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The diffeomorphism type of manifolds with almost maximal volume

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The smallest r so that a metric r–ball covers a metric space M is called the radius of M. The volume of a metric r-ball in the space form of constant curvature k is an upper bound for the volume of any Riemannian manifold with sectional curvature $\geq k$ and radius $\leq r$. We show that when such a manifold has volume almost equal to this upper bound, it is diffeomorphic to a sphere or a real projective space.

1. Introduction

Any closed Riemannian n–manifold M has a lower bound, $k \in \mathbb{R}$, for its sectional curvature. This gives an upper bound for the volume of any metric ball $B(x,r) \subset M$,

$$\operatorname{vol} B(x,r) \leq \operatorname{vol} \mathcal{D}^n_k(r),$$

where $\mathcal{D}^n_k(r)$ is an r–ball in the n–dimensional, simply connected space form of constant curvature k. If rad M is the smallest number r such that a metric r–ball covers M, it follows that

$$\operatorname{vol} M \leq \operatorname{vol} \mathcal{D}^n_k(\operatorname{rad} M). \tag{1.0.1}$$

The invariant rad M is known as the *radius* of M and can alternatively be defined as

$$\operatorname{rad} M = \min_{p\in M} \max_{x \in M} \operatorname{dist}(p,x).$$

In the event that $\operatorname{vol} M$ is almost equal to $\operatorname{vol} \mathcal{D}^n_k(\operatorname{rad} M)$, we determine the diffeomorphism type of M.

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Main Theorem. *Given $n \in \mathbb{N}, k \in \mathbb{R}$, and $r > 0$, there is an $\varepsilon > 0$ so that every closed Riemannian n–manifold M with*

$$\begin{aligned} &\sec M \geq k, \\ &\text{rad } M \leq r, \text{ and} \\ &\text{vol } M \geq \text{vol } \mathcal{D}_k^n(r) - \varepsilon \end{aligned} \tag{1.0.2}$$

is diffeomorphic to S^n or $\mathbb{R}P^n$.

This generalizes Part 1 of Theorem A in [8], where Grove and Petersen classified these manifolds up to homeomorphism. They also showed that for any $\varepsilon > 0$ and $M = S^n$ or $\mathbb{R}P^n$, there are Riemannian metrics that satisfy (1.0.2), except when $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$. Thus Inequality (1.0.1) is optimal, except when $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$.

For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, Grove and Petersen also computed the optimal upper volume bound for the class of manifolds M with

$$\sec M \geq k \quad \text{and} \quad \text{rad } M \leq r. \tag{1.0.3}$$

It is strictly less than vol $\mathcal{D}_k^n(r)$, [8]. For $k > 0$ and $r \in \left(\frac{1}{2}\frac{\pi}{\sqrt{k}}, \frac{\pi}{\sqrt{k}}\right)$, manifolds satisfying (1.0.3) with almost maximal volume are already known to be diffeomorphic to spheres [10]. The main theorem in [17] gives the same result when $r = \frac{\pi}{\sqrt{k}}$.

For $k > 0$ and $r = \frac{\pi}{\sqrt{k}}$, the maximal volume vol $\mathcal{D}_1^n\left(\frac{\pi}{\sqrt{k}}\right)$ is realized by the n-sphere with constant curvature k. For $k > 0$ and $r = \frac{\pi}{2\sqrt{k}}$, the maximal volume vol $\mathcal{D}_1^n\left(\frac{\pi}{2\sqrt{k}}\right)$ is realized by $\mathbb{R}P^n$ with constant curvature k. Apart from these cases, there are no Riemannian manifolds M satisfying (1.0.3) and vol $M =$ vol $\mathcal{D}_k^n(r)$. Rather, the maximal volume is realized by one of the following two types of Alexandrov spaces [8].

Definition 1.1. (Purse) Let $R : \mathcal{D}_k^n(r) \to \mathcal{D}_k^n(r)$ be reflection in a totally geodesic hyperplane H through the center of $\mathcal{D}_k^n(r)$. The Purse, $P_{k,r}^n$, is the quotient space

$$\mathcal{D}_k^n(r) / \{v \sim R(v)\}, \text{ provided } v \in \partial \mathcal{D}_k^n(r).$$

Alternatively we let $\left\{\frac{1}{2}\mathcal{D}_k^n(r)\right\}^+ \cup \left\{\frac{1}{2}\mathcal{D}_k^n(r)\right\}^- = D_k^n(r)$ be the decomposition of $\mathcal{D}_k^n(r)$ into the two half disks on either side of H. Then $P_{k,r}^n$ is isometric to the double of $\left\{\frac{1}{2}\mathcal{D}_k^n(r)\right\}^+$. In particular, $P_{k,r}^n$ is homeomorphic to S^n.

Definition 1.2. (Crosscap) The constant curvature k Crosscap, $C_{k,r}^n$, is the quotient of $\mathcal{D}_k^n(r)$ obtained by identifying antipodal points on the boundary. Thus $C_{k,r}^n$ is homeomorphic to $\mathbb{R}P^n$. There is a canonical metric on $C_{k,r}^n$ that makes this quotient map a submetry. The universal cover of $C_{k,r}^n$ is the double of $\mathcal{D}_k^n(r)$. If we write this double as $\boldsymbol{DD}_k^n(r) \equiv \mathcal{D}_k^n(r)^+ \cup_{\partial\mathcal{D}_k^n(r)^\pm} \mathcal{D}_k^n(r)^-$, then the free involution

$$A : \boldsymbol{DD}_k^n(r) \longrightarrow \boldsymbol{DD}_k^n(r)$$

that gives the covering map $\boldsymbol{DD}_k^n(r) \longrightarrow C_{k,r}^n$ is

$$A : (x,+) \longmapsto (-x,-),$$

where the sign in the second entry indicates whether the point is in $\mathcal{D}_k^n(r)^+$ or $\mathcal{D}_k^n(r)^-$.

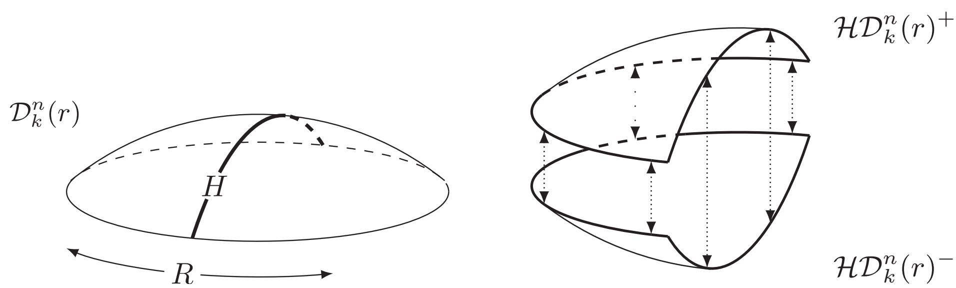

Figure 1: Two equivalent constructions of $P_{1,r}^2$

Let $\{M_i\}_{i=1}^\infty$ be a sequence of closed n-manifolds with sec $M \geq k$, rad $M \leq r$, and $\{\text{vol} M_i\}$ converging to vol $\mathcal{D}_k^n(r)$, where $r \leq \frac{\pi}{2\sqrt{k}}$ if $k > 0$. Grove and Petersen showed that $\{M_i\}$ has a subsequence that converges to either the crosscap, $C_{k,r}^n$, or the purse, $P_{k,r}^n$, in the Gromov-Hausdorff topology [8]. Our main theorem follows by combining this with the following *diffeomorphism stability theorems.*

Theorem 1.3. *Let* $\{M_\alpha\}_{\alpha=1}^{\infty}$ *be a sequence of closed Riemannian* n*–manifolds with* $\sec M_\alpha \geq k$ *so that*

$$M_\alpha \longrightarrow P^n_{k,r}$$

in the Gromov-Hausdorff topology. Then all but finitely many of the M_α*s are diffeomorphic to* S^n*.*

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in the Gromov-Hausdorff topology. Then all but finitely many of the M_α*s are diffeomorphic to* $\mathbb{R}P^n$*.*

Theorem 1.4 follows directly from Theorem 6.1 in [15], as all points in $C^n_{k,r}$ are $(n, 0)$–strained. So this paper is devoted to the proof of Theorem 1.3. Our proof of Theorem 1.3 is related to an alternative proof of Theorem 1.4 which was included in the original version of this paper and is also in [21].

Remark 1.5. One can get Theorem 1.3 for the case $k = 1$ and $r > \operatorname{arccot}\left(\frac{1}{\sqrt{n-3}}\right)$ as a corollary of Theorem C in [11]. Theorem 1.4 when $k = 1$ and $r = \frac{\pi}{2}$ follows from the main theorem in [25] and the fact that $C^n_{1,\frac{\pi}{2}}$ is $\mathbb{R}P^n$ with constant curvature 1.

In [8], Grove and Petersen proved the topological stability theorems that are analogous to Theorems 1.3 and 1.4. Perelman has since proved a much more general Topological Stability Theorem, which in particular implies the following.

Topological Stability Theorem: *Let* $\{M_\alpha\}_\alpha$ *be a sequence of closed Riemannian* n*–manifolds with sectional curvature* $\geq k$*. If the Gromov-Hausdorff limit of* $\{M_\alpha\}_\alpha$ *is* X *and* $\dim(X) = n$*, then all but finitely many of the* M_α*'s are homeomorphic to* X, [12, 19].

In a similar way, Theorems 1.3 and 1.4 would follow from an affirmative answer to the following open question.

Diffeomorphism Stability Question: *Let* $\{M_\alpha\}_\alpha$ *be a sequence of closed Riemannian* n*–manifolds with sectional curvature* $\geq k$*. If the Gromov-*

Hausdorff limit of $\{M_\alpha\}_\alpha$ *is* X *and* $\dim(X) = n$, *then are all but finitely many of the* M_α*'s diffeomorphic to each other* [11]*?*

An affirmative answer to the Diffeomorphism Stability Question would also provide generalizations of Cheeger's Finiteness Theorem and the Diameter Sphere Theorem [3], [4], [9], [11].

Definition 1.6. Let $\mathcal{M}_k(n)$ be the class of closed Riemannian n–manifolds with sectional curvature $\geq k$. A compact, n–dimensional $X \in \text{closure}(\mathcal{M}_k(n))$ is called *diffeomorphically stable* if for any sequence $\{M_\alpha\}_{\alpha=1}^{\infty} \subset \mathcal{M}_k(n)$ with $M_\alpha \longrightarrow X$, in the Gromov–Hausdorff topology, all but finitely many of the M_αs are diffeomorphic to each other.

Together, Theorem 1.3 and Corollary E of [10] say that purses and the so-called "lemons" of [8] are diffeomorphically stable. These are the only known diffeomorphically stable limit spaces having a space of directions that is Gromov–Hausdorff far from the unit sphere.

The proof of Theorem 1.3 starts with the simple observation that the purse, $P^n_{k,r}$, can be topologically identified with the disjoint union of $D^{n-1} \times S^1$ and $S^{n-2} \times D^2$ glued together via the identity map of their common boundary $S^{n-2} \times S^1$. (See figure 2) Using this, we show that if $\{M_\alpha\}_\alpha$ is as in Theorem 1.3, then for α sufficiently large, M_α is diffeomorphic to the disjoint union of $D^{n-1} \times S^1$ and $S^{n-2} \times D^2$ glued together via a diffeomorphism f of $S^{n-2} \times S^1$. That is, M_α is diffeomorphic to

$$D^{n-1} \times S^1 \cup_f S^{n-2} \times D^2. \tag{1.6.1}$$

We show, moreover, that the diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ satisfies

$$p_{n-2} \circ f = p_{n-2},$$

where

$$p_{n-2} : S^{n-2} \times S^1 \longrightarrow S^{n-2}$$

is projection to the first factor.

Notice that a diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ so that $p_{n-2} \circ f = p_{n-2}$ gives rise to an element of $\pi_{n-2}\left(\text{Diff}_+\left(S^1\right)\right)$. If two such diffeomorphisms give the same homotopy class, then the construction (1.6.1) yields diffeomorphic manifolds (cf. [11]). Since the group of orientation preserving diffeomorphisms of the circle deformation retracts to $SO(2)$, it follows that M_α is diffeomorphic to S^n for all α sufficiently large.

To construct the decomposition (1.6.1) we start with the observation that the singularities of $P^n_{k,r}$ occur along a constant curvature sphere of codimension 2, that we call $\mathcal{S}^{n-2}$. The construction of $P^n_{k,r}$ also allows us to view $\mathcal{S}^{n-2}$ as the boundary of $\mathcal{D}^{n-1}_k(r)$. As in [17], we then write coordinate functions f_i of $\mathcal{D}^{n-1}_k(r)$ in terms of distance functions from points of $\mathcal{S}^{n-2}$. The formulas for these coordinate functions also make sense on $P^n_{k,r}$ and, as in [17], restrict to an isometric embedding of $\mathcal{S}^{n-2}$ into $\mathbb{R}^{n-1}$. Since the f_i's are written in terms of distance functions, they have lifts, f^α_i, to the M_α's. Using these lifts, we define

$$\Psi^\alpha : M_\alpha \longrightarrow \mathbb{R}^{n-1}$$
$$\Psi^\alpha = \left(f^\alpha_1, f^\alpha_2, \ldots, f^\alpha_{n-1}\right).$$

We then show that the restriction of Ψ^α to a subset $E^\alpha_D \subset M_\alpha$ is a trivial S^1–bundle over D^{n-1}, and the restriction of Ψ^α to $M \setminus \mathrm{int}\,(E^\alpha_D)$ is a trivial D^2–bundle over S^{n-2}. In other words, $E^\alpha_D \cong D^{n-1} \times S^1$ and $M \setminus \mathrm{int}\,(E^\alpha_D) \cong S^{n-2} \times D^2$, as in (1.6.1).

Remark 1.7. It was shown in [11] that the presence of a decomposition of the form $M_\alpha = D^{n-1} \times S^1 \cup_f S^{n-2} \times D^2$ with $p_{n-2} \circ f = p_{n-2}$ has an equivalent formulation in terms of the Gromoll Filtration of the group of exotic n–spheres. We review the details of this alternative formulation at the beginning of Section 4.

Section 2 introduces notations and conventions. Section 3 is a review of necessary tools from Alexandrov geometry, and Theorem 1.3 is proven in Section 4.

Throughout the remainder of the paper, we assume, without loss of generality, by rescaling if necessary, that $k = -1, 0$ or 1.

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2. Conventions and notations

Recall that an Alexandrov space is a complete, locally compact, intrinsic metric space with a lower curvature bound in the triangle comparison sense. We will assume a basic familiarity with Alexandrov spaces, including, but not limited to, [1]. We list here several conventions that will be used freely throughout.

Let X be an n–dimensional Alexandrov space and $x, p, y \in X$. We call minimal geodesics in X *segments* and denote by px a segment in X with endpoints p and x. We let Σ_p and T_pX denote the space of directions and tangent cone at p, respectively. For a geodesic direction $v \in T_pX$, we let γ_v be the segment whose initial direction is v. Following [20], we let $\Uparrow_x^p \subset \Sigma_x$ denote the set of directions of segments from x to p, and we let $\uparrow_x^p \in \Uparrow_x^p$ be the direction of a single segment from x to p. We let $\sphericalangle(x, p, y)$ denote the angle of a hinge formed by px and py and $\tilde{\sphericalangle}(x, p, y)$ denote the corresponding comparison angle.

Following [17], we let $\tau : \mathbb{R}^k \to \mathbb{R}_+$ be any function that satisfies

$$\lim_{x_1, \ldots, x_k \to 0} \tau\left(x_1, \ldots, x_k\right) = 0,$$

and abusing notation, we let $\tau : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$ be any function that satisfies

$$\lim_{x_1, \ldots, x_k \to 0} \tau\left(x_1, \ldots, x_k | y_1, \ldots, y_n\right) = 0,$$

provided $y_1, \ldots, y_n$ remain fixed. When making an estimate with a function τ, we implicitly assert the existence of such a function for which the estimate holds.

For $p \in X$ and $r > 0$, we set

$$B\left(p, r\right) \equiv \left\{x \in X \mid \operatorname{dist}\left(x, p\right) < r\right\}.$$

3. Basic tools from alexandrov geometry

Strainers, as defined in [1], form the core of the calculus arguments used to prove our main theorem. To motivate them let $\{v_i\}_{i=1}^n$ be an orthonormal basis for $\mathbb{R}^n$. Notice that the gradients of the distance functions from the v_is are orthonormal at 0 and almost orthonormal in a neighborhood N of 0. Thus the map $f : N \to \mathbb{R}^n$,

$$f(x) = \left(\operatorname{dist}\left(v_1, x\right), \operatorname{dist}\left(v_2, x\right), \ldots, \operatorname{dist}\left(v_n, x\right)\right)$$

is a bi-Lipschitz embedding with Lipschitz constants that converge to 1 as N gets smaller.

By exponentiating an orthonormal basis, it is easy to re-create these data around a point in a Riemannian manifold. This plus the fact that comparison angles are continuous leads us to the definition of strainers.

Definition 3.1. Let X be an Alexandrov space. A point $x \in X$ is said to be (n, δ, r)–strained by the strainer $\{(a_i, b_i)\}_{i=1}^n \subset X \times X$ provided that for all $i \neq j$ we have

$$\begin{gathered} \widetilde{\sphericalangle}\,(a_i, x, b_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}\,(a_i, x, b_i) > \pi - \delta, \\ \widetilde{\sphericalangle}\,(a_i, x, a_j) > \tfrac{\pi}{2} - \delta, \quad \widetilde{\sphericalangle}\,(b_i, x, b_j) > \tfrac{\pi}{2} - \delta, \text{ and} \\ \min_{i=1,\ldots,n} \{\operatorname{dist}(\{a_i, b_i\}, x)\} > r. \end{gathered}$$

We say $B \subset X$ is (n, δ, r)–strained with strainer $\{(a_i, b_i)\}_{i=1}^n$ provided every point $x \in B$ is (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$.

The following is observed in [26].

Proposition 3.2. *Let X be a compact n-dimensional Alexandrov space. Then the following are equivalent:*
1. There is a (sufficiently small) $\eta > 0$ so that for every $p \in X$,

$$\operatorname{dist}_{G-H}\left(\Sigma_p, S^{n-1}\right) < \eta.$$

2. There is a (sufficiently small) $\delta > 0$ and an $r > 0$ such that X is covered by finitely many (n, δ, r)–strained neighborhoods.

Theorem 3.3. *([1] Theorem 9.4) Let X be an n–dimensional Alexandrov space with curvature bounded from below. Let $p \in X$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Provided δ is small enough, there is a $\rho > 0$ such that the map $f : B(p, \rho) \to \mathbb{R}^n$ defined by*

$$f(x) = (\operatorname{dist}(a_1, x), \operatorname{dist}(a_2, x), \ldots, \operatorname{dist}(a_n, x))$$

is a bi-Lipschitz embedding with Lipschitz constants in $(1 - \tau(\delta, \rho), 1 + \tau(\delta, \rho))$.

If B is (n, δ, r)–strained by $\{a_i, b_i\}_{i=1}^n$, any choice of $2n$–directions, $\left\{\left(\uparrow_x^{a_i}, \uparrow_x^{b_i}\right)\right\}_{i=1}^n$, where $x \in B$, will be called a set of straining directions for Σ_x. As in, [1] and [26], we say an Alexandrov space Σ with $\operatorname{curv} \Sigma \geq 1$ is globally (m, δ)-strained by pairs of subsets $\{A_i, B_i\}_{i=1}^m$ provided

$$\begin{gathered} |\operatorname{dist}(a_i, b_j) - \tfrac{\pi}{2}| < \delta, \quad \operatorname{dist}(a_i, b_i) > \pi - \delta, \\ |\operatorname{dist}(a_i, a_j) - \tfrac{\pi}{2}| < \delta, \quad |\operatorname{dist}(b_i, b_j) - \tfrac{\pi}{2}| < \delta \end{gathered}$$

for all $a_i \in A_i$, $b_i \in B_i$ and $i \neq j$.

Theorem 3.4. *([1], Theorem 9.5, cf. also [17], Section 3) Let* Σ *be an* $(n-1)$*–dimensional Alexandrov space with curvature* ≥ 1. *Suppose* Σ *is globally strained by* $\{A_i, B_i\}$. *There is a map* $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ *so that* $\Psi : \Sigma \to S^{n-1}$ *defined by*

$$\Psi(x) = \tilde{\Psi} \circ (\operatorname{dist}(A_1, x), \operatorname{dist}(A_2, x), \ldots, \operatorname{dist}(A_n, x))$$

is a bi-Lipschitz homeomorphism with Lipschitz constants in $(1-\tau(\delta), 1+\tau(\delta))$.

Remark 3.5. The description of $\tilde{\Psi} : \mathbb{R}^n \longrightarrow S^{n-1}$ in [1] is explicit but is geometric rather than via a formula. Combining the proof in [1] with a limiting argument, one can see that the map Ψ can be given by

$$\Psi(x) = \left(\sum \cos^2(\operatorname{dist}(A_i, x))\right)^{-1/2} \times (\cos(\operatorname{dist}(A_1, x)), \ldots, \cos(\operatorname{dist}(A_n, x))).$$

Next we state a powerful lemma showing that for a $(1, \delta, r)$–strained neighborhood, angle and comparison angle almost coincide for geodesic hinges with one side in the neighborhood and the other reaching a strainer.

Lemma 3.6. *([1], Lemma* 5.6*) Let* $B \subset X$ *be* $(1, \delta, r)$*–strained by* (y_1, y_2). *For any* $x, z \in B$,

$$|\tilde{\sphericalangle}(y_1, x, z) + \tilde{\sphericalangle}(y_2, x, z) - \pi| < \tau(\delta, \operatorname{dist}(x, z)\,|r).$$

In particular, for $i = 1, 2$,

$$|\sphericalangle(y_i, x, z) - \tilde{\sphericalangle}(y_i, x, z)| < \tau(\delta, \operatorname{dist}(x, z)\,|r).$$

Corollary 3.7. *Let* $B \subset X$ *be* $(1, \delta, r)$*–strained by* (a, b). *Let* $\{X^\alpha\}_{\alpha=1}^\infty$ *be a sequence of Alexandrov spaces with* $\operatorname{curv} X^\alpha \geq k$ *such that* $X^\alpha \longrightarrow X$. *For* $x, z \in B$, *suppose that* $a^\alpha, b^\alpha, x^\alpha, z^\alpha \in X^\alpha$ *converge to* a, b, x, *and* z, *respectively. Then*

$$|\sphericalangle(a^\alpha, x^\alpha, z^\alpha) - \sphericalangle(a, x, z)| < \tau\left(\delta, \frac{1}{\alpha}, \operatorname{dist}(x, z)\ \Big|\ r\right).$$

Proof. The convergence $X^\alpha \longrightarrow X$ implies that

$$\left|\tilde{\sphericalangle}\left(a^\alpha, x^\alpha, z^\alpha\right) - \tilde{\sphericalangle}\left(a, x, z\right)\right| < \tau\left(\frac{1}{\alpha} \mid \operatorname{dist}\left(x, z\right), r\right).$$

Combined with the previous lemma,

$$\begin{aligned}
&\left|\sphericalangle\left(a^\alpha, x^\alpha, z^\alpha\right) - \sphericalangle\left(a, x, z\right)\right| \\
&\leq \left|\sphericalangle\left(a^\alpha, x^\alpha, z^\alpha\right) - \tilde{\sphericalangle}\left(a^\alpha, x^\alpha, z^\alpha\right)\right| \\
&\quad + \left|\tilde{\sphericalangle}\left(a^\alpha, x^\alpha, z^\alpha\right) - \tilde{\sphericalangle}\left(a, x, z\right)\right| + \left|\tilde{\sphericalangle}\left(a, x, z\right) - \sphericalangle\left(a, x, z\right)\right| \\
&\leq 2\tau\left(\delta, \frac{1}{\alpha}, \operatorname{dist}\left(x, z\right) | r\right) + \tau\left(\frac{1}{\alpha} \mid \operatorname{dist}\left(x, z\right), r\right) \\
&= \tau\left(\delta, \frac{1}{\alpha}, \operatorname{dist}\left(x, z\right) \mid r\right).
\end{aligned}$$

□

Lemma 3.8. *Let $B \subset X$ be (n, δ, r)–strained by $\{(a_i, b_i)\}_{i=1}^n$. Let $\{X^\alpha\}_{\alpha=1}^\infty$ have* $\operatorname{curv} X^\alpha \geq k$, *and suppose that $X_\alpha \longrightarrow X$. Let $\{(\gamma_{1,\alpha}, \gamma_{2,\alpha})\}_{\alpha=1}^\infty$ be a sequence of geodesic hinges in the X^α that converge to a geodesic hinge (γ_1, γ_2) with vertex in B. Then*

$$\left|\sphericalangle\left(\gamma_{1,\alpha}'(0), \gamma_{2,\alpha}'(0)\right) - \sphericalangle\left(\gamma_1'(0), \gamma_2'(0)\right)\right| < \tau\left(\delta, 1/\alpha \mid l\left(\gamma_1\right), l\left(\gamma_2\right), r\right),$$

where $l(\gamma_i)$ is the length of γ_i.

Remark 3.9. Note that without the strainer, $\liminf_{\alpha\to\infty}\sphericalangle(\gamma_{1,\alpha}'(0), \gamma_{2,\alpha}'(0)) \geq \sphericalangle(\gamma_1'(0), \gamma_2'(0))$ [1, 7].

Proof. Apply the previous corollary with $x^\alpha = \gamma_{1,\alpha}(0)$, $z^\alpha = \gamma_{1,\alpha}(\varepsilon)$, $x^\alpha \to x$, and $z^\alpha \to z$ to conclude

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma_{1,\alpha}'(0)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma_1'(0))\right| < \tau\left(\delta, \frac{1}{\alpha}, \operatorname{dist}\left(x, z\right) \mid r\right).$$

Similar reasoning with $x^\alpha = \gamma_{2,\alpha}(0)$, $z^\alpha = \gamma_{2,\alpha}(\varepsilon)$, $x = \lim_{\alpha\to\infty} x^\alpha$, and $z = \lim_{\alpha\to\infty} z^\alpha$ gives

$$\left|\sphericalangle(\Uparrow_{x^\alpha}^{a_i^\alpha}, \gamma_{2,\alpha}'(0)) - \sphericalangle(\Uparrow_x^{a_i}, \gamma_2'(0))\right| < \tau\left(\delta, \frac{1}{\alpha}, \operatorname{dist}\left(x, z\right) \mid r\right).$$

Since $\operatorname{dist}(x, z)$ may be as small as we please, the result then follows from Theorem 3.4. □

4. Purse stability

We start this section with a review of Gromoll groups. We then state Theorem 4.2 and show that it implies Theorem 1.3. The bulk of this section is devoted to the proof of Theorem 4.2.

Recall that a twisted n–sphere, Σ^n, is a compact smooth manifold that admits a Morse function f with exactly two critical points. The gradient flow of f allows us to decompose Σ^n as the union of two n–disks. In [14], Kervaire and Milnor showed that the twisted n–spheres form a group Γ^n under connected sum. Gromoll showed that there is a filtration

$$\{e\} \subset \Gamma^n_{n-1} \subset \cdots \subset \Gamma^n_1 = \Gamma^n$$

by subgroups, which are now called Gromoll groups [5]. Rather than using the definition of the Γ^n_qs from [5], we use the equivalent notion from Theorem D in [11].

Definition 4.1. Let

$$f : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1} \times S^{n-q}$$

be a diffeomorphism that satisfies

$$p_{q-1} \circ f = p_{q-1}, \tag{4.1.1}$$

where

$$p_{q-1} : S^{q-1} \times S^{n-q} \longrightarrow S^{q-1}$$

is projection to the first factor. Then Γ^n_q consists of those smooth manifolds that are diffeomorphic to

$$D^q \times S^{n-q} \cup_f S^{q-1} \times D^{n-q+1}. \tag{4.1.2}$$

Theorem 4.2. *Let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed, Riemannian n–manifolds with*

$$\sec M^\alpha \geq k$$

so that

$$M_\alpha \longrightarrow P^n_{k,r}$$

in the Gromov-Hausdorff topology. Then for α sufficiently large, $M_\alpha \in \Gamma^n_{n-1}$.

It is known that Γ^n_{n-1} is trivial for all n. Given this fact, Theorem 1.3 implies Theorem 4.2. To see why Γ^n_{n-1} is trivial, we first point out that $\Gamma^n = \{e\}$ for $n = 1, 2, 3$, [16]. So we may assume that $n \geq 4$. Next notice that a diffeomorphism $f : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1$ so that $p_{n-2} \circ f = p_{n-2}$ gives rise to an element of $\pi_{n-2}\left(\mathrm{Diff}_+\left(S^1\right)\right)$. If two such diffeomorphisms give the same homotopy class, then the construction (4.1.2) yields diffeomorphic manifolds (cf. [11]). Since the group of orientation preserving diffeomorphisms of the circle deformation retracts to $SO(2)$, it follows that for $n \geq 4$, $\Gamma^n_{n-1} = \{e\}$, as desired.

4.1. The model submetry

View $P^n_{k,r}$ as the double of the half disk $\left\{\frac{1}{2}\mathcal{D}^n_k(r)\right\}^+$,

$$P^n_{k,r} \equiv \mathrm{Double}\left(\left\{\frac{1}{2}\mathcal{D}^n_k(r)\right\}^+\right),$$

and let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed, Riemannian n–manifolds with

$$\sec M^\alpha \geq k$$

and

$$\mathrm{dist}_{GH}\left(M_\alpha, P^n_{k,r}\right) < \frac{1}{\alpha}.$$

Our Model Submetry

$$\Psi : P^n_{k,r} \longrightarrow \mathbb{R}^{n-1} \tag{4.2.1}$$

is the restriction to either half disk of orthogonal projection to the totally geodesic hyperplane $H \subset \mathcal{D}^n_k(r)$ that defines $P^n_{k,r}$.

In this subsection, we describe the Model Submetry in terms of distance functions on the Purse. This will enable us, in the next subsection, to approximate Ψ by maps $\Psi_\alpha : M_a \longrightarrow \mathbb{R}^{n-1}$ that inherit much of the regularity of Ψ. The inherited regularity is established in the paper's final subsection in Corollary 4.11 and Lemma 4.15. It will allow us to decompose M_a as the union of a trivial D^2–bundle and a trivial circle bundle (see the circle and disk bundle lemmas, 4.3,4.4, below). The proof of Theorem 4.2 is completed by showing that M_α is the union of these two bundles glued together on their common boundary via a diffeomorphism that satisfies Equation (4.1.1).

To describe Ψ in terms of distance functions we use

$$H^n \equiv \Big\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \\ -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2 = -1,\ x_0 > 0 \Big\}$$

as our model for hyperbolic space. We write $\mathcal{S}_k^n$ for any of $H^n \subset \mathbb{R}^{n+1}$, $\{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1}$, or $S^n \subset \mathbb{R}^{n+1}$. We denote the standard basis for $\mathbb{R}^{n+1}$ by $\{e_0, e_1, \ldots, e_n\}$, and we identify $\mathcal{D}_k^n(r)$ with

$$\mathcal{D}_k^n(r) \equiv \begin{cases} \left\{ z \in H^n \subset \mathbb{R}^{n+1} \,\middle|\, \mathrm{dist}_{H^n}(e_0, z) \le r \right\} & \text{if } k = -1 \\ \left\{ z \in \{e_0\} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \,\middle|\, \mathrm{dist}_{\mathbb{R}^{n+1}}(e_0, z) \le r \right\} & \text{if } k = 0 \\ \left\{ z \in S^n \subset \mathbb{R}^{n+1} \,\middle|\, \mathrm{dist}_{S^n}(e_0, z) \le r \right\} & \text{if } k = 1. \end{cases}$$

Set

$$p_0 = e_0,$$

and for $i \in \{1, 2, \ldots, n-1\}$, set

$$p_i \equiv \begin{cases} \cosh(r) e_0 + \sinh(r) e_i & \text{if } k = -1 \\ e_0 + r e_i & \text{if } k = 0 \\ \cos(r) e_0 - \sin(r) e_i & \text{if } k = 1. \end{cases} \tag{4.2.2}$$

We let the totally geodesic hyperplane $H \subset \mathcal{D}_k^n(r)$ that defines $P_{k,r}^n$ be the one containing $p_0, p_1, \ldots, p_{n-1}$. We denote the singular subset of $P_{k,r}^n$ by $\mathcal{S}^{n-2}$, that is, $\mathcal{S}^{n-2}$ is the copy of the $(n-2)$-sphere which is the boundary of the $(n-1)$–disk $\mathcal{D}_k^n(r) \cap H$. Thus $\{p_i\}_{i=1}^{n-1} \subset \mathcal{S}^{n-2}$.

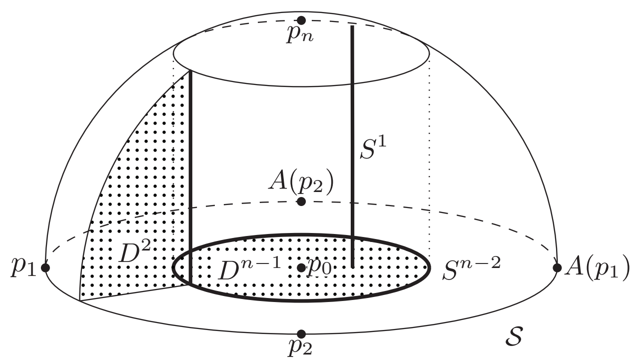

Figure 2: One side of $P_{k,r}^n$ for $n = 3$ and $k = 0$.

Since the antipodal map $A : \mathcal{D}^n_k(r) \longrightarrow \mathcal{D}^n_k(r)$ commutes with the reflection R in H, it induces a well-defined involution of $A_P : P^n_{k,r} \longrightarrow P^n_{k,r}$. Note that A_P restricts to the antipodal map of $\mathcal{S}^{n-2}$ and fixes the circle at maximal distance from $\mathcal{S}^{n-2}$. For simplified notation, we will write A for the restriction of A_P to $\mathcal{S}^{n-2}$.

For $i \in \{1, 2, \ldots, n-1\}$, set

$$f_i(x) \equiv h_k \circ \operatorname{dist}(A(p_i), x) - h_k \circ \operatorname{dist}(p_i, x)$$

where $h_k : \mathbb{R} \to \mathbb{R}$ is defined as

$$h_k(x) \equiv \begin{cases} \frac{1}{2\sinh r}\cosh(x) & \text{if } k = -1 \\ \frac{x^2}{4r} & \text{if } k = 0 \\ \frac{1}{2\sin r}\cos(x) & \text{if } k = 1. \end{cases}$$

The functions $\{f_i\}_{i=1}^{n-1}$ are then restrictions of $(n-1)$–coordinate functions of $\mathbb{R}^{n+1}$ to $\mathcal{D}^n_k(r) \subset \mathcal{S}^n_k$. In particular, the f_is are the coordinate functions of the Model Submetry $\Psi : P^n_{k,r} \longrightarrow \mathbb{R}^{n-1}$ from (4.2.1), that is,

$$\Psi = (f_1, f_2, \ldots, f_{n-1}).$$

It follows that $\Psi|_{\mathcal{S}^{n-2}}$ is the inclusion of $\mathcal{S}^{n-2}$ into $\mathcal{S}^n_k \subset \mathbb{R}^{n+1}$.

To construct our decompositions of the M_αs into $D^{n-1} \times S^1$ and $S^{n-2} \times D^2$, we next approximate the Model Submetry by maps $\Psi^\alpha_d : M^\alpha \longrightarrow \mathbb{R}^{n-1}$.

4.2. Approximating the model submetry

Let $\{M^\alpha\}_{\alpha=1}^\infty$ be a sequence of closed, Riemannian n–manifolds with

$$\sec M^\alpha \geq k$$

and

$$\operatorname{dist}_{GH}\left(M_\alpha, P^n_{k,r}\right) < \frac{1}{\alpha}.$$

Let $A : M^\alpha \longrightarrow M^\alpha$ denote any map that is Gromov-Hausdorff close to $A : P^n_{k,r} \longrightarrow P^n_{k,r}$.

We define approximations $f_{i,d}^{\alpha} : M^{\alpha} \longrightarrow \mathbb{R}$ of the f_is by

$$f_{i,d}^{\alpha}(x) = \frac{1}{\mathrm{vol}\left(B\left(A\left(p_i^{\alpha}\right), d\right)\right)} \int_{z \in B(A(p_i^{\alpha}), d)} h_k \circ \mathrm{dist}\,(z, x) - \frac{1}{\mathrm{vol}\left(B\left(p_i^{\alpha}, d\right)\right)} \int_{z \in B(p_i^{\alpha}, d)} h_k \circ \mathrm{dist}\,(z, x)\,.$$

We let $\Psi_d^{\alpha} : M^{\alpha} \longrightarrow \mathbb{R}^{n-1}$ be defined by

$$\Psi_d^{\alpha} = (f_{1,d}^{\alpha}, \ldots, f_{n-1,d}^{\alpha}).$$

4.3. The bundle decomposition

To prove Theorem 4.2 we decompose the M_{α}s as the union of a trivial circle bundle, $D^{n-1} \times S^1$, and a trivial disk bundle, $S^{n-2} \times D^2$, which we describe in this subsection.

We identify $\mathbb{R}^{n-1}$ with

$$\mathbb{R}^{n-1} \equiv \mathrm{span}\,\{e_1, \ldots, e_{n-1}\}\,.$$

For small $\varepsilon > 0$, we set

$$\begin{aligned} E_D\,(r-\varepsilon) &\equiv (\Psi)^{-1}\,(D^{n-1}(0, r-\varepsilon)), \\ E_D^{\alpha}\,(r-\varepsilon) &\equiv (\Psi_d^{\alpha})^{-1}\,(D^{n-1}(0, r-\varepsilon)), \\ E_A\,(r-\varepsilon) &\equiv (\Psi)^{-1}\,(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \text{ and} \\ E_A^{\alpha}\,(r-\varepsilon) &\equiv (\Psi_d^{\alpha})^{-1}\,(\overline{A^{n-1}(0, r-\varepsilon, 2r)}), \end{aligned}$$

where $\overline{A^{n-1}(0, r-\varepsilon, 2r)}$ is the closed annulus in $\mathbb{R}^{n-1}$ centered at 0 with inner radius $r-\varepsilon$ and outer radius $2r$, and $D^{n-1}(0, r-\varepsilon)$ is the closed ball in $\mathbb{R}^{n-1}$ centered at 0 with radius $r-\varepsilon$.

The next two Lemmas give us the desired bundle decomposition of the M_{α}s. Hence together they imply Theorem 4.2.

Circle Bundle Lemma 4.3. *For any sufficiently small $\varepsilon > 0$,*

$$\Psi_d^{\alpha} : E_D^{\alpha}\,(r-\varepsilon) \longrightarrow D^{n-1}(0, r-\varepsilon)$$

is a trivial S^1–bundle, provided α is sufficiently large and d is sufficiently small.

Let $\mathrm{pr} : \overline{A^{n-1}(0, r-\varepsilon, 2r)} \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$ be radial projection and set

$$g \equiv \mathrm{pr} \circ \Psi : E_A(r-\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$
$$g_d^\alpha \equiv \mathrm{pr} \circ \Psi_d^\alpha : E_A^\alpha(r-\varepsilon) \to \partial\left(D^{n-1}(0, r-\varepsilon)\right).$$

Disk Bundle Lemma 4.4. *There is an $\varepsilon > 0$ so that*

$$g_d^\alpha : E_A^\alpha(r-\varepsilon) \longrightarrow \partial\left(D^{n-1}(0, r-\varepsilon)\right)$$

is a trivial D^2–bundle over $\partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$, provided α is sufficiently large and d is sufficiently small.

Proof of Theorem 4.2 assuming the circle and disk bundle lemmas. To simplify notation, set $D^{n-1} = D^{n-1}(0, r-\varepsilon)$ and $\partial D^{n-1} = S^{n-2} = \partial D^{n-1}(0, r-\varepsilon)$. Let

$$\begin{array}{ccc} E_D^\alpha(r-\varepsilon) & \xrightarrow{\Phi_D} & D^{n-1} \times S^1 \\ & {\scriptstyle \Psi_d^\alpha} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & S^{n-2} & \end{array} \tag{4.4.1}$$

and

$$\begin{array}{ccc} E_A^\alpha(r-\varepsilon) & \xrightarrow{\Phi_A} & S^{n-2} \times D^2 \\ & {\scriptstyle g_d^\alpha} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & S^{n-2} & \end{array} \tag{4.4.2}$$

be trivializations of Ψ_d^α and g_d^α.

By the circle and disk bundle lemmas,

$$M_\alpha = E_D^\alpha(r-\varepsilon) \cup_{\Phi_A \circ \Phi_D|^{-1}_{\partial\left(D^{n-1}\times S^1\right)}} E_A^\alpha(r-\varepsilon)$$

with $E_D^\alpha(r-\varepsilon) \cong D^{n-1} \times S^1$, $E_A^\alpha(r-\varepsilon) \cong S^{n-2} \times D^2$, and $E_D^\alpha(r-\varepsilon) \cap E_A^\alpha(r-\varepsilon) \cong S^{n-2} \times S^1$. So we only need to verify that the gluing map satisfies

$$p_1 \circ \Phi_A \circ \Phi_D|^{-1}_{\partial(D^{n-1}\times S^1)} = p_1,$$

where $p_1 : S^{n-2} \times S^1 \longrightarrow S^{n-2}$ is projection onto the first factor.

Observe that

$$(4.4.3)\qquad \begin{aligned} g_d^\alpha|_{\partial(E_D^\alpha(r-\varepsilon))} &= \mathrm{pr}\circ \Psi_d^\alpha|_{\partial(E_D^\alpha(r-\varepsilon))}, \text{ by the definition of } g_d^\alpha \\ &= \Psi_d^\alpha|_{\partial(E_A^\alpha(r-\varepsilon))}, \end{aligned}$$

since pr is a retraction onto $\Psi_d^\alpha\left(\partial\left(E_A^\alpha\left(r-\varepsilon\right)\right)\right) = \partial D^{n-1}(0, r-\varepsilon)$. Thus

$$\begin{aligned} p_1\circ\Phi_A\circ\Phi_D|^{-1}_{\partial(D^{n-1}\times S^1)} &= g_d^\alpha\circ\Phi_D|^{-1}_{\partial(D^{n-1}\times S^1)}, \text{ by 4.4.2} \\ &= \Psi_d^\alpha\circ\Phi_D|^{-1}_{\partial(D^{n-1}\times S^1)}, \text{ by 4.4.3} \\ &= p_1, \text{ by 4.4.1,} \end{aligned}$$

as desired. □

Before proving the Circle and Disk Bundle Lemmas we establish some preliminary machinery.

Since every space of directions of $P^n_{k,r}$ contains an isometrically embedded, totally geodesic copy of S^{n-3}, and every space of directions of $P^n_{k,r}\setminus\mathcal{S}^{n-2}$ is isometric to S^{n-1}, we get the following (cf Proposition 3.2).

Proposition 4.5. *There are $r,\delta>0$ so that every point in the purse $P^n_{k,r}$ has a neighborhood B that is $(n-2,\delta,r)$–strained.*

For any neighborhood U of $\mathcal{S}^{n-2}$, there are $r,\delta>0$ so that every point in $P^n_{k,r}\setminus U$ has a neighborhood B that is (n,δ,r)–strained.

Remark 4.6. For $x\in\mathcal{S}^{n-2}$, the strainer $\{(a_i,b_i)\}_{i=1}^{n-2}$ can be chosen to lie in $\mathcal{S}^{n-2}$.

Because the $f_i : P^n_{k,r}\longrightarrow\mathbb{R}$ are coordinate functions, $\Psi|_{\mathcal{D}^n_k(r)\cap H}$ differs from the identity by translation by e_0. Using this we prove the following.

Proposition 4.7. *There is a neighborhood U of $\mathcal{S}^{n-2}\subset P^n_{k,r}$ so that for any family of open sets $U^\alpha\subset M^\alpha$ with $U^\alpha\to U$, $g_d^\alpha|_{U^\alpha}$ is a submersion, provided α is sufficiently large and d is sufficiently small.*

We will show that our bundle lemmas hold for any $\varepsilon>0$ such that

$$\Psi^{-1}\left(\overline{A^{n-1}(0,r-\varepsilon,r)}\right)\subset U.$$

Since $\{f_i\}_{i=1}^{n-1}$ are the $(n-1)$–coordinate functions for the standard embedding of $\mathcal{S}^{n-2}\subset\mathbb{R}^{n-1}+e_0$, we have

Lemma 4.8. *There is a $\lambda > 0$ so that for all $v \in T\mathcal{S}^{n-2}$, there is a j so that the j^{th}–component function of g satisfies*

$$|D_v (g_j)| > \lambda |v| . \tag{4.8.1}$$

Moreover, there is a $\rho > 0$ so that for all $x \in B(p_i, \rho) \cup B(A(p_i), \rho)$ and all $v \in T_x\mathcal{S}^{n-2}$, the index j in Inequality (4.8.1) can be chosen to be different from i.

To lift Lemma 4.8 to the M^αs, we need an analog of $T\mathcal{S}^{n-2}$ within each M^α, or better, a notion of g_d^α–almost horizontal for each $U^\alpha \subset M^\alpha$. To achieve this, cover $\mathcal{S}^{n-2}$ by a finite number of $(n-2, \delta, \rho)$–strained neighborhoods $B \subset P^n_{k,r}$ with strainers $\{(a_i, b_i)\}_{i=1}^{n-2} \subset \mathcal{S}^{n-2}$. Let U be the union of this finite collection, and let $U^\alpha \subset M^\alpha$ converge to U.

Given $x^\alpha \in U^\alpha$, we now define a g_d^α–almost horizontal space at x^α as follows. Let B^α be a $(n-2, \delta, \rho)$–strained neighborhood for x^α with strainers $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{n-2}$ that converge

$$\left(B^\alpha, \{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{n-2}\right) \longrightarrow \left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right), \tag{4.8.2}$$

where $\left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right)$ is part of our finite collection of $(n-2, \delta, \rho)$–strained neighborhoods for points in $\mathcal{S}^{n-2} \subset P^n_{k,r}$. We set

$$H^{g_d^\alpha}_{x^\alpha} \equiv \operatorname{span}_{i \in \{1,\ldots,n-2\}} \left\{ \uparrow^{a_i^\alpha}_{x^\alpha} \right\},$$

where $\uparrow^{a_i^\alpha}_{x^\alpha}$ is the direction of *some* segment from x^α back to a_i^α. The definition of $H^{g_d^\alpha}_{x^\alpha}$ depends on the choice of neighborhood, the choice of strainers, and the choice of the directions $\uparrow^{a_i^\alpha}_{x^\alpha}$.

Regardless of these choices, $H^{g_d^\alpha}_{x^\alpha}$ satisfies the following Lemma, which follows from Corollary 3.7.

Lemma 4.9. *Let $\{(a_i, b_i)\}_{i=1}^{n-2}$ and $\{(a_i^\alpha, b_i^\alpha)\}_{i=1}^{n-2}$ be as in (4.8.2). For $\rho > 0$,*

$$x \in U \setminus \{B(p_j, \rho) \cup B(A(p_j), \rho)\},$$

and $x^\alpha \in U^\alpha \setminus \left\{B\left(p_j^\alpha, \rho\right) \cup B\left(A\left(p_j^\alpha\right), \rho\right)\right\}$ with $\operatorname{dist}(x^\alpha, x) < \frac{1}{\alpha}$, we have

$$\left| D_{\uparrow^{a_i^\alpha}_{x^\alpha}} f^\alpha_{j,d} - D_{\uparrow^{a_i}_{x}} f_j \right| < \tau\left(\delta, \frac{1}{\alpha}, d \middle| \rho\right).$$

The following is a corollary of Lemma 4.8.

Corollary 4.10. *Let* $\left(B, \{(a_i, b_i)\}_{i=1}^{n-2}\right) \subset U$ *be as in (4.8.2). There are* $\lambda, \varepsilon > 0$ *so that for* $x \in B$ *and* $v \in \Sigma_x$ *with*

$$\sum_{i=1}^{n-2} \cos \sphericalangle \left(v, \uparrow_x^{a_i}\right) > 1 - \varepsilon, \tag{4.10.1}$$

there is a j *so that the* j^{th}*–component function of* g *satisfies*

$$\left|D_v\left(g_j\right)\right| > \lambda \left|v\right|, \tag{4.10.2}$$

provided $\mathrm{diam}\,(B)$ *is sufficiently small.*

Moreover, there is a $\rho > 0$ *so that for all* $x \in B\left(p_i, \rho\right) \cup B\left(A\left(p_i\right), \rho\right),$ *the index* j *in (4.10.2) can be chosen to be different from* i.

Since directions v that satisfy (4.10.1) are almost horizontal for $\mathrm{pr} : \overline{A^{n-1}(0, r-\varepsilon, 2r)} \to \partial\left(D^{n-1}(0, r-\varepsilon)\right) = S^{n-2}$. Lemma 4.9 and Corollary 4.10 give us the following result.

Corollary 4.11. *There is a* $\lambda > 0$ *so that for all* $x^\alpha \in U^\alpha$ *and all* $v \in H_{x^\alpha}^{g_d^\alpha}$, *there is a* j *so that the* j^{th}*–component function of* g_d^α *satisfies*

$$\left|D_v\left(\left(g_d^\alpha\right)_j\right)\right| > \lambda \left|v\right|,$$

provided U *and* d *are sufficiently small and* α *is sufficiently large. In particular,* $g_d^\alpha|_{U^\alpha}$ *is a submersion.*

Proposition 4.7 follows from Corollary 4.11.

Let $p_n \in \mathcal{D}_k^n(r)$ be as in (4.2.2). That is,

$$p_n \equiv \begin{cases} \cosh(r)e_0 + \sinh(r)e_n & \text{if } k = -1 \\ e_0 + re_n & \text{if } k = 0 \\ \cos(r)e_0 - \sin(r)e_n & \text{if } k = 1. \end{cases}$$

Let $Q : \mathcal{D}_k^n(r) \longrightarrow P_{k,r}^n$ be the quotient map. We abuse notation and refer to $Q\left(p_n\right)$ as p_n. We define $f_n : P_{k,r}^n \to \mathbb{R}$ by

$$f_n(x) \equiv h_k \circ \mathrm{dist}\left(p_n, x\right) - h_k \circ \mathrm{dist}\left(p_0, x\right).$$

With a slight modification of the proof of Proposition 3.2, we get

Lemma 4.12. *There are $\delta, r > 0$ so that for all $x \in E_D\left(r - \frac{\varepsilon}{2}\right)$, there is an (n, δ, r)–strainer $\{(a_i, b_i)\}_{i=1}^n$ for a neighborhood of x with*

$$\{(a_i, b_i)\}_{i=1}^{n-1} \subset f_n^{-1}(l)$$

for some $l \in \mathbb{R}$.

We cover $E_D\left(r - \frac{\varepsilon}{2}\right)$ by a finite number of such (n, δ, r)–strained sets and make the following definition.

Definition 4.13. For $x \in E_D\left(r - \frac{\varepsilon}{2}\right)$, set

$$H_x^{\Psi} \equiv \mathrm{span}_{i \in \{1, \ldots, n-1\}} \{\uparrow_x^{a_i}\},$$

where $\{(a_i, b_i)\}_{i=1}^{n-1}$ is as in the previous lemma.

Since $\Psi : E_D\left(r - \frac{\varepsilon}{2}\right) \longrightarrow D^{n-1}\left(r - \frac{\varepsilon}{2}\right)$ is simply orthogonal projection, we have

Lemma 4.14. *There is a $\lambda > 0$ so that for all $x \in E_D\left(r - \frac{\varepsilon}{2}\right)$ and all $v \in H_x^{\Psi}$, there is an i so that*

$$|D_v f_i| > \lambda |v|.$$

To lift this lemma to the M^{α}s, we need a notion of Ψ_d^{α}–almost horizontal for each M^{α}. Given $z^{\alpha} \in E_D^{\alpha}\left(r - \frac{\varepsilon}{2}\right)$, we define a Ψ_d^{α}–almost horizontal space at z^{α} as follows. Let B^{α} be an (n, δ, r)–strained neighborhood for z^{α} with strainers $\{(a_i^{\alpha}, b_i^{\alpha})\}_{i=1}^n$ that converge to $\{(a_i, b_i)\}_{i=1}^n$, where $(B, \{(a_i, b_i)\}_{i=1}^n)$ is part of our finite collection of (n, δ, r)–strained neighborhoods for points in $E_D\left(r - \frac{\varepsilon}{2}\right)$ that comes from Lemma 4.12. We set

$$H_{z^{\alpha}}^{\Psi_d^{\alpha}} \equiv \mathrm{span}_{i \in \{1, \ldots, n-1\}} \left\{\uparrow_{z^{\alpha}}^{a_i^{\alpha}}\right\},$$

where $\uparrow_{z^{\alpha}}^{a_i^{\alpha}}$ is the direction of *some* segment from z^{α} back to a_i^{α}. Regardless of these choices, $H_{z^{\alpha}}^{\Psi_d^{\alpha}}$ satisfies the following lemma, whose proof is nearly identical to the proof of Corollary 4.11.

Lemma 4.15. *There is a* $\lambda > 0$ *so that for all* $z^{\alpha} \in E_D^{\alpha}\left(r - \frac{\varepsilon}{2}\right)$ *and all* $v \in H_{z^{\alpha}}^{\Psi_d^{\alpha}}$, *there is an* $i \in \{1, \ldots, n-1\}$ *so that*

$$\left| D_v f_{i,d}^{\alpha} \right| > \lambda \left| v \right|,$$

provided α *is sufficiently large and* d *is sufficiently small. In particular,* $\Psi_d^{\alpha}|_{E_0^{\alpha}(\varepsilon/2)}$ *is a submersion.*

Proposition 4.16. $E_A^{\alpha}(r - \varepsilon)$ *is homeomorphic to* $S^{n-2} \times D^2$, *and* $E_D^{\alpha}(r - \varepsilon)$ *is homeomorphic to* $D^{n-1} \times S^1$, *provided* α *is sufficiently large and* d *is sufficiently small.*

Proof. First we show that $E_D^{\alpha}(r - \varepsilon)$ is connected. By the Stability Theorem [12], we have homeomorphisms $h_{\alpha} : P_k^n(r) \longrightarrow M^{\alpha}$ that are also Gromov–Hausdorff approximations (cf. [6], [8] and [19]). Thus for α sufficiently large, we have

$$E_D^{\alpha}(r - \varepsilon) \subset h_{\alpha}\left(E_D\left(r - \frac{\varepsilon}{2} \right) \right).$$

Define $\rho^{\alpha} : M^{\alpha} \longrightarrow \mathbb{R}$ by

$$\rho^{\alpha}(x) \equiv \left| \Psi_d^{\alpha}(x) \right|.$$

Since $\Psi_d^{\alpha}|_{E_D^{\alpha}\left(r - \frac{\varepsilon}{2}\right)}$ is a submersion, it follows that ρ^{α} does not have critical points on $E_D^{\alpha}\left(r - \frac{\varepsilon}{2}\right) \setminus E_D^{\alpha}(r - 2\varepsilon)$.

There is a one-to-one correspondence between the flow lines of $\nabla \rho^{\alpha}$ and the boundary of $E_D^{\alpha}(r - \varepsilon)$. Since each point of $h_{\alpha}\left(E_D\left(r - \frac{\varepsilon}{2}\right)\right)$ is on precisely one flow line, the flow lines of $\nabla \rho^{\alpha}$ give a continuous map from $h_{\alpha}\left(E_D\left(r - \frac{\varepsilon}{2}\right)\right)$ onto $E_D^{\alpha}(r - \varepsilon)$. In particular, $E_D^{\alpha}(r - \varepsilon)$ is connected.

Since the domain of $\Psi_d^{\alpha}|_{E_D^{\alpha}(r-\varepsilon)}$ is compact, $\Psi_d^{\alpha}|_{E_D^{\alpha}(r-\varepsilon)}$ is proper. Since it is also a submersion, it is a fiber bundle with contractible base $D^{n-1}(0, r - \varepsilon)$. Since the fiber is 1–dimensional and the total space is connected, we conclude that $E_D^{\alpha}(r - \varepsilon)$ is homeomorphic to $D^{n-1} \times S^1$. Since $E_D\left(r - \frac{\varepsilon}{2}\right)$ is also homeomorphic to $D^{n-1} \times S^1$, there is a homeomorphism $h_0 : E_D\left(r - \frac{\varepsilon}{2}\right) \longrightarrow E_D^{\alpha}\left(r - \frac{\varepsilon}{2}\right)$ so that

$$\begin{array}{ccc} E_0(r - \frac{\varepsilon}{2}) & \xrightarrow{\quad h_0 \quad} & E_0^{\alpha}(r - \frac{\varepsilon}{2}) \\ & {\scriptstyle \Psi_d} \searrow \quad \swarrow {\scriptstyle \Psi_d^{\alpha}} & \\ & D^{n-1} & \end{array}$$

commutes. Using the Strong Gluing Theorem ([12], Theorem 4.10), and the fact that Ψ_d^α converges to Ψ as $\alpha \to \infty$ and $d \to 0$, we choose the homeomorphism $h_0 : E_D\left(r - \frac{\varepsilon}{2}\right) \longrightarrow E_D^\alpha\left(r - \frac{\varepsilon}{2}\right)$ so that it is a $\tau\left(\frac{1}{\alpha}\right)$-Gromov-Hausdorff approximation.

Applying the Gluing Theorem again, we construct a homeomorphism $h : P_k^n(r) \longrightarrow M^\alpha$ so that

$$h = \begin{cases} h_0 & \text{on } E_D(r-\varepsilon) \\ h_\alpha & \text{on } E_A\left(r - \frac{\varepsilon}{4}\right). \end{cases}$$

It follows that $h(E_A(r-\varepsilon)) = E_A^\alpha(r-\varepsilon)$. Since $E_A(r-\varepsilon)$ is homeomorphic to $S^{n-2} \times D^2$, the result follows. □

We are now in a position to prove the Disk and Circle Bundle Lemmas.

Proof of the Disk Bundle Lemma. By Proposition 4.7, $g_d^\alpha : E_A^\alpha(r-\varepsilon) \longrightarrow \partial D^{n-1}(0, r-\varepsilon) = S^{n-2}$ is a submersion. Since the domain of g_d^α is compact, g_d^α is proper. So g_d^α is a fiber bundle with two-dimensional fiber F. From the long exact homotopy sequence and Proposition 4.16, we conclude that F is a 2–disk. The orientations of $E_A^\alpha(r-\varepsilon) \cong S^{n-2} \times D^2$ and S^{n-2} together induce an orientation on the fibers of $E_A^\alpha(r-\varepsilon)$. The oriented 2–disk bundles over S^{n-2} are classified by $\pi_{n-3}\left(\mathrm{Diff}_+\left(D^2\right)\right)$, where $\mathrm{Diff}_+\left(D^2\right)$ is the group of orientation preserving diffeomorphisms of D^2. By Theorem 1 of [13], $\pi_{n-3}\left(\mathrm{Diff}_+\left(D^2\right)\right) \cong \pi_{n-3}(SO(2)) \cong \{0\}$, unless $n = 4$. So for $n \neq 4$, every D^2–bundle over S^{n-2} is trivial.

When $n = 4$, $E_A(r-\varepsilon)$ is a D^2–bundle over S^2 whose total space is homeomorphic to $S^2 \times D^2$. The D^2–bundles over S^2 are precisely those whose corresponding unit circle bundles are lens spaces. (See for example [23], page 135.) Since the total space of $E_A(r-\varepsilon)$ is homeomorphic to $S^2 \times D^2$, it follows that $E_A(r-\varepsilon)$ is trivial in all cases, completing the proof of the Disk Bundle Lemma. □

Proof of the Circle Bundle Lemma. Since $\Psi_d^\alpha|_{E_0^\alpha(\varepsilon)}$ is a proper submersion,

$$\left(E_D^\alpha(r-\varepsilon), \Psi_d^\alpha\right)$$

is a fiber bundle over $D^{n-1}(0, r-\varepsilon)$ with one-dimensional fiber F. Since $E_D^\alpha(r-\varepsilon)$ is also homeomorphic to $D^{n-1} \times S^1$, it follows that the fiber is S^1. The base is contractible, so the bundle is trivial. □

This completes the proofs of Theorem 4.2, Theorem 1.3, and the Main Theorem.

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