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A sphere theorem for three dimensional manifolds with integral pinched curvature

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In [3], we proved a number of optimal rigidity results for Riemannian manifolds of dimension greater than four whose curvature satisfies an integral pinching. In this article, we use the same integral Bochner technique to extend the results in dimension three. Then, by using the classification of closed three-manifolds with non-negative scalar curvature and a few topological considerations, we deduce optimal sphere theorems for three-dimensional manifolds with integral pinched curvature.

1. Introduction

A celebrated result of R. Hamilton is the classification of closed three dimensional manifolds (M^3, g) endowed with a Riemannian metric with non-negative Ricci curvature (see [21] for metrics with positive Ricci curvature and [22] for the case of non-negative Ricci curvature). The result is that *such Riemannian manifold* (M^3, g) *is*

- *either diffeomorphic to a spherical space form: there is finite group* $\Gamma \subset \mathrm{SO}(4)$ *acting freely on* $\mathbb{S}^3$ *such that* $M^3 = \mathbb{S}^3/\Gamma$,
- *or a flat manifold,*
- *or isometric to a quotient of the Riemannian product* $\mathbb{R} \times \mathbb{S}^2$ *where* $\mathbb{S}^2$ *is endowed with a metric of non-negative Gaussian curvature.*

This classification has been obtained with the Ricci flow and this result is certainly the first main milestone in the success of the Ricci flow.

In dimension four, a similar result has been obtained by C. Margerin [27]: *a closed* 4*-manifold* M^4 *carrying a Riemannian metric with positive scalar curvature and whose curvature tensor satisfies*

$$\|W_g\|^2 + \frac{1}{2}\|\mathring{Ric}_g\|^2 \leq \frac{1}{24}\,\mathrm{Scal}_g^2 \tag{1.1}$$

is

- *either isometric to* $\mathbb{P}^2(\mathbb{C})$ *endowed with the Fubini-Study metric* g_{FS}*,*
- *or isometric to a quotient of the Riemannian product* $\mathbb{R} \times \mathbb{S}^3$ *where* $\mathbb{S}^3$ *is endowed with a round metric of constant sectional curvature.*
- *or diffeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{P}^4(\mathbb{R})$*.*

In the inequality (1.1), W_g denotes the Weyl tensor of the metric g and

$$\overset{\circ}{Ric}_g = \mathrm{Ric}_g - \frac{1}{4}\,\mathrm{Scal}_g\, g$$

is the traceless Ricci tensor. The norms of the curvature tensors are taken by considering them as symmetric operators on differential forms, for instance with the Einstein summation convention we have: $\|W_g\|^2 = \frac{1}{4} W_{ijk\ell} W^{ijk\ell}$. In fact, the above curvature pinching (1.1) implies that the Ricci curvature of g is non-negative (see [3, Section 2]).

This result was a generalization of the classification of closed Riemannian 4-manifold (M^4, g) with positive curvature operator in [22]. Thanks to the work of C. Böhm and B. Wilking [2], this classification is now valid in all dimensions and has been generalized by S. Brendle and R. Schoen to others pinching conditions [5, 6].

These rigidity results, among many others in Riemannian geometry, involve what is called a "pointwise curvature pinching" hypothesis. The curvature is supposed to satisfy some constraint at every point of the Riemannian manifold, and strong restrictions on the topology of the manifold follow.

Some of these results have been extended to manifolds that only satisfy the constraint in an average sense, i.e. that satisfy an integral curvature pinching. For instance, Margerin's result was extended by A. Chang, M. Gursky and P. Yang in [11, 12]. They show that *if* (M^4, g) *is a closed Riemannian manifold with positive Yamabe invariant satisfying*

$$\int_M \left(\|W_g\|^2 + \frac{1}{2}\|\overset{\circ}{Ric}_g\|^2 \right) dv_g \leq \frac{1}{24} \int_M \mathrm{Scal}_g^2\, dv_g, \tag{1.2}$$

then (M^4, g) *is*

- *either conformally equivalent to* $\mathbb{P}^2(\mathbb{C})$ *endowed with the Fubini-Study metric,*

- *or conformally equivalent to a quotient of the Riemannian product* $\mathbb{R} \times \mathbb{S}^3$ *where* $\mathbb{S}^3$ *is endowed with a round metric of constant sectional curvature.*
- *or diffeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{P}^4(\mathbb{R})$.

We recall that the *Yamabe invariant* of a closed Riemannian manifold (M^n, g) is the conformal invariant defined as:

$$Y(M^n, [g]) = \inf_{\substack{\tilde{g}=e^f g \\ f \in \mathcal{C}^\infty(M)}} \mathrm{vol}(M, \tilde{g})^{\frac{2}{n}-1} \int_M \mathrm{Scal}_{\tilde{g}}\, dv_{\tilde{g}}.$$

In fact, all the hypotheses of the theorem are conformally invariant: by using the Chern-Gauss-Bonnet formula, the condition (1.2) is equivalent to

$$\int_M \|W_g\|^2 dv_g \leq 4\pi^2 \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M.

In dimension three, integral versions of the result of Hamilton have also been proved. For instance according to G. Catino and Z. Djadli [9] or Y. Ge, C.-S. Lin and G. Wang [16], *a closed* $3-$*manifold* (M^3, g) *with positive scalar curvature such that*

$$\int_M \|\overset{\circ}{Ric}_g\|^2 dv_g \leq \frac{1}{24} \int_M \mathrm{Scal}_g^2\, dv_g \tag{1.3}$$

is diffeomorphic to a spherical space form.

However, this result is not optimal, and does not contain a characterization of the equality case.

The strategy of the proof of these two integral pinching sphere theorems is to solve a fully nonlinear PDE in order to find a conformally invariant metric $\bar{g} = e^{2f} g$ that satisfies Margerin's pointwise pinching in dimension 4 or that has positive Ricci curvature in dimension 3. Then the conclusion follows from the original pointwise versions of the respective theorems.

In [3], we used a Bochner method to extend a theorem of M. Gursky in dimension four ([19]):

Theorem 1.1. *If* (M^n, g) *is a Riemannian manifold of dimension greater than four with positive Yamabe constant such that*

$$\left(\int_M \left\|\overset{\circ}{Ric}_g\right\|^{n/2} dv_g\right)^{4/n} \leq \frac{1}{n(n-1)} \mathrm{Y}(M^n, [g])^2, \tag{1.4}$$

then

- *either its first Betti number* $b_1(M^n)$ *vanishes,*
- *or equality is attained in* (1.4), $b_1(M^n) = 1$, *and there is an Einstein manifold* (N^{n-1}, h) *with positive scalar curvature such that* (M^n, g) *is isometric (or conformally equivalent in dimension four) to a quotient of the Riemannian product:* $\mathbb{R} \times N^{n-1}$.

The proof of the first part of the result is essentially the following: we first prove that the strict integral pinching implies that a certain Schrödinger operator $\Box_g$ (see its definition in Section 3) is positive, then we prove by a Bochner method that the positivity of the operator forces harmonic forms to vanish. We rewrote it in this way in Section 3.

However, the Bochner method we used didn't extend to the three-dimensional case. The result is in fact false for three manifolds and we provide in Proposition 3.7 an example of a Riemannian metric g on $\mathbb{S}^1 \times \mathbb{S}^2$ which satisfies

$$\left(\int_{\mathbb{S}^1 \times \mathbb{S}^2} \left\|\mathring{Ric}_g\right\|^{3/2} dv_g\right)^{2/3} < \frac{1}{\sqrt{6}} \mathrm{Y}(\mathbb{S}^1 \times \mathbb{S}^2, [g]).$$

It follows that in dimension three, Theorem 1.1 is false. In this article, we obtain similar results for three manifolds, with a pinching involving the operator norm of the Schouten tensor $A_g = \mathrm{Ric}_g - \frac{1}{4} \mathrm{Scal}_g\, g$:

$$|||A_g||| = \max_{v \in T_xM \setminus \{0\}} \frac{|A_g(v,v)|}{|v|^2},$$

and we deduce sphere theorems for three dimensional manifolds that satisfy integral pinchings.

We obtain the following theorem:

Theorem A. *If* (M^3, g) *is a closed Riemannian manifold whose Schouten tensor satisfies*

$$\left(\int_M |||A_g|||^{\frac{3}{2}}\, dv_g\right)^{\frac{2}{3}} \leq \frac{1}{4} Y(M, [g])$$

then

- *either* M^3 *is diffeomorphic to a spherical space form:* $M^3 \simeq \mathbb{S}^3/\Gamma$.
- *or* (M^3, g) *is flat,*

- *or* (M^3, g) *is conformally equivalent to a compact quotient of* $(\mathbb{R} \times \mathbb{S}^2, (dt)^2 + h)$ *where* h *is a round metric on* $\mathbb{S}^2$.

Alternative pinching results are presented in Section 5. We first prove that when the pinching holds, the Schrödinger operator $\square_g$ is non-negative. If $\square_g$ is positive, it is also positive on all finite covers of the manifold, so we obtain that the first Betti number of all finite covers of the manifold vanishes. In dimension three, this is in fact sufficient to characterize the quotients of the sphere, according to the classification of closed Riemannian manifolds with non-negative scalar curvature.

Our proof uses three main ingredients, the first and major one is the classification of closed Riemannian three dimensional manifolds with positive scalar curvature initiated by R. Schoen and S.-T. Yau [35], M. Gromov and H.-B. Lawson [17], and achieved by the fundamental work of G. Perelman ([30–32]). The second one is the Bochner's type argument that we used in [3] (Section 3) and the last one is a topological observation about the virtual Betti number of a connected sum (Section 2).

In a certain extent, our argument is similar to the new proof of the conformal sphere theorem of A. Chang, M. Gursky and P. Yang found recently by B.-L. Chen and X.-P. Zhu [14]. They used a modified version of the Yamabe invariant, invented by M. Gursky ([20]), in order to apply their classification of closed 4-manifolds carrying a Riemannian metric with positive isotropic curvature ([23] and [13]).

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2. Topological considerations

Theorem 2.1. *Let* M^n *be a closed manifold of dimension* $n \geq 3$ *admitting a connected sum decomposition:*

$$M = X_1 \# X_2$$

where X_1 *and* X_2 *are not simply connected and where* $\pi_1(M)$ *is residually finite, then* M *has a finite normal covering* $\widehat{M} \to M$ *with positive first Betti number:*

$$b_1(\widehat{M}) \geq 1.$$

Remark 2.2. We recall that a group π is residually finite if for any finite subset $A \subset \pi \setminus \{e\}$ there is a normal subgroup $\Gamma \triangleleft \pi$ with finite index such that

$$A \cap \Gamma = \emptyset.$$

A result[1] of K.-W. Gruenberg shows that a free product $\pi = \Gamma_1 * \Gamma_2$ of residually finite groups, (both Γ_j are residually finite) is residually finite (see [18] or [26]).

Proof. Let $\Sigma \subset M$ be an embedded $(n-1)$-sphere such that

$$M \setminus \Sigma = (X_1 \setminus \mathbb{B}^n) \cup (X_2 \setminus \mathbb{B}^n)\,,$$

and let $p \in \Sigma$. Then

$$\pi_1(M, p) = \pi_1(X_1, p) \star \pi_1(X_2, p).$$

We choose two non-trivial loops $c_i \colon [0,1] \to X_i \setminus \mathbb{B}^n$ based at p (hence $c_i(0) = c_i(1) = p$). We can assume that each c_i is smooth on $[0,1]$, that $c_1'(0) = -c_1'(1) = -c_2'(0) = c_2'(1)$ is not tangent to Σ and that

$$\forall t \in (0,1),\ \ c_i(t) \notin \Sigma.$$

Then, we define the loop $\gamma \colon [0,1] \to M$ by

$$\gamma(t) = \begin{cases} c_1(2t) & \text{if } t \in [0, 1/2] \\ c_2(2t-1) & \text{if } t \in [1/2, 1], \end{cases}$$

and we consider $[\gamma] = [c_1 \star c_2] \in \pi_1(M, p)$.

Since $\pi_1(M, p)$ is assumed to be residually finite, we can find a normal subgroup of finite index $\Gamma \subset \pi_1(M, p)$ which does not contain $[c_1]$ nor $[c_2]$, and we can consider the quotient of the universal cover $\widetilde{M} \to M$ by Γ:

$$\pi \colon \widehat{M} = \widetilde{M}/\Gamma \to M.$$

If $\widehat{M}$ is not orientable, we can replace it by its orientation double cover; indeed the orientation double cover of $\widehat{M}$ is a quotient of $\widetilde{M}$ by a normal subgroup $\Lambda \subset \Gamma$ of index 2, in particular $[c_1]$ and $[c_2]$ are not in Λ. In the following, we assume that $\widehat{M}$ is oriented.

[1]We are grateful to F. Laudenbach for these references.

We consider a lift $\widehat{\gamma}:\mathbb{R}\to\widehat{M}$ of the continuous path $t\mapsto\gamma\,(t \bmod 1)$, i.e. $\widehat{\gamma}$ is a path such that

$$\pi\left(\widehat{\gamma}(t)\right)=\gamma\,(t \bmod 1)\,.$$

Since the covering $\pi\colon\widehat{M}\to M$ is finite of order ℓ, the set $\pi^{-1}\{\Sigma\}$ is a finite union of 2-spheres:

$$\pi^{-1}\{\Sigma\}=\Sigma_1\cup\cdots\Sigma_\ell$$

and the set $\pi^{-1}\{p\}=\{p_1,\dots,p_\ell\}$ contains the set $\{\widehat{\gamma}(t)\,,\,t\in\frac{1}{2}\mathbb{N}\}$, hence $\widehat{\gamma}$ is periodic.

Let $\widehat{\Sigma}$ be the lift of Σ such that $\widehat{\gamma}(0)\in\widehat{\Sigma}$ and let $\tau>0$ be the first positive time such that $\widehat{\gamma}(\tau)=\widehat{\gamma}(0)$. We know by construction that $\tau\in\frac{1}{2}\mathbb{N}$, and since we have assumed that $[c_1]$ and $[c_2]$ are not in Γ, we also have $\tau\neq\frac{1}{2}$; it also follows that for all $t\in(0,\tau)$,

$$\widehat{\gamma}(t)\notin\widehat{\Sigma}.$$

If we now define $\widehat{\gamma}_{\mathrm{red}}\colon\mathbb{R}/\tau\mathbb{Z}\to\widehat{M}$ by

$$\widehat{\gamma}_{\mathrm{red}}(t)=\widehat{\gamma}(t \bmod \tau\mathbb{Z}),$$

then the intersection number between $\widehat{\gamma}_{\mathrm{red}}$ and $\widehat{\Sigma}$ is ±1, hence

$$H^1(\widehat{M},\mathbb{Z})\neq\{0\}.$$

$\square$

Remark 2.3. Using a deep result of W. Lück, we can give another, more analytical, proof. We can equipped M with a Riemannian metric g. Let $\widetilde{M}\to M$ be the universal cover of M. According to the main result of the paper [25], we only need to show that the universal cover of M carries some non-trivial L^2 harmonic 1-forms. But a lift of Σ to $\widetilde{M}$ separated $\widetilde{M}$ into two unbounded connected components (because X_1 and X_2 are non-simply connected), hence $\widetilde{M}$ has at least two ends. Moreover the injectivity radius of $(\widetilde{M},g)$ is positive. According to Brooks [7], if $\pi_1(M)$ is non-trivial, then the Laplace operator acting on functions on $\widetilde{M}$ has a spectral gap, hence by [10, Proposition 5.1], $\widetilde{M}$ carries a non-constant harmonic function h with L^2 gradient $dh\in L^2$. Hence the result holds when $\pi_1(M)$ is not amenable. But[2] a non-trivial free product $\pi_1(X_1,p)\star\pi_1(X_2,p)$ is amenable only for $\mathbb{Z}_2\star\mathbb{Z}_2$; as $\mathbb{Z}_2\star\mathbb{Z}_2=\mathbb{Z}\rtimes\mathbb{Z}_2$ contains a normal subgroup of index 2 isomorphic to $\mathbb{Z}$,

[2]We are grateful to S. Tapie for explaining this to us. See [28, Lemma 2.28] for a proof.

we see that in this remaining case, M will have a two fold cover with first Betti number equals to 1.

3. A Bochner result

In this section, we prove a Bochner result, which was almost contained in [24] and [3]: if the operator $\Box_g := \Delta_g + \frac{n-2}{n-1}\rho_1$ is positive, the first Betti number must vanish, and the equality case is characterized.

3.1. Preliminaries

We consider a closed connected Riemannian manifold (M^n, g) of dimension $n \geq 3$.

We denote by ρ_1 the lowest eigenvalue of the Ricci tensor of g. The function $\rho_1 \colon M \to \mathbb{R}$ satisfies

$$\forall x \in M,\ \forall v \in T_xM,\ \mathrm{Ric}_g(v,v) \geq \rho_1(x) g(v,v)\,.$$

Then, we denote by $\Box_g$ the Schrödinger operator

$$\Box_g := \Delta_g + \frac{n-2}{n-1}\rho_1.$$

It is non-negative if for any smooth function φ,

$$\int_M \left(|\nabla\varphi|^2 + \frac{n-2}{n-1}\rho_1\varphi^2 \right) dv_g \geq 0,$$

or equivalently if its lowest eigenvalue is non-negative.

Definition 3.1. If (N,h) is a Riemannian manifold, $f \colon N \to N$ is an isometry of (N,h) and $\eta \colon \mathbb{R} \to \mathbb{R}_+^*$ is a positive smooth τ-periodic function, we define the *twisted warped product* $\mathbb{S}^1 \underline{\times}_\eta N$ as the quotient of the warped product $(\mathbb{R} \times N, dt^2 + \eta^2(t)h)$ by the cyclic group generated by the isometry

$$(t,x) \mapsto (t+\tau, f(x))$$

where $f \colon N \to N$ is a diffeomorphism of N and τ is a positive number. If f is isotopic to the identity map then this twisted product is diffeomorphic to $\mathbb{S}^1 \times N$.

Remark 3.2. Our definition, is similar to the definition of twisted product of oriented Riemannian manifolds, given for instance by C. Bär in [1, Section 2].

Finally, we prove the following two preliminary results, which will be used in the next section:

Lemma 3.3. *Let (M^n, g) be a closed Riemannian manifold and let $\pi : \widehat{M} \to M$ be a cover of M.*

If the operator $\Box_g$ is non-negative, then the operator $\Box_{\pi^* g}$ is non-negative.

Proof. Let $u \in C^\infty(M)$ be an eigenfunction associated to the lowest eigenvalue of $\Box_g$. We can suppose that u is positive. Then the function $u \circ \pi$ is a positive eigenfunction for $\Box_{\pi^* g}$. By a principle due to W. F. Moss and J. Piepenbrink and D. Fisher-Colbrie and R. Schoen, ([15, 29] or [33, lemma 3.10]), we know that the bottom of the spectrum of $\Box_{\pi^* g}$ is non-negative. □

Lemma 3.4. *Let (M^n, g) be a closed Riemannian manifold. If ξ is a non-trivial harmonic one-form (i.e. $d\xi = 0$ and $\delta\xi = 0$), then the function $u = |\xi|^{\frac{n-2}{n-1}}$ satisfies in the weak sense*

$$\Box_g u + f_\xi u = 0$$

where

$$f_\xi = \begin{cases} \frac{\mathrm{Ric}(\xi,\xi)}{|\xi|^2} - \rho_1 + \frac{1}{|\xi|^2}\left(|\nabla\xi|^2 - \frac{n}{n-1}\left|d|\xi|\right|^2\right) & \text{where } \xi \neq 0 \\ 0 & \text{where } \xi = 0 \end{cases}$$

is non-negative on M.

Proof. The harmonic one-form ξ satisfies both the Bochner equation

$$\langle \nabla^*\nabla\xi | \xi\rangle + \mathrm{Ric}(\xi, \xi) = 0,$$

and the refined Kato inequality ([37, lemma 2], [4], [8])

$$\frac{n}{n-1}\left|d|\xi|\right|^2 \leq |\nabla\xi|^2 .$$

According to basic calculations (see [3], Section 6), the function $u_\epsilon = (|\xi|^2 + \varepsilon^2)^{\frac{n-2}{2(n-1)}}$ satisfies

$$\Delta_g u_\varepsilon + \frac{n-2}{n-1}(\rho_1 + f_\xi)\,|\xi|^2\, u_\varepsilon^{-\frac{n}{n-2}} = 0$$

And since $|\xi|^2 \, u_\varepsilon^{-\frac{n}{n-2}} = u\,.\left(\frac{|\xi|^2}{|\xi|^2+\varepsilon^2}\right)^{\frac{n}{n-1}} \leq u$, the function u satisfies

$$\Box_g u + f_\xi u = 0,$$

in the weak sense. □

3.2. The Bochner result

We can reformulate a part of the Bochner result in [24] or [3] as follows:

Proposition 3.5. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 3$. If the operator*

$$\Box_g := \Delta_g + \frac{n-2}{n-1}\rho_1$$

is non-negative then

- *either the first Betti number of M vanishes:* $\mathrm{b}_1(M^n) = 0$,
- *or M^n is isometric to a twisted warped product $\mathbb{S}^1 \underline{\times}_\eta N^{n-1}$ endowed with a warped product metric*

$$(dt)^2 + \eta^2(t)h$$

where (N^{n-1}, h) is a closed Riemannian manifold with non-negative Ricci curvature.

Moreover, in the second case, if the Ricci curvature of (N^{n-1}, h) vanishes somewhere, then the function η is constant and the metric g is locally a Riemannian product.

Proof. We assume that the operator $\Box_g$ is non-negative. If $b_1(M) > 0$, according to the deRham isomorphism and the Hodge theorem, we can find a non-trivial harmonic one-form $\xi \in \mathcal{C}^\infty(T^*M)$ with integral periods:

$$d\xi = 0,\ \delta\xi = 0 \quad \text{and} \quad \forall \gamma \in \pi_1(M)\,, \int_\gamma \xi \in \mathbb{Z}.$$

According to Lemma 3.4, the function $u := |\xi|^{\frac{n-2}{n-1}}$ satisfies in the weak sense

$$\Box_g u + f_\xi u = 0.$$

Integrating by parts, it follows that

$$\int_M \left(|\nabla u|^2 + \frac{n-2}{n-1}\rho_1 u^2 \right) dv_g \leq -\int_M f_\xi u^2 dv_g \leq 0.$$

But as $\Box_g$ is non-negative, we must have

$$\int_M \left(|\nabla u|^2 + \frac{n-2}{n-1}\rho_1 u^2 \right) dv_g = 0,$$

and therefore $f_\xi = 0$. It implies that at any point x where $\xi(x) \neq 0$, equality is attained in the refined Kato inequality, and ξ is an eigenvector for the lowest eigenvalue of the Ricci tensor.

As equality is attained in the refined Kato inequality, we know (see [3, Proposition 5.1]) that the normal cover $(\widehat{M}, \widehat{g})$ associated to the kernel of the morphism

$$\begin{array}{ccc} \pi_1(M) & \to & \mathbb{Z} \\ \gamma & \mapsto & \int_\gamma \xi, \end{array}$$

is isometric to a warped product:

$$(\widehat{M}, \widehat{g}) \simeq (\mathbb{R} \times N^{n-1}, (dt)^2 + \eta^2(t)h),$$

where (N^{n-1}, h) is a closed Riemannian manifold.

It follows that (M, g) is a compact quotient of $(\mathbb{R} \times N^{n-1}, (dt)^2 + \eta^2(t)h)$ by a cyclic group, generated by some isometry f of $(\widehat{M}, \widehat{g})$.

Moreover, still according to [3, Proposition 5.1], the pullback $\pi^*\xi$ of ξ on $\widehat{M}$ is colinear to the derivative of the function

$$\Phi(t, x) = \int_0^t \frac{dr}{\eta(r)^{n-1}}.$$

Therefore, $\pi^*\xi$ is colinear to $\frac{dt}{\eta(t)^{n-1}}$, and it implies that dt is an eigenvector associated to the lowest eigenvalue of the Ricci tensor.

We will now prove that the isometry f can be written as:

$$f(t, x) = (t + \tau, \varphi(x)),$$

where $\varphi\colon (N, h) \to (N, h)$ is an isometry and τ is a real number.

As f is an isometry and as it preserves the one-form $\pi^*\xi$, both the line generated by ∇t and its orthogonal TN are invariant by its differential. It follows that there exist two functions u and φ such that $f(t, x) = (u(t), \varphi(x))$.

Moreover, the function u must satisfy $(u'(t))^2 = 1$, and since $f^*(\pi^*\xi) = \pi^*\xi$, we have $u'(t) = 1$. It follows that there exists $\tau \neq 0$ such that $u(t) = t + \tau$.

Then, since f is an isometry, we obtain that the map $\varphi\colon x \mapsto \varphi(x)$ is a conformal equivalence: for all $t \in \mathbb{R}$,

$$\varphi^* h = \frac{\eta^2(t)}{\eta^2(u(t))} h.$$

But since N is compact, by integrating the associated volume forms over N, we obtain that for all $t \in \mathbb{R}$, $\frac{\eta^2(t)}{\eta^2(u(t))} = 1$, hence $\eta(u(t)) = \eta(t)$ and φ is an isometry.

We finally prove that the Ricci tensor of h is non-negative and that the function η must be constant if the Ricci tensor has a zero eigenvalue somewhere.

Let write the Ricci tensor of the warped product metric $(dt)^2 + \eta^2(t)h$:

$$\mathrm{Ric} = \begin{pmatrix} -(n-1)\frac{\eta''}{\eta} & 0 \\ 0 & \frac{\mathrm{Ric}_h}{\eta^2} - \frac{(n-2)(\eta')^2 + \eta''\eta}{\eta^2}\mathrm{Id} \end{pmatrix}.$$

By writing that dt is an eigenvector associated to the lowest eigenvalue of the Ricci tensor, we obtain that

$$\mathrm{Ric}_h \geq -(n-2)\left(\eta''\eta - (\eta')^2\right) h = -(n-2)\eta^2 \left(\ln(\eta)\right)'' h,$$

and since η is $|\tau|$-periodic, $(\ln(\eta))''$ must vanish for some $t \in \mathbb{R}$, hence Ric_h is non-negative.

Moreover, when the Ricci tensor of h has a zero eigenvalue somewhere, then the function $\eta''\eta - (\eta')^2 = \eta^2 \left(\ln(\eta)\right)''$ must vanish, and as the function η is $|\tau|$-periodic, η must be constant. $\square$

Corollary 3.6. *Let (M^3, g) be a closed Riemannian manifold. If the operator*

$$\Box_g := \Delta_g + \frac{1}{2}\rho_1$$

is non-negative then

- *either the first Betti number of any finite normal cover of M^3 vanishes,*
- *(M^3, g) is isometric to a compact quotient of $\mathbb{R} \times \mathbb{S}^2$ endowed with a warped product metric $(dt)^2 + \eta^2(t)h$, where h is a metric of non-negative Gaussian curvature on $\mathbb{S}^2$,*

- *or* (M^3, g) *is flat.*

Moreover, in the second case, if the Gaussian curvature of $(\mathbb{S}^2, h)$ *vanishes somewhere, then the function* η *is constant and* (M^3, g) *is isometric to a quotient of the Riemannian product* $(\mathbb{R} \times \mathbb{S}^2, (dt)^2 + h)$.

Proof. We apply Proposition 3.5 to any finite normal cover of M^3, using Lemma 3.3.

If there exists a finite normal cover of $\bar{M}^3$ with positive first Betti number, then according to Proposition 3.5, the finite cover is isometric to a twisted warped product $\mathbb{S}^1 \underline{\times}_\eta N^2$, endowed with a warped product metric

$$(dt)^2 + \eta^2(t)h$$

where h has non-negative Ricci curvature, hence non-negative Gaussian curvature.

If (N^2, h) is flat, then according to Proposition 3.5, the function η is constant and (M^3, g) is flat.

If (N^2, h) is not flat then N^2 is diffeomorphic to $\mathbb{S}^2$ or to $\mathbb{P}^2(\mathbb{R})$. And the universal cover $\widetilde{M}$ of M^3 is isometric to the warped product $(\mathbb{R} \times \mathbb{S}^2,\ (dt)^2 + \eta^2(t)h)$. $\square$

3.3. A counterexample to Theorem 1.1 in dimension three

In [3], we have shown that, for $n \geq 4$, the pinching (1.4) implies that the operator $\square_g := \Delta_g + \frac{n-2}{n-1}\rho_1$ is non-negative, and the strict version of the pinching implies that the operator is positive.

In this subsection, we prove that this is false if $n = 3$:

Proposition 3.7. *There is a Riemannian metric* g *on* $\mathbb{S}^1 \times \mathbb{S}^2$ *such that*

$$\int_{\mathbb{S}^1 \times \mathbb{S}^2} \left\| \mathring{Ric}_g \right\|^{3/2} dv_g < \left(\frac{1}{\sqrt{6}} \, \mathrm{Y}(\mathbb{S}^1 \times \mathbb{S}^2, [g]) \right)^{\frac{3}{2}}. \tag{3.1}$$

In particular, in dimension 3, this integral pinching does not imply the vanishing of the first Betti number.

Proof. On $\mathbb{R}/(\pi\mathbb{Z}) \times \mathbb{S}^2$, we consider the warped product metric

$$g_\varepsilon = (dr)^2 + \left(\varepsilon^2 + \sin^2(r)\right) h$$

where h is the metric of constant Gaussian curvature 1 on $\mathbb{S}^2$ and ε is a positive number. We will show that for ε small enough, the strict inequality (3.1) holds.

The metric g_ε is conformally equivalent to the Riemannian product

$$\left(\mathbb{R}/(\ell\mathbb{Z}) \times \mathbb{S}^2, (d\theta)^2 + h\right),$$

where

$$\ell = \int_0^\pi \frac{dr}{\sqrt{\varepsilon^2 + \sin^2(r)}} \sim -2\log(\varepsilon).$$

In particular, the Yamabe constant of the metric g_ε goes to $6\operatorname{vol}(\mathbb{S}^3)^{\frac{2}{3}}$ when ε goes to 0 (see [34]).

It follows that

$$\lim_{\varepsilon\to 0}\left(\frac{1}{\sqrt{6}}\,\mathrm{Y}(\mathbb{S}^1 \times \mathbb{S}^2, [g_\varepsilon])\right)^{\frac{3}{2}} = 6^{\frac{3}{4}} 2\pi^2. \tag{3.2}$$

On the other hand, the Ricci tensor of the warped product g_ε is

$$\mathrm{Ric}_{g_\varepsilon} = \begin{pmatrix} -2\frac{\eta''}{\eta} & 0 \\ 0 & \frac{1-(\eta')^2-\eta''\eta}{\eta^2}\,\mathrm{Id} \end{pmatrix},$$

where $\eta^2(r) = \varepsilon^2 + \sin^2(r)$. We find in particular that

$$\left\|\mathring{Ric}_{g_\varepsilon}\right\|^2 = \frac{2}{3}\left(\frac{1-(\eta')^2+\eta''\eta}{\eta^2}\right)^2,$$

where

$$\frac{1-(\eta')^2+\eta''\eta}{\eta^2} = \varepsilon^2 \frac{2\cos^2(r)}{(\varepsilon^2+\sin^2(r))^2}.$$

Therefore,

$$\int_{\mathbb{S}^1\times\mathbb{S}^2} \left\|\mathring{Ric}_{g_\varepsilon}\right\|^{3/2} dv_{g_\varepsilon} = \left(\frac{2}{3}\right)^{\frac{3}{4}} \varepsilon^3 4\pi 2^{\frac{3}{2}} \int_0^\pi \frac{\cos^3(r)}{(\varepsilon^2+\sin^2(r))^2} dr.$$

But

$$\begin{aligned}\varepsilon^3 \int_0^{\pi} \frac{\cos^3(r)}{(\varepsilon^2+\sin^2(r))^2} dr &= 2\varepsilon^3 \int_0^1 \frac{1-\tau^2}{(\varepsilon^2+\tau^2)^2} d\tau \\ &= 2\int_0^{\frac{1}{\varepsilon}} \frac{1-\varepsilon^2 t^2}{(1+t^2)^2} dt = 2\int_0^{+\infty} \frac{1}{(1+t^2)^2} dt + \mathcal{O}(\varepsilon^2) \\ &= \frac{\pi}{2} + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Hence

$$\int_{\mathbb{S}^1\times\mathbb{S}^2} \left\| \overset{\circ}{Ric}_{g_\varepsilon} \right\|^{3/2} dv_{g_\varepsilon} = \left(\frac{2}{3}\right)^{\frac{3}{4}} 2^{\frac{3}{2}} 2\pi^2 + \mathcal{O}(\varepsilon^2) = \left(\frac{8}{3}\right)^{\frac{3}{4}} 2\pi^2 + \mathcal{O}(\varepsilon^2).$$

We see that for $\varepsilon > 0$ small enough, the metric g_ε satisfies the strict pinching (3.1). □

Remark 3.8. If, on $\mathbb{R}/(\pi\mathbb{Z}) \times \mathbb{S}^{n-1}$, we consider the warped product metric

$$g_\varepsilon = (dr)^2 + \left(\varepsilon^2 + \sin^2(r)\right) h,$$

where h is the metric of constant sectional curvature 1 on $\mathbb{S}^{n-1}$, then we obtain

$$\lim_{\varepsilon\to 0} \left(\frac{1}{\sqrt{n(n-1)}} \mathrm{Y}(\mathbb{S}^1 \times \mathbb{S}^{n-1}, [g_\varepsilon]) \right)^{\frac{n}{2}} = (n(n-1))^{\frac{n}{4}} \operatorname{vol}(\mathbb{S}^n)$$

and

$$\lim_{\varepsilon\to 0} \left(\int_{\mathbb{S}^1\times\mathbb{S}^{n-1}} \left\| \overset{\circ}{Ric}_{g_\varepsilon} \right\|^{n/2} dv_{g_\varepsilon} \right) = \left(\frac{n-1}{n}\right)^{\frac{n}{4}} (n-2)^n 2^{\frac{n}{2}} \operatorname{vol}(\mathbb{S}^n).$$

But for $n > 3$, $\left(\frac{n-1}{n}\right)^{\frac{n}{4}} (n-2)^n 2^{\frac{n}{2}} > (n(n-1))^{\frac{n}{4}}$.

4. A sphere theorem for three-dimensional manifolds

In this section, we show how to obtain a sphere theorem for integral pinched manifolds based on the classification of three-manifolds with non-negative scalar curvature.

Lemma 4.1. *If (M, g) is a compact manifold of dimension 3 or 4 such that $\square_g$ is non-negative, then we can find a metric $\tilde{g}$ which is conformally equivalent to g and has non-negative scalar curvature.*

Proof. The inequality $\mathrm{Scal}_g \geq n\rho_1$ is always true. Therefore, if the operator $\Box_g$ is non-negative then the operator

$$\Delta_g + \frac{n-2}{n(n-1)}\,\mathrm{Scal}_g$$

is also non-negative. In dimension 3 and 4, it implies that the Yamabe operator

$$\Delta_g + \frac{n-2}{4(n-1)}\,\mathrm{Scal}_g$$

is non-negative.

As a result, if $\Box_g$ is non-negative on a compact manifold of dimension 3 or 4, we can find a metric $\tilde{g} = u^{\frac{4}{n-2}} g$ conformally equivalent to g which has non-negative scalar curvature. □

But closed three manifolds which carry a metric with non-negative scalar curvature have been classified. After the results of R. Schoen and S.-T. Yau [35] or M. Gromov and H.-B. Lawson [17], this classification is a consequence of the solution of the Poincaré conjecture by G. Perelman ([30–32]). The Ricci flow with surgeries of Perelman also provides a direct proof of this classification.

If M^3 is a closed oriented three dimensional manifold which carries a metric of non-negative scalar curvature, then either M^3 carries a flat metric or it admits a connected sum decomposition

$$M^3 = S_1 \# \cdots \# S_r \# \ell \left(\mathbb{S}^1 \times \mathbb{S}^2\right) \,,$$

where each S_j with $j \geq 1$ is diffeomorphic to a spherical space form $S_j \simeq \mathbb{S}^3/\Gamma_j$, Γ_j being a a finite subgroup (possibly trivial) of SO(4) acting freely on $\mathbb{S}^3$.

As a consequence, we obtain:

Proposition 4.2. *If M^3 is a closed manifold which carry a metric of non-negative scalar curvature, then*

- *either M^3 is diffeomorphic to a spherical space form $M^3 \simeq \mathbb{S}^3/\Gamma$,*
- *or M^3 carries a flat metric,*
- *or M^3 has a finite normal cover with positive first Betti number.*

Proof. We consider the oriented cover $\bar{M}^3$ of M^3.

Then either $\bar{M}^3$ carries a flat metric and it is covered by a torus, hence M^3 also carry a flat metric, or it admits a connected sum decomposition

$$\bar{M}^3 = S_1 \# \cdots \# S_r \# \ell \left(\mathbb{S}^1 \times \mathbb{S}^2\right) ,$$

where each S_j has a finite fundamental group Γ_j and is diffeomorphic to a spherical space form $S_j \simeq \mathbb{S}^3/\Gamma_j$.

Let see what happens if M^3 doesn't carry a flat metric and if the first cohomology group of all finite normal covers of $\bar{M}^3$ vanishes. In this case, according to Theorem 2.1 and Remark 2.2, and since $\mathbb{S}^2 \times \mathbb{S}^1$ has a non-zero first Betti number, we must have $(\ell, r) = (0, 1)$ in the decomposition, hence M^3 is a spherical space form. □

According to this, we can prove the following theorem:

Theorem 4.3. *If (M^3, g) is a closed manifold such that the operator*

$$\Box_g := \Delta_g + \frac{1}{2}\rho_1$$

is non-negative, then

- *either M^3 is diffeomorphic to a spherical space form $M^3 \simeq \mathbb{S}^3/\Gamma$,*
- *or (M^3, g) is a flat Riemannian manifold,*
- *or (M^3, g) is isometric to a compact quotient of $\mathbb{R} \times \mathbb{S}^2$ endowed with a warped product metric $(dt)^2 + \eta^2(t)h$, where h is a metric of non-negative Gaussian curvature on $\mathbb{S}^2$.*

Moreover, in the third case, if the Gaussian curvature of $(\mathbb{S}^2, h)$ vanishes somewhere, then the function η is constant and (M^3, g) is isometric to a quotient of the Riemannian product $(\mathbb{R} \times \mathbb{S}^2, (dt)^2 + h)$.

Proof. It follows from Lemma 4.1, Proposition 4.2 and Corollary 3.6. The only thing that remains to be proved is that if M^3 carries a flat metric, then (M^3, g) is flat. And when M^3 carries a flat Riemannian metric, there is a finite cover $\pi\colon \mathbb{T}^3 \to M$. Then, the operator $\Box_{\pi^* g}$ is non-negative and Corollary 3.6 implies that $\pi^* g$ is Ricci flat hence flat. □

Remark 4.4. We can be more precise in the third case of the conclusion of Theorem 4.3.

The compact quotients of $\mathbb{R} \times \mathbb{S}^2$ are classified. According to [36, Section 4, *geometry of* $\mathbb{S}^2 \times \mathbb{R}$], they are diffeomorphic:

- either to $\mathbb{S}^1 \times \mathbb{S}^2 = \left(\mathbb{R} \times \mathbb{S}^2\right)/\Gamma$ where Γ is generated by $\alpha\colon (t,\theta) \mapsto (t+1,\theta)$,
- or to $\mathbb{S}^1 \times \mathbb{P}^2(\mathbb{R}) = \left(\mathbb{R} \times \mathbb{S}^2\right)/\Gamma_1$ where Γ_1 is generated by $\alpha\colon (t,\theta) \mapsto (t+1,\theta)$ and by $\alpha_1\colon (t,\theta) \mapsto (t,-\theta)$,
- or to $\left(\mathbb{R} \times \mathbb{S}^2\right)/\Gamma_2$ where Γ_2 is generated by $\beta\colon (t,\theta) \mapsto (t+1/2,-\theta)$,
- or to $\mathrm{SO}(3)\#\,\mathrm{SO}(3) = \left(\mathbb{R} \times \mathbb{S}^2\right)/\Gamma_3$ where Γ_3 is generated:
  either by $\sigma_1\colon (t,\theta) \mapsto (-t,-\theta)$ and $\sigma_2\colon (t,\theta) \mapsto (1-t,-\theta)$, or by σ_1 and α.

The corresponding statement for the third case of the conclusion of Theorem 4.3 is that the manifold (M^3,g) is isometric to a quotient of $\mathbb{R} \times \mathbb{S}^2$ by a group Γ, where $\mathbb{R} \times \mathbb{S}^2$ is endowed with a warped product metric $(dt)^2 + \eta^2(t)h$, h being a metric of non-negative Gaussian curvature on $\mathbb{S}^2$, and where the group Γ is generated:

- either by an isometry $\alpha : (t,\theta) \mapsto (t+\tau, f(\theta))$ where f is an orientation-preserving isometry of $(\mathbb{S}^2, h)$,
- or by two isometries $\alpha : (t,\theta) \mapsto (t+\tau, f(\theta))$ and $\alpha_1 : (t,\theta) \mapsto (t, f_1(\theta))$ where f (resp. f_1) is an orientation-preserving (resp. orientation-reversing) isometry of $(\mathbb{S}^2, h)$,
- or by an isometry $\beta : (t,\theta) \mapsto (t+\tau, f(\theta))$ where f is an orientation-reversing isometry of $(\mathbb{S}^2, h)$,
- or by two isometries $\sigma_1 : (t,\theta) \mapsto (-t, f_1(\theta))$ and $\sigma_2 : (t,\theta) \mapsto (\tau - t, f_2(\theta))$ where f_1 and f_2 are orientation-reversing isometries of $(\mathbb{S}^2, h)$.

Remark 4.5. If we compare Theorem 4.3 with Hamilton's classification of manifolds with non-negative Ricci curvature, we see that Theorem 4.3 also allows warped product metrics in the third case, in addition to Riemannian products.

This is easily understood by the fact that the condition of non-negative $\Box_g$ operator is slightly more general than the condition of non-negative Ricci curvature (a manifold with non-negative Ricci curvature has a non-negative ρ_1, hence a non-negative operator $\Box_g$).

Moreover, the assumption on $\Box_g$ is not a pointwise condition, as is the assumption that the Ricci curvature is non-negative at every point of the manifold, but it captures the positivity of the Ricci curvature in an average sense.

We will see in the next section a number of conditions under which the operator $\Box_g$ is non-negative. They are expressed as integral pinching assumptions on the curvature, as opposed to pointwise pinching assumptions such as the non-negativity of the Ricci tensor.

5. Conditions for the operator $\Box_g$ to be non-negative

We give several integral pinching conditions under which the operator $\Box_g$ is non-negative and deduce the sphere theorems implied by Theorem 4.3.

5.1. With a pinching involving the Schouten tensor

In dimension 3, the Riemann curvature tensor of a Riemannian metric g has the following decomposition:

$$\mathrm{Rm}_g = A_g \wedge g,$$

where $\wedge$ is the Kulkarni-Nomizu product and A_g is the Schouten tensor, which satisfies

$$A_g = \mathring{Ric}_g + \frac{1}{12}\,\mathrm{Scal}_g\, g = \mathrm{Ric}_g - \frac{1}{4}\,\mathrm{Scal}_g\, g.$$

We denote by $a_1 \leq a_2 \leq a_3$ the eigenvalues of the Schouten tensor. Then

$$\rho_1 = a_1 + \frac{1}{4}\,\mathrm{Scal}_g$$

and therefore

$$\Box_g = \Delta_g + \frac{1}{8}\,\mathrm{Scal}_g + \frac{1}{2} a_1 = \frac{1}{8} L_g + \frac{1}{2} a_1,$$

where $L_g = \Delta_g + \frac{1}{8}\,\mathrm{Scal}_g$ is the Yamabe operator.

We recall that the Yamabe constant $Y(M,[g])$ is the best possible constant in the Sobolev inequality:

$$\forall \varphi \in \mathcal{C}^\infty(M), \quad Y(M,[g])\,\|\varphi\|^2_{L^6} \leq \int_M \left(8|d\varphi|^2 + \mathrm{Scal}_g\,\varphi^2\right) dv_g. \tag{5.1}$$

Another interpretation of the Yamabe constant is given in more geometrical terms:

$$Y(M,[g]) = \inf_{\substack{\tilde g = e^f g\\ f\in C^\infty(M)}} \mathrm{vol}(M,\tilde g)^{-\frac{1}{3}} \int_M \mathrm{Scal}_{\tilde g}\, dv_{\tilde g}.$$

We define $(a_1)_- = \max\{-a_1, 0\}$ and the operator norm $|||A_g|||$ of the Schouten tensor by:

$$|||A_g|||\,(x) = \max_{v \in T_xM\setminus\{0\}} \frac{|g(A_g v, v)|}{g(v,v)}$$

The following inequality holds:

$$(a_1)_- \leq |||A_g|||$$

and equality is attained if and only if $a_1 + a_3 \leq 0$.

Proposition 5.1. *If (M^3, g) is a closed Riemannian manifold whose Schouten tensor satisfies*

$$\|(a_1)_-\|_{L^{\frac{3}{2}}} \leq \frac{1}{4} Y(M,[g]) \qquad \textit{or} \qquad \||A_g|\|_{L^{\frac{3}{2}}} \leq \frac{1}{4} Y(M,[g])$$

then the operator $\Box_g$ is non-negative.

Moreover, if a function $\varphi \neq 0$ satisfies $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g = 0$, then it is a solution of the Yamabe equation on (M^3, g).

Proof. Since $\rho_1 = a_1 + \frac{1}{4}\,\mathrm{Scal}_g$, we get from the Sobolev's type inequality (5.1) and the Hölder inequality that

$$\begin{aligned} (5.2) \qquad & 8 \int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g \\ & \geq \int_M \left(8|d\varphi|^2 + \mathrm{Scal}_g\,\varphi^2 - 4\left((a_1)_-\right)\varphi^2\right) dv_g \\ & \geq \left(Y(M,[g]) - 4\,\|(a_1)_-\|_{L^{\frac{3}{2}}}\right)\,\|\varphi\|^2_{L^6}\,, \end{aligned}$$

hence $\Box_g$ is non-negative as soon as the inequality $\|(a_1)_-\|_{L^{\frac{3}{2}}} \leq \frac{1}{4}Y(M,[g])$ holds.

If φ satisfies $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g = 0$, then it must also attain equality in (5.1), hence it is a solution of the Yamabe equation $8\Delta_g\varphi + \mathrm{Scal}_g\,\varphi = Y(M,[g])\varphi^5$. □

According to Theorem 4.3, we obtain

Theorem 5.2. *If (M^3, g) is a closed Riemannian manifold whose Schouten tensor satisfies*

$$\|(a_1)_-\|_{L^{\frac{3}{2}}} \leq \frac{1}{4} Y(M, [g]) \qquad or \qquad |||A_g|||_{L^{\frac{3}{2}}} \leq \frac{1}{4} Y(M, [g])$$

then

- *either (M^3, g) is a flat Riemannian manifold,*
- *or M^3 is diffeomorphic to a spherical space form $M \simeq \mathbb{S}^3/\Gamma$,*
- *or (M^3, g) is conformally equivalent to a compact quotient of the Riemannian product $\mathbb{R} \times \mathbb{S}^2$ where $\mathbb{S}^2$ is endowed with a round metric.*

Proof. The third case corresponds to the equality case in Proposition 3.5. In that case, the function $u = \sqrt{|\xi|} = \eta^{-1}$ satisfies $\int_M \left(|du|^2 + \frac{1}{2}\rho_1 u^2 \right) dv_g = 0$, and according to Proposition 5.1, it must satisfy the Yamabe equation $8\Delta_g u + \mathrm{Scal}_g\, u = Y(M, [g]) u^5$.

But since $u = \eta^{-1}$ only depends on t and since

$$\mathrm{Scal}_g = \frac{1}{\eta^2(t)} \left(\mathrm{Scal}_h - 4\eta''(t)\eta(t) - 2\left(\eta'(t)\right)^2 \right),$$

we see that Scal_h only depends on t, hence must be constant. It follows that $(\mathbb{S}^2, h)$ has constant Gaussian curvature.

Then, we can write that the lift of g is isometric to $\eta^2(t)\left(\left(\frac{dt}{\eta(t)}\right)^2 + h \right)$, which is isometric to $\phi^2(r)\left((dr)^2 + h\right)$, a metric conformally equivalent to the Riemannian product of $\mathbb{R}$ and a round sphere.

Moreover, as the Riemannian metric $u^4 g = \frac{1}{\eta^2(t)}\left(\left(\frac{dt}{\eta(t)}\right)^2 + h \right)$ has to be a Yamabe minimizer on $(M, [g])$, the analysis of R. Schoen in [34] implies that the function η is uniquely determined. □

5.2. With a pinching involving the traceless Ricci tensor

Let r_1 be the lowest eigenvalue of the traceless Ricci tensor defined by

$$\mathring{Ric}_g := \mathrm{Ric}_g - \frac{1}{3} \mathrm{Scal}_g\, g.$$

We always have

$$r_1^2 \leq \frac{2}{3} \|\mathring{Ric}_g\|^2$$

where $\|\mathring{Ric}_g\|$ is the Hilbert-Schmidt norm of the traceless Ricci tensor, and equality occurs only when the spectrum of $\mathring{Ric}_g$ is r_1 and $-\frac{r_1}{2}$ with multiplicity two.

Then we have

$$\Box_g = \Delta_g + \frac{1}{6}\operatorname{Scal}_g + \frac{1}{2} r_1.$$

We introduce the lowest eigenvalue $\mu(g)$ of the operator

$$4\Delta_g + \operatorname{Scal}_g \ .$$

This quantity has the remarkable property of being non-increasing along the Ricci flow (see [30]). Moreover if $\mu(g) \geq 0$ then $Y(M,[g]) \geq 0$.

Proposition 5.3. *If (M^3, g) is a closed Riemannian manifold with non-negative Yamabe constant and whose traceless Ricci tensor satisfies*

$$\|r_1\|_{L^3} \leq \frac{1}{3}\sqrt{Y(M,[g])\,\mu(g)} \qquad \textit{or} \qquad \|\mathring{Ric}_g\|_{L^3} \leq \frac{1}{\sqrt{6}}\sqrt{Y(M,[g])\,\mu(g)},$$

then the operator $\Box_g$ is non-negative.

Moreover, if a function $\varphi \neq 0$ satisfies $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g = 0$, then it is constant and the metric g is a Yamabe minimizer and (M^3, g) has non-negative Ricci curvature.

Proof. For any $\varphi \in C^\infty(M)$ we have by Hölder inequality

$$\begin{aligned} &Y(M,[g])\,\mu(g)\,\|\varphi\|^4_{L^3} \leq Y(M,[g])\,\|\varphi\|^2_{L^6}\,\mu(g)\,\|\varphi\|^2_{L^2} \\ &\leq \left(\int_M \left[8|d\varphi|^2 + \operatorname{Scal}_g \varphi^2\right] dv_g\right)\left(\int_M \left[4|d\varphi|^2 + \operatorname{Scal}_g \varphi^2\right] dv_g\right) \end{aligned} \tag{5.3}$$

Then, by using the inequality $\sqrt{AB} \leq \frac{1}{2}(A+B)$, we obtain the following Sobolev inequality : for all φ in $C^\infty(M)$,

$$\begin{aligned} &\sqrt{Y(M,[g])\,\mu(g)}\,\|\varphi\|^2_{L^3} \\ &\leq \left(\int_M \left[8|d\varphi|^2 + \operatorname{Scal}_g \varphi^2\right] dv_g\right)\left(\int_M \left[4|d\varphi|^2 + \operatorname{Scal}_g \varphi^2\right] dv_g\right) \\ &\leq \int_M \left[6|d\varphi|^2 + \operatorname{Scal}_g \varphi^2\right] dv_g. \end{aligned} \tag{5.4}$$

Finally, since $\rho_1 = r_1 + \frac{1}{3}\operatorname{Scal}_g$, we obtain

$$\begin{aligned} 6\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g &\geq \int_M \left(6|d\varphi|^2 + \operatorname{Scal}_g \varphi^2 - 3r_1\varphi^2\right) dv_g \\ &\geq \left(\sqrt{Y(M,[g])\,\mu(g)} - 3\,\|r_1\|_{L^3}\right) \|\varphi\|^2_{L^3}\,, \end{aligned}$$

If there exists φ such that $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g = 0$, then equality must be attained in inequalities (5.3) and (5.4). Since the equality $\sqrt{AB} = \frac{1}{2}(A+B)$ is only attained when $A = B$, it follows that φ must be constant, and equality in (5.3) implies that equality is attained in (5.1) for a constant function, hence g is a Yamabe minimizer.

Finally $0 = \langle \Box_g\varphi, \varphi\rangle = \int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right) dv_g$ and $\Box_g$ is non-negative, φ is an eigenfunction of $\Box_g$.Therefore we have $0 = \Box_g\varphi = \Delta_g\varphi + \frac{1}{2}\rho_1\varphi = \frac{1}{2}\rho_1\varphi$, hence $\rho_1 = 0$.It follows that (M^3, g) has non-negative Ricci curvature. □

According to Theorem 4.3, we obtain

Theorem 5.4. *Let (M^3, g) be a closed Riemannian manifold with non-negative Yamabe constant. If the traceless Ricci tensor satisfies*

$$\|r_1\|_{L^3} \leq \frac{1}{3}\sqrt{Y(M,[g])\,\mu(g)} \qquad \text{or} \qquad \|\mathring{Ric}_g\|_{L^3} \leq \frac{1}{\sqrt{6}}\sqrt{Y(M,[g])\,\mu(g)},$$

then

- *either (M^3, g) is a flat Riemannian manifold,*
- *or M^3 is diffeomorphic to a spherical space form $M \simeq \mathbb{S}^3/\Gamma$,*
- *or (M^3, g) is isometric to a compact quotient of the Riemannian product $\mathbb{R} \times \mathbb{S}^2$ where $\mathbb{S}^2$ is endowed with a round metric.*

Proof. The third case corresponds to the equality case in Proposition 3.5. In that case, the function $u = \sqrt{|\xi|} = \eta^{-1}$ satisfies $\int_M \left(|du|^2 + \frac{1}{2}\rho_1 u^2\right) dv_g = 0$. According to Proposition 5.3, g is a Yamabe minimizer and the function u is constant.

As it was the case in the proof of Theorem 5.2, it implies that $(\mathbb{S}^2, h)$ has constant Gaussian curvature, and it implies that the metric on $\mathbb{R} \times \mathbb{S}^2$ is a Riemannian product. □

5.3. When the scalar curvature is non-negative

We denote by κ_g the minimum of the scalar curvature:

$$\kappa_g = \min_{x\in M} \mathrm{Scal}_g(x).$$

Proposition 5.5. *If (M^3, g) is a closed Riemannian manifold with non-negative scalar curvature whose traceless Ricci tensor satisfies*

$$\|r_1\|_{L^2} \leq \frac{1}{3} Y^{\frac{3}{4}}(M,[g])\,\kappa_g^{\frac{1}{4}} \qquad or \qquad \|\mathring{Ric}_g\|_{L^2} \leq \frac{1}{\sqrt{6}} Y^{\frac{3}{4}}(M,[g])\,\kappa_g^{\frac{1}{4}}$$

then the operator $\Box_g$ is non-negative.

Moreover, if a function $\varphi \neq 0$ satisfies $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right)dv_g = 0$, then it is constant and the metric g is a Yamabe minimizer and (M^3, g) has non-negative Ricci curvature.

Proof. By using the same idea we have

$$\begin{aligned} Y^{\frac{3}{4}}(M,[g])\,\kappa_g^{\frac{1}{4}}\,\|\varphi\|_{L^4}^2 &\leq \left(Y(M,[g])\,\|\varphi\|_{L^6}^2\right)^{\frac{3}{4}} \left(\kappa_g\,\|\varphi\|_{L^2}^2\right)^{\frac{1}{4}} \\ &\leq \left(Y(M,[g])\,\|\varphi\|_{L^6}^2\right)^{\frac{3}{4}} \left(\int_M \mathrm{Scal}_g\,\varphi^2 dv_g\right)^{\frac{1}{4}} \\ &\leq \left(\int_M \left[8|d\varphi|^2 + \mathrm{Scal}_g\,\varphi^2\right] dv_g\right)^{\frac{3}{4}} \left(\int_M \mathrm{Scal}_g\,\varphi^2 dv_g\right)^{\frac{1}{4}} \end{aligned}$$

Then, using the inequality $A^{\frac{3}{4}}B^{\frac{1}{4}} \leq \frac{3}{4}A + \frac{1}{4}B$, we obtain

$$\forall \varphi \in C^\infty(M), \quad Y^{\frac{3}{4}}(M,[g])\,\kappa_g^{\frac{1}{4}}\,\|\varphi\|_{L^4}^2 \leq \int_M \left[6|d\varphi|^2 + \mathrm{Scal}_g\,\varphi^2\right] dv_g.$$

Finally, we have

$$\begin{aligned} 6\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2 dv_g\right) &\geq \int_M \left(6|d\varphi|^2 + \mathrm{Scal}_g\,\varphi^2 - 3r_1\varphi^2\right) dv_g \\ &\geq \left(Y^{\frac{3}{4}}(M,[g])\,\kappa_g^{\frac{1}{4}} - 3\,\|r_1\|_{L^2}\right)\,\|\varphi\|_{L^3}^2\,, \end{aligned}$$

If there exists φ such that $\int_M \left(|d\varphi|^2 + \frac{1}{2}\rho_1\varphi^2\right)dv_g = 0$, then $A^{\frac{3}{4}}B^{\frac{1}{4}} = \frac{3}{4}A + \frac{1}{4}B$, hence $A = B$ and φ is constant, and equality in (5.1) implies that g is a Yamabe minimizer. $\Box$

In the same way as we proved Theorem 5.4, we obtain:

Theorem 5.6. *Let* (M^3, g) *be a closed Riemannian manifold with nonnegative scalar curvature and let* $\kappa = \min_{x\in M} \mathrm{Scal}_g(x)$. *If the traceless Ricci tensor satisfies*

$$\|r_1\|_{L^2} \leq \frac{1}{3} Y^{\frac{3}{4}}(M,[g])\,\kappa^{\frac{1}{4}} \qquad or \qquad \|\mathring{Ric}_g\|_{L^2} \leq \frac{1}{\sqrt{6}} Y^{\frac{3}{4}}(M,[g])\,\kappa^{\frac{1}{4}}$$

then

- *either* (M^3, g) *is a flat Riemannian manifold,*
- *or* M^3 *is diffeomorphic to a spherical space form* $M \simeq \mathbb{S}^3/\Gamma$,
- *or* (M^3, g) *is isometric to a compact quotient of the Riemannian product* $\mathbb{R} \times \mathbb{S}^2$ *where* $\mathbb{S}^2$ *is endowed with a round metric.*

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