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Homotopy properties of horizontal path spaces and a theorem of Serre in subriemannian geometry

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We discuss homotopy properties of endpoint maps for *horizontal* path spaces, i.e. spaces of curves on a manifold M whose velocities are constrained to a subbundle $\Delta \subset TM$ in a nonholonomic way. We prove that for every $1 \leq p < \infty$ these maps are Hurewicz fibrations with respect to the $W^{1,p}$ topology on the space of trajectories.

We prove that the space of horizontal curves joining any two points (with the induced $W^{1,p}$ topology) has the homotopy type of a CW-complex and its inclusion into the standard path space (i.e. with no nonholonomic constraints) is a homotopy equivalence. We derive topological implications on the local structure of these spaces (even near abnormal curves, whose possible existence is not excluded from our constructions).

We consider indeed the more general class of *affine* control systems, for which the above theorems hold for all $1 \leq p < p_c$ (here $p_c > 1$ depends only the *step* of the system).

We study critical points of geometric costs for these affine control systems, proving that if the base manifold is compact and there are no abnormal trajectories, then the number of their critical points is infinite (we use Lusternik-Schnirelmann category combined with the Hurewicz property). In the special case where the control system is *subriemannian* this result can be read as the corresponding version of Serre's theorem.

1. Introduction

In this paper we study homotopy properties of the set of those curves on a manifold M whose velocities are constrained in a nonholonomic way (these curves are called *horizontal*). The nonholonomic constraint is made explicit by requiring that the curves should be tangent to a totally nonintegrable distribution (for example a contact distribution, whose horizontal curves are called *legendrian*). More generally we will allow *affine* constraints, by considering a set of vector fields $\mathcal{F} = \{X_0, X_1, \ldots, X_d\}$ and defining a horizontal curve $\gamma : I = [0, 1] \to M$ to be an *absolutely continuous* curve (hence differentiable almost everywhere) solving the equation:

$$\dot{\gamma} = X_0(\gamma) + \sum_{i=1}^{d} u_i X_i(\gamma), \quad \gamma(0) = x \tag{1}$$

for functions $u_1, \ldots, u_d$ called *controls* ($x \in M$ is a point that we fix from the very beginning).

The vector field X_0 is special (it plays the role of a “drift”) and in many interesting cases, like the subriemannian, it is assumed to be zero; the remaining vector fields satisfy the totally nonintegrable *Hörmander* condition: a finite number of their iterated brackets should span the whole tangent space TM (this is also called the *bracket generating* condition).

The regularity we impose on the controls determines the topology on the space Ω of all horizontal curves (called also *trajectories*). In this paper we will assume $u = (u_1, \ldots, u_d) \in L^p(I, \mathbb{R}^d)$ for some $1 < p < \infty$ (thus we consider the $W^{1,p}$ topology on the space of trajectories). The correspondence between a curve and its controls defines local coordinates on Ω, which in turn becomes a Banach manifold modeled on $L^p = L^p(I, \mathbb{R}^d)$ (in fact this manifold is just the open subset of L^p consisting of all controls whose corresponding trajectory is defined on the whole interval I, see the Appendix of this paper or [23] for more details); as a byproduct of this identification we will often replace a curve with the d-tuple of controls describing it in local coordinates.

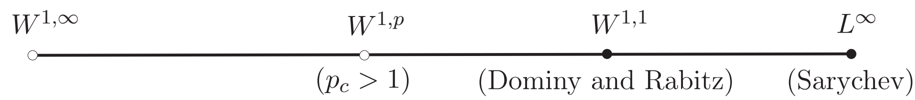

Figure 1: A picture of the continuous inclusions (from left to right) of the various $W^{1,p}([0,1])$ spaces.

The *endpoint map* is the map that associates to each trajectory its final point:

$$F : \Omega \to M \quad \gamma \mapsto \gamma(1).$$

This map is differentiable (smooth in the $W^{1,2}$ case [2]), and the set:

$$\Omega(y) = F^{-1}(y)$$

with the induced topology coincides with the set of horizontal curves joining x to y. In the riemannian case, these spaces are well understood and their topological properties are related to those of the manifold M via the *path fibration* (see [9, 15]), which in our setting we discuss below.

The uniform convergence topology on Ω has been studied in [25] and the $W^{1,1}$ in [13]. For the scopes of calculus of variations the case $W^{1,p}$ with $p > 1$ is especially interesting as the analysis becomes more pleasant: for example the p-th power of the L^p norm becomes a C^1 function and one can apply classical techniques from critical point theory to many problems of interest. Also, it is worth recalling that already in the subriemannian case not all topologies on Ω are equivalent a priori: for example in the $W^{1,\infty}$ case the so-called *rigidity* phenomenon appear: some curves might be isolated (up to reparametrization) in the $W^{1,\infty}$ topology [10].

The key property for studying the topology of horizontal path spaces is the homotopy lifting property for the endpoint map. Our first result generalizes the main results from [13, 25], proving that there exists $p_c > 1$ (depending on $\mathcal{F}$) such that endpoint map is a *Hurewicz fibration* for the $W^{1,p}$ topology for all $1 \leq p < p_c$ (i.e. F has the homotopy lifting property with respect to any space for these topologies).

Theorem (The endpoint map is a Hurewicz fibration). *There exists an interval $[1, p_c) \subseteq [1, \infty)$ (depending on $\mathcal{F}$), such that if $p \in [1, p_c)$ the Endpoint map $F : \Omega \to M$ is a Hurewicz fibration for the $W^{1,p}$ topology on Ω. Moreover if $X_0 = 0$ then $p_c = \infty$.*

It is remarkable that the subriemannian case ($X_0 = 0$) has the Hurewicz fibration property for all $1 \leq p < \infty$, as in general if $X_0 \neq 0$ the endpoint map can fail to have the homotopy lifting property for some finite $p < \infty$, as shown in the next example [5].

Example 1. Consider $M = \mathbb{R}^2$ with coordinates (x_1, x_2) and:

$$X_0 = x_1^2 \partial_{x_2}, \quad X_1 = \partial_{x_1}, \quad X_2 = x_1^k \partial_{x_2}, \quad k \geq 3$$

We consider on Ω the function $u \mapsto J(u) = \|u\|_2^2$, which is continuous for every $p \geq 2$ for the $W^{1,p}$ topology. Let us also consider the function $c_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$c_1(x, y) = \inf\{J(\gamma) \in \Omega \mid \gamma(0) = x,\, \gamma(1) = y\}.$$

In [5, Proposition 2.1] it is proved that there exists $K > 0$ such that for all $w \in \mathbb{R}$ and all $z < 0$:

$$c_1((0, w), (0, w)) = 0 \quad \text{and} \quad c_1((0, w), (0, z)) \geq K.$$

Consider now the path $g_s = (0, -s)$ and let $u_0 \in \Omega$ be a lift for g_0 (i.e. $F(u_0) = g_0$). Now this path (a homotopy of inclusions of a single point) cannot be lifted: an existence of such a lift would be a continuous path u_s on Ω with $u_s \in \Omega(g_s)$, and in particular:

$$\lim_{s \to 0} J(u_s) = 0,$$

which contradicts the fact that $J|_{\Omega(g_s)} \geq K > 0$ for all $s > 0$.

Our proof of the previous theorem is much inspired from [13, 25] and in fact consists in a simple (but important) modification of the proof from [25]. This theorem has important consequences for the topology of fibers of the endpoint map.

Theorem (The homotopy type of the fiber). *Any two fibers of the endpoint map, endowed with the $W^{1,p}$ topology ($p < p_c$), are homotopy equivalent. Moreover each fiber $\Omega(y)$ has the homotopy type of a CW-complex and its inclusion in the ordinary space of curves (i.e. curves without the nonholonomic constraint) is a homotopy equivalence.*

We should stress at this point that the space $\Omega(y)$ might be highly singular, because of the possible existence of *abnormal curves* (curves γ such that the differential $d_\gamma F$ is not a submersion). The existence of these curves *is not* excluded in our setting. It is remarkable that even if abnormal curves might influence the differential structure of $\Omega(y)$, still its homotopy remains very controlled: *any* two fibers of F are homotopy equivalent, regardless them being singular or regular, and what is known for the homotopy of the standard loop space can be deduced also for our horizontal one.

Corollary (Some topological implications). *For every $k \in \mathbb{N}$, every $1 \leq p < p_c$ and every $y \in M$ the following isomorphism between homotopy groups holds for the $W^{1,p}$ topology:*

$$\pi_k(\Omega(y)) \simeq \pi_{k+1}(M).$$

Moreover if M is compact and simply connected, then the Lusternik-Schnirelmann category of the space $\Omega(y)$ is infinite.

Once there is some information available for the topology of $\Omega(y)$, it can be used to study critical points of functionals, the classical example being the study of geodesics between two points. A celebrated theorem of Serre [27] states that if a riemannian manifold M is compact, then every two points are joined by infinitely many geodesics; the proof of this theorem essentially uses the topology of $\Omega(y)$ to force the existence of critical points of the Energy functional, which in the riemannian case are exactly geodesics.

More generally one can study critical points of the p-Energy $J_p : u \mapsto \|u\|_p^p$ on $\Omega(y)$ for affine control systems on *regular fibers* $\Omega(y)$: as long as $1 < p < p_c$ this function is C^1 (Lemma 9) and when restricted to $\Omega(y)$ it satisfies the Palais-Smale condition (Proposition 10). These two properties allow to use classical results to force the existence of critical points.

Theorem (On the critical points for the p-Energy). *Let y be a regular value for the endpoint map of the control system* (1), $1 < p < p_c$ *and consider $f = J_p|_{\Omega(y)}$. If the base manifold M is compact then f has infinitely many critical points.*

As a corollary, we thus obtain a subriemannian version of the Serre's theorem: given x and any regular point y for the endpoint map on a compact subriemannian manifold there are infinitely many geodesics connecting them. In some cases (e.g. contact or fat distributions) the assumption of y being a regular value may be dropped: in these situations there are no abnormal curves other than the trivial ones, and our arguments are essentially not affected (here the fact that $\Omega(y)$ has the homotopy type of a CW-complex plays a crucial role, see the end of the proof of Theorem 16).

1.1. Related work

The problem of understanding the topology of the space of maps with some restrictions on their differential goes back to the works on immersions of S. Smale [28], for the case of curves on a manifold the author considers spherical-type constraints on the velocities (i.e. immersions and regular homotopies). Hurewicz properties for endpoint maps of affine control systems were studied first by A. V. Sarychev [25] for the uniform convergence topology and by J. Dominy and H. Rabitz [13] for the $W^{1,1}$ topology. The quantitative study of the interaction between the topology of the horizontal loop space and the set of geodesics was initiated by the second author together with A. Agrachev and A. Gentile in [4]. In the contact case a "local" version of Serre's theorem was investigated by the second author and L. Rizzi in [20] (the authors perform an asymptotic count of the number of geodesics between two point on a contractible contact manifold, using the relation between a subriemannian manifold and its nilpotent approximation).

1.2. Structure of the paper

Section 2.1 is devoted to the proof of the Hurewicz fibration property (Theorem 4): the crucial ingredient is the construction of a continuous cross-section for the endpoint map (Proposition 2). The topological implications are discussed in Section 2.2. In Section 3 we study critical points of geometric costs: the Palais-Smale property is proved in Proposition 10 and applications via Lusternik-Schnirelmann method are discussed in Section 3.2. The subriemannian case is discussed in Section 4. The Appendix contains some additional technical results, mostly known to experts.

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2. Homotopy properties of the endpoint map

2.1. Some preliminary results

Lemma 1. *Let $0 < \beta < \frac{p}{p-1}$ and for every $j = 1, \ldots, N$ define the map $\rho_j : \mathbb{R}^N \to L^p([0,\infty))$ by: $\rho_j(r) = 0$ if $r_j = 0$ and $\rho_j(r) = \chi_j r_j |r_j|^{-\beta}$ otherwise (χ_j is the characteristic function of the interval $[|r_{j-1}|^\beta, |r_{j-1}|^\beta + |r_j|^\beta]$ and $r_0 = 0$). Then the map ρ_j is continuous.*

Proof. The only needed verification is continuity at zero:

$$\begin{aligned}\lim_{r_j \to 0} \|\chi_j r_j |r_j|^{-\beta}\|_p &= \lim_{r_j \to 0} \left(\int_{|r_{j-1}|^\beta}^{|r_{j-1}|^\beta + |r_j|^\beta} \left| r_j |r_j|^{-\beta} \right|^p dt \right)^{1/p} \\ &= \lim_{r_j \to 0} |r_j|^{\frac{\beta + p - \beta p}{p}} = 0\end{aligned}$$

since $\beta + p - \beta p > 0$. □

Proposition 2 (The cross-section). *Given the manifold M and the family of vector fields $\mathcal{F}$, there exists an interval $[1, p_c) \subset [1, \infty)$ such that for every $1 \leq p < p_c$ every point in M has a neighborhood W and a continuous map:*

$$\begin{aligned}\hat{\sigma} : W \times W &\to L^p([0,\infty), \mathbb{R}^d) \times \mathbb{R} \\ (x, y) &\mapsto (\sigma(x,y), T(x,y))\end{aligned}$$

such that $F_x^{T(x,y)}(\sigma(x,y)) = y$ and $\hat{\sigma}(x,x) = (0,0)$ for every $x, y \in W$. Moreover, if $X_0 = 0$ then $p_c = \infty$.

Proof. We first work out the case $X_0 = 0$ and $p > 1$ (the case $p = 1$ and $X_0 = 0$ is a special case of [13, Lemma 1], whose notation we follow closely).

Given the vector fields $\{Y_1, \dots, Y_k\}$ define inductively $Q^1(Y_1) = e^{Y_1}$ and:

$$Q^\nu(Y_1, \dots, Y_\nu) = e^{Y_\nu} \circ Q^{\nu-1}(Y_1, \dots, Y_{\nu-1}) \circ e^{-Y_\nu} \circ (Q^{\nu-1}(Y_1, \dots, Y_{\nu-1}))^{-1}, \quad \nu \geq 1.$$

Given a real number r we define also:

$$P^\nu(Y_1, \dots, Y_\nu, r) = Q^\nu(rY_1, \dots, rY_\nu).$$

It follows from the Baker-Campbell-Hausdorff formula that, for r sufficiently small,

$$P^\nu(Y_1, \dots, Y_\nu, r^{1/\nu}) = e^{r\,\mathrm{ad}Y_\nu \cdots \mathrm{ad}Y_2 Y_1 + \text{higher order terms in } r}.$$

The bracket generating condition on $\mathcal{F}$ implies that (see [18, Section 2.1] or the proof of [13, Lemma 1]) every point in M has a neighborhood W and a continuous[1] map $\phi : W \times W \to \mathbb{R}^n$ such that $\phi(x, x) = 0$ for all $x \in W$ and:

$$\left(\prod_{k=1}^{n} P^{\nu_k}(X_{k_1}, \dots, X_{k_{\nu_k}}, \phi_k(x, y))\right)(x) = y \quad \forall x, y \in W. \tag{2}$$

Now we notice that the product in (2) can be written as:

$$\left(\prod_{k=1}^{n} P^{\nu_k}(X_{k_1}, \dots, X_{k_{\nu_k}}, \phi_k(x, y))\right) = \prod_{j=1}^{N} e^{\phi_{a_j}(x,y) X_{b_j}}$$

where N is a given number and $a_j, b_j \in \{1, \dots, d\}$ for $j = 1, \dots, N$ (these numbers are fixed and depend on the neighborhood W only).

Given $p > 1$ choose β satisfying the hypothesis of Lemma 1. Using the notation of Lemma 1 we can now interpret $y = (\prod_{j=1}^{N} e^{\phi_{k_j}(x,y) X_{k_j}})(x)$ as the solution at time:

$$T(x, y) = \sum_{j=1}^{N} |\phi_{k_j}(x, y)|^\beta$$

[1]The k-th component of $\phi = (\phi_1, \dots, \phi_n)$ is the ν_k-th root of a C^1 function.

of the control problem with initial datum $y(0) = x$ and control:

$$\sigma(x,y) = \Bigg(\sum_{\{j \,|\, k_j = 1\}} \rho_j(\phi_{k_j}(x,y)), \sum_{\{j \,|\, k_j = 2\}} \rho_j(\phi_{k_j}(x,y)), \dots, \sum_{\{j \,|\, k_j = d\}} \rho_j(\phi_{k_j}(x,y)) \Bigg).$$

By Lemma 1 it follows that the map $\hat{\sigma} = (\sigma, T)$ defined in this way is continuous: each component is the sum of compositions of continuous functions ($T(x,y)$ is continuous since $\beta > 0$) and $\hat{\sigma}(x,x) = (0,0)$.

For the case $X_0 \neq 0$ we notice that the proof of [13, Lemma 1] produces indeed the continuity of the cross section for some $1 < p < p_c$ (as we will see, a lower bound for p_c in this case is given by $\sigma/(\sigma-1)$, where σ is the step of the distribution $\mathcal{F}$). We simply check the needed details. The sequence of exponentials (2) now has to be replaced with [13, Equation 6.a] (using the same notation as the mentioned paper):

$$\left(\prod_{k=1}^{n} R^{\nu_k}(X_0, X_{k_1}, \dots, X_{k_{\nu_k}}, \pm\phi_{k_j}(x,y), \phi_{k_j}(x,y), \dots, \phi_{k_j}(x,y)) \right). \tag{3}$$

The construction in [13] works in such a way that given $\alpha > \nu_k/2$, using BCH formula, R^{ν_k} can be written as the exponential of a series of terms from $\{\phi_{k_1}^{2\alpha} X_0, \dots, \phi_{k_\nu}^{2\alpha} X_0, \phi_{k_1} X_{k_1}, \dots, \phi_{k_\nu} X_{k_\nu}\}$ and their Lie brackets. We choose thus $\alpha > \sigma/2$ which guarantees $\alpha > \nu_k/2$ for all $k = 1, \dots, n$. The product in (3) can thus be regarded as the solution at time $T = \sum_j \nu_k \phi_{k_j}^{2\alpha}$ of a control problem with initial datum $y(0) = x$ and locally constant controls $\sigma = (\sigma_1, \dots, \sigma_d)$ taking values on an interval of length $\phi_{k_j}^{2\alpha}$. The continuity of the final time T follows from the fact that $\alpha > 0$; for the continuity of the corresponding σ we argue as in [13, Appendix C]. Each component of σ is the concatenation of some fixed number of locally constant controls (some of them can possibly be zero) each one defined on an interval of length $\phi_{k_j}^{2\alpha}$ and taking a value proportional to $\phi_{k_j}^{1-2\alpha}$. Then it is enough to check the continuity of this control at zero for the L^p-topology. If we choose $p < \frac{2\alpha}{2\alpha-1}$ then:

$$\lim_{\phi_{k_j} \to 0} \int_c^{c+\phi_{k_j}^{2\alpha}} \left| \phi_{k_j}^{1-2\alpha} \right|^p dt = \lim_{\phi_{k_j} \to 0} \phi_{k_j}^{2\alpha + p - 2p\alpha} = 0.$$

(Notice in particular that, because of the way we chose α, a lower bound for p_c is given by $\sigma/(\sigma-1)$.) □

Proposition 3 (Rescaled concatenation). *Let $p \in [1,\infty)$, then the map $\mathcal{C} : L^p(I) \times L^p([0,+\infty)) \times \mathbb{R} \to L^p(I)$ defined below is continuous:*

$$\mathcal{C}(u,v,T)(t) = \begin{cases} (T+1)u(t(T+1)) & 0 \le t < \dfrac{1}{T+1} \\ (T+1)v((T+1)t-1), & \dfrac{1}{T+1} < t \le 1. \end{cases}$$

Moreover (extending the definition componentwise to controls with value in $\mathbb{R}^d$) we also have $F_x^{1+T}(u*v) = F_x^1(\mathcal{C}(u,v,T))$ for every $x \in M$ (here $u*v$ denotes the usual concatenation).

Proof. As $L^p(I) \times L^p([0,+\infty)) \times \mathbb{R}$ is a metric space, it is sufficient to prove that if $(u_k,v_k,T_k) \to (u,v,T)$, then $\|\mathcal{C}(u_k,v_k,T_k) - \mathcal{C}(u,v,T)\|_p \to 0$.

Assume for simplicity that $T_k \ge T$ (we can split the sequence $\{T_k\}_{k\in\mathbb{N}}$ into two monotone subsequences and work the case $T_k \le T$ separately, it is completely analogous). Start with:

$$\begin{aligned}(4)\qquad &\|\mathcal{C}(u_k,v_k,T_k) - \mathcal{C}(u,v,T)\|_p^p \\ &= \int_0^{1/(T_k+1)} |(T_k+1)u_k(t(T_k+1)) - (T+1)u(t(T+1))|^p dt \\ &\quad + \int_{1/(T_k+1)}^{1/(T+1)} |(T_k+1)v_k(t(T_k+1)-1) - (T+1)u(t(T+1))|^p dt \\ &\quad + \int_{1/(1+T)}^{1} |(T_k+1)v_k(t(T_k+1)-1) - (T+1)v(t(T+1)-1)|^p dt.\end{aligned}$$

Fix $\varepsilon > 0$ and let g be a smooth function compactly supported on $[0,3/2)$ such that $\|g-u\|_p \le \varepsilon$. Observe that for k sufficiently large we have $\|u_k - g\|_p \le \|u-u_k\|_p + \varepsilon \le 2\varepsilon$. We can bound the first integral in (4) as:

$$(5)\quad \int_0^{1/(T_k+1)} |(T_k+1)u_k(t(T_k+1)) - (T+1)u(t(T+1))|^p dt$$
$$\leq 2^{2(p-1)}\Big(\int_0^{1/(T_k+1)} |(T_k+1)u_k(t(T_k+1)) - (T_k+1)g(t(T_k+1))|^p dt$$
$$+\int_0^{1/(T_k+1)} |(T_k+1)g(t(T_k+1)) - (T+1)g(t(T+1))|^p dt$$
$$+\int_0^{1/(T_k+1)} |(T+1)g(t(T+1)) - (T+1)u(t(T+1))|^p dt\Big)$$
$$\leq 2^{2(p-1)}\Big(|T_k+1|^{p-1}\|u_k-g\|_p^p + |T+1|^{p-1}\|u-g\|_p^p$$
$$+\int_0^{1/(T_k+1)} |(T_k+1)g(t(T_k+1)) - (T+1)g(t(T+1))|^p dt\Big).$$

Since g is uniformly continuous in $[0,1]$, the last integral in (5) can also be made as small as we wish as $k\to\infty$ as it is evident from:

$$\int_0^{1/(T_k+1)} |(T_k+1)g(t(T_k+1)) - (T+1)g(t(T+1))|^p dt$$
$$\leq 2^{p-1}\Big(\int_0^{1/(T_k+1)} |(T_k+1)g(t(T_k+1)) - (T_k+1)g(t(T+1))|^p$$
$$+\int_0^{1/(T_k+1)} |(T_k+1)g(t(T+1)) - (T+1)g(t(T+1))|^p\Big).$$

The third integral in (4) is formally the same as the one just handled; a similar reasoning proves that it goes to zero as $k\to\infty$. We are left to deal with the middle one. In this case as $k\to\infty$ by the dominated convergence theorem we have both

$$\int_{1/(T_k+1)}^{1/(T+1)} |v_k(t(T_k+1)-1)|^p dt = |T_k+1|^{p-1}\int_0^{(T_k+1)/(T+1)} |v(z)|^p dz \to 0$$

and

$$\int_{1/(T_k+1)}^{1/(T+1)} |(T+1)u(t(T+1))|^p dt = |T+1|^{p-1}\int_{(T+1)/(T_k+1)}^{1} |u(z)|^p dz \to 0.$$

Finally this yields:

$$\begin{aligned}&\int_{1/(T_k+1)}^{1/(T+1)} |(T_k+1)v_k(t(T_k+1)-1)-(T+1)u(t(T+1))|^p dt\\ &\leq 2^{p-1}\Bigg(|T_k+1|^{p-1}\int_0^{(T_k-t)/(T+1)} |v(z)|^p dz\\ &\qquad + |T+1|^{p-1}\int_{(T+1)/(T_k+1)}^{1} |u(z)|^p dz\Bigg),\end{aligned}$$

and with this we can eventually conclude that:

$$\lim_{k\to\infty} \|\mathcal{C}(u_k, v_k, T_k) - \mathcal{C}(u, v, T)\|_p^p = 0.$$

□

2.2. The Hurewicz fibration property and its consequences

Theorem 4. *There exists an interval*[2] $[1, p_c) \subseteq [1, \infty)$, *such that if* $p \in [1, p_c)$ *the Endpoint map* $F : \Omega \to M$ *is a Hurewicz fibration for the* $W^{1,p}$ *topology on* Ω. *Moreover if* $X_0 = 0$ *then* $p_c = \infty$.

Remark 1. In general the family of vector fields $\{X_1, \ldots, X_d\}$ generating the distribution cannot be chosen such that $d = \mathrm{rank}(\Delta)$ (some topological obstructions might occur), unless we restrict to a small contractible neighborhood in M. The correspondence $A : L^2(I, \mathbb{R}^d) \to \Omega$ associating to a control its trajectory might not be injective, but still it is a Hurewicz fibration: the fibers of this map are convex sets and the map $\mu : \Omega \to L^2(I, \mathbb{R}^d)$ giving the *minimal control* [2] is a continuous section of this fibration (the reader is referred to [19] for a detailed discussion of this map). In particular, the Hurewicz fibration property for $F \circ A$ implies the Hurewicz fibration property for F and we can reduce to study the case $F : L^p(I, \mathbb{R}^d) \to M$ (this is the definition we considered, using the control system in (1)).

Proof. Recall that *Hurewicz fibration* means that F has the homotopy lifting property with respect to every space Z. By Hurewicz uniformization theorem [16], it is enough to show that the homotopy lift property holds locally, i.e. every point $x \in M$ has a neighborhood W such that $F|_{F^{-1}(W)}$ has the homotopy lifting property with respect to any space.

[2]Depending on $(M, X_0, X_1, \ldots, X_d)$.

The case $p = 1$ is proved in [13], thus let $1 < p < p_c$, W and $\hat{\sigma}$ be given as in Proposition 2. Consider a continuous map $g : Z \times I \to W$ and a lift $\tilde{g}_0 : Z \to \Omega$ such that $F(\tilde{g}_0(z)) = g(z, 0)$ for all $z \in Z$. We define the lifting homotopy $\tilde{g} : Z \times I \to \Omega$ by:

$$\tilde{g}(z, s) = \mathcal{C}(\tilde{g}_0(z), \underbrace{\sigma(g(z, 0), g(z, s)), T(g(z, 0), g(z, s))}_{\hat{\sigma}(g(z,0),g(z,s))})$$

(here $\mathcal{C}$ is defined as in Proposition 3 componentwise).

The defined function $\tilde{g}$ is the composition of continuous functions (by Propositions 2 and 3). Moreover by the second assertions in Propositions 2 and 3:

$$F(\tilde{g}(z, s)) = g(z, s) \quad \forall (z, s) \in Z \times I,$$

which proves the claim. □

2.3. The homotopy type of the fibers

As a consequence of Theorem 4 all fibers of F (even the singular fibers) have the same homotopy type [29]. Moreover, by the long exact homotopy sequence of Hurewicz fibrations [29] one also obtains the following isomorphisms between homotopy groups:

$$\pi_k(\Omega(y)) \simeq \pi_{k+1}(M) \quad \forall k \geq 0 \tag{6}$$

In the case the domain of the Hurewicz fibration is contractible we can be more precise about the homotopy type of the fiber.

Theorem 5. *For every $p < p_c$ and $y \in M$ the space $\Omega(y)$ with the $W^{1,p}$ topology has the homotopy type of a CW-complex. In particular the inclusion $\Omega(y) \hookrightarrow \Omega(y)_{std}$ in the standard loop space with the $W^{1,p}$ topology is a homotopy equivalence.*

Proof. First we recall that given the Hurewicz fibration $F : \mathcal{U} \to M$ (in fact any Hurewicz fibration with Ω contractible), then any two fibers are homotopy equivalent to:

$$\Omega M = \{\text{loop spaces in } M \text{ based at } x \text{ with the compact-open topology}\}.$$

The Hurewicz fibration condition is indeed equivalent [7] to the existence of a map:

$$\lambda : \{(u, \omega) \in \mathcal{U} \times M^I \mid F(u) = \omega(0)\} \to \mathcal{U}^I$$

where[3] the map λ satisfies:

$$\lambda(u, \omega)(0) = u \quad \text{and} \quad F(\lambda(u, \omega)(t)) = \omega(t).$$

The map $\eta : \Omega M \to F^{-1}(y)$ defined by $\eta(\omega) = \lambda(x, \omega)(1)$ is proved to be a homotopy equivalence in [13, Lemma 2].

The inclusion $i : \Omega(x)_{\text{std}} \hookrightarrow \Omega M$ is a weak homotopy equivalence: the corresponding Hurewicz fibrations of endpoint maps for $\Omega(x)_{\text{std}}$ with the $W^{1,p}$ and ΩM with the compact open topology give rise to two long exact sequence of homotopy groups; the map i induces an isomorphism between these long exact sequences.

The space ΩM has the homotopy type of a CW-complex [22, Corollary 2] and $\Omega(x)_{\text{std}}^{1,p}$ also have the homotopy type of a CW-complex, since it is a Banach manifold modeled on a metrizable space. In particular [22, Lemma 1] the weak homotopy equivalence $\Omega(x)_{\text{std}} \hookrightarrow \Omega M$ is indeed a homotopy equivalence.

Finally, $\Omega(y)$ has the homotopy type of a CW-complex, since:

$$\Omega(y) \sim \Omega(x) \sim \Omega M \sim \Omega(x)_{\text{std}}$$

(the first homotopy equivalence follows from the fact that all fibers of a Hurewicz fibration have the same homotopy type) and consequently:

$$\Omega(y) \hookrightarrow \Omega(y)_{\text{std}} \quad \text{is a homotopy equivalence.}$$

□

Corollary 6. *If the base manifold M is compact and simply connected, then for every $p < p_c$ (where p_c is given by Theorem 4) and every $y \in M$ the Lusternik-Schnirelmann category of the space $\Omega(y)$ with respect to the $W^{1,p}$ topology is infinite.*

Proof. Let $1 < p < p_c$ be given by Theorem 4. Then $\Omega(y)$ and $\Omega(y)_{\text{std}}$ are homotopy equivalent by the previous theorem (no matter the $W^{1,p}$ topology,

[3] for a topological space X we denoted by X^I the space of paths in $\omega : I \to X$ endowed with the compact-open topology.

as long as $p < p_c$). Since the cup length of the $W^{1,2}$-*ordinary* loop space of a compact simply connected manifold is infinite (see [26, Corollary 20] or the classical work of Serre [27]), so it is for $\Omega(y)$ with the $W^{1,p}$-topology. The cup-length is a lower bound for the Lusternik-Schnirelmann category, hence the result follows. □

A sufficiently small neighborhood of a nonsingular point in $\Omega(y)$ looks like a Hilbert space (hence it is contractible), but the structure near an abnormal curve can be fairly more complicated. This local structure is sharpened by the following result.

Corollary 7. *Every point $\gamma \in \Omega(y)$ (in particular an abnormal curve) has a neighborhood U such that the inclusion $U \hookrightarrow \Omega(y)$ is homotopic to a constant.*

Proof. Since $\Omega(y)$ has the homotopy type of a CW-complex by Theorem 5 above, then the result follows from [14, Proposition 5.1.2]. □

3. Critical points of geometric costs

3.1. The regularity of the Energy

For $p > 1$ we define the p-Energy $J_p : L^p(I, \mathbb{R}^d) \to \mathbb{R}$ by (for simplicity we omit to make explicit the dependence of J_p on p, when it will be clear from the context):

$$J_p(u) = \sum_{i=1}^{d} \|u_i\|_p^p, \quad u = (u_1, \dots, u_d).$$

To simplify notations below we will simply denote $L^p = L^p(I, \mathbb{R}^d)$, also we will omit the subscript notation for $u = (u_1, \dots, u_d)$ when not needed (the corresponding equations should thus be interpreted componentwise).

We will need the following result on Nemytskii operators.

Theorem 8 (Theorem 2.2 [3]). *Let $g : I \times \mathbb{R} \to \mathbb{R}$ be a function such that (i) the function $v \mapsto g(t, v)$ is continuous for almost every $t \in I$; (ii) the function $t \mapsto g(t, v)$ is measurable for all $v \in \mathbb{R}$. Assume also there exists*

$a, b > 0$ such that:

$$|g(t,v)| \leq a + b|v|^{\alpha}, \quad \alpha = \frac{p}{q}.$$

Then the map $u(\cdot) \mapsto g(\cdot, u(\cdot))$ (a Nemytskii operator*) is continuous from $L^p(I)$ to $L^q(I)$.*

As a corollary we derive the following elementary lemma.

Lemma 9. *The map $u \mapsto u|u|^{p-2}$ is a continuous map from $L^p(I)$ to $L^{\frac{p}{p-1}}(I)$. In particular, if y is a regular value of the Endpoint map, then $f = J\big|_{\Omega(y)}$ is a C^1 function.*

Proof. The continuity of $u \mapsto u|u|^{p-2}$ is immediate from the previous Theorem. Now, if y is a regular value of the Endpoint, the differential $d_u f$ coincides with $d_u J|_{T_u\Omega(y)}$ thus to prove that it is differentiable with continuous derivative it is enough to prove it for J. The differential $d_u J$ as a linear functional on $L^p(I, \mathbb{R}^d)$ is easily computed to be (componentwise):

$$\langle d_u J, h \rangle = \int_0^1 pu(t)|u(t)|^{p-2} h(t)dt, \quad \text{for all } h \in L^p,$$

i.e. $d_u J = pu|u|^{p-2} \in L^q = (L^p)^*$, then the result is clear from the previous claim. □

Proposition 10 (Palais-Smale condition). *Let y be a regular value of the Endpoint map and $p > 1$. Then the function $f = J|_{\Omega(y)}$ satisfies the Palais-Smale condition, i.e. any sequence $\{\gamma_k\}_{k \in \mathbb{N}} \subset \Omega(y)$ on which f is bounded and such that $d_{\gamma_k} f \to 0$ has a convergent subsequence.*

Proof. Consider the differential $d_u F$ of the endpoint map at a point u. Using the notations of Theorem 22 we can write it, for any $v \in L^p$ as:

$$(d_u F)v = \int_0^1 M_u(1) M_u(s)^{-1} B_u(s) v(s) ds.$$

Denote by $w_1(t;u), \ldots, w_n(t;u)$ the rows of the matrix $M_u(1)M_u(t)^{-1}B_u(t)$; notice that for $j = 1, \ldots, d$ we have $w_j(\cdot; u) \in L^q$. If $u \in \Omega(y)$, then we can write:

$$T_u\Omega(y) = \ker d_uF = \operatorname{span}\{w_1(\cdot;u), \dots, w_n(\cdot;u)\}^\perp;$$

as the latter is a linear subspace, we also deduce:

$$T_u\Omega(y)^\perp = \operatorname{span}\{w_1(\cdot;u), \dots, w_n(\cdot;u)\}.$$

In particular, for any $u \in \Omega(y)$, $T_u\Omega(y)$ is a closed subspace of codimension n in L^p and therefore it is complemented, i.e. there exists a closed and finite dimensional subspace W_u such that

$$L^p = T_u\Omega(y) \oplus W_u; \tag{7}$$

finally, observe that there exists a continuous linear projection $\pi_u : L^p \to W_u$ subordinated to this splitting, that is $\ker(\pi_u) = T_u\Omega(y)$, see [11, Chapter 2].

Let now $\{u_k\}_{k\in\mathbb{N}} \subset \Omega(y)$ be a bounded sequence such that $d_{u_k}f \to 0$. Since $d_uf = (d_uJ)|_{T_u\Omega(y)}$ then by definition of the projections π_{u_k} we have:

$$\langle d_{u_k}J, (\mathrm{Id} - \pi_{u_k})v\rangle \to 0, \quad \forall v \in L^p.$$

The space L^p is uniformly convex, hence reflexive by the Milman-Pettis theorem; the sequence $\{u_k\}$ is bounded by assumption and invoking Banach-Alaoglu we deduce the existence of a subsequence $\{u_{k_l}\}_{l\in\mathbb{N}}$ and $\overline{u} \in L^p$ such that $u_{k_l} \rightharpoonup \overline{u}$. Furthermore, observe that if $q = p^* = \frac{p}{p-1}$ is the conjugate exponent of p, then:

$$d_uJ = pu|u|^{p-2} \Rightarrow \|d_uJ\|_q^q = \|u\|_p^p.$$

By the above discussion, up to subsequences, we may thus assume that $\|u_k\|_p < C$ and $u_k \rightharpoonup \overline{u}$ in L^p. There exists then $K \in \mathbb{N}$ sufficiently large so that for any norm-one $v \in L^p$ and $k > K$ the following holds:

$$|\langle d_{u_k}J, \pi_{u_k}(v)\rangle| \le |\langle d_{u_k}J, v\rangle| + |\langle d_{u_k}J, v - \pi_{u_k}(v)\rangle| < C + 1. \tag{8}$$

It is well-known [11, Section 3] that the splitting in (7) induces a dual splitting on L^q, namely for any $u \in \Omega(y)$ we have

$$L^q = (T_u\Omega(y))^* \oplus W_u^*;$$

moreover the adjoint operator $\pi_{u_k}^*$ is still a projection with kernel $W_{u_k}^\perp$ and range $(T_{u_k}\Omega(y))^\perp \cong W_{u_k}^* \cong L^q/W_{u_k}^\perp = \operatorname{span}\{w_1(\cdot;u_k), \dots, w_n(\cdot;u_k)\}$. In particular, (8) shows that

$$\|\pi^*_{u_k}(d_{u_k}J)\|_q < C+1, \quad \forall k > K.$$

Write:

$$\pi^*_{u_k}(d_{u_k}J) = \sum_{j=1}^{n} a_{j,k} w_j(\cdot; u_k);$$

since the projections have finite ranges, and all norms are equivalent on finite-dimensional spaces, by the above we deduce that there exists $C' > 0$ so that

$$\sum_{j,l} a_{j,k}a_{l,k}\langle w_j(\cdot;u_k), w_l(\cdot;u_k)\rangle = \|\pi^*_{u_k}(d_{u_k}J)\|_2^2 < C'. \tag{9}$$

Because of Lemma 24 and Theorem 19 and the fact that $u_k \to \overline{u}$ weakly in L^p, then for every $j = 1, \ldots, n$ the function $w_j(\cdot; u_k) : [0,1] \to \mathbb{R}^d$ converges strongly (and hence in any L^p norm) to a function $\overline{w}_j : [0,1] \to \mathbb{R}^d$. Also, $F(\overline{u}) = y$ and since y is a regular value, then $\{\overline{w}_1, \ldots, \overline{w}_n\}$ is a linearly independent set.

By (9) we have $\sum_{j,l} a_{j,k}a_{l,k}\langle \overline{w}_j, \overline{w}_l\rangle < C'$, which tells the sequence:

$$\left\{ z_k = \sum_j a_{j,k}\overline{w}_j \right\}_{k\in\mathbb{N}} \subset \operatorname{span}\{\overline{w}_1, \ldots, \overline{w}_n\} \quad \text{is bounded.}$$

Since $\operatorname{span}\{\overline{w}_1, \ldots, \overline{w}_n\}$ is finite dimensional we can then assume $z_k \to \overline{z}$; since $\{\overline{w}_1, \ldots, \overline{w}_n\}$ is a linearly independent set then the sequences $\{a_{j,k}\}_{k\in\mathbb{N}}$ for $j = 1, \ldots, n$ are bounded and we can assume they converge. Consequently also $\pi^*_{u_k}(d_{u_k}J) \to \overline{z}$ (all this up to subsequences).

Finally we have:

$$\begin{aligned}\lim_{k\to\infty} \|d_{u_k}J - \overline{z}\|_q \leq \lim_{k\to\infty} &\left(\|d_{u_k}J - \pi^*_{u_k}(d_{u_k}J)\|_q\right) \\ &+ \lim_{k\to\infty}\left(\|\pi^*_{u_k}(d_{u_k}J) - \overline{z}\|_q\right) = 0.\end{aligned}$$

This proves that $u_k|u_k|^{p-2} = d_{u_k}J \xrightarrow{L^q} \overline{z}$ (up to subsequences), and the result follows now from the next Lemma 11. □

Lemma 11. *Let* $\{u_n\}_{n\in\mathbb{N}} \subset L^p$ *such that:*

$$u_n|u_n|^{p-2} \xrightarrow{L^q} z.$$

Then $u_n \xrightarrow{L^p} z|z|^{(2-p)/(p-1)}$.

Proof. Consider the Nemytskii operator $N : L^q \to L^p$ defined by $v \mapsto v|v|^{(2-p)/(p-1)}$. Since:

$$\left| v|v|^{\frac{2-p}{p-1}} \right| \leq |v|^{\frac{1}{p-1}} = |v|^{\frac{p}{p-1}\cdot\frac{1}{p}}$$

then $N \in C^0(L^q, L^p)$ by Theorem 8. In particular $u_n = N(u_n|u_n|^{p-2}) \xrightarrow{L^p} N(z)$, and the claim follows. □

3.2. Critical points

Theorem 12. *Let* y *be a regular value for the endpoint map of the control system* (1), $1 < p < p_c$ *(where* p_c *is given by Theorem 4) and consider* $f = J_p|_{\Omega(y)}$. *Then* f *has infinitely many critical points.*

Proof. The first part of the proof follows the lines of the classical argument. Assume first that the fundamental group of M is infinite. Then by (6) $\Omega(y)$ has infinitely many components. Lemma 9 tells that f is C^1 and Proposition 10 that it satisfies the Palais-Smale condition. Assume that one component of $\Omega(y)$ does not contain any critical point of f. Then we can apply the deformation lemma [12, Lemma 3.2] and conclude that f needs to be unbounded from below, which is in contradiction with the definition $f = J_p|_{\Omega(y)} \geq 0$.

Assume now the fundamental group of M is finite. Let us call $r : \overline{M} \to M$ the universal covering map. Then $\overline{M}$ is also compact, and the structure $\mathcal{F}$ can be lifted to a structure $\overline{\mathcal{F}} = \{\overline{X}_0, \dots, \overline{X}_d\}$ by setting:

$$d_{\overline{x}} r \overline{X}_i(\overline{x}) = X_i(r(\overline{x})).$$

Let $\overline{x}$ be a lift of x and $\{\overline{y}_1, \dots, \overline{y}_k\}$ be the lifts of y (here $k = \#\pi_1(M)$, the number of sheets of the covering map). Denote by $\overline{\Omega}$ the set of horizontal curves on $\overline{M}$ leaving from $\overline{x}$, by $\overline{F}$ the corresponding endpoint map and by $\overline{\Omega}(\overline{y})$ the set of horizontal curves on $\overline{M}$ between $\overline{x}$ and $\overline{y} \in \overline{M}$. We denote by $\overline{r} : \overline{\Omega} \to \Omega$ the smooth map that associates to a horizontal trajectory $\overline{\gamma}$ on $\overline{M}$

the trajectory $r \circ \overline{\gamma}$ on M. Notice that in coordinates this map is the identity maps on controls (hence it is a local diffeomorphism), and in particular:

$$J(\overline{\gamma}) = J(\overline{r}(\overline{\gamma})).$$

Moreover, by construction the following diagram is commutative:

$$\begin{array}{ccc} \overline{\Omega}(\overline{y}) & \xrightarrow{\ \overline{r}\ } & \Omega(y) \\ \overline{F}\big\downarrow & & \big\downarrow F \\ \overline{M} & \xrightarrow{\ r\ } & M \end{array}$$

and since r and $\overline{r}$ are local diffeomorphism, then $\overline{y}$ is a regular value of $\overline{F}$.

If we prove the statement for $\overline{M}$, then we are done: in fact given a critical point $\overline{u}$ for the geometric cost $\overline{f} = J|_{\overline{\Omega}(\overline{y})}$ then $\overline{r}(\overline{u})$ is a critical point for f (hence we would obtain an infinite numbers of distinct critical points for f). To see this fact let us use the Lagrange multiplier formulation: $\overline{u}$ is a critical point of $\overline{f}$ if and only if there exists $\overline{\lambda} \in T^*_{\overline{y}}\overline{M}$ such that:

$$\overline{\lambda} \circ d_{\overline{u}}\overline{F} = d_{\overline{u}}J.$$

Using the commutativity of the above diagram, and the fact that r is a local diffeomorphism we see that this implies the existence of a $\lambda \in T^*_y M$ such that

$$\lambda \circ d_{\overline{r}(\overline{u})}F \circ d_{\overline{u}}\overline{r} = d_{\overline{r}(\overline{u})}J \circ d_{\overline{u}}\overline{r} : \tag{10}$$

in fact

$$\begin{aligned} d_{\overline{r}(\overline{u})}J \circ d_{\overline{u}}\overline{r} &= d_{\overline{u}}J \\ &= \overline{\lambda} \circ d_{\overline{u}}\overline{F} \\ &= \overline{\lambda} \circ d_{r(\overline{F}(\overline{u}))}r^{-1} \circ d_{\overline{F}(\overline{u})}r \circ d_{\overline{u}}\overline{F} \\ &= \lambda \circ d_{\overline{r}(\overline{u})}F \circ d_{\overline{u}}\overline{r}. \end{aligned}$$

On the other hand, being $\overline{r}$ a local diffeomorphism, $d_{\overline{u}}\overline{r}$ is also an isomorphism of vector spaces; consequently simplifying it from (10) we can write:

$$\lambda \circ d_{\overline{r}(\overline{u})} F = d_{\overline{r}(\overline{u})} J$$

which tells exactly that $\overline{r}(\overline{u})$ is a critical point for f.

We are left with the case M compact and *simply connected*. Let y be a regular value of the endpoint map and consider the horizontal path space $\Omega(y)$ endowed with the $W^{1,p}$ topology (recall that we are assuming $1 < p < p_c$ with p_c given by Theorem 4). Since y is a regular value of the Endpoint map, $\Omega(y)$ is a smooth Banach manifold modeled on $L^p = L^p([0,1], \mathbb{R}^d)$ (here d is the rank of the distribution). The function f is C^1 (by Lemma 9) and it satisfies the Palais-Smale condition (by Proposition 10 above), hence the results follows from Corollary 6 and the following Proposition.

Proposition 13 (Corollary 3.4 from [12]). *Let $\Omega(y)$ be Banach manifold and $f \in C^1(\Omega(y), \mathbb{R})$ bounded from below and satisfying the Palais-Smale condition. Then f has at least as many critical points as the Lusternik-Schnirelmann category of $\Omega(y)$.* □

4. The subriemannian case

In this section we discuss applications of the previous results to the subriemannian case, in particular we will always make the assumption $X_0 = 0$.

4.1. Geodesics

Given two points x, y in a subriemannian manifold M, a *subriemannian geodesic* is a curve $\gamma : I \to M$ satisfying the following properties: (i) it is absolutely continuous; (ii) its derivative (which exists almost everywhere) belongs to the subriemannian distribution; (iii) it is parametrized by constant speed; (iv) $\gamma(0) = x$ and $y(1) = y$; (v) it is locally length minimizer, i.e. for every $t \in [0,1]$ there exists $\delta(t) > 0$ such that $\gamma|_{[t-\delta(t), t+\delta(t)]}$ has minimal length among all horizontal curves joining $\gamma(t - \delta(t))$ with $\gamma(t + \delta(t))$.

Proposition 14. *Let y be a regular value of the Endpoint map centered at x. For every $p > 1$ all critical points of $f = J_p\big|_{\Omega(y)}$ are subriemannian geodesics joining x to y.*

Proof. First let us notice that curves that are *locally* J_p-minimizers are parametrized by constant speed and are locally length minimizer (the proof of this fact is the same as the classical proof for $p = 2$ as in [21, Section 12] and essentially uses the fact that $\left(\int |u|\right)^p \leq \int |u|^p$ with equality if and only if $|u| \equiv c$). Also, being locally length minimizer and parametrized by constant speed implies that *globally* the parametrization is with constant speed.

Let us consider the equation for $u \in L^p$ to be a critical point of $f = J_p|_{F^{-1}(y)}$ (using Lagrange multipliers rule):

$$\exists \lambda \in T_y^* M \quad \text{such that} \quad \lambda \circ d_u F = pu|u|^{p-2}. \tag{11}$$

In particular since a critical point u of f is a *local* length minimizer (this can be seen by considering variations of only a small portion of the corresponding curve), we must have $|u| \equiv c > 0$ and we can rewrite (11) as:

$$\exists \eta = \frac{\lambda}{pc} \in T_y^* M \quad \text{such that} \quad \eta \circ d_u F = u,$$

which is the equation for the critical points of J_2 on $\Omega(y)$.

Thus if y is a regular value of the Endpoint map, the critical points of J_2 and J_p on $\Omega(y)$ are the same; since critical points of $J_2\big|_{\Omega(y)}$ are subriemannian geodesics joining x to y (see [2, Section 4]), the result follows. □

As a corollary of Proposition 14 and Theorem 12, we obtain the subriemannian version of Serre's theorem.

Theorem 15 (Subriemannian Serre's Theorem). *If y is a regular value of the endpoint map centered at a point x in a compact* subriemannian *manifold, the set of subriemannian geodesics joining x and y is infinite.*

4.2. The contact case

In the contact case we can remove from the subriemannian Serre's theorem the *regularity* assumption on the two points. In fact the same proof works in the slightly more general case of *fat* distributions (see [23] for more details on these distributions), as the only property that we are going to use is that there are no nontrivial abnormal curves.

Theorem 16. *For every two points on a compact, contact subriemannian manifold the set of subriemannian geodesics joining them is infinite.*

Proof. We prove that J_p (with $p > 1$) has infinitely many critical points when restricted to each $\Omega(y)$. Because of Theorem 12 the only case that we have to cover is the case the final point y is the same point as the initial point x (in which case it is not a regular value for F).

Recall that on a contact manifold there are no *nontrivial* abnormal extremals (i.e. critical points of the Endpoint map), see [2, 23], the trivial one being the one with zero control.

The case when the base manifold is not simply connected can be treated as in the proof of Theorem 12: if the fundamental group is infinite, then only one of the infinitely many components of $\Omega(x)$ contains the zero control; if the fundamental group is finite, we pass to the universal cover (which is still compact) and notice that the projection of a geodesic is still a geodesic (no matter if it is a singular point of the Endpoint map, as in the subriemannian case geodesics are locally length minimizers and length is preserved by projection).

Thus we assume our manifold M is compact and simply connected. Consider $\tilde{F}$, the restriction to $L^p\backslash\{0\}$ of the Endpoint map centered at x. Then $\tilde{F}^{-1}(x)$ is a smooth Banach manifold and:

$$\Omega(x) = \tilde{F}^{-1}(x) \cup \{0\}$$

($\Omega(x)$ has its only singularity at zero).

We prove that the Lusternik-Schnirelmann category of $\tilde{F}^{-1}(x)$ is infinite. Combining this with the fact that the p-Energy $f : \tilde{F}^{-1}(x) \to \mathbb{R}$ is C^1 and satisfies Palais-Smale for every level $c > 0$ (by [19, Theorem 19]), implies that f has infinitely many critical points.

Assume that the Lusternik-Schnirelmann category of $\tilde{F}^{-1}(x)$ is finite and let $U_1, \ldots, U_k$ be contractible open sets covering $\tilde{F}^{-1}(x)$. By Corollary 7 there exists an open neighborhood U_0 of the zero control (the singular point of $\Omega(x)$) such that the inclusion $U_0 \hookrightarrow \Omega(x)$ is homotopic to a constant map. As a consequence $\{U_0, \ldots, U_k\}$ would be an open cover of $\Omega(x)$ made of sets contractible in $\Omega(x)$, hence Lusternik-Schnirelmann category of $\Omega(x)$ would be finite, contradicting Corollary 6. □

5. Appendix

In this section we collect a list of technical results that we use in the proofs. Most of these results are well known to experts, but it is often not easy to find an appropriate reference. Some proofs are adaptations from [30] to the general case $p \in (1, \infty)$.

Lemma 17 (Gronwall inequality). *Assume $\varphi : [0,T] \to \mathbb{R}$ to be a bounded nonnegative measurable function, $\alpha : [0,T] \to \mathbb{R}$ to be a nonnegative integrable function and $B : [0,T] \to \mathbb{R}$ to be non decreasing such that*

$$\varphi(t) \leq B(t) + \int_0^t \alpha(\tau)\varphi(\tau)d\tau, \quad \forall t \in [0,T];$$

then

$$\varphi(t) \leq B(t)e^{\int_0^t \alpha(\tau)d\tau}, \quad \forall t \in [0,T].$$

Proposition 18. *Let $T > 0$ be fixed. Then the domain of the endpoint map is open in $L^p([0,T],\mathbb{R}^d)$.*

Proof. The strategy of the proof consists in showing that if v belongs to a sufficiently small neighborhood of u in $L^p([0,T],\mathbb{R}^d)$, then the corresponding trajectories γ_u and γ_v remain uniformly close. It is not restrictive to prove the theorem for small $T > 0$, which in turn allows us to work inside a coordinate chart. Also, we assume that the vector fields X_i, $i = 0,1,\dots,d$ have compact support in $\mathbb{R}^n$; Lemma 3.2 in the aforementioned paper yields that they are therefore globally Lipschitzian. For any $t \in [0,T]$ we have the following:

$$\begin{aligned}\|\gamma_u(t) - \gamma_v(t)\| \leq \Bigg\| &\int_0^t (X_0(\gamma_u(\tau)) - X_0(\gamma_v(\tau)))d\tau \\ &+ \int_0^t \sum_{i=1}^d v_i(\tau)(X_i(\gamma_u(\tau)) - X_i(\gamma_v(\tau)))d\tau \\ &- \int_0^t \sum_{i=1}^d (v_i(\tau) - u_i(\tau))X_i(\gamma_u(\tau))d\tau \Bigg\| \\ \leq C &\int_0^t \left(1 + \sum_{i=1}^d |u_i(\tau)|\right) \|\gamma_u(\tau) - \gamma_v(\tau)\| d\tau + h_v(t),\end{aligned}$$

with

$$h_v(t) = \left\| \int_0^t \sum_{i=1}^d (v_i(\tau) - u_i(\tau))X_i(\gamma_u(\tau))d\tau \right\|.$$

By Hölder inequality we obtain

$$h_v(t) \leq C' T^{1/q} \|u - v\|_p, \quad \forall t \in [0,T];$$

moreover we deduce that for any $\varepsilon > 0$ there exists a neighborhood U of u in $L^p([0,T],\mathbb{R}^d)$ such that $h_v(t) \leq \varepsilon$, for any $v \in U$ and $t \in [0,T]$. We conclude using Gronwall inequality that

$$\|\gamma_u(t) - \gamma_u(v)\| \leq \varepsilon e^{C(T+T^{1/q}K)}, \quad \forall t \in [0,T].$$

□

Theorem 19. *Let $u = (u_1,\ldots,u_d) \in L^p([0,T],\mathbb{R}^d)$ be a control in the domain of the endpoint map F, and let γ_u be the corresponding solution to (7). Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^p([0,T],\mathbb{R}^d)$. If $u_n \overset{L^p}{\rightharpoonup} u$, then for n large enough γ_{u_n} is well-defined on $[0,T]$ and moreover γ_{u_n} converges to γ_u, uniformly on $[0,T]$.*

Proof. It suffices to prove the proposition when T is close to zero; this in turn permits to work in a coordinate chart, that is we may suppose the vector fields X_i to have compact support in $\mathbb{R}^n$. Moreover, let K be a compact neighborhood of x such that there exists $C > 0$ for which

$$\|X_i(z_1) - X_i(z_2)\| \leq C\|z_1 - z_2\|$$

holds for any $z_1, z_2 \in K$ and any $i = 0,1,\ldots,d$. For all $t \in [0,T]$ we have:

$$\begin{aligned}
\|\gamma_u(t) - \gamma_{u_n}(t)\| \leq & \int_0^t \|(X_0(\gamma_u(\tau)) - X_0(\gamma_{u_n}(\tau))\|d\tau \\
& + \int_0^t \sum_{i=1}^d |u_{n,i}(\tau)|\|X_i(\gamma_u(\tau)) - X_i(\gamma_{u_n}(\tau))\|d\tau \\
& + \int_0^t \sum_{i=1}^d |u_{n,i}(\tau) - u_i(\tau)|\|X_i(\gamma_u(\tau))\|d\tau \\
\leq & \, C\int_0^1 \left(1 + \sum_{i=1}^d |u_{n,i}(\tau)|\right) \|\gamma_u(\tau) - \gamma_{u_n}(\tau)\|d\tau + h_n(t),
\end{aligned}$$

where

$$h_n(t) = \int_0^t \sum_{i=1}^d |u_{n,i}(\tau) - u_i(\tau)|\|X_i(\gamma_u(\tau))\|d\tau.$$

The uniform boundedness principle of Banach and Steinhaus ensures that $\sup_{n\in\mathbb{N}} \|u_n\|_p \leq M$; if we can prove that h_n tends *uniformly* on $[0,T]$ to the zero function, then we would finish the argument using the Gronwall inequality.

Observe that h_n tends pointwise to the zero function; it is also uniformly $1/q$-Hölderian, where $q = \frac{p}{p-1}$, indeed if $L = \sup_i \sup_{p\in\mathbb{R}^n} \|X_i(p)\|$ we have

$$\begin{aligned}\|h_n(t_1) - h_n(t_2)\| &\leq L \int_{t_1}^{t_2} \sum_{i=1}^{d} (|u_{n,i}(\tau)| + |u_i(\tau)|) d\tau \\ &\leq L(M + \|u\|_p)|t_1 - t_2|^{1/q}.\end{aligned}$$

The proof is then concluded by the next lemma. □

Lemma 20 (Uniform convergence of Hölderian maps). *Let $\{f_k\}_{k\in\mathbb{N}} : [a, b] \to \mathbb{R}^n$ be a uniformly α-Hölderian sequence of functions which converges pointwise to a limit function f. Then f is α-Hölderian and $f_k \to f$ uniformly on $[a, b]$.*

Proof. The relation $\|f_k(x) - f_k(y)\| \leq M|x - y|^\alpha$ immediately yields that the limit function f is also α-Hölderian.

Next, let $\varepsilon > 0$ be arbitrary and let accordingly $\rho = \left(\frac{\varepsilon}{3M}\right)^{1/\alpha}$. As $[a, b]$ is compact, it can be covered by a finite collection $\{B_i\}_{i=1}^l$ of balls of radius ρ, whose centers will be denoted by x_i; this means that for any $x \in [a, b]$ there exists $i \in \{1, \ldots, l\}$ such that $|x - x_i| \leq \rho$. Let $K \in \mathbb{N}$ be such that $\|f_k(x_i) - f(x_i)\| \leq \varepsilon/3$ for all $i = 1, \ldots, l$ if $k > K$. The following holds true for $k \in \mathbb{N}$ sufficiently large:

$$\begin{aligned}\|f_k(x) - f(x)\| &\leq \|f_k(x) - f_k(x_i)\| + \|f_k(x_i) - f(x_i)\| + \|f(x_i) - f(x)\| \\ &\leq 2M|x - x_i|^\alpha + \frac{\varepsilon}{3} \leq \varepsilon,\end{aligned}$$

and this finishes the proof. □

We turn now to the issue of the differentiability of the endpoint map F, i.e. we want to determine its Fréchet differential and prove some of its continuity properties.

Proposition 21. *Let u be in the domain of the endpoint map*

$$F : L^p([0, T], \mathbb{R}^d)$$

and let γ_u be the associated trajectory. Then for any bounded neighborhood U of u in $L^p([0, T], \mathbb{R}^d)$, there exists a constant $C = C(U)$ such that whenever

$v, w \in U$ and $t \in [0,T]$ we have

$$\|\gamma_v(t) - \gamma_w(t)\| \leq C\|v - w\|_p.$$

Proof. Using (7) we derive the following estimate

$$\begin{aligned}\|\gamma_v(t) - \gamma_w(t)\| \leq & \sum_{i=1}^{d} \int_0^t |v_i - w_i| \|X_i(\gamma_v(s))\| ds \\ & + \int_0^t \|X_0(\gamma_v(s)) - X_0(\gamma_w(s))\| ds \\ & + \sum_{i=1}^{d} \int_0^t |w_i| \|X_i(\gamma_v(s)) - X_i(\gamma_w(s))\| ds.\end{aligned} \tag{12}$$

Theorem 19 ensures that γ_v and γ_w take values in a compact K which depends just on U[4]; as $X_0, X_1, \dots, X_d$ are smooth, we have the existence of a constant M such that for all $v, w \in U$ and for all $i \in 1, \dots, d$ there holds

$$\begin{aligned}\|X_i(\gamma_v)\| &\leq M, \\ \|X_i(\gamma_v) - X_i(\gamma_w)\| &\leq M\|\gamma_v - \gamma_w\|, \quad \forall t \in [0,T];\end{aligned}$$

lastly we may assume that U is contained in a ball of radius R, that is $\|w\|_p \leq R$ for all $w \in U$. We proceed with the estimate in (12) as

$$\begin{aligned}\|\gamma_v(t) - \gamma_w(t)\| \leq & B\|v - w\|_p \\ & + M \int_0^t \left(1 + \sum_{i=1}^{d} |w_i|\right) \|\gamma_v(s) - \gamma_w(s)\| ds, \quad \forall t \in [0,T],\end{aligned}$$

where $B = MT^{1/q}$; finally, Gronwall inequality yields

$$\|\gamma_v(t) - \gamma_w(t)\| \leq B e^{M(T + RT^{1/q})} \|v - w\|_p, \quad \forall t \in [0,T].$$

□

[4]By Banach-Alaoglu U is sequentially weakly compact, hence weakly compact by the Eberlein-Smulian theorem. On the other hand theorem 19 implies that for any $\varepsilon > 0$, whenever u, v belong to a sufficiently small open set, $\|\gamma_u(t) - \gamma_v(t)\| \leq \varepsilon$ on $[0,T]$. The statement follows since whenever we cover U with a collection of open sets of arbitrary small size, we may always extract a finite subcover and then proceed via the triangular inequality.

We fix now some notations used in the next theorem: let $A_u(t) = dX_0(\gamma_u) + \sum_{i=1}^d u_i dX_i(\gamma_u)$, $B_u(t) = (X_1(\gamma_u), \dots, X_d(\gamma_u))$, and let M_u be the $n \times n$ matrix solution of $M_u' = A_u M_u$ satisfying $M_u(0) = I$; we have

Theorem 22 (Differentiability of the endpoint map). *The endpoint map F is L^p-Fréchet differentiable; its differential at u is the linear map $dF(u) : L^p \to \mathbb{R}^n$ defined by*

$$(d_u F)v = \int_0^T M_u(T) M_u(s)^{-1} B_u(s) v(s) ds.$$

Proof. Let $u \in L^p([0,T], \mathbb{R}^d)$ be fixed in the domain of F. Let us consider a neighborhood U of u in L^p; without loss of generality we may assume that there exists $R > 0$ such that $\|v\|_p \le R$ for any $v \in U$. Let γ_u and γ_{u+v} be the solutions to (7) with respect to the controls u and $u+v$ respectively. We have

$$\begin{aligned} \dot\gamma_{u+v} - \dot\gamma_u = X_0(\gamma_{u+v}) - X_0(\gamma_u) + \sum_{i=1}^d v_i X_i(\gamma_{u+v}) \\ + \sum_{i=1}^d u_i (X_i(\gamma_{u+v}) - X_i(\gamma_u)). \end{aligned} \tag{13}$$

For all $i = 0, 1, \dots, d$ there hold the expansions

$$\begin{aligned} X_i(\gamma_{u+v}) - X_i(\gamma_u) &= dX_i(\gamma_u)(\gamma_{u+v} - \gamma_u) \\ &\quad + \int_0^1 (1-t) d^2 X_i(t\gamma_u + (1-t)\gamma_{u+v})(\gamma_{u+v} - \gamma_u, \gamma_{u+v} - \gamma_u) dt, \\ X_i(\gamma_{u+v}) &= X_i(\gamma_u) + \int_0^1 (1-t) dX_i(t\gamma_u + (1-t)\gamma_{u+v})(\gamma_{u+v} - \gamma_u) dt; \end{aligned}$$

plug the above into (13) to rewrite that equation as

$$\dot\omega = A_u \omega + B_u v + \xi, \tag{14}$$

where $\omega(t) = \gamma_{u+v}(t) - \gamma_u(t)$ and

$$\begin{aligned}
\xi(t) = &\sum_{i=1}^{d} v_i(t) \int_0^1 (1-s)dX_i(s\gamma_u + (1-s)\gamma_{u+v})(\gamma_{u+v} - \gamma_u)ds \\
&+ \int_0^1 (1-s)d^2X_0(s\gamma_u + (1-s)\gamma_{u+v})(\gamma_{u+v} - \gamma_u, \gamma_{u+v} - \gamma_u)ds \\
&+ \sum_{i=1}^{d} u_i(t) \int_0^1 (1-s)d^2X_i(s\gamma_u + (1-s)\gamma_{u+v})(\gamma_{u+v} - \gamma_u, \gamma_{u+v} - \gamma_u)ds.
\end{aligned}$$

We have $\|v\|_p \le R$ for all $v \in U$; the previous proposition and the estimate

$$\begin{aligned}
\|s\gamma_u(s) + (1-s)\gamma_{u+v}(s)\| &\le \|\gamma_u(s)\| + (1-s)\|\gamma_{u+v}(s) - \gamma_u(s)\| \\
&\le \|\gamma_u(s)\| + CR
\end{aligned}$$

imply that there exists a compact $K \subset \mathbb{R}^n$ such that $s\gamma_u(s)+(1-s)\gamma_{u+v}(s) \in K$ for any $s \in [0,1]$ and any $v \in U$. Since the X_i are smooth, again by the proposition above we have we the estimate

$$\|\xi(t)\| \le c_1\|v\|_p \sum_{i=1}^{d} |v_i(t)| + c_2\|v\|_p^2 \left(1 + \sum_{i=1}^{d} |u_i(t)|\right).$$

We solve (14) to obtain

$$\omega(t) = \int_0^t M_u(t)M_u(s)^{-1}B_u(s)v(s)ds + \int_0^t M_u(t)M_u(s)^{-1}\xi(s)ds;$$

in particular for $t = T$

$$\begin{aligned}
(15) \qquad &\left\|\gamma_{u+v}(T) - \gamma_u(T) - \int_0^T M_u(T)M_u(s)^{-1}(s)B_u(s)v(s)ds\right\| \\
&\le C'\left(c_1\|v\|_p \int_0^T \sum_{i=1}^{d} |v_i(s)|ds + c_2\|v\|_p^2 \int_0^T (1 + \sum_{i=1}^{d} |u_i(s)|)ds\right) \\
&\le C'\left(c_1T^{1/q} + c_2(T + \|u\|_pT^{1/q})\right)\|v\|_p^2.
\end{aligned}$$

The map

$$\mathcal{F}_u : L^p \ni v \mapsto \int_0^T M_u(T)M_u(s)^{-1}B_u(s)v(s)ds \in \mathbb{R}^n$$

is evidently linear and by (15) also continuous. It then follows that the endpoint map F is differentiable at u and $d_u F u = \mathcal{F}_u$. □

Theorem 23. *Let $u = (u_1, \dots, u_d) \in L^p([0,T], \mathbb{R}^d)$ be a control in the domain of the endpoint map F. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^p([0,T], \mathbb{R}^d)$ such that $u_n \overset{L^p}{\rightharpoonup} u$ for some $u \in L^p([0,T], \mathbb{R}^d)$. Then $d_{u_n}F \to d_u F$.*

The proof of this theorem needs a series of preliminary lemmas; for $s \in [0,T]$, set $N_u(s) = M_u(T)M_u(s)^{-1}$. Since $N_u(s)M_u(s) = M_u(T)$, upon differentiation and using the definition of M_u, we obtain $N_u'(s)M_u(s) + N_u(s)A_u(s)M_u(s) = 0$, that is

$$N_u'(s) = -N_u(s)A_u(s), \quad N_u(T) = I.$$

Lemma 24. *Let $\{u_n\}_{n\in\mathbb{N}}$ and u be as in the statement of theorem 23. Then $N_{u_n} \to N_u$ uniformly on $[0,T]$.*

Proof.

$$\begin{aligned}
(16)\qquad & N_u(t) - N_{u_n}(t) \\
&= \int_0^t \Bigg(N_{u_n}(s)(dX_0(\gamma_{u_n}(s)) + \sum_{i=1}^d u_{n,i} dX_i(\gamma_{u_n}(s))) \\
&\qquad - N_u(s)(dX_0(\gamma_u(s)) + \sum_{i=1}^d u_i(s) dX_i(\gamma_u(s))) \Bigg) ds \\
&= \int_0^t \Bigg((N_{u_n}(s) - N_u(s)) dX_0(\gamma_u(s)) + N_u(s)(dX_0(\gamma_{u_n}(s)) \\
&\qquad - dX_0(\gamma_u(s))) + (N_{u_n}(s) - N_u(s)) \sum_{i=1}^d u_{n,i}(s) dX_i(\gamma_{u_n}(s)) \\
&\qquad + N_u(s) \sum_{i=1}^d u_{n,i}(s)(dX_i(\gamma_{u_n}(s)) - dX_i(\gamma_u(s))) \\
&\qquad + N_u(s) \sum_{i=1}^d (u_{n,i}(s) - u_i(s)) dX_i(\gamma_u(s)) \Bigg) ds.
\end{aligned}$$

By virtue of theorem 19, $\gamma_{u_n} \to \gamma_u$ uniformly on $[0,T]$; moreover if

$$h_n(t) = \int_0^1 N_u(s) \sum_{i=1}^d (u_{n,i}(s) - u_i(s)) dX_i(\gamma_u(s)) ds,$$

then $\|h_n\| \to 0$ uniformly on $[0,T]$ by lemma 20: indeed the sequence $\{h_n\}_{n\in\mathbb{N}}$ is $1/q$- Hölderian and converges pointwise to 0, moreover the factor $N_u(s)$ does not depend on n. Then (16) can be estimated for n sufficiently large as

$$\|N_u(t) - N_{u_n}(t)\| \leq C \int_0^t \|N_u(s) - N_{u_n}(s)\| ds + \varepsilon,$$

and the theorem follows using the Gronwall inequality, as desired. □

Proof of theorem 23. Theorem 22 yields that the differential of the endpoint map at the point w has the form

$$(d_w F)v = \int_0^T N_w(s) B_w(s) v(s) ds.$$

We know from theorem 19 that $\gamma_{u_n} \to \gamma_u$ uniformly on $[0,T]$; then $B_{u_n} \to B_u$ uniformly on $[0,T]$. As lemma 24 shows that also $N_{u_n} \to N_u$ uniformly on $[0,T]$, we deduce that

$$d_{u_n} F v \to d_u F v$$

uniformly on $[0,T]$, for any $v \in L^p([0,T], \mathbb{R}^d)$, and this finishes the proof. □

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