operator and a self-adjoint operator respectively. The eigenvalues $\lambda_1(p) \leq \cdots \leq \lambda_n(p)$ of $\mathscr{W}$ are called the principal curvatures of $X(M^n)$ at $X(p)$. The mean curvature is given by

$$H := \mathrm{tr}\mathscr{W} = h^i_i = \sum_{i=1}^{n} \lambda_i,$$

the squared norm of the second fundamental form by

$$\left|A\right|^2 := \mathrm{tr}(\mathscr{W}^t\mathscr{W}) = h^i_j h^j_i = h^{ij}h_{ij} = \sum_{i=1}^{n} \lambda_i^2,$$

and Gauß-Kronecker curvature by

$$K := \det(\mathscr{W}) = \det\{h^i_j\} = \frac{\det\{h_{ij}\}}{\det\{g_{ij}\}} = \prod_{i=1}^{n} \lambda_i.$$

More generally, the mth elementary symmetric functions E_m are given by

$$E_m(\lambda) = \sum_{1\leq i_1<\cdots<i_m\leq n} \lambda_{i_1}\cdots\lambda_{i_m}, \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n,$$

and the mth mean curvatures H_m are given by (1.4). Since H_m is homogeneous of degree m, the speed F is homogeneous of degree $m\beta$ in the curvatures λ_i. Denote the vector $(\lambda_1, \ldots, \lambda_n)$ of $\mathbb{R}^n$ by λ and the positive cone by $\Gamma_+ \subset \mathbb{R}^n$, i.e.

$$\Gamma_+ = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i > 0, \forall\ i\}.$$

It is clear that H, K, H_m, F may be viewed as functions of λ, or as functions of A, or as functions of $\mathscr{W}$, or also functions of space and time on M_t. Since the differentiability properties of these functions are the same in our setting, we do not distinguish between these notions and write always these functions the same letters in all cases. We use the notation

$$\dot{F}^i := \frac{\partial F}{\partial \lambda_i}, \quad \dot{F}^{ij} := \frac{\partial F}{\partial h_{ij}}, \quad \text{and} \quad \dot{F}^j_i := \frac{\partial F}{\partial h^i_j}.$$

If B, C are matrices, we write

$$\dot{F}(B) := \dot{F}^j_i B^i_j \quad \text{and} \quad \ddot{F}(B, C) := \frac{\partial^2 F}{\partial h^j_i \partial h^l_k} B^j_i C^l_k.$$

Finally, if $F \in C^2(\Gamma_+)$ is concave, then F is also concave as a curvature function depending on $\{h^i_j\}$.

We note some important properties of H_m (see [19] for a simple derivation).

Lemma 2.1. *Let $1 \leq m \leq n$ be fixed.*

i) *The mth roots $H_m^{1/m}$ are concave in Γ_+.*

ii) *For all i, $\frac{\partial H_m}{\partial \lambda_i}(\lambda) > 0$, where $\lambda \in \Gamma_+$.*

iii) $H_m^{1/m} \leq \dfrac{H}{n}$*; equivalently,* $F \leq \left(\dfrac{H}{n}\right)^{m\beta}$.

iv) $\mathrm{tr}(\dot{F}) \geq m\,\beta\,F^{1-\frac{1}{m\beta}}$.

v) *H_m, as a function of h^i_j, is also a homogeneous polynomial of degree m; in addition, as a function on M, it satisfies $\nabla_j\left(\dfrac{\partial H_m}{\partial h^i_j}\right) = 0$ for any $i \in \{1, \ldots, n\}$, where ∇ is the covariant derivative on M.*

The following algebraic property proved by Schulze in ([51], Lemma 2.5) will be needed in the later sections.

Lemma 2.2. *For any $\varepsilon > 0$ assume that $\lambda_i \geq \varepsilon H > 0$, $i = 1, \ldots, n$, at some point of an n-dimensional hypersurface. Then at the same point there exists a $\delta = \delta(\varepsilon, n) > 0$ such that*

$$\frac{n\left|A\right|^2 - H^2}{H^2} \geq \delta\left(\frac{1}{n^n} - \frac{K}{H^n}\right).$$

Consider the functions as in [17]:

$$\mathrm{s}_\kappa(x) = \frac{\sinh(ax)}{a}, \qquad \mathrm{c}_\kappa(r) = \mathrm{s}'_\kappa(x),$$
$$\mathrm{ta}_\kappa(x) = \frac{\mathrm{s}_\kappa(x)}{\mathrm{c}_\kappa(x)}, \qquad \mathrm{co}_\kappa(x) = \frac{1}{\mathrm{ta}_\kappa(x)}.$$

Denote r_p the function "distance to p" in $\mathbb{H}^{n+1}_\kappa$ and use the notation $\partial_{r_p} = \bar{\nabla} r_p$. and denote the component of ∂_{r_p} by $\partial^\top_{r_p}$ tangent to M_t, which satisfies $\partial^\top_{r_p} = \nabla(r_p|_{M^n})$. Define the inner radius ρ_- and the circumradius

radius ρ_+ by

$$\rho_+(t) = \inf\{r : B_r(q) \text{ encloses } M_t \text{ for some } q \in \mathbb{H}^{n+1}_\kappa\},$$
$$\rho_-(t) = \sup\{r : B_r(q) \text{ is enclosed by } M_t \text{ for some } q \in \mathbb{H}^{n+1}_\kappa\},$$

where $B_r(q)$ is the geodesic ball of radius r with center at q. The following well-known result for h-convex hypersurfaces in $\mathbb{H}^{n+1}_\kappa$ will be applied in later sections.

Lemma 2.3. *Let Ω be a compact h-convex domain, o the center of an inball of Ω, ρ_- its inner radius, and ρ_+ its circumradius radius. Furthermore let $\tau := \mathrm{ta}_\kappa(\frac{\rho_-}{2})$, then*

i) *the maximal distance* $\max d(o, \partial\Omega)$ *between o and the points in $\partial\Omega$ satisfies the inequality*

$$\max d(o, \partial\Omega) \leq \rho_- + a\frac{\ln(1+\sqrt{\tau})^2}{1+\tau} < \rho_- + a\ln 2.$$

ii) *For any interior point p of Ω, $\langle \nu, \partial_{r_p}\rangle \geq a\mathrm{ta}_\kappa(\mathrm{dist}((p, \partial\Omega))$, where* dist *denotes the distance in the ambient space $\mathbb{H}^{n+1}_\kappa$.*

iii) *There exists a constant $C = C(a) > 0$ such that*

$$\rho_+ \leq C\left(\rho_- + \sqrt{\rho_-}\right).$$

Proof. See ([14], Theorem 3.1) for the proof of i) and ii) in the Lemma. As a consequence of i) and ii), iii) has been proved by Makowski (see ([41], Theorem 5.2). $\square$

In our analysis we need some a priori estimates on the Hölder norms of the solutions to elliptic and parabolic partial differential equations in Euclidean spaces. We recall that, in the case of a function depending on space and time, there is a suitable definition of Hölder norm which is adapted to the purposes of parabolic equations (see e.g. [40]). In addition to the standard Schauder estimates for linear equations, we use in the paper some more recent results which are collected here. The estimates below hold for suitable classes of weak solutions; for the sake of simplicity, we state them in the case of a smooth classical solution, which is enough for our purposes.

Given $r > 0$, we denote by B_r the ball of radius $r > 0$ in $\mathbb{R}^n$ centered at the origin. First we recall a well known result due to Krylov and Safonov, which applies to linear parabolic equations of the form

$$\left(a^{ij}(x,t)D_iD_j + b^i(x,t)D_i + c(x,t) - \frac{\partial}{\partial t}\right) u = f \tag{2.1}$$

in $B_r \times [0,T]$, for some $T > 0$. We assume that $a^{ij} = a^{ji}$ and that a^{ij} is uniformly elliptic; that is, there exist two constants λ, $\Lambda > 0$ such that

$$\lambda|v|^2 \leq a^{ij}(x,t)v_iv_j \leq \Lambda|v|^2 \tag{2.2}$$

for all $v \in \mathbb{R}^n$ and all $(x,t) \in B_r \times [0,T]$. Then the following estimate holds [36, Theorem 4.3]:

Theorem 2.4. *Let $u \in C^2(B_r \times [0,T])$ be a solution of* (2.1)*, where the coefficients are measurable, satisfy* (2.2) *and*

$$|b^i|,\ |c| \leq K_1 \qquad \text{for all } \ i = 1,\dots,n,$$

for some $K_1 > 0$. Then, for any $0 < r' < r$ and any $0 < \delta < T$ we have

$$\|u\|_{C^\alpha(B_{r'}\times[\delta,T])} \leq C\left(\|u\|_{C(B_r\times[0,T])} + \|f\|_{L_\infty(B_r\times[0,T])}\right)$$

for some constants $C > 0$ and $\alpha \in (0,1)$ depending on n, λ, Λ, K_1, r, r' and δ.

Next we quote a result for fully nonlinear elliptic equations, which is due to Caffarelli. We consider the equation

$$F(D^2u(x), x) = f(x), \qquad x \in B_r. \tag{2.3}$$

Here $F : \mathcal{S} \times B_r \to \mathbb{R}$, where $\mathcal{S}$ is the set of the symmetric $n \times n$ matrices. The nonlinear operator F is called uniformly elliptic if there exist $\Lambda \geq \lambda > 0$ such that

$$\lambda||B|| \leq F(A+B,x) - F(A,x) \leq \Lambda||B|| \tag{2.4}$$

for any $x \in B_r$ and any pair $A, B \in \mathcal{S}$ such that B is nonnegative definite.

Theorem 2.5. *Let $u \in C^2(B_r)$ be a solution of* (2.3)*, where F is continuous and satisfies* (2.4)*. Suppose in addition that F is concave with respect to*

D^2u for any $x \in B_r$. Then there exists $\bar{\alpha} \in (0,1)$ with the following property: if, for some $K_2 > 0$ and $\alpha \in (0,\bar{\alpha})$, we have that $f \in C^\alpha(\Omega)$ and that

$$F(A,x) - F(A,y) \leq K_2|x-y|^\alpha(||A||+1), \qquad x,y \in B_r,\ A \in \mathcal{S},$$

then, for any $0 < r' < r$, we have the estimate

$$\|u\|_{C^{2+\alpha}(B_{r'})} \leq C(||u||_{C(B_r)} + ||f||_{C^\alpha(B_r)} + 1)$$

where $C > 0$ only depends on n, λ, Λ, K_2, r and r'.

The above result follows from Theorem 3 in [15] (see also Theorem 8.1 in [16] and the remarks thereafter). It generalizes, by a perturbation method, a previous estimate, due to Evans and Krylov, about equations with concave dependence on the hessian. In contrast with Evans-Krylov result (see e.g. inequality (17.42) in [28]), Theorem 2.4 gives an estimate in terms of the C^α-norm of f rather than the C^2-norm, and this is essential for our purposes.

Finally, we recall an interior Hölder estimate, due to Di Benedetto and Friedman [23, Theorem 1.3], for solutions of the degenerate parabolic equation

$$\frac{\partial v}{\partial t} - D_i\left(a^{ij}(x,t,Dv)D_j v^d\right) = f(x,t,v,Dv), \tag{2.5}$$

being $d > 1$.

Theorem 2.6. *Let $v \in C^2(B_r \times [0,T])$ be a nonnegative solution of (2.5), where a^{ij} satisfies (2.2). Let $c_1, c_2, N > 0$ be such that*

$$|f(x,t,v,Dv)| \leq c_1|Dv^d| + c_2,$$

and

$$\sup_{0<t<T} \|v(\cdot,t)\|^2_{L^2(B_r)} + \|Dv^d\|^2_{L^2(B_r\times[0,T])} \leq N.$$

Then for any $0 < \delta < T$ and $0 < r' < r$, we have

$$\|v\|_{C^\alpha(B_{r'}\times[\delta,T])} \leq C,$$

for suitable $C > 0, \alpha \in (0,1)$ depending only on $n, N, \lambda, \Lambda, \delta, c_1, c_2, r$ and r'.

3. Short time existence and evolution equations

This section first considers short time existence for the initial value problem (1.1).

Theorem 3.1. *Let* $\mathrm{X}_0 : M^n \to \mathbb{H}^{n+1}_\kappa$ *be a smooth closed hypersurface with the mean curvature is strictly bounded from below by na everywhere. Then there exists a unique smooth solution* $\mathrm{X_t}$ *of problem* (1.1)*, defined on some time interval* $[0, T)$*, with* $T > 0$.

Proof. We can argue exactly as in [19, Theorem 3.1]. Although the assumptions on the initial hypersurface and the ambient background space in that paper are different, the proof applies to our case as well. □

Proceeding now exactly as in [27, 31, 41] we derive some evolution equations on M_t from the basic equation (1.1).

Proposition 3.2. *For the ambient space* $N^{n+1} = \mathbb{H}^{n+1}_\kappa$*, on any solution* M_t *of* (1.1) *the following hold:*

$$\partial_t g = 2(\bar{F} - F)A, \tag{3.1}$$

$$\partial_t g^{-1} = -2(\bar{F} - F)g^{-1}\mathscr{W}, \tag{3.2}$$

$$\partial_t \nu = \mathrm{X}_*(\nabla F), \tag{3.3}$$

$$\partial_t(\mathrm{d}\mu_t) = (\bar{F} - F)H\mathrm{d}\mu_t, \tag{3.4}$$

$$\begin{aligned}\partial_t A = {} & \Delta_{\dot{F}} A + \ddot{F}(\nabla_{\cdot}\mathscr{W}, \nabla_{\cdot}\mathscr{W}) + \left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) + a^2 \mathrm{tr}(\dot{F})\right] A \\ & + \left[\bar{F} - (m\beta + 1)F\right] A\mathscr{W} + a^2\left[\bar{F} - (m\beta + 1)F\right] g,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\partial_t \mathscr{W} = {} & \Delta_{\dot{F}} \mathscr{W} + \ddot{F}(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) + a^2 \mathrm{tr}(\dot{F})\right] \mathscr{W} \\ & - \left[\bar{F} + (m\beta - 1)F\right] \mathscr{W}^2 + a^2\left[\bar{F} - (m\beta + 1)F\right] Id.\end{aligned} \tag{3.6}$$

Proof. The first fourth evolution equations under (1.1) follow from straightforward computation as in §3 of [31], and valid for in arbitrary Riemannian manifolds.

The evolution of A can be calculated from the definition of A:

$$\begin{aligned}\partial_t h_{ij} &= -\frac{\partial}{\partial t}\langle \bar{\nabla}_{\mathrm{X}_*(\partial_i)}\mathrm{X}_*(\partial_j), \nu\rangle. \\ &= -\langle \bar{\nabla}_{\mathrm{X}_*(\partial_t)}\bar{\nabla}_{\mathrm{X}_*(\partial_i)}\mathrm{X}_*(\partial_j), \nu\rangle - \left\langle \bar{\nabla}_{\mathrm{X}_*(\partial_i)}\mathrm{X}_*(\partial_j), \frac{\partial \nu}{\partial t}\right\rangle. \\ &= -\langle \bar{\nabla}_{\mathrm{X}_*(\partial_i)}\bar{\nabla}_{\mathrm{X}_*(\partial_t)}\mathrm{X}_*(\partial_j), \nu\rangle - \bar{R}\left(\mathrm{X}_*(\partial_i), \mathrm{X}_*(\partial_t), \mathrm{X}_*(\partial_j), \nu\right) \\ &\quad - \langle \bar{\nabla}_{\mathrm{X}_*(\partial_i)}\mathrm{X}_*(\partial_j), \mathrm{X}_*(\nabla F)\rangle\end{aligned}$$

$$
\begin{aligned}
&= -\langle \bar\nabla_{X_*(\partial_i)} \bar\nabla_{X_*(\partial_j)}((\bar F - F)\nu), \nu\rangle + (F - \bar F)\bar R_{i0j0} - \nabla_{\nabla_{\partial_i}\partial_j} F \\
&\partial_i\partial_j F - \nabla_{\partial_i}\partial_j F - (F - \bar F)\langle \bar\nabla_{X_*(\partial_i)} X_*(\partial_k), \nu\rangle h_j^k - (\bar F - F)\bar R_{i0j0} \\
&= \mathrm{Hess}_\nabla F(\partial_i, \partial_j) + (\bar F - F) h_{ik} h_j^k - (\bar F - F)\bar R_{i0j0}.
\end{aligned}
$$

Note that the definition of $\dot F$ and $\ddot F$ allow us to write $\mathrm{Hess}_\nabla F$ as follows:

$$
\begin{aligned}
\mathrm{Hess}_\nabla F(\partial_i, \partial_j) &= \nabla_i \nabla_j F = \nabla_i (\dot F_k^l \nabla_j h_l^k) \\
&= \dot F_k^l \nabla_i \nabla_j h_l^k + \ddot F_{k\ m}^{l\ n} \nabla_i h_n^m \nabla_j h_l^k \\
&= \dot F^{kl} \nabla_i \nabla_j h_{kl} + \ddot F_{k\ m}^{l\ n} \nabla_i h_n^m \nabla_j h_l^k.
\end{aligned}
$$

Recall a form of Simons' identity [52], which is a consequence of the Gauß and Codazzi equations:

$$
\begin{aligned}
\nabla_i \nabla_j h_{kl} = {}& \nabla_k \nabla_l h_{ij} + h_{ij} h_{kp} h_l^p - h_{ip} h_l^p h_{kj} + h_{il} h_{kp} h_j^p - h_{ip} h_j^p h_{kl} \\
&+ \bar R_{ikjp} h_l^p + \bar R_{iklp} h_j^p + \bar R_{plkj} h_i^p + \bar R_{pjli} h_k^p + \bar R_{0k0l} h^{ij} - \bar R_{0i0j} h_{kl}^p \\
&+ \bar\nabla_i \bar R_{0lkj} + \bar\nabla_k \bar R_{0jli},
\end{aligned}
$$

where ν is arranged to be e_0. Therefore

$$
\begin{aligned}
\partial_t h_{ij} = {}& \dot F^{kl} \nabla_k \nabla_l h_{ij} + \ddot F_{k\ m}^{l\ n} \nabla_i h_n^m \nabla_j h_l^k \\
&+ \dot F^{kl} \big\{ h_{ij} h_{kp} h_l^p - h_{ip} h_l^p h_{kj} + h_{il} h_{kp} h_j^p - h_{ip} h_j^p h_{kl} \\
&+ \bar R_{ikjp} h_l^p + \bar R_{iklp} h_j^p + \bar R_{plkj} h_i^p + \bar R_{pjli} h_k^p + \bar R_{0k0l} h^{ij} - \bar R_{0i0j} h_{kl}^p \\
&+ \bar\nabla_i \bar R_{0lkj} + \bar\nabla_k \bar R_{0jli} \big\} + (\bar F - F) h_{ik} h_j^k - (\bar F - F)\bar R_{i0j0}.
\end{aligned}
\tag{3.7}
$$

Also note that in our case where the background space is a hyperbolic space, the ambient space is locally symmetric ($\bar\nabla\bar{\mathrm R} = 0$) and the Riemannian curvature tensor takes the form

$$
\bar{\mathrm R}_{\alpha\beta\gamma\delta} = -a^2 \left(\bar g_{\alpha\gamma} \bar g_{\beta\delta} - \bar g_{\alpha\delta} \bar g_{\beta\gamma} \right). \tag{3.8}
$$

Since F is a homogeneous function of the Weingarten map $\mathscr W$ of degree $m\beta$, then

$$
\dot F \mathscr W = m\beta F \tag{3.9}
$$

Then, the relations (3.8) with $\bar\nabla\bar{\mathrm R} = 0$ and (3.9) apply to (3.7) to give:

$$
\begin{aligned}
\partial_t h_{ij} = {}& \dot F^{kl} \nabla_k \nabla_l h_{ij} + \ddot F_{k\ m}^{l\ n} \nabla_i h_n^m \nabla_j h_l^k + \dot F_k^l h_p^k h_l^p h_{ij} + a^2 \dot F_k^k h_{ij} \\
&+ \big[\bar F - (m\beta + 1) F\big] h_{ik} h_j^k + a^2 \big[\bar F - (m\beta + 1) F\big] g_{ij}.
\end{aligned}
$$

Hence in compact notation we have (3.5).

Finally recalling that $\mathscr{W} = g^{-1}A$ we have

$$\begin{aligned}\partial_t \mathscr{W} &= g^{-1}\partial_t A + \partial_t g^{-1} A \\ &= \Delta_{\dot F}\mathscr{W} + \ddot F(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \left[\mathrm{tr}_{\dot F}(A\mathscr{W}) + a^2\mathrm{tr}(\dot F)\right]\mathscr{W} \\ &\quad + \left[\bar F + (m\beta - 1)F\right]\mathscr{W}^2 + a^2\left[\bar F - (m\beta+1)F\right] Id - 2(\bar F - F)\mathscr{W}^2 \\ &= \Delta_{\dot F}\mathscr{W} + \ddot F(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \left[\mathrm{tr}_{\dot F}(A\mathscr{W}) + a^2\mathrm{tr}(\dot F)\right]\mathscr{W} \\ &\quad - \left[\bar F + (m\beta - 1)F\right]\mathscr{W}^2 + a^2\left[\bar F - (m\beta+1)F\right] Id,\end{aligned}$$

which is (3.6). □

In the next theorem, we derive the evolution of any homogeneous function of the Weingarten map $\mathscr{W}$ defined on an evolving hypersurface M_t of $\mathbb{H}^{n+1}_\kappa$ under the flow (1.1).

Theorem 3.3. *If G is a homogeneous function of the Weingarten map $\mathscr{W}$ of degree α, then the evolution equation of G under the flow* (1.1) *in $\mathbb{H}^{n+1}_\kappa$ is the following*

$$\begin{aligned}\partial_t G = \Delta_{\dot F} G - \dot F \ddot G(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \dot G \ddot F(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) \\ + \alpha\left[\mathrm{tr}_{\dot F}(A\mathscr{W}) + a^2\mathrm{tr}(\dot F)\right] G - \left[\bar F + (m\beta - 1)F\right]\dot G\mathscr{W}^2 \\ + a^2\left[\bar F - (m\beta+1)F\right]\mathrm{tr}(\dot G).\end{aligned}$$

Proof. The definition of $\dot G$ and $\ddot G$ allow us to write $\mathrm{Hess}_\nabla G$ as follows:

$$\mathrm{Hess}_\nabla G = \dot G\, \mathrm{Hess}_\nabla \mathscr{W} + \ddot G(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}),$$

which gives

$$\Delta_{\dot F} G = \dot F g^{-1}\,\mathrm{Hess}_\nabla G = \dot G\,\Delta_{\dot F}\mathscr{W} + \dot F \ddot G(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathcal{W}).$$

Therefore, by (3.6)

$$\begin{aligned}\partial_t G &= \dot G \partial_t \mathscr{W} \\ &= \dot G \Delta_{\dot F}\mathscr{W} + \dot G\ddot F(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \left[\mathrm{tr}_{\dot F}(A\mathscr{W}) + a^2\mathrm{tr}(\dot F)\right]\dot G\mathscr{W} \\ &\quad - \left[\bar F + (m\beta - 1)F\right]\dot G\mathscr{W}^2 + a^2\left[\bar F - (m\beta+1)F\right]\mathrm{tr}(\dot G) \\ &= \Delta_{\dot F} G - \dot F\ddot G(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) + \dot G\ddot F(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W}) \\ &\quad + \alpha\left[\mathrm{tr}_{\dot F}(A\mathscr{W}) + a^2\mathrm{tr}(\dot F)\right] G - \left[\bar F + (m\beta - 1)F\right]\dot G\mathscr{W}^2 \\ &\quad + a^2\left[\bar F - (m\beta+1)F\right]\mathrm{tr}(\dot G),\end{aligned}$$

where Euler's relation $\dot{G}\mathscr{W} = \alpha G$ is used in the last line. □

An immediate application of the theorem above is to obtain the evolution equations for H, and F.

Proposition 3.4. *For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_{\kappa}$, on any solution M_t of* (1.1) *the following hold:*

$$\partial_t H = \Delta_{\dot{F}} H + \mathrm{tr}\big[\ddot{F}(\nabla_{\cdot}\mathscr{W}, \nabla^{\cdot}\mathscr{W})\big] + \big[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) + a^2\mathrm{tr}(\dot{F})\big]\, H - \big(\bar{F} + (m\beta - 1)F\big)|A|^2 + na^2\big[\bar{F} - (m\beta + 1)F\big], \tag{3.10}$$

$$\partial_t F = \Delta_{\dot{F}} F + (F - \bar{F})\left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2\mathrm{tr}(\dot{F})\right]. \tag{3.11}$$

For the proof of the main theorem, as mentioned in the introduction, it is convenient for us to define some suitable perturbations of the second fundamental form. Define the shifted second fundamental form

$$\tilde{h}_{ij} = h_{ij} - ag_{ij}.$$

Denote $\tilde{A}$ (resp. $\tilde{\mathscr{W}}$) the matrix whose entries are $\tilde{h}_{ij}$ (resp. $\tilde{h}^i_j$), Then $\tilde{\lambda}_i$ given by

$$\tilde{\lambda}_i = \lambda_i - a, \quad i \in 1, \ldots, n,$$

are the eigenvalues of $\tilde{\mathscr{W}}$. Denote the elementary symmetric functions of the $\tilde{\lambda}_i$ by $\tilde{E}_r, 1 \le r \le n$. From the definition it follows that

$$\begin{aligned}
\tilde{H} &= \mathrm{tr}\tilde{\mathscr{W}} = \tilde{E}_1 = \sum_{i=1}^{n} \tilde{\lambda}_i = H - na, \\
\big|\tilde{A}\big|^2 &= \mathrm{tr}(\tilde{\mathscr{W}}^t\tilde{\mathscr{W}}) = \sum_{i=1}^{n} \tilde{\lambda}_i^2 = \big|A\big|^2 + na^2 - 2Ha, \\
\tilde{K} &= \det\tilde{\mathscr{W}} = \det\{\tilde{h}^i_j\} = \prod_{i=1}^{n} \tilde{\lambda}_i.
\end{aligned}$$

It is easy to check that

$$\nabla_k \tilde{h}_{ij} = \nabla_k h_{ij},$$

and therefore the Codazzi equation holds for $\nabla_k \tilde{h}_{ij}$.

The following theorem is easily obtained from (3.1), (3.2), (3.5) and (3.6) by the definitions of $\tilde{A}$ and $\tilde{\mathscr{W}}$.

Theorem 3.5. *For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_\kappa$, on any solution M_t of* (1.1) *the following hold:*

$$\partial_t \tilde{A} = \Delta_{\dot{F}} \tilde{A} + \ddot{F}(\nabla_\cdot \tilde{\mathscr{W}}, \nabla_\cdot \tilde{\mathscr{W}}) + \big[\bar{F} - (m\beta+1)F\big] A\tilde{\mathscr{W}} + a\big[(m\beta+1)F - \bar{F}\big]\tilde{A} + \mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\, A. \tag{3.12}$$

$$\partial_t \tilde{\mathscr{W}} = \Delta_{\dot{F}} \tilde{\mathscr{W}} + \ddot{F}(\nabla_\cdot \tilde{\mathscr{W}}, \nabla^\cdot \tilde{\mathscr{W}}) + \big[(1-m\beta)F - \bar{F}\big] \mathscr{W}\tilde{\mathscr{W}} + a\big[(m\beta+1)F - \bar{F}\big]\tilde{\mathscr{W}} + \mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\, \mathscr{W}. \tag{3.13}$$

Proof. By (3.1) and (3.5)

$$\begin{aligned}
\partial_t \tilde{A} &= \partial_t A - a\,\partial_t g \\
&= \Delta_{\dot{F}} A + \ddot{F}(\nabla_\cdot \mathscr{W}, \nabla_\cdot \mathscr{W}) + \big[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) + a^2 \mathrm{tr}(\dot{F})\big]\, A \\
&\quad + \big[\bar{F} - (m\beta+1)F\big] A\mathscr{W} + a^2\big[\bar{F} - (m\beta+1)F\big] g - 2a(\bar{F} - F)A \\
&= \Delta_{\dot{F}} \tilde{A} + \ddot{F}(\nabla_\cdot \tilde{\mathscr{W}}, \nabla_\cdot \tilde{\mathscr{W}}) + \big[\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}}) + 2am\beta\, F\big]\, A \\
&\quad + \big[\bar{F} - (m\beta+1)F\big] A\tilde{\mathscr{W}} + a\big[\bar{F} - (m\beta+1)F\big] A \\
&\quad + a^2\big[\bar{F} - (m\beta+1)F\big] g - 2a(\bar{F} - F)A \\
&= \Delta_{\dot{F}} \tilde{A} + \ddot{F}(\nabla_\cdot \tilde{\mathscr{W}}, \nabla_\cdot \tilde{\mathscr{W}}) + \big[\bar{F} - (m\beta+1)F\big] A\tilde{\mathscr{W}} \\
&\quad + a\big[(m\beta+1)F - \bar{F}\big]\tilde{A} + \mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\, A,
\end{aligned}$$

where the third line follows by the relation

$$\mathrm{tr}_{\dot{F}}(A\mathscr{W}) = \mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}}) + 2am\beta\, F - a^2 \mathrm{tr}(\dot{F}).$$

Then (3.12) and (3.2) together imply (3.13). ☐

The evolution equation (3.13) of $\tilde{\mathscr{W}}$ applies to give the evolution of any homogeneous function of the $\tilde{\mathscr{W}}$ defined on an evolving hypersurface M_t of $\mathbb{H}^{n+1}_\kappa$ under the flow (1.1).

Theorem 3.6. *If P is a homogeneous function of the shifted Weingarten map $\tilde{\mathscr{W}}$ of degree γ , then the evolution equation of P under the flow* (1.1) *in $\mathbb{H}^{n+1}_\kappa$ is the following*

$$\begin{aligned}
\partial_t P = \Delta_{\dot{F}} P - \dot{F}\ddot{P}(\nabla_\cdot \mathscr{W}, \nabla^\cdot \mathscr{W}) + \dot{P}\ddot{F}(\nabla_\cdot \mathscr{W}, \nabla^\cdot \mathscr{W}) + \big[(1-m\beta)F - \bar{F}\big]\dot{P}\tilde{\mathscr{W}}^2 \\
+ 2a\,\gamma\big(F - \bar{F}\big)P + \mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\dot{P}\mathscr{W}.
\end{aligned}$$

An immediate application of the theorem above is to obtain the evolution equations for $\tilde{H}$, and $\tilde{H}^n$ and $\tilde{K}$.

Proposition 3.7. *For the ambient space* $N^{n+1} = \mathbb{H}^{n+1}_{\kappa}$*, on any solution* M_t *of* (1.1) *the following hold:*

$$\begin{aligned}\partial_t \tilde H &= \Delta_{\dot F}\tilde H + \mathrm{tr}\big[\ddot F(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W})\big] - \big(\bar F + (m\beta-1)F\big)|\tilde A|^2 \\ &\quad + 2a(F-\bar F)\tilde H + \mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})H,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\partial_t \tilde H^n &= \Delta_{\dot F}\tilde H^n - n(n-1)\tilde H^{n-2}|\nabla\tilde H|^2_{\dot F} + n\tilde H^{n-1}\mathrm{tr}\big[\ddot F(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W})\big] \\ &\quad + n\big((1-m\beta)F-\bar F\big)\tilde H^{n-1}|\tilde A|^2 \ + 2an(F-\bar F)\tilde H^n \\ &\quad + n\,\mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})\tilde H^n + an^2\mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})\tilde H^{n-1},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\partial_t \tilde K &= \Delta_{\dot F}\tilde K - \dot F\ddot{\tilde K}(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W}) + \dot{\tilde K}\ddot F(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W}) \\ &\quad + \big[(1-m\beta)F-\bar F\big]\dot{\tilde K}\tilde{\mathscr W}^2 + 2a\,n\big(F-\bar F\big)\tilde K \\ &\quad + \mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})\dot{\tilde K}\mathscr W.\end{aligned} \tag{3.16}$$

Furthermore, (3.16) can be rewritten as

Lemma 3.8. *For the ambient space* $N^{n+1} = \mathbb{H}^{n+1}_{\kappa}$*, on any solution* M_t *of* (1.1) *the following holds:*

$$\begin{aligned}\partial_t \tilde K &= \Delta_{\dot F}\tilde K - \frac{(n-1)}{n}\frac{\left|\nabla\tilde K\right|^2_{\dot F}}{\tilde K} + \frac{\tilde K}{\tilde H^2}\left|\tilde H\nabla\tilde{\mathscr W} - \tilde{\mathscr W}\nabla\tilde H\right|^2_{\dot F,\tilde b} \\ &\quad - \frac{\tilde H^{2n}}{n\tilde K}\left|\nabla(\tilde K\tilde H^{-n})\right|^2_{\dot F} + \tilde K\,\mathrm{tr}_{\tilde b}\left(\ddot F(\nabla\tilde{\mathscr W},\nabla\tilde{\mathscr W})\right) \\ &\quad + \big[(1-m\beta)F-\bar F\big]\tilde K\tilde H + 2a\,n\big(F-\bar F\big)\tilde K \\ &\quad + n\mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})\tilde K + a\mathrm{tr}_{\dot F}(\tilde A\tilde{\mathscr W})\tilde K\mathrm{tr}(\tilde b),\end{aligned} \tag{3.17}$$

where $\tilde b := \tilde{\mathscr W}^{-1}$.

Proof. Note that

$$\dot{\tilde K} = \tilde K\tilde b, \tag{3.18}$$

this implies

$$\dot{\tilde K}\tilde{\mathscr W}^2 = \tilde K\tilde H, \tag{3.19}$$

and

$$\dot{\tilde K}\ddot F(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W}) = \tilde K\tilde b\ddot F(\nabla_{\cdot}\tilde{\mathscr W},\nabla^{\cdot}\tilde{\mathscr W}) = \tilde K\,\mathrm{tr}_{\tilde b}\left(\ddot F(\nabla\tilde{\mathscr W},\nabla\tilde{\mathscr W})\right). \tag{3.20}$$

A direct calculation as for example in Lemma 3.2 of [20] gives

$$-\dot{F}\ddot{\tilde{K}}(\nabla_{\cdot}\tilde{\mathscr{W}},\nabla^{\cdot}\tilde{\mathscr{W}}) = -\frac{\left|\nabla\tilde{K}\right|^2_{\dot{F}}}{\tilde{K}} - \tilde{K}\,\mathrm{tr}_{\dot{F}}\left(\nabla\tilde{b}\nabla\tilde{\mathscr{W}}\right) \tag{3.21}$$

and

$$\begin{aligned}-\tilde{K}\,\mathrm{tr}_{\dot{F}}\left(\nabla\tilde{b}\nabla\tilde{\mathscr{W}}\right) &= \frac{\tilde{K}}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} \\ &\quad + \frac{\left|\nabla\tilde{K}\right|^2_{\dot{F}}}{n\tilde{K}} - \frac{\tilde{H}^{2n}}{n\tilde{K}}\left|\nabla(\tilde{K}\tilde{H}^{-n})\right|^2_{\dot{F}}.\end{aligned} \tag{3.22}$$

Therefore, identities (3.19), (3.20), (3.21) and (3.22) together apply to (3.16) to give (3.17). □

4. Preserving pinching

To control the pinching of the principal curvatures along the flow (1.1) of Euclidean spaces, Schulze, in [51], following an idea of Tso [54], looked at a test function $Q = K/H^n$, which was also considered in [19]. An analogous quantity which is the quotient $\tilde{Q} = \tilde{K}/\tilde{H}^n$ is more natural for our flow. By the arithmetic-geometric mean inequality, $\tilde{Q} \le 1/n^n$ on M_t and equality holds at a point in M_t if and only if $\tilde{\lambda}_1 = \cdots = \tilde{\lambda}_n$, i.e, $\lambda_1 = \cdots = \lambda_n$ at the point. Thus, the only hypersurfaces such that $\tilde{Q} = 1/n^n$ are the geodesic spheres. The rest of this section consists of showing the inequality $\tilde{Q} \ge C > 0$ remains under the evolution.

Lemma 4.1. *For the ambient space $N^{n+1} = \mathbb{H}^{n+1}_\kappa$, on any solution M_t of* (1.1) *the following holds:*

$$\begin{aligned}\partial_t\tilde{Q} = \Delta_{\dot{F}}\tilde{Q} &+ \frac{(n+1)}{n\tilde{H}^n}\left\langle\nabla\tilde{Q},\nabla\tilde{H}^n\right\rangle_{\dot{F}} - \frac{(n-1)}{n\tilde{K}}\left\langle\nabla\tilde{Q},\nabla\tilde{K}\right\rangle_{\dot{F}} \\ &- \frac{\tilde{H}^n}{n\tilde{K}}\left|\nabla\tilde{Q}\right|^2_{\dot{F}} + \frac{\tilde{Q}}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} \\ &+ \tilde{Q}\,\mathrm{tr}_{\tilde{b}-\frac{n}{\tilde{H}}Id}\left(\ddot{F}(\nabla\tilde{\mathscr{W}},\nabla\tilde{\mathscr{W}})\right) \\ &+ \left[(m\beta-1)F + \bar{F}\right]\frac{\tilde{Q}}{\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2\right) \\ &+ a\tilde{Q}\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\left(\mathrm{tr}(\tilde{b}) - \frac{n^2}{\tilde{H}}\right).\end{aligned} \tag{4.1}$$

Proof. By (3.15) and (3.17)

$$
\begin{aligned}
\partial_t \tilde{Q} &= \frac{1}{\tilde{H}^n}\partial_t \tilde{K} - \frac{1}{\tilde{H}^{2n}}\partial_t \tilde{H}^n \\
&= \frac{\Delta_{\dot{F}}\tilde{K}}{\tilde{H}^n} - \frac{\tilde{K}}{\tilde{H}^{2n}}\Delta_{\dot{F}}\tilde{H}^n - \frac{(n-1)}{n}\frac{\left|\nabla \tilde{K}\right|^2_{\dot{F}}}{\tilde{K}\tilde{H}^n} \\
&\quad - \frac{\tilde{Q}}{n}\left|\nabla \tilde{Q}\right|^2_{\dot{F}} + n(n-1)\frac{\tilde{Q}}{\tilde{H}^2}\left|\nabla \tilde{H}\right|^2_{\dot{F}} \\
&\quad + \frac{\tilde{Q}}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} + \tilde{Q}\,\mathrm{tr}_{\tilde{b}-\frac{n}{\tilde{H}}Id}\left(\ddot{F}(\nabla\tilde{\mathscr{W}},\nabla\tilde{\mathscr{W}})\right) \\
&\quad + \left[(m\beta-1)F + \bar{F}\right]\frac{\tilde{Q}}{\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2\right) \\
&\quad + a\tilde{Q}\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\left(\mathrm{tr}(\tilde{b}) - \frac{n^2}{\tilde{H}}\right).
\end{aligned} \tag{4.2}
$$

Furthermore, the first derivative and second derivative term in (4.2) can be computed as follows, the equality

$$
\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right) = \frac{\nabla\tilde{K}}{\tilde{H}^n} - \frac{\tilde{K}}{\tilde{H}^{2n}}\nabla\tilde{H}^n
$$

implies

$$
\begin{aligned}
\Delta_{\dot{F}}\left(\frac{\tilde{K}}{\tilde{H}^n}\right) &= \frac{\Delta_{\dot{F}}\tilde{K}}{\tilde{H}^n} - 2\frac{\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle_{\dot{F}}}{\tilde{H}^{2n}} \\
&\quad + 2\frac{\tilde{K}}{\tilde{H}^{3n}}|\nabla\tilde{H}^n|^2_{\dot{F}} - \frac{\tilde{K}}{\tilde{H}^{2n}}\Delta_{\dot{F}}\tilde{H}^n,
\end{aligned} \tag{4.3}
$$

$$
\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{H}^n\right\rangle_{\dot{F}} = \frac{\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle_{\dot{F}}}{\tilde{H}^n} - \frac{\tilde{K}}{\tilde{H}^{2n}}|\nabla\tilde{H}^n|^2_{\dot{F}}, \tag{4.4}
$$

and

$$
\left\langle\nabla\left(\frac{\tilde{K}}{\tilde{H}^n}\right),\nabla\tilde{K}\right\rangle_{\dot{F}} = \frac{|\nabla\tilde{K}|^2_{\dot{F}}}{\tilde{H}^n} - \frac{\tilde{K}}{\tilde{H}^{2n}}\left\langle\nabla\tilde{H}^n,\nabla\tilde{K}\right\rangle_{\dot{F}}. \tag{4.5}
$$

From (4.3), (4.4) and (4.5), it follows

$$(4.6)\qquad \begin{aligned} &\frac{\Delta_{\dot F}\tilde K}{\tilde H^n}-\frac{\tilde K}{\tilde H^{2n}}\Delta_{\dot F}\tilde H^n-\frac{(n-1)}{n}\frac{\left|\nabla\tilde K\right|^2_{\dot F}}{\tilde K\tilde H^n}\\ &=\Delta_{\dot F}\left(\frac{\tilde K}{\tilde H^n}\right)+\frac{(n+1)}{n\tilde H^n}\left\langle\nabla\left(\frac{\tilde K}{\tilde H^n}\right),\nabla\tilde H^n\right\rangle_{\dot F}\\ &\quad-\frac{(n-1)}{n\tilde K}\left\langle\nabla\left(\frac{\tilde K}{\tilde H^n}\right),\nabla\tilde K\right\rangle_{\dot F}-n(n-1)\frac{\tilde K}{\tilde H^{n+2}}\left|\nabla\tilde H\right|^2_{\dot F}. \end{aligned}$$

Thus, equation (4.6) applies to (4.2) to give (4.1). □

In order to apply the maximum principle to (4.1) and show that $\min_{p\in M_t}\tilde Q(p,t)$ is non-decreasing in time some preliminary inequalities are needed in the sequel. The following elementary property is a consequence of ([19], Lemma 4.2) (see also [20] and [51]).

Lemma 4.2. *Given $\varepsilon\in(0,1/n)$, there exists a constant $C=C(\varepsilon,n)\in(0,1/n^n)$ such that, for any $\tilde\lambda=(\tilde\lambda_1,\dots,\tilde\lambda_n)\in\mathbb{R}^n$ with $\tilde\lambda_i>0$ for all $i=1,\dots,n$,*

$$\tilde K(\tilde\lambda)>C\tilde H^n(\tilde\lambda),$$

then, we have

$$\tilde\lambda_1>\varepsilon\tilde H(\tilde\lambda).$$

The following estimate which is a stronger version of Lemma 2.3 (ii) in [31] can be viewed as a generalization of Cabezas-Rivas and Miquel in [19].

Lemma 4.3. *If $\tilde H>0$ and the inequality $\tilde{\mathscr W}>\varepsilon\tilde H Id$ is valid with some $\varepsilon>0$ at a point on a hypersurface immersed in $\mathbb{H}^{n+1}_\kappa$, then $\varepsilon\le 1/n$ and*

$$\left|\tilde H\nabla\tilde{\mathscr W}-\tilde{\mathscr W}\nabla\tilde H\right|^2\ge\frac{n-1}{2}\varepsilon^2\tilde H^2\left|\nabla\tilde{\mathscr W}\right|^2.$$

Proof. The proof of the Lemma can be argued exactly as in ([19], Lemma 4.1), only by defining $\tilde{\mathscr W}:=\mathscr W-a\,Id$ at a point on a hypersurface immersed in $\mathbb{H}^{n+1}_\kappa$. □

Also as in [19], the preceding two lemmas allow us to prove the pinching estimate for our flow, which is one of the key steps in the proof of our main result.

Theorem 4.4. *There exists a constant* $C^* = C(n, m, \beta) \in (0, 1/n^n)$ *with the following property: if* $X : M \times (0,T) \to \mathbb{H}^{n+1}_\kappa$*, with* $t \in (0,T)$*, is a smooth solution of* (1.1)–(1.2)*, with* F *given by* (1.4) *for some* $\beta > 1/m$*, such that*

- *the initial immersion* X_0 *satisfies* (1.6) *with the constant* C^**,*
- *the solution* $M_t = X(M,t)$ *satisfies* $\tilde{H} > 0$ *for all times* $t \in (0,T)$*,*

then the minimum of $\tilde{K}/\tilde{H}^n$ *on* M_t *is nondecreasing in time.*

Proof. The assumption $\tilde{H} > 0$ on initial hypersurface ensures that the quotient $\tilde{Q}$ is well-defined for $t \in (0,T)$. For proof of the theorem, it is sufficient to prove that the minimum of $\tilde{Q}$ (denote by $\tilde{\mathcal{Q}}$) is nondecreasing in time. First, by (1.6), $\tilde{\lambda}_1 > 0$ on M_t for $t = 0$, then this implies that $\tilde{\lambda}_1 > 0$ on M_t for $t \in (0,T)$. In fact, suppose to the contrary that there exists a first time $t_0 > 0$ at which $\tilde{\lambda}_1 = 0$ at some point, then $\tilde{\mathcal{Q}}(t_0) = 0$. On the other hand, since the theorem holds in the h-convex case, $\tilde{\mathcal{Q}}(t)$ is nondecreasing in $(0, t_0)$, so it cannot decrease from C^* to zero which gives a contradiction. Now applying the maximum principle to equation (4.1) for $\tilde{\mathcal{Q}}$ gives

$$
\begin{aligned}
\partial_t \tilde{Q} \geq\ & \frac{\tilde{Q}}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} + \tilde{Q}\,\mathrm{tr}_{\tilde{b}-\frac{n}{\tilde{H}}Id}\left(\ddot{F}(\nabla\tilde{\mathscr{W}},\nabla\tilde{\mathscr{W}})\right) \\
& + \left[(m\beta-1)F + \bar{F}\right]\frac{\tilde{Q}}{\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2\right) \\
& + a\tilde{Q}\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\left(\mathrm{tr}(\tilde{b}) - \frac{n^2}{\tilde{H}}\right). \\
\geq\ & \tilde{Q}\left\{\frac{1}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} - \left|\tilde{b} - \frac{n}{\tilde{H}}Id\right|\left|\ddot{F}(\nabla\tilde{\mathscr{W}},\nabla\tilde{\mathscr{W}})\right|\right. \\
& + \left[(m\beta-1)F + \bar{F}\right]\frac{1}{\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2\right) \\
& \left. + a\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\left(\mathrm{tr}(\tilde{b}) - \frac{n^2}{\tilde{H}}\right)\right\}.
\end{aligned}
\tag{4.7}
$$

The various terms appearing here can be estimated as follows, as in [19, Theorem 4.3]. The h-convexity of M_t implies that the third term of RHS in inequality (4.7) can be dropped with the strictly h-convexity on M_t. The last term can also be dropped by the arithmetic-harmonic mean inequality,

$$\sum_{i=1}^{n} \tilde{b}_i^i - \frac{n^2}{\tilde{H}} \geq 0$$

on M_t. It remains to estimate the first two terms of RHS in inequality (4.7), now proceeding exactly as in [19], [20] and [51], choose orthonormal frame which diagonalizes $\tilde{\mathscr{W}}$ so that

$$\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} = \sum_{i,m,n} \dot{F}^i \frac{1}{\tilde{\lambda}_m}\frac{1}{\tilde{\lambda}_n}\left(\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right)^2 \geq \frac{1}{\tilde{H}^2}\sum_{i,m,n}\dot{F}^i\left(\tilde{H}\nabla_i\tilde{h}^n_m - \tilde{h}^n_m\nabla_i\tilde{H}\right)^2 \tag{4.8}$$

where $\tilde{\lambda}_m \leq \tilde{H}$ was used in the last inequality by strictly h-convexity of M_t, i.e., $\tilde{\lambda}_m > 0$ for any m. Now the property that each $\dot{F}^i$ is positive in the interior of the positive cone can be used. More precisely, for any $\varepsilon \in (0, 1/n]$, we set

$$\Xi_\varepsilon := \{\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in \mathbb{R}^n \ : \ \min_{1\leq i\leq n}\tilde{\lambda}_i \geq \varepsilon(\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_n) > 0\},$$

and

$$W_1(\varepsilon) = \min\{\dot{F}^i(\tilde{\lambda}) \ : \ 1 \leq i \leq n, \ \tilde{\lambda} \in \Xi_\varepsilon, |\tilde{\lambda}| = 1\}.$$

By homogeneity of $\dot{F}^i$ with degree $m\beta - 1$ and Lemma 2.1 ii), exactly as in the formula at the top of p.453 of [19], the following inequality holds:

$$\dot{F}^i(\tilde{\lambda}) \geq W_1(\varepsilon)|\tilde{\lambda}|^{m\beta-1}, \qquad \tilde{\lambda} \in \Xi_\varepsilon,$$

where $W_1(\varepsilon)$ is an increasing positive function of ε. This estimation, h-convexity of a hypersurface and Lemma 4.3 together imply that the inequality (4.8) can be estimated as follows:

$$\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} \geq \frac{n-1}{2}W_1(\varepsilon)\varepsilon^2|\tilde{\mathscr{W}}|^{m\beta-1}\left|\nabla\tilde{\mathscr{W}}\right|^2, \tag{4.9}$$

for some $\varepsilon \in (0, 1/n)$.

The term $\left|\ddot{F}(\nabla\tilde{\mathscr{W}}, \nabla\tilde{\mathscr{W}})\right|$ is smooth as long as $\tilde{\lambda}_i > 0$ for any i, homogeneous of degree $m\beta - 2$ in $\tilde{\lambda}_i$ and quadratic in $\nabla\tilde{\mathscr{W}}$. Thus the following estimation of the term $\left|\ddot{F}(\nabla\tilde{\mathscr{W}}, \nabla\tilde{\mathscr{W}})\right|$ can be derived as in [19, inequality (4.7)]: For any $\varepsilon \in (0, 1/n)$, there exists a constant $W_2(\varepsilon)$ such that, at any point where $\tilde{\mathscr{W}} \geq \varepsilon\tilde{H}Id$,

$$\left|\ddot{F}(\nabla\tilde{\mathscr{W}}, \nabla\tilde{\mathscr{W}})\right| \leq W_2(\varepsilon)|\tilde{\mathscr{W}}|^{m\beta-2}\left|\nabla\tilde{\mathscr{W}}\right|^2, \tag{4.10}$$

where $W_2(\varepsilon)$ is decreasing in ε.

A next step is to show that $\left|\tilde{b} - \frac{n}{\tilde{H}} Id\right|$ is small if the principal curvatures are pinched enough. It is clear that

$$\left|\tilde{b} - \frac{n}{\tilde{H}} Id\right| \le \sqrt{n} \max\left\{\left(\frac{1}{\tilde{\lambda}_1} - \frac{n}{\tilde{H}}\right), \left(\frac{n}{\tilde{H}} - \frac{1}{\tilde{\lambda}_n}\right)\right\}.$$

Since for some $\varepsilon \in (0, 1/n)$

$$\tilde{\lambda}_1 \ge \varepsilon \tilde{H}, \tag{4.11}$$

then

$$\frac{1}{\tilde{\lambda}_1} - \frac{n}{\tilde{H}} \le \frac{1 - \varepsilon n}{\varepsilon \tilde{H}}. \tag{4.12}$$

On other hand, (4.11) gives

$$\tilde{\lambda}_n \le (1 - (n-1)\varepsilon)\, \tilde{H} \tag{4.13}$$

which implies that

$$\frac{n}{\tilde{H}} - \frac{1}{\tilde{\lambda}_n} \le \frac{(n-1)\,(1 - n\varepsilon)}{\tilde{H}\,(1 - (n-1)\varepsilon)}. \tag{4.14}$$

This combines with estimate (4.12) to give

$$\left|\tilde{b} - \frac{n}{\tilde{H}} Id\right| \le \frac{\mathscr{N}(\varepsilon)}{\tilde{H}}, \tag{4.15}$$

where

$$\mathscr{N}(\varepsilon) = \begin{cases} \dfrac{\sqrt{n}(1 - \varepsilon n)}{\varepsilon}, & 0 < \varepsilon \le \dfrac{1}{2(n-1)}, \\ \dfrac{\sqrt{n}(n-1)\,(1 - n\varepsilon)}{(1 - (n-1)\varepsilon)}, & \dfrac{1}{2(n-1)} < \varepsilon < \dfrac{1}{n}. \end{cases}$$

Thus, the inequalities $\tilde{H} < H$, $\left|H\right|^2 \le n\left|\mathscr{W}\right|^2$, estimations (4.8), (4.9), (4.10) and (4.15) together give:

$$\begin{aligned} &\frac{1}{\tilde{H}^2}\left|\tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H}\right|^2_{\dot{F},\tilde{b}} - \left|\tilde{b} - \frac{n}{\tilde{H}} Id\right| \left|\ddot{F}(\nabla\tilde{\mathscr{W}}, \nabla\tilde{\mathscr{W}})\right| \\ &\ge \frac{1}{\tilde{H}}|\mathscr{W}|^{m\beta - 2}\left|\nabla\tilde{\mathscr{W}}\right|^2 \left(\frac{(n-1)}{2\sqrt{n}} W_1(\varepsilon)\varepsilon^2 - W_2(\varepsilon)\mathscr{N}(\varepsilon)\right). \end{aligned} \tag{4.16}$$

To achieve our purpose by application of the maximum principle, it is necessary that $\mathscr{F}(\varepsilon) := \left(\frac{(n-1)}{2\sqrt{n}} W_1(\varepsilon)\varepsilon^2 - W_2(\varepsilon)\mathscr{N}(\varepsilon)\right)$ is non-negative on M_t.

In fact, $\mathscr{N}(\varepsilon)$ is a strictly decreasing function of ε; in addition, $\mathscr{N}(\varepsilon)$ is arbitrarily large as ε goes to zero and tends to zero as ε goes to $1/n$ by definition, $W_1(\varepsilon)$ is increasing and $W_2(\varepsilon)$ is decreasing. Therefore, $\mathscr{F}(\varepsilon)$ is a strictly increasing function of ε, it is negative as ε goes to zero and positive as ε goes to $1/n$. So there exists a unique value $\varepsilon_0 \in (0, 1/n)$ such that

$$\mathscr{F}(\varepsilon_0) = 0. \tag{4.17}$$

By Lemma 4.2 there exists a constant $C^* \in (0, 1/n^n)$ satisfies $\tilde{Q}(\tilde{\lambda}) > C^*$ such that $\tilde{\lambda}_1 > \varepsilon \tilde{H}(\tilde{\lambda})$ with a $\varepsilon_0 \in (0, 1/n)$ given by (4.17). Thus, if $\tilde{Q} > C^* \geq 0$ everywhere on the initial hypersurface, applying the maximum principle for $\tilde{Q}$ implies that $\partial_t \tilde{\mathcal{Q}} \geq 0$, i.e., $\tilde{\mathcal{Q}}$ is nondecreasing in time. This guarantees that $\tilde{Q} > C^*$ is preserved under the H_m^β-flow in $\mathbb{H}_\kappa^{n+1}$. □

Theorem 4.4 asserts that inequality $\tilde{Q} > C^*$ holds for all $t \in [0, T)$, furthermore, the definition of C^* together with Lemma 4.2 shows that

$$\tilde{\lambda}_i \geq \varepsilon_0 \tilde{H} \quad \text{on } M \times [0, T) \quad \text{for each } i, \tag{4.18}$$

where ε_0 is given by (4.17), which implies

$$\lambda_i \geq \varepsilon_0 H \quad \text{on } M \times [0, T) \quad \text{for each } i. \tag{4.19}$$

5. Upper bound on F

In this section uniform bounds from above on the speed for the flow and for the curvature of the hypersurface are derived, depending only on the initial data. The bounds on curvatures together with the estimates in the next section will imply the long time existence of the flow by well-known arguments. In order to achieve this, the method is to study the evolution under (1.1) of the function

$$Z_t = \frac{F}{\Phi - \epsilon}. \tag{5.1}$$

Here $\Phi = \mathrm{s}_\kappa(r_p)\langle \nu, \partial_{r_p} \rangle$, which could be seen as "support function" of M^n in $\mathbb{H}_\kappa^{n+1}$, and ϵ is a constant to be chosen later. The method used to obtain these bounds is very robust, and applies to the Gauss curvature flow in [54], the flow with a general class of speeds in [3], volume-preserving anisotropic mean curvature flow [7], the mixed volume preserving mean curvature flows in [44], mixed volume preserving curvature flow in [45], volume preserving mean curvature flow in the hyperbolic space in [17] and volume preserving flow by powers of the mth mean curvature in [19].

Given a function $f : \mathbb{R} \to \mathbb{R}$, $f(r_p)$ will mean $f \circ r_p$. An extension of [17, Lemma 2.1] will be needed later.

Lemma 5.1. *In* $\mathbb{H}^{n+1}_\kappa$,

$$\langle \bar{\nabla}_X \partial_{r_p}, Y\rangle = \bar{\nabla}^2 r_p(X,Y) = \begin{cases} 0 & \text{if } \ \mathrm{X} = \partial_{\mathrm{r_p}} \\ \mathrm{co}_\kappa(r_p)\langle X, Y\rangle & \text{if } \ \langle \mathrm{X}, \partial_{\mathrm{r_p}}\rangle = 0, \end{cases} \tag{5.2}$$

$$\bar{\Delta}_{\dot{F}} r_p = \mathrm{tr}(\dot{F})\mathrm{co}_\kappa(r_p). \tag{5.3}$$

Moreover, if $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is a* C^2 *function,*

$$\bar{\Delta}_{\dot{F}}(f(r_p)) = f''(r_p)|\partial_{r_p}|^2_{\dot{F}} + f'(r_p)\ \bar{\Delta}_{\dot{F}} r_p. \tag{5.4}$$

And, for the restriction of r_p *to a hypersurface* M *of* $\mathbb{H}^{n+1}_\kappa$*, one has*

$$\Delta_{\dot{F}} r_p = -\mathrm{tr}_{\dot{F}}(\mathscr{W})\langle \nu, \partial_{r_p}\rangle + \mathrm{co}_\kappa(r_p)\left(\mathrm{tr}(\dot{F})\ - |\partial^\top_{r_p}|^2_{\dot{F}}\right). \tag{5.5}$$

$$\begin{aligned} \Delta_{\dot{F}}(f(r_p)) &= f''(r_p)\ |\partial^\top_{r_p}|^2_{\dot{F}} + f'(r_p)\ \Delta_{\dot{F}} r_p \\ &= (f''(r_p) - f'(r_p)\ \mathrm{co}_\kappa(r_p))|\partial^\top_{r_p}|^2_{\dot{F}} \\ &\quad +\ f'(r_p)\ (\mathrm{tr}(\dot{F})\ \mathrm{co}_\kappa(r_p) - \mathrm{tr}_{\dot{F}}(\mathscr{W})\langle \nu, \partial_{r_p}\rangle). \end{aligned} \tag{5.6}$$

Proof. First (5.2) and (5.3) follow from [46] (see also [29]), and (5.4) follows from a direct calculation. On the other hand, the Gauß and Codazzi equations give the following:

$$\begin{aligned} \mathrm{Hess}_{\bar{\nabla}} r_p(X,Y) &= \bar{\nabla}^2 r_p(X,Y) = \langle \bar{\nabla}_X \partial_{r_p}, Y\rangle \\ &= \langle \nabla_X \nabla r_p, Y\rangle + A(X,Y)\left\langle \partial_{r_p}, \nu\right\rangle \\ &= \nabla^2 r_p(X,Y) + A(X,Y)\left\langle \partial_{r_p}, \nu\right\rangle \\ &= \mathrm{Hess}_{\nabla} r_p(X,Y) + A(X,Y)\left\langle \partial_{r_p}, \nu\right\rangle, \end{aligned}$$

Combining this with (5.2) gives (5.5). (5.5) gives (5.6) by a direct calculation. □

Corollary 5.2. *For $t \in [0,T)$ and any constant ϵ,*

$$\begin{aligned}(5.7)\qquad \partial_t Z = \Delta_{\dot F} Z + \frac{2\left\langle \nabla Z, \nabla \Phi\right\rangle_{\dot F}}{\Phi-\epsilon} - \frac{\bar F}{\Phi-\epsilon}\Big(\mathrm{tr}_{\dot F}(A\mathscr{W}) - a^2 \mathrm{tr}(\dot F)\Big)\\ - \mathrm{c}_\kappa(r)\frac{Z}{\Phi-\epsilon}\bar F - \epsilon\frac{Z}{\Phi-\epsilon}\mathrm{tr}_{\dot F}(A\mathscr{W}) - a^2\mathrm{tr}(\dot F)Z\\ + (1+m\beta)\mathrm{c}_\kappa(r)Z^2.\end{aligned}$$

Proof. Using (1.1) and (5.2) a direct calculation gives

$$(5.8)\qquad \bar\nabla_t(\mathrm{s}_\kappa(r_p)\partial_{r_p}) = \mathrm{c}_\kappa(r_p)(\bar F - F)\nu,$$

which implies that

$$(5.9)\qquad \partial_t\Phi = \mathrm{s}_\kappa(r_p)\langle\partial_{r_p}, \nabla F\rangle + \mathrm{c}_\kappa(r_p)(\bar F - F)$$

by combining (3.4). On the other hand, a direct calculation gives

$$\begin{aligned}(5.10)\qquad \Delta_{\dot F}\Phi = \left\langle \nu, \partial_{r_p}\right\rangle \Delta_{\dot F}\mathrm{s}_\kappa(r_p) + 2\left\langle \nabla \mathrm{s}_\kappa(r_p), \nabla\langle\nu,\partial_{r_p}\rangle\right\rangle_{\dot F}\\ + \mathrm{s}_\kappa(r_p)\Delta_{\dot F}\langle\nu,\partial_{r_p}\rangle.\end{aligned}$$

Taking $f = \mathrm{s}_\kappa$ and using (5.6) give

$$(5.11)\quad \Delta_{\dot F}\left(\mathrm{s}_\kappa(r_p)\right) = -\frac{1}{\mathrm{s}_\kappa(r_p)}|\partial_{r_p}^\top|^2_{\dot F} - \mathrm{c}_\kappa(r_p)\ \mathrm{tr}_{\dot F}(\mathscr{W})\langle\nu,\partial_{r_p}\rangle + \mathrm{tr}(\dot F)\frac{\mathrm{c}_\kappa^2}{\mathrm{s}_\kappa}(r_p).$$

Choosing a frame $\{e_i\}$ at p which is normal to ν and tangent to M_t. Direct computations having into account (5.2) give

$$\begin{aligned}(5.12)\qquad &\langle\nabla \mathrm{s}_\kappa(r_p), \nabla\langle\partial_{r_p},\nu\rangle\rangle_{\dot F}\\ &= -\frac{\mathrm{c}_\kappa(r_p)^2}{\mathrm{s}_\kappa}(r_p)\left\langle\partial_{r_p},\nu\right\rangle|\partial_{r_p}^\top|^2_{\dot F} + \mathrm{c}_\kappa(r_p)\ \dot F^i_j\ A(\partial_{r_p}^\top, \langle\partial_{r_p}^\top, e^j\rangle e_i).\end{aligned}$$

Since

$$(5.13)\qquad \Delta_{\dot F}\langle\nu,\partial_{r_p}\rangle = \langle\nu,\bar\Delta_{\dot F}\partial_{r_p}\rangle + \langle\bar\Delta_{\dot F}\nu,\partial_{r_p}\rangle + 2\langle\bar\nabla\nu,\bar\nabla\partial_{r_p}\rangle_{\dot F},$$

$$\begin{aligned}(5.14)\qquad \langle\nu,\bar\nabla_i\bar\nabla_j\partial_{r_p}\rangle = &\frac{1}{\mathrm{s}_\kappa^2(r_p)}\langle\partial_{r_p},e_i\rangle\langle\partial_{r_p},e_j\rangle\langle\partial_{r_p},\nu\rangle\\ &- \mathrm{co}_\kappa(r_p)h_{ij} - \mathrm{co}_\kappa^2(r_p)\ g_{ij}\ \langle\nu,\partial_{r_p}\rangle\\ &+ 2\ \mathrm{co}_\kappa^2(r_p)\langle\partial_{r_p},e_i\rangle\langle\partial_{r_p},e_j\rangle\langle\partial_{r_p},\nu\rangle\\ &+ \mathrm{co}_\kappa(r_p)\ h_{ij}\langle\nu,\partial_{r_p}\rangle^2,\end{aligned}$$

$$(5.15)\qquad \langle\bar\nabla_j\nu,\bar\nabla_i\partial_{r_p}\rangle = \mathrm{co}_\kappa(r_p)\ h_{ij} - \mathrm{co}_\kappa(r_p)\ h(\partial_{r_p}^\top, \langle\partial_{r_p}^\top, e_j\rangle e_i),$$

and

$$\langle \bar{\nabla}_i \bar{\nabla}_j \nu, \partial_{r_p} \rangle = \langle \partial_{r_p}, e_k \rangle \ \bar{\nabla}_k (h_{ij}) - \langle \nu, \partial_{r_p} \rangle h_i^k h_{kj}, \tag{5.16}$$

we have by combining (5.13), (5.14), (5.15) and (5.16)

$$\begin{aligned} \Delta_{\dot{F}} \langle \partial_{r_p}, \nu \rangle = & \frac{1}{\mathrm{s}_\kappa^2(r_p)} \left\langle \partial_{r_p}, \nu \right\rangle |\partial_{r_p}^\top|_{\dot{F}}^2 \ + \mathrm{co}_\kappa(r_p) \ \mathrm{tr}_{\dot{F}}(\mathscr{W}) \\ & - \mathrm{tr}(\dot{F}) \ \mathrm{co}_\kappa^2(r_p) \langle \partial_{r_p}, \nu \rangle + 2 \ \mathrm{co}_\kappa^2(r_p) \ \langle \nu, \partial_{r_p} \rangle |\partial_{r_p}^\top|_{\dot{F}}^2 \\ & + \mathrm{co}_\kappa(r_p) \langle \nu, \partial_{r_p} \rangle^2 \ \mathrm{tr}_{\dot{F}}(\mathscr{W}) \\ & - 2 \ \mathrm{co}_\kappa(r_p) \ \dot{F}_j^i \ A(\partial_{r_p}^\top, \langle \partial_{r_p}^\top, e^j \rangle e_i) \\ & + \langle \partial_{r_p}^\top, \nabla F \rangle - \left\langle \partial_{r_p}, \nu \right\rangle \ \mathrm{tr}_{\dot{F}}(A\mathscr{W}). \end{aligned} \tag{5.17}$$

From (5.10), (5.11), (5.12) and (5.17) it follows

$$\Delta_{\dot{F}} \Phi = \mathrm{co}_\kappa(r_p) \ \mathrm{tr}_{\dot{F}}(\mathscr{W}) + \mathrm{s}_\kappa(r_p) \langle \partial_{r_p}, \nabla F \rangle - \Phi \ \mathrm{tr}_{\dot{F}}(A\mathscr{W}).$$

Combining this with (5.9) yields

$$\partial_t \Phi = \Delta_{\dot{F}} \Phi + \Phi \ \mathrm{tr}_{\dot{F}}(A\mathscr{W}) + \mathrm{c}_\kappa(r_p) \left(\bar{F} - F - \mathrm{tr}_{\dot{F}}(\mathscr{W}) \right). \tag{5.18}$$

From (5.18), (3.11) and (5.1), it follows

$$\begin{aligned} \partial_t Z = & \frac{1}{\Phi - \epsilon} \Big(\Delta_{\dot{F}} F + (F - \bar{F}) \left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2 \mathrm{tr}(\dot{F}) \right] \Big) \\ & - \frac{F}{(\Phi - \epsilon)^2} \left(\Delta_{\dot{F}} \Phi + \Phi \ \mathrm{tr}_{\dot{F}}(A\mathscr{W}) + \mathrm{c}_\kappa(r_p) \left(\bar{F} - F - \mathrm{tr}_{\dot{F}}(\mathscr{W}) \right) \right). \end{aligned} \tag{5.19}$$

Another computation leads to

$$\Delta_{\dot{F}} Z = \frac{\Delta_{\dot{F}} F}{\Phi - \epsilon} - \frac{F \Delta_{\dot{F}} \Phi}{(\Phi - \epsilon)^2} - 2 \frac{1}{\Phi - \epsilon} \langle \nabla Z, \nabla \Phi \rangle_{\dot{F}}. \tag{5.20}$$

Inserting (5.20) into (5.19), a few more computations having into account $\mathrm{tr}_{\dot{F}}(\mathscr{W}) = m\beta F$ by Euler's relation gives the desired evolution equation (5.7) of Z. □

To get a finite and independent of t upper bound for Z by application of the maximum principle, firstly it is necessary to get bounds for r_p and $\langle \partial_{r_p}, \nu \rangle$. The following estimate on r_p for the preserving volume mean curvature flow in [17] is also valid in our case with the help of Lemma 2.3 i).

Lemma 5.3. *[17] Let ψ be the inverse of the function $s \mapsto \mathrm{vol}(\mathbb{S}^n)\int_0^s \mathrm{s}(\ell)d\ell$ and ξ the inverse function of $s \mapsto s + a\ln\frac{\left(1+\sqrt{\mathrm{ta}_\kappa(\frac{s}{2})}\right)^2}{1+\mathrm{ta}_\kappa(\frac{s}{2})}$. If $V_0 = \mathrm{vol}(\Omega_0)$ and $\rho_-(t)$ is the inradius of Ω_t, then*

$$\xi(\psi(V_0)) \leq \rho_-(t) \leq \psi(V_0), \tag{5.21}$$

for every $t \in [0, T)$.

An immediate consequence of the lemma above and Lemma 2.3 i) is

Corollary 5.4. *For every $t \in [0, T)$, if $p, q \in \Omega_t$, then*

$$\mathrm{dist}(p, q) < 2(\psi(V_0) + a\ln 2). \tag{5.22}$$

Now, if $p_{t_0} \in \Omega_t$ for an arbitrary fixed $t_0 \in [0, T)$, then using (5.22) gives an upper bound $r_{p_{t_0}}(x) \leq 2(\psi(V_0) + a\ln 2)$ for every $x \in M_t$. Thus, for an upper bound on F, it is necessary to show that a geodesic ball with fixed center remains inside the evolving Ω_t for a short time.

Lemma 5.5. *If $B(p_{t_0}, \rho_{t_0}) \subset \Omega_{t_0}$ for some $t_0 \in [0, T)$, where $\rho_{t_0} = \rho_-(t_0)$ is the inradius of M_{t_0}, then there exists some constant $\tau = \tau(a, n, m, \beta, V_0) > 0$ such that $B(p_{t_0}, \rho_{t_0}/2) \subset \Omega_t$ for every $t \in [t_0, \min\{t_0 + \tau, T\})$.*

Proof. Proceeding similarly as in [17, Lemma 8], our procedure is to compare the deformation of M_t by the equation (1.1) with a geodesic sphere shrinking under the H_m^β-flow.

For convenience, let $r_B(t)$ be the radius at time t of a geodesic sphere $\partial B(p_{t_0}, r_B(t))$ centered at p_{t_0}, evolving under H_m^β-flow and with the initial condition $r_B(t_0) = \rho_{t_0}$. The radius of the evolving geodesic sphere $\partial B(p_{t_0}, r_B(t))$ satisfies

$$\frac{\mathrm{d}r_B(t)}{\mathrm{d}t} = -\,\mathrm{co}_\kappa^{m\beta}(r_B(t)). \tag{5.23}$$

with the initial condition $r_B(t_0) = \rho_{t_0}$, this ODE has solution

$$\int_{\rho_{t_0}}^{r} \mathrm{ta}_\kappa^{m\beta}(s)\mathrm{d}s = -(t - t_0). \tag{5.24}$$

Denote $\mathcal{F}(r) := \int_{\rho_{t_0}}^{r} \mathrm{ta}_\kappa^{m\beta}(s)ds$. Since $\mathcal{F}(r)$ is an increasing function in r, then for $t \geq t_0$, $r_B(t) \geq \rho_{t_0}/2$ if and only if

$$t \leq t_0 + \int_{\rho_{t_0}/2}^{\rho_{t_0}} \mathrm{ta}_\kappa^{m\beta}(s)\mathrm{d}s.$$

On the other hand, let $\mathcal{G}(s) = \int_{s/2}^{s} \mathrm{ta}_\kappa^{m\beta}(u)\mathrm{d}u$, since $s \mapsto \mathrm{ta}_\kappa(s)$ is increasing, then

$$\frac{\mathrm{d}\mathcal{G}(s)}{\mathrm{d}s} > 0$$

which shows that $\mathcal{G}(s)$ is an increasing function in s. Now using (5.21), this gives that if

$$t - t_0 \leq \int_{\xi(\psi(V_0))/2}^{\xi(\psi(V_0))} \mathrm{ta}_\kappa^{m\beta}(s)\mathrm{d}s =: \tau, \tag{5.25}$$

then

$$r_B(t) \geq \rho_{t_0}/2. \tag{5.26}$$

For any $x \in M$, let $r(x,t) = r_{p_{t_0}}(X_t(x))$. From (1.1), it follows

$$\frac{\mathrm{d}r}{\mathrm{d}t} = (\bar{F}(t) - F)\left\langle \nu_t, \partial_{r_{p_{t_0}}} \right\rangle. \tag{5.27}$$

If $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^2 function, set $f(x,t) = \varphi(r(x,t)) - \varphi(r_B(t))$, from (5.23) and (5.27), it follows

$$\partial_t f = \varphi'(r_{p_{t_0}})\ (\bar{F}(t) - F)\left\langle \nu_t, \partial_{r_{p_{t_0}}} \right\rangle + \varphi'(r_B)\mathrm{co}_\kappa^{m\beta}(r_B). \tag{5.28}$$

On the other hand, from (5.6), it follows

$$\begin{aligned} \Delta f &= \Delta_t(\varphi(r_{p_{t_0}})) \\ &= (\varphi''(r_{p_{t_0}}) - \varphi'(r_{p_{t_0}})\ \mathrm{co}_\kappa(r_{p_{t_0}})|\partial_{r_p}^\top|^2 \\ &\quad + \varphi'(r_{p_{t_0}})\ (n\ \mathrm{co}_\kappa(r_{p_{t_0}}) - H\langle \nu, \partial_{r_{p_{t_0}}} \rangle). \end{aligned} \tag{5.29}$$

Therefore, (5.28) can be rewritten as

$$\begin{aligned}\partial_t f = &\frac{F}{H}\Delta f + \varphi'(r_{p_{t_0}})\left\langle \nu_t, \partial_{r_{p_{t_0}}}\right\rangle \bar{F}(t) \\ &+ \varphi'(r_B)\mathrm{co}_\kappa^{m\beta}(r_B) - n\frac{F}{H}\varphi'(r_{p_{t_0}})\mathrm{co}_\kappa(r_{p_{t_0}}) \\ &+ \frac{F}{H}\left[\varphi'(r_{p_{t_0}})\ \mathrm{co}_\kappa(r_{p_{t_0}}) - \varphi''(r_{p_{t_0}})\right]|\partial_{r_p}^\top|^2.\end{aligned} \tag{5.30}$$

Taking $\varphi'(u) = \mathrm{ta}_\kappa(u)$ in (5.30) gives

$$\begin{aligned}\partial_t f = &\frac{F}{H}\Delta f + \mathrm{ta}_\kappa(r_{p_{t_0}})\left\langle \nu_t, \partial_{r_{p_{t_0}}}\right\rangle \bar{F}(t) + \mathrm{co}_\kappa^{m\beta-1}(r_B) \\ &- n\frac{F}{H} + \frac{F}{H}\left(1 - \frac{1}{\mathrm{c}_\kappa^2}(r_{p_{t_0}})\right)|\partial_{r_p}^\top|^2.\end{aligned} \tag{5.31}$$

Now, set $t_1 = \inf\{t > t_0;\ p_{t_0} \notin \Omega_t\}$. Because Ω_t is h-convex, 2.3 ii) implies $\left\langle \nu_t, \partial_{r_{p_{t_0}}}\right\rangle \geq 0$ for any $t \in [t_0, t_1]$. Thus, (5.31) combines with Lemma 2.1 iii) and the initial condition to give

$$\begin{cases} \partial_t f \geq \frac{F}{H}\Delta f + \mathrm{co}_\kappa^{m\beta-1}(r_B) - \left(\frac{H}{n}\right)^{m\beta-1}, \\ f(x, t_0) = \varphi(r(x, t_0)) - \varphi(\rho_{t_0}) \geq 0. \end{cases} \tag{5.32}$$

Next, set $\mathbb{r}(t) := \min_{x\in M} r(x, t)$ for any $t \in [t_0, t_1]$ and $\Theta(t) := \{x \in M \mid r(x, t) = \mathbb{r}(t)\}$. Applying the minimum of f to (5.32) gives

$$\begin{cases} \partial_t f_{\min} \geq \mathrm{co}_\kappa^{m\beta-1}(r_B) - \left(\frac{H_{\max}}{n}\right)^{m\beta-1}, \\ f_{\min}(t_0) \geq 0. \end{cases} \tag{5.33}$$

Here $H_{\max} = \max_{x\in M_t} H(x)$. Note that any point where the minimum of f is attained is the point where the minimum of r is attained for any $t \in [t_0, t_1]$, and at the point the hypersurface is tangent to an inball of radius $\mathbb{r}(t)$, which implies that $H_{\max} = n\mathrm{co}_\kappa(\mathbb{r})$ on any point of $\Theta(t)$. Thus, using a standard comparison principle we conclude that

$$f(x, t) \geq 0 \tag{5.34}$$

for any $t \in [t_0, t_1]$ as long as $f(x, t)$ is well defined for $t \in [0, T)$, and it follows from (5.24) it is positive for $t \in [t_0, t_0 + \int_0^{\xi(\psi(V_0))} \mathrm{ta}_\kappa^{m\beta}(s)\mathrm{d}s) (\supset [t_0, t_0 + \tau))$. Then $f(x, t) \geq 0$ for any $t \in [t_0, \min\{t_0 + \tau, T, t_1\})$.

To complete the proof, assume that $t_1 < \min\{t_0 + \tau, T\}$. By (5.34),

$$r(x, t_1 - \zeta) \geq r_B(t_1 - \zeta) \text{ for all } \zeta \in (0, t_1 - \tau].$$

Hence by (5.26),

$$r(x, t_1) = \lim_{\zeta \to 0^+} r(x, t_1 - \zeta) \geq r_B(t_1) \geq \rho_{t_0}/2,$$

which is a contradiction with $r(x, t_1) = r_{p_0}(t_1) = 0$ by definition of t_1. Therefore, $t_1 \geq \min\{t_0 + \tau, T\}$, which, together with (5.34) and (5.26), implies

$$\mathbb{r}(t) \geq \rho_{t_0}/2 \quad \text{on } [t_0, \min\{t_0 + \tau, T\}),$$

which concludes the proof. □

The above lemma assists us by allowing us to consider a uniform bound on the speed of the flow.

Corollary 5.6. *For $t \in [0, T)$,*

$$F(\cdot, t) < C_1 = C_1(n, m, \beta, a, M_0), \tag{5.35}$$

moreover,

$$H_m(\cdot, t) < C_2 := C_1^{1/\beta}. \tag{5.36}$$

Proof. For any fixed $t_0 \in [0, T)$, let p_{t_0} and ρ_{t_0} be as in Lemma 5.5. Then by Corollary 5.4 and Lemma 5.5, on the hypersurface M_t for every $t \in [t_0, \min\{t_0 + \tau, T\})$

$$D_1 := \frac{\xi(\psi(V_0))}{2} \leq r_{p_{t_0}} \leq \xi(\psi(V_0)) =: D_2.$$

Moreover, having into account Lemma 2.3 ii),

$$\Phi = \mathrm{s}_\kappa(r_{p_{t_0}}) \left\langle \nu, \partial_{r_{p_{t_0}}} \right\rangle \geq a\mathrm{s}_\kappa(D_1)\mathrm{ta}_\kappa(D_1).$$

Then, taking the constant $\epsilon = a\mathrm{s}_\kappa(D_1)\mathrm{ta}_\kappa(D_1)/2$ leads to

$$\Phi - \epsilon \geq \epsilon > 0, \tag{5.37}$$

on the same time interval, which ensures $Z_t = \frac{F}{\Phi - \epsilon}$ is well-defined.

Let us go back to equation (5.7), since strict h-convexity holds for each M_t, F, $\bar{F}$ and $\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2\mathrm{tr}(\dot{F})$ are all positive, which together with (5.37), the two terms containing $\bar{F}$ and the term $a^2\mathrm{tr}(\dot{F})Z$ can be neglected. Furthermore, note that F is homogeneous of degree $m\beta$, Euler's relation and (4.19) together give the following

$$\mathrm{tr}_{\dot{F}}(A\mathscr{W}) = \dot{F}^i\lambda_i^2 \geq \varepsilon_0 H\dot{F}^i\lambda_i = \varepsilon_0 m\beta HF.$$

Now from the above remark,

$$\partial_t Z \leq \Delta_{\dot{F}} Z + \frac{2\left\langle \nabla Z, \nabla\Phi\right\rangle_{\dot{F}}}{\Phi - \epsilon} - \epsilon\varepsilon_0 m\beta HZ^2 + (1+m\beta)\mathrm{c}_\kappa(D_2)Z^2. \tag{5.38}$$

On the other hand, from (5.37) and Lemma 2.1 iii) it follows

$$Z \leq \frac{F}{\epsilon} \leq \frac{1}{\epsilon}\left(\frac{H}{n}\right)^{m\beta}.$$

Applying this to (5.38) gives

$$\partial_t Z \leq \Delta_{\dot{F}} Z + \frac{2\left\langle \nabla Z, \nabla\Phi\right\rangle_{\dot{F}}}{\Phi - \epsilon} + \left((1+m\beta)\mathrm{c}_\kappa(D_2) - \epsilon^{1+\frac{1}{m\beta}} nm\beta\varepsilon_0 Z^{\frac{1}{m\beta}}\right) Z^2.$$

Assume that in $(\bar{x}, \bar{t})$, $\bar{t} \in [t_0, \min\{t_0+\tau, T\})$, Z attains a big maximum $C \gg 0$ for the first time. Then

$$Z(\bar{x}, \bar{t}) \geq C(\Psi - \epsilon)(\bar{x}, \bar{t}) \geq \epsilon C,$$

which gives a contradiction if

$$C > \max_{x\in M^n}\left\{Z(x, t_0), \frac{1}{\epsilon}\left(\frac{\mathrm{c}_\kappa(D_2)(m\beta+1)}{n\varepsilon_0\epsilon m\beta}\right)^{m\beta}\right\}.$$

Thus,

$$Z_t(x) \leq \max_{x\in M^n}\left\{Z(x, t_0), \frac{1}{\epsilon}\left(\frac{\mathrm{c}_\kappa(D_2)(m\beta+1)}{n\varepsilon_0\epsilon m\beta}\right)^{m\beta}\right\},$$

on $[t_0, \min\{t_0+\tau, T\})$.

From the definition of Z_t, the upper bound D_2 of ρ_t it follows

$$F(x,t) \leq (\mathrm{s}_\kappa(D_2) - \epsilon) \max_{x\in M^n} \left\{ Z(x,t_0), \frac{1}{\epsilon}\left(\frac{\mathrm{c}_\kappa(D_2)(m\beta+1)}{n\varepsilon_0 \epsilon m\beta}\right)^{m\beta} \right\},$$

on $[t_0, \min\{t_0+\tau, T\})$. Since t_0 is arbitrary, and τ does not depend on t_0, this implies

$$\begin{aligned} F(x,t) &\leq (\mathrm{s}_\kappa(D_2) - \epsilon) \max_{x\in M^n} \left\{ Z(x,t_0), \frac{1}{\epsilon}\left(\frac{\mathrm{c}_\kappa(D_2)(m\beta+1)}{n\varepsilon_0 \epsilon m\beta}\right)^{m\beta} \right\} \\ &=: C_1(n,m,\beta,a,M_0), \end{aligned}$$

on $[0,T)$, and so (5.36) follows by the definition of F. □

Inserting the estimate (5.35) into (1.2) immediately gives the following

Corollary 5.7. *For $t \in [0,T)$,*

$$\bar{F}(t) < C_1. \tag{5.39}$$

Hence the speed of the evolving hypersurfaces is bounded.

Corollary 5.8. *For $t \in [0,T)$,*

$$\left| \frac{\partial}{\partial t}\mathrm{X}\,(p,t) \right| < C_3 := 2C_1. \tag{5.40}$$

The curvature of M_t also remains bounded.

Corollary 5.9. *For $t \in [0,T)$,*

$$\left|\mathscr{W}\right| < H \leq C_4. \tag{5.41}$$

Proof. The homogeneity of F, (4.19) and the inequality Lemma 2.1 iv) imply that

$$m\beta F = \dot{F}\lambda_i \geq \varepsilon_0 H \mathrm{tr}(\dot{F}) \geq \varepsilon_0 H m\beta F^{1-\frac{1}{m\beta}}.$$

Thus, by (5.35)

$$H \leq \frac{1}{\varepsilon_0} F^{\frac{1}{m\beta}} \leq \frac{1}{\varepsilon_0} C_1^{\frac{1}{m\beta}} =: C_4,$$

and so with the h-convexity of M_t

$$\left|\mathscr{W}\right| < C_4.$$

□

6. Long time existence

In this section, it is shown that solution of the initial value problem (1.1) with the pinching condition (1.6) exists for all positive times. As usual, the first step is to obtain suitable bounds on the solution on any finite time interval $[0, T)$, which guarantees the problem (1.1) has a unique solution on the time interval such that the solution converges to a smooth hypersurface M_T as $t \to T$. Thus, it is necessary to show that the solution remains uniformly convex on the finite time interval which ensure the parabolicity assumption of (1.1).

First it is to show the preserving h-convexity of the evolving hypersurface M_t. Recall that Theorem 4.4 and Lemma 4.2 together imply the uniform h-convexity of M_t, however, comparing with the initial assumptions of Theorem 1.6, there is a priori assumption $\tilde{H} > 0$ in Theorem 4.4. As Cabezas-Rivas and Sinestrari mentioned in [19], note that for small times such an assumption holds due to the smoothness of the flow for small times and the initial pinching condition (1.6), but it is possible that at some positive time both $\min \tilde{K}$ and $\min \tilde{H}$ tend to zero such that $\tilde{K}/\tilde{H}^n$ remains bounded. Thus, to exclude such a possibility, following [19], it is necessary to complement Theorem 4.4 by establishing positivity of $\tilde{H}$ for the finite time.

Lemma 6.1. *Under the hypotheses of Theorem 4.4,*

$$\tilde{H} > 0 \qquad \textit{for all times } t \in [0, T). \tag{6.1}$$

Proof. Let us go back to the evolution equation (3.11) of F,

$$\partial_t F = \Delta_{\dot{F}} F + (F - \bar{F}) \left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2 \mathrm{tr}(\dot{F}) \right].$$

Since under the hypotheses of Theorem 4.4, the evolving hypersurfaces M_t remains h-convex for every $t \in [0, T)$, i.e. $\lambda_i \geq a$ on $[0, T)$, taking a normal coordinate system at a point where $\mathscr{W}$ is diagonal, we have

$$\left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2 \mathrm{tr}(\dot{F}) \right] = \sum_{i=1}^{n} \dot{F}^i (\lambda_i^2 - a^2) \geq 0.$$

Then we have the following computation of F:

$$\begin{aligned}\partial_t F &\geq \Delta_{\dot{F}} F - \bar{F}\sum_{i=1}^{n}\dot{F}^i(\lambda_i^2 - a^2) = \Delta_{\dot{F}} F - \bar{F}\sum_{i=1}^{n}\dot{F}^i(\lambda_i + a)(\lambda_i - a)\\ &\geq \Delta_{\dot{F}} F - C_5\sum_{i=1}^{n}\dot{F}^i(\lambda_i - a)\\ &= \Delta_{\dot{F}} F - C_5\sum_{i=1}^{n}\dot{F}^i(\lambda)\lambda_i + C_5\sum_{i=1}^{n}\dot{F}^i(\lambda)a,\end{aligned}$$

where we have used the estimates (5.35) and (5.39) in the last line, and $C_5 = C_1(C_4 + a)$.

Now by Lemma 2.1 iv) we have $\sum_{i=1}^n \dot{F}^i = \mathrm{tr}(\dot{F}) \geq m\,\beta\,F^{1-\frac{1}{m\beta}}$. Thus, from the above evolution of F we have

$$\begin{aligned}\partial_t F &\geq \Delta_{\dot{F}} F - C_5 m\beta F + C_5 m\,\beta\,F^{1-\frac{1}{m\beta}}a\\ &\geq \Delta_{\dot{F}} F - C_5 m\beta F + C_5 m\beta a^{m\beta}\\ &= \Delta_{\dot{F}} F - C_5 m\beta(F - a^{m\beta}),\end{aligned}$$

where we use the fact that F is homogeneous of degree $m\beta$ and that $F^{1-\frac{1}{m\beta}}(\lambda) \geq (a^{m\beta})^{1-\frac{1}{m\beta}}$, since $\lambda \geq a$ and $1 - \frac{1}{m\beta} \geq 0$. Now, letting $f = F - a^{m\beta}$, the function f satisfies

$$\partial_t f \geq \Delta_{\dot{F}} f - C_5 m\beta f.$$

The parabolic maximum principle and the fact $f \geq 0$ now give

$$f(\cdot,t) \geq f(\cdot,0)e^{-C_6 t},$$

where $C_6 = C_5 m\beta$. This also implies that for all times $t \in [0,T)$, $f > 0$. Thus,

$$F(\cdot,t) \geq \left(a^{m\beta} + f(\cdot,0)e^{-C_6 t}\right).$$

By Lemma 2.1 iii),

$$H(\cdot,t) \geq n\,(F(\cdot,t))^{\frac{1}{m\beta}} \geq n\left[f(\cdot,0)e^{-C_6 t} + a^{m\beta}\right]^{\frac{1}{m\beta}} > na,$$

which implies $\tilde{H} > 0$. □

Corollary 6.2. *Let* $X : M \times [0, T_{\max}) \to \mathbb{H}^{n+1}_\kappa$ *be the solution of* (1.1) *with an initial value which satisfies the pinching condition* (1.6)*. Then, the hypersurfaces* M_t *are uniformly h-convex on any finite time interval; that is, for any* $T < +\infty$, $T \le T_{\max}$*, we have*

$$\inf_{M\times[0,T)} \tilde{\lambda}_i > 0, \qquad \forall i = 1, \ldots, n.$$

Therefore, Theorem 4.4 is valid also without the hypothesis that $\tilde{H} > 0$ *for* $t \in (0, T)$*. The same holds for the other results that have been obtained until here under the same assumptions of Theorem 4.4.*

Proof. The conclusion follows the argument as in [19, Corollary 6.2], only with obvious change of λ_i by $\tilde{\lambda}_i = \lambda_i - a$. □

Preserving h-convexity of the evolving hypersurface leads to the following lower bound on the global term $\bar{F}(t)$.

Corollary 6.3. $\bar{F}(t) \ge a^{m\beta}$ *for all* $t \ge 0$.

Since our flow is different from volume preserving mean curvature flow, we cannot follow the induction argument of Hamilton as in [17, 30–33, 43, 44], etc, to obtain uniform estimates on all orders of curvature derivatives and hence smoothness and convergence of the M_t for the flow (1.1). Instead we use a more PDE theoretic approach, following an argument similar to the one in [19].

Before proceeding further, we adopt a local graph representation for a h-convex hypersurface as in [17]. For each fixed t_0, let p_{t_0} be a center of an inball of Ω_{t_0}, and $\mathbb{S}^n$ the unit sphere in $T_{p_{t_0}}\mathbb{H}^{n+1}_\kappa$. For each t, since M_t is h-convex, there exists a function $r : \mathbb{S}^n \to \mathbb{R}^+$ such that M_t can be written as a map $X_t : \mathbb{S}^n \to \mathbb{H}^{n+1}_\kappa$ satisfying

$$X_t(x) = \exp_{p_{t_0}} r(t, u(t, x))u(t, x), \tag{6.2}$$

where $u(t, x) = \dfrac{\exp^{-1}_{p_{t_0}} X_t(x)}{r_{p_{t_0}}(X_t(x))}$ and $r(t, u(t, x)) = r_{p_{t_0}}(X_t(x))$. At least, from Lemma 5.5 there exists some constant $\tau = \tau(a, n, m, \beta, V_0) > 0$ such that for $t \in [t_0, \min\{t_0 + \tau, T\})$ (near to t_0), $p_{t_0} \in \Omega_t$, and so the map $u_t : M \to \mathbb{S}^n \subset T_{p_{t_0}}\mathbb{H}^{n+1}_\kappa$ defined by $u_t(x) = u(t, x)$ is a diffeomorphism. On the other hand, the map

$$\breve{X}_t(x) = \exp_{p_{t_0}} r(t, u(t_0, x))\ u(t_0, x) \tag{6.3}$$

is another parametrization of M_t. Incorporating a tangential diffeomorphism $\chi_t = u_{t_0}^{-1} \circ u_t : M \to M$ into the flow (1.1) to ensure that this parametrization is preserved, that is, if X_t is a solution of (1.1), $\breve{X}_t$ satisfies the equation

$$\left\langle \partial_t \breve{X}_t, \nu_t \right\rangle = \bar{F}_t - F_t. \tag{6.4}$$

$\breve{X}_t$ can be considered as a map $\mathbb{S}^n$ into $\mathbb{H}_\kappa^{n+1}$ by using the diffeomorphism $u_{t_0}^{-1}$, i.e.,

$$\breve{X}_t(u) = \exp_{p_{t_0}} r(t, u)\ u \quad \text{for every } u \in \mathbb{S}^n, \tag{6.5}$$

where $r(u) = r_{p_{t_0}}(\breve{X}_t(u))$ is a function on $\mathbb{S}^n$. For any local orthonormal frame $\{e_i\}$ of $\mathbb{S}^n$, let D be the Levi-Civita connection on $\mathbb{S}^n$, a basis $\{\breve{e}_i\}$ of the tangent space to M_t is given by

$$\breve{e}_i = \breve{X}_{t*} e_i = D_i(r)\partial_{r_{p_{t_0}}} + \mathrm{s}_\kappa(r)\tau_s e_i, \qquad 1 \le i \le n, \tag{6.6}$$

where τ_s denotes the parallel transport along the geodesic starting from p_{t_0} in the direction of u, and until $\exp_{p_{t_0}} r(u)u$. By using Lemma 2.3 and (5.21) Cabezas-Rivas and Miquel (see p.2078 in [17]) proved that

$$\left|\breve{X}_{t*} e_i\right| < \frac{\mathrm{s}_\kappa(\psi(V_0) + a \ln 2)}{a\ \mathrm{ta}_\kappa(\xi(\psi(V_0)))}.$$

Furthermore, (6.6) implies

$$\left|e_i(r)\right| \le \left|\breve{X}_{t*} e_i\right|.$$

Therefore, both the first derivatives of $\breve{X}_t$ and r are bounded independently of t. The outward unit normal vector of $M_{\hat{t}}$ can be expressed as

$$\nu = \frac{1}{|\xi|}\left(\mathrm{s}_\kappa(r)\partial_{r_p} - \sum_{i=1}^n D_i r e_i\right) \tag{6.7}$$

with

$$\left|\xi\right| = \sqrt{\mathrm{s}_\kappa^2(r) + |Dr|^2}.$$

After a standard computation, the second fundamental form of M_t can be expressed as

$$h_{ij} = -\frac{1}{|\xi|}\Big(\mathrm{s}_\kappa(r)D_jD_ir - \mathrm{s}_\kappa^2(r)\mathrm{c}_\kappa(r)\sigma_{ij} - 2\mathrm{c}_\kappa(r)D_irD_jr\Big), \tag{6.8}$$

and the metric g_{ij} is

$$g_{ij} = D_irD_jr + \mathrm{s}_\kappa^2(r)\sigma_{ij}. \tag{6.9}$$

From this, the inverse metric can be expressed as

$$g^{ij} = \frac{1}{\mathrm{s}_\kappa^2(r)}\left(\sigma^{ij} - \frac{1}{|\xi|^2}D^irD^jr\right), \tag{6.10}$$

where $D^ir = \sigma^{ij}D_jr$. Then equations (6.8) and (6.10) imply that

$$h_j^i = -\frac{1}{|\xi|\mathrm{s}_\kappa(r)}\left[\frac{1}{\mathrm{s}_\kappa(r)}\left(D_jD^ir - \frac{D_jD_lrD^irD^lr}{|\xi|^2}\right) - \mathrm{c}_\kappa(r)\left(\delta_j^i + \frac{D^irD_jr}{|\xi|^2}\right)\right] \tag{6.11}$$

and

$$H = -\frac{1}{|\xi|\mathrm{s}_\kappa(r)}\left(\Delta_{\mathbb{S}}r - \frac{1}{|\xi|^2}\nabla_{\mathbb{S}}^2r(Dr,Dr)\right) + \frac{\mathrm{c}_\kappa(r)}{|\xi|}\left(n + \frac{|Dr|^2}{|\xi|^2}\right). \tag{6.12}$$

Using (6.9) and (6.10) the Christoffel symbols have the expression:

$$\begin{aligned}\Gamma_{ij}^k = {}& \frac{1}{\mathrm{s}_\kappa^2(r)}\Big[D_iD_jrD_lr + \mathrm{s}_\kappa(r)\mathrm{c}_\kappa(r)\big(D_ir\sigma_{lj} + D_jr\sigma_{il} - D_lr\sigma_{ij}\big)\Big] \\ & \times\left(\sigma^{kl} - \frac{1}{|\xi|^2}D^krD^lr\right).\end{aligned} \tag{6.13}$$

Lemma 6.4. *Let $\phi := H_m^{1/m}$ and*

$$\hat\Gamma = \{\lambda = (\lambda_1,\dots,\lambda_n) \ : \ \kappa_1 \le H(\lambda) \le \kappa_2, \quad \min_{1\le i\le n}\lambda_i \ge \varepsilon H(\lambda)\},$$

which is a compact symmetric subset of the positive cone Γ_+ for two positive constants κ_1 and κ_2. There exist constants $m_2 > m_1 > 0$ depending only on

n, M_0 such that for every $t \in [0,T)$ and $x \in M_t$ there holds

$$m_1 \leq \frac{\partial \phi}{\partial \lambda_i}(\lambda) \leq m_2, \qquad i = 1, \ldots, n,\ \lambda \in \hat{\Gamma},$$

as long as the hypersurfaces M_t are strictly h-convex.

Proof. Since $\frac{\partial \phi}{\partial \lambda_i}(\lambda) > 0$ for any $\lambda \in \hat{\Gamma}$, and $\hat{\Gamma}$ is compact, there exist $m_2 > m_1 > 0$ such that

$$m_1 \leq \frac{\partial \phi}{\partial \lambda_i}(\lambda) \leq m_2, \qquad i = 1, \ldots, n,\ \lambda \in \hat{\Gamma}.$$

□

Lemma 6.5. *Let $M \subset \mathbb{H}^{n+1}_\kappa$ be an embedded hypersurface satisfying at every point $D_3 < H < D_4$, $\lambda_1 \geq \varepsilon H$ for given positive constants D_3, D_4, ε. Given any $p \in M$, let r be a local graph representation of M over a unit ball $\mathbb{S}^n \subset T_pM$. Then r satisfies*

$$||r||_{C^{2,\alpha}(\mathbb{S}^n)} \leq C_7(1 + ||F||_{C^{\alpha}(\mathbb{S}^n)})$$

for some $C_7 > 0$ and $0 < \alpha < 1$ depending only on n, D_3, D_4, ε and the parameters β, m in the definition of F.

Proof. We prove the lemma in exactly the same way as [19, Lemma 6.3]. Recalling $\phi = H_m^{1/m}$ and Lemma 2.1 i), ϕ is concave in Γ_+. Then in this case, the Bellman's extension $\bar{\phi}$ of ϕ takes the form

$$\bar{\phi}(\bar{\lambda}) := \inf_{\lambda \in \hat{\Gamma}} \left[\phi(\lambda) + D\phi(\lambda)\left(\bar{\lambda} - \lambda\right)\right]$$

for any $\bar{\lambda} \in \Gamma_+$. Notice that ϕ is homogeneous of degree one, the extension simplifies to

$$\bar{\phi}(\bar{\lambda}) = \inf_{\lambda \in \hat{\Gamma}} D\phi(\lambda)\,\bar{\lambda}.$$

The Bellman extension preserves concavity, by definition and homogeneity, since it is the infimum of linear functions. Importantly, $\bar{\phi}$ coincides with ϕ on $\hat{\Gamma}$ by homogeneity of ϕ. Furthermore, using the definition of $\bar{\phi}$ and Lemma 6.4, $\bar{\phi}$ is uniformly elliptic, that is

$$m_1|\bar{\eta}| \leq \bar{\phi}(\bar{\lambda} + \bar{\eta}) - \bar{\phi}(\bar{\lambda}) \leq \sqrt{n} m_2 |\bar{\eta}|, \quad \text{for all } \bar{\lambda}, \bar{\eta} \in \mathbb{R}^n,\ \bar{\eta} \geq 0.$$

Now the hypotheses imply that the principal curvatures of M at every point are contained between two fixed positive constants. So M can be written as a local graph representation r over a unit ball $\mathbb{S}^n \subset T_pM$ at a given point $p \in M$ with $\|r\|_{C^2}$ bounded in terms of D_4. Let us consider the function $\bar{\phi}(\bar{\lambda}(u))$, where $\bar{\lambda}(u)$ are the principal curvatures of M at the point $(u, r(u))$. Since $\bar{\lambda}(u)$ are the eigenvalues of a matrix depending on Dr, D^2r in view of (6.11), $\bar{\phi}(\bar{\lambda}(u))$ can be expressed as $\bar{\Phi}(Dr(u), D^2r(u))$ for a suitable function $\bar{\Phi} = \bar{\Phi}(u, A)$, with $(u, A) \in \mathbb{S}^n \times \mathcal{S}$, $\mathcal{S}$ being the set of symmetric $n \times n$ matrices. The dependence of $\bar{\Phi}$ on A is related to the dependence of $\bar{\Phi}$ on $\bar{\lambda}$. In fact, it is well known that the concavity of $\bar{\Phi}$ with respect to $\bar{\lambda}$ implies the concavity of $\bar{\Phi}$ with respect to D^2r, and that ellipticity on $\bar{\phi}$ implies the ellipticity condition (2.4) for $\bar{\Phi}$. In addition, $\bar{\Phi}$ is homogeneous of degree one with respect to D^2r. Furthermore, if we set $G(u, A) := \bar{\Phi}(Dr(u), A)$ and $f(u) = \bar{\Phi}(Dr(u), D^2r(u))$, the above argument implies that the elliptic equation for r

$$G(D^2r(u), u) = f(u), \qquad u \in \mathbb{S}^n,\ r \in C^2(\mathbb{S}^n)$$

satisfies the conditions of Theorem 2.5. This theorem asserts that there exists $\alpha \in (0, 1)$ such that

$$\|r\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C(1 + \|f\|_{C^\alpha(\mathbb{S}^n)})$$

where C depends on $n, D_3, D_4, \varepsilon, \beta$.

Our assumptions say that $\bar{\lambda}(u)$ belongs for every u to the set $\hat{\Gamma}$ where $\bar{\phi}$ and $\bar{\Phi}$ coincide. Thus, f coincides with $H_m^{1/m} = F^{1/\beta m}$ at $\bar{\lambda}(u)$. Therefore according to our assumption, the uniform bounds on the curvatures both from below and above imply that F is bounded from below and above by two positive constants depending only on $n, D_3, D_4, \varepsilon, \beta$. Then $||F^{1/\beta m}||_{C^\alpha}$ is estimated by $||F||_{C^\alpha}$ times a constant depending only on these quantities. This finishes the proof of Lemma 6.5. □

Theorem 6.6. *Let M_t be a solution of* (1.1)*, defined on any finite time interval* $[0, T)$*, with initial condition satisfying* (1.6)*. Then, for any* $0 < t_0 < T$*,* $\alpha \in (0, 1)$ *and every natural number* k*, there exist constant* C_8*, depending on* n, m, β, a, M_0 *and* $C_{9,k} = C_9(n, m, \beta, a, M_0, k)$ *such that*

$$\|F\|_{C^\alpha(M \times (t_0, T])} \leq C_8, \tag{6.14}$$

$$\|r\|_{C^k(M \times (t_0, T])} \leq C_{9,k}. \tag{6.15}$$

Proof. For each fixed $t_0 \in [0, T)$, M_t can be locally reparameterized as graphs over the unit sphere $\mathbb{S}^n$ with center p_{t_0} in $T_{p_{t_0}}\mathbb{H}^{n+1}_\kappa$ as (6.5). Then, from (6.4), (6.5) and (6.7), a short computation yields that the distance function r on $\mathbb{S}^n$ satisfies the following parabolic PDE

$$\partial_t r = \mathrm{s}_\kappa^{-1}(r)\big|\xi\big|(\bar{F}_t - F_t), \tag{6.16}$$

where the outward normal vector length $\big|\xi\big|$ is given by the expression (6.7). The function $F = H_m^\beta$ in the coordinate system under consideration is a function of $D^2 r$ and Dr. The right hand side of (6.16) is a fully nonlinear operator, furthermore, observe that (6.16) can be rewritten as

$$\begin{aligned}\partial_t r &= -\mathrm{s}_\kappa^{-1}(r)\big|\xi\big|H_m^\beta + \mathrm{s}_\kappa^{-1}(r)\big|\xi\big|\bar{F}_t,\\ &= \mathrm{s}_\kappa^{-1}(r)H_m^{\beta-1/m}\left(\left(-\big|\xi\big|\right)^m H_m\right)^{1/m} + \mathrm{s}_\kappa^{-1}(r)\big|\xi\big|\bar{F}_t.\end{aligned} \tag{6.17}$$

Since $H_m^{\beta-1/m}$ is larger than $a^{m\beta-1}$ and bounded above by $C_2^{\beta-\frac{1}{m}}$ from (5.36), and r can be bounded independently of t, this implies that $\mathrm{s}_\kappa^{-1}(r)H_m^{\beta-1/m}$ are also uniformly Hölder continuous functions. Then, from (6.11) this ensures that (6.16) is a linear, strictly parabolic partial differential equation. However, the higher order regularity does not follow by the general theory of Krylov and Safonov [36] because the operator is not a concave function of $D^2 u$. Here, we use instead the arguments in [19] who follow the procedure of [8, 45, 53], which consists of proving first regularity in space at a fixed time and then regularity in time.

The first step is to derive C^α-estimate (6.14) of F. In the coordinate system under consideration, (3.11) can be rewritten as

$$\partial_t F = a^{ij} D_i D_j F + b^i D_i F + e\,F, \qquad (x,t) \in \mathbb{S}^n \times J, \tag{6.18}$$

where $J = [t_0, \min\{t_0 + \tau, T\})$, and the coefficients are given by

$$a^{ij} = \beta\, H_m^{\beta-1}\, c^{ij}, \quad b^i = -\beta\, H_m^{\beta-1}\, c^{lj}\Gamma_{lj}^i \tag{6.19}$$

and

$$e = \beta\,(F - \bar{F})\, H_m^{-1}\left(\mathrm{tr}_c(A\mathscr{W}) - a^2 tr(c)\right).$$

Here

$$c = \{c^{ij}\} = \left\{\frac{\partial H_m}{\partial h_i^k} g^{kj}\right\}.$$

Since a^{ij}, b^i and $e\,F$ are uniformly bounded on the curvatures both from above on any finite time interval in view of (5.41) and from below by h-convexity of M_t, equation (6.18) is uniformly parabolic with uniformly bounded coefficients. Then applying Theorem 2.4 to (6.18) gives that for any $0 < r' < 1$ and $J' = J - t_0$ there exist some constants $D_6 > 0$ and $\alpha \in (0,1)$ depending on n, m, β, a, M_0 such that

$$\|F\|_{C^\alpha(B_{r'} \times J')} \le D_5 \|F\|_{C^0(M\times[0,T))} \le D_6. \tag{6.20}$$

Therefore, covering M_t by finite many graphs on balls of radius r' can give (6.14).

Furthermore, for any fixed time $t \in [t_0, T)$, applying (6.14) to Lemma 6.5 on M_t implies that

$$\|r\|_{C^{2,\alpha}(M)} \le D_7 := D_7(n, m, \beta, a, M_0, k).$$

From this estimate on $r(\cdot, t)$ for any fixed t, following the procedure of [8, §3.3, 3.4] or [53, Theorem 2.4] to equation (6.16), a $C^{2,\alpha}$ estimate for r with respect to both space and time can be obtained. Once $C^{2,\alpha}$ regularity is established, standard parabolic theory yields uniform C^k estimates (6.15) for any integer $k > 2$, which implies uniform C^k estimates (6.15) for any integer $k \ge 1$ with the fact that r and its first order derivatives are uniformly bounded. □

Theorem 6.7. *If M be a closed n-dimensional smooth manifold and $X_0 : M \to \mathbb{H}^{n+1}_\kappa$ be an immersion pinched in the sense of* (1.6)*, then the solution M_t of* (1.1) *with initial condition X_0, exists, is smooth and satisfies at every point* (1.6) *on* $[0, \infty)$.

Proof. As we know, the preserving pinching condition (1.6) and smoothness throughout the interval of existence follows form Theorem 4.4, and Lemma 6.1.

On the other hand, it is clear from the expression (6.5) for $\breve{X}_t$ that all the higher order derivatives of $\breve{X}_t$ are bounded if and only if the corresponding derivatives of r are bounded. Thus, uniform C^k estimate (6.15) of r implies that all the derivatives of $\breve{X}_t$ are also uniformly bounded. So the smoothness of $\breve{X}_t$ implies the same smoothness of the reparametrization X_t of $\breve{X}_t$ given by (6.2).

It remains to show that the interval of existence is infinite. Suppose to the contrary there is a maximal finite time T beyond which the solution cannot be extended. Then the evolution equation (1.1) implies that

$$\|\mathrm{X}(\mathrm{p}, \sigma) - \mathrm{X}(\mathrm{p}, \tau)\|_{C^0(X_0)} \le \int_\tau^\sigma \left|\bar{F} - F\right|(p, t)\, \mathrm{d}t$$

for $0 \leq \tau \leq \sigma < T$. By (5.35) and (5.39), $\mathrm{X}(\cdot, t)$ tends to a unique continuous limit $\mathrm{X}(\cdot, T)$ as $t \to T$. In order to conclude that $\mathrm{X}(\cdot, T)$ represents a hypersurface M_T, next under this assumption and in view of the evolution equation (3.1) the induced metric g remains comparable to a fix smooth metric $\tilde{g}$ on M^n:

$$\left|\frac{\partial}{\partial t}\left(\frac{g(u,u)}{\tilde{g}(u,u)}\right)\right| = \left|\frac{\partial_t g(u,u)}{g(u,u)}\frac{g(u,u)}{\tilde{g}(u,u)}\right| \leq 2|H||h|_g\frac{g(u,u)}{\tilde{g}(u,u)},$$

for any non-zero vector $u \in TM^n$, so that ratio of lengths is controlled above and below by exponential functions of time, and hence since the time interval is bounded, there exists a positive constant C_{10} such that

$$\frac{1}{C_{10}}\tilde{g} \leq g \leq C_{10}\tilde{g}.$$

Then the metrics $g(t)$ for all different times are equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g(T)$ which is continuous and also equivalent by following Hamilton's ideas in [30]. Therefore using the smoothness of the hypersurfaces M_t and interpolation,

$$\begin{aligned}
&\|\mathrm{X}(\mathrm{p},\sigma) - \mathrm{X}(\mathrm{p},\tau)\|_{C^k(X_0)} \\
&\leq C\|\mathrm{X}(\mathrm{p},\sigma) - \mathrm{X}(\mathrm{p},\tau)\|^{1/2}_{C^0(X_0)}\|\mathrm{X}(\mathrm{p},\sigma) - \mathrm{X}(\mathrm{p},\tau)\|^{1/2}_{C^{2k}(X_0)} \\
&\leq C|\sigma - \tau|^{1/2},
\end{aligned}$$

so the sequence $\{\mathrm{X(t)}\}$ is a Cauchy sequence in $C^k(X_0)$ for any k. Therefore M_t converges to a smooth limit hypersurface M_T which must be a compact embedded hypersurface in $\mathbb{H}^{n+1}_\kappa$. Finally, applying the local existence result, the solution M_T can be extended for a short time beyond T, since there is a solution with initial condition M_T, contradicting the maximality of T. This completes the proof of Theorem 6.7. □

7. Exponential convergence to a geodesic sphere

Observe that, to finish the proof of Theorem 1.6, it remains to deal with the issues related to the convergence of the flow (1.1): It is proved that solutions of equation (1.1) with initial conditions (1.6) converge, exponentially in the C^∞-topology, to a geodesic sphere in $\mathbb{H}^{n+1}_\kappa$ as t approaches infinity.

The first step is to show that, if a smooth limit exists, it has to be a geodesic sphere of $\mathbb{H}^{n+1}_\kappa$. To address this, let

$$f = \frac{1}{n^n} - \frac{\tilde{K}}{\tilde{H}^n}.$$

and show that the principal curvatures come close together when time tends to infinity. Then as remarked in Section 4, $f \geq 0$ with equality holding only at umbilic points, which is the value assumed on a sphere. The following Lemma is an immediate consequence of the evolution equation (4.1) of $\tilde{Q}$.

Lemma 7.1. *The quantity f evolves under* (1.1) *satisfies*

$$\begin{aligned}
\partial_t f = {} & \Delta_{\dot{F}} f + \frac{(n+1)}{n\tilde{H}^n}\left\langle \nabla f, \nabla \tilde{H}^n \right\rangle_{\dot{F}} \\
& - \frac{(n-1)}{n\tilde{K}}\left\langle \nabla f, \nabla \tilde{K} \right\rangle_{\dot{F}} - \frac{\tilde{H}^n}{n\tilde{K}}|\nabla f|^2_{\dot{F}} \\
& - \left(\frac{\tilde{Q}}{\tilde{H}^2}\left| \tilde{H}\nabla\tilde{\mathscr{W}} - \tilde{\mathscr{W}}\nabla\tilde{H} \right|^2_{\dot{F},\tilde{b}} + \tilde{Q}\,\mathrm{tr}_{\tilde{b}-\frac{n}{\tilde{H}}Id}\left(\ddot{F}(\nabla\tilde{\mathscr{W}}, \nabla\tilde{\mathscr{W}}) \right) \right) \\
& - \left[(m\beta - 1)F + \bar{F} \right] \frac{\tilde{Q}}{\tilde{H}}\left(n\left|\tilde{A}\right|^2 - \tilde{H}^2 \right) \\
& - a\tilde{Q}\mathrm{tr}_{\dot{F}}(\tilde{A}\tilde{\mathscr{W}})\left(\mathrm{tr}(\tilde{b}) - \frac{n^2}{\tilde{H}} \right).
\end{aligned} \tag{7.1}$$

Corollary 7.2. *Under the conditions of Theorem 1.6,*

$$\begin{aligned}
\partial_t f \leq {} & \Delta_{\dot{F}} f + \frac{(n+1)}{n\tilde{H}^n}\left\langle \nabla f, \nabla \tilde{H}^n \right\rangle_{\dot{F}} - \frac{(n-1)}{n\tilde{K}}\left\langle \nabla f, \nabla \tilde{K} \right\rangle_{\dot{F}} \\
& - \frac{\tilde{H}^n}{n\tilde{K}}|\nabla f|^2_{\dot{F}} - C_{11}\tilde{H}f,
\end{aligned} \tag{7.2}$$

where $C_{11} = \delta a^{m\beta} C^*$.

Proof. Applying the similar argument as in Theorem 4.4, Corollary 6.3, inequality (1.6) and Lemma 2.2 to (7.1) gives the conclusion. □

In order to obtain the convergence of f, we have to apply the maximum principal to (7.2). Thus, unlike Lemma 6.1, here we have to get a positive lower bound for $\tilde{H}$ independence of t (not only just for finite t).

Lemma 7.3. *There exist constants* $C_{12} = C_{12}(n, a, m, \beta, M_0) > 0$ *and* $\tau > 0$ *such that for all* $t \in [\tau, \infty)$

$$\tilde{H} \geq C_{12}. \tag{7.3}$$

Proof. We derive exactly as in [3, Lemma 7.7]. First, we adopt a local graph representation as in Lemma 5.5. Let $p_{t_0} \in \Omega_t$ for an arbitrary fixed $t_0 \in [0, \infty)$ as in Lemma 5.5. Let $x_t \in M_t$ be a point in contact with an enclosing sphere of radius ρ_t. Then by Lemma 5.5 and the proof of Corollary 5.6, we know that there is a constant τ such that

$$D_1 < \frac{\rho_{t_0}}{2} < \rho_t < D_2$$

on the interval $[t_0, t_0 + \tau)$. This implies that on the same interval

$$\sup_{x \in M_t} F(x) \geq F(x_t) \geq \mathrm{co}_\kappa^{m\beta}(\rho_t) \geq \mathrm{co}_\kappa^{m\beta}(D_1). \tag{7.4}$$

Let us go back to Equation (6.18). Setting $\theta = F - a^{m\beta}$, we see that the evolution equation of θ has the form:

$$\partial_t \theta = a^{ij} D_i D_j \theta + b^i D_i \theta + \tilde{e}\, \theta, \quad (x, t) \in \mathbb{S}^n \times [t_0, t_0 + \tau), \tag{7.5}$$

where a^{ij} and b^i are given by (6.19), and

$$\tilde{e} = \frac{(F - \bar{F}) \left[\mathrm{tr}_{\dot{F}}(A\mathscr{W}) - a^2 \mathrm{tr}(\dot{F}) \right]}{F - a^{m\beta}}.$$

The coefficients a^{ij} and b^i are obviously uniformly bounded in view of Theorem 6.6. The following shows that the coefficient $\tilde{e}$ is also measurable.

$$|\tilde{e}| \leq \frac{C_{13} \sum_{i=1}^n \dot{F}^i (\lambda_i - a)}{F - a^{m\beta}} \leq \frac{C_{13} m\beta F^{1 - \frac{1}{m\beta}} (F^{\frac{1}{m\beta}} - a)}{F - a^{m\beta}} < +\infty,$$

where we have used Lemma 2.1 iv) and the degree of the homogeneous function F in the second inequality. Again by the h-convexity of M_t, a^{ij} is uniformly bounded from below by a positive constant. This implies that (7.5) is uniformly parabolic with uniformly bounded coefficients.

Reasoning like in [45, Page 147] and [3, Lemma 7.7], one can apply the Krylov Harnack inequality ([35, Section 4.2]) to (7.5) together with (7.4) to deduce the existence of a positive constant c such that

$$\inf_{t_0+\tau} \theta \geq c \left(\mathrm{co}_\kappa^{m\beta}(D_1) - a^{m\beta} \right).$$

Since t_0 is arbitrary, this gives the following result

$$F - a^{m\beta} \geq c \left(\mathrm{co}_\kappa^{m\beta}(D_1) - a^{m\beta} \right)$$

on $[0, \infty)$. From Lemma 2.1 iii), it follows

$$H \geq n \left[c \left(\mathrm{co}_\kappa^{m\beta}(D_1) - a^{m\beta} \right) + a^{m\beta} \right]^{\frac{1}{m\beta}},$$

which implies

$$\tilde{H} \geq C_{12},$$

where

$$C_{12} = n \left\{ \left[c \left(\mathrm{co}_\kappa^{m\beta}(D_1) - a^{m\beta} \right) + a^{m\beta} \right]^{\frac{1}{m\beta}} - a \right\} > 0.$$

□

Proposition 7.4. *Under the conditions of Theorem 1.6, the rate of convergence of f to 0 as $t \to \infty$ is exponential.*

Proof. Applying the similar argument as in Theorem 4.4 and Lemma 7.3 to (7.1) gives

$$\partial_t f_{\max}(t) \leq -C_{11} C_{12} f_{\max}(t).$$

which implies that

$$f_{\max}(t) \leq C_{14} \mathrm{e}^{-C_{15} t},$$

where $C_{14} = f_{\max}(0)$, $C_{15} = C_{11} C_{12}$. This proves the proposition. □

Now if t_0 can be taken big enough so that M_t can be represented as graph r for $[t_0, \infty)$, then since all the derivatives of r are uniformly bounded, applying Arzel-Ascoli Theorem we conclude that the solution $r(t, \cdot\,)$ of (6.3) starting at $r_{p_{t_0}}$ is defined on $[t_0, \infty)$ and converges to a unique function r_∞. This implies that $\breve{X}_t(x) = \exp_{p_{t_0}} r(t, u(t_0, x))\; u(t_0, x)$ solves (6.3) and converges to $u \mapsto \exp_{p_{t_0}} r_\infty(u) u$. Therefore, the reparametrization X_t of $\breve{X}_t$ given by (6.2) has the same convergence properties. That is $X_\infty : \mathbb{S}^n \to \mathbb{H}_\kappa^{n+1}$ and $r_\infty : \mathbb{S}^n \to \mathbb{R}_+$ satisfying $X_\infty(u) = \exp_{p_0} r_\infty(u) u$ which implies that X_∞

is an immersion and, since the convergence is smooth and all the hypersurfaces $X_t(\mathbb{S}^n)$ satisfy (1.6), we can assure that $\mathcal{S} = X_\infty(\mathbb{S}^n)$ must be a compact embedded hypersurface and satisfy (1.6); in addition, X_t is a solution of (1.1) starting at $r_{p_{t_0}}$, and, by uniqueness, X_t coincides on $[t_0, \infty)$ with the solution of (1.1).

On the other hand, Proposition 7.4 says that all points on $\mathcal{S}$ are umbilic points. In conclusion, the only possibility is that $\mathcal{S}$ represents a geodesic sphere in $\mathbb{H}^{n+1}_\kappa$ and, by the volume-preserving properties of the flow, such sphere has to enclose the same volume as the initial condition $X_0(M)$.

Finally, from Proposition 7.4 we can conclude with the standard arguments as in [51, Theorem 3.5], [19, Theorem 7.3] and [41, Corollary 7.3] that the flow converges exponentially to the geodesic sphere $\mathcal{S}$ in $\mathbb{H}^{n+1}_\kappa$ in the C^∞-topology.

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